ARCHIVES OF MECHANICS
Arch. Mech. **77** (1), 3–28, 2025, DOI: 10.24423/aom.4544

Potential method in the theory of Moore–Gibson–Thompson thermoporoelasticity

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In the present paper the linear theory of Moore–Gibson–Thompson thermoporoelasticity is considered and the basic boundary value problems (BVPs) of steady vibrations are investigated. Namely, the fundamental solution of the system of steady vibration equations are constructed explicitly by elementary functions and its basic properties are established. The formula of integral representation of regular vectors is obtained. The surface and volume potentials are introduced and their basic properties are given. Then, some helpful singular integral operators are defined for which the symbolic determinants and indexes are calculated. The BVPs of steady vibrations are reduced to the equivalent singular integral equations. Finally, the existence theorems for classical solutions of the aforementioned BVPs are proved with the help of the potential method and the theory of singular integral equations.

Key words: thermoporoelasticity, fundamental solution, steady vibrations, existence theorems, potential method.

1. Introduction

From the 60s of the last century, new theories of thermoelasticity based on the non-Fourier law began to be proposed and intensively studied. In particular, because according to Fourier's classical law, the heat propagation in the medium occurs at an infinite speed, which is a paradox from the physical point of view, Cattaneo [1, 2] and Vernotte [3] presented the heat propagation law by introducing a positive relaxation parameter. On the basis of this law, Lord and Shulman [4] developed the theory of thermoelasticity (LS thermoelasticity), which is a generalization of Biot's [5] classical theory of thermoelasticity based on the Fourier law.

Furthermore, various theories of generalized thermoelasticity were constructed, among which the three theories of Green and Naghdi [6–8] are noteworthy. These theories are based on an entropy-balance equation and in which the thermal-displacement variable is introduced. The linear version of the first of these theories is identical to the classical theory of thermoelasticity. The second theory (GN type II thermoelasticity) is specific and does not involve energy dissipation, while the third one (GN type III thermoelasticity) proposes a more general theory. A wide historical information on the non-Fourier law of heat conduction and the basic results obtained in the generalised thermoelasticity theories are given in the books by Ignaczak and Ostoja-Starzewski [9], Straughan [10] and in the review papers by Hetnarski and Ignaczak [11], Chandrasekharaiah [12, 13], Joseph and Preziosi [14, 15].

Latterly, Quintanilla [16] developed a new theory of thermoelasticity based on the Moore–Gibson–Thompson [17, 18] law of heat propagation (MGT thermoelasticy), which turned out to be more general than the aforementioned thermoelasticity theories. For this reason, the theory of MGT thermoelasticity has attracted a lot of attention from scientists, and various interesting and practical problems of this theory are currently being investigated. The main results obtained in this direction are presented in the series of papers by Bazarra *et al.* [19], Florea and Bobe [20], Jangid and Mukhopadhyay [21], Marin *et al.* [22], Ostoja-Starzewski and Quintanilla [23], Quintanilla [24], Singh and Mukhopadhyay [25], and the references therein.

On the other hand, since the 40s of the last century, the creation of theories of porous bodies and their intensive research began. Namely, in the works of Biot [26, 27], two different theories of poroelasticity are presented. In the first of which (see [26]), the governing quasi-static equations are written with respect to the displacement vector and the change of fluid pressure in the pores, and in the second (see [27]) the dynamic equations of poroelasticity are written with respect to the displacement vectors for the skeleton of porous material and fluid in the pore network. Nowadays, these Biot's poroelasticity theories have been generalized to account for various mechanical effects and the structure of multi-porosity materials. Among them are theories of thermoporoelasticity (or porothermoelasticity) based on Fourier's law. The main results obtained in these theories and an extensive review of the literature are presented in the monographs by Cheng [28], Ichikawa and Selvadurai [29], Selvadurai and Suvorov [30], Straughan [31, 32], Svanadze [33] and Wang [34].

Recently, Svanadze [35] has proposed the linear model of MGT thermoporoelasticity and the governing equations of motion and steady vibrations are introduced. On the one hand, this model is a natural extension of MGT thermoelasticity (see [16]) for porous materials based on Darcy's law of fluid flow, and on the other hand, it is also a natural generalization of Biot's [26] poroelasticity theory using the MGT law of heat conduction. In addition, in the paper [35], the uniqueness theorems for classical solutions of the BVPs of steady vibrations in the theory of MGT thermoporoelaticity are proved.

The aim of this paper is to prove the existence theorems for the classical solutions of the basic BVPs of steady vibrations in the MGT thermoporoelasticity theory [35] using the potential method (boundary integral equations method).

This work is articulated as follows. In Section 2, the governing field equations of steady vibrations of the MGT thermoporoelasticity theory are given. The different specific cases of the constitutive parameters are discussed and the systems of steady vibrations in the following 5 theories are established: the classical thermoporoelasticity, Lord–Shulman thermoporoelasticity, Green–Naghdi types II and III thermoporoelasticity, and Moore–Gibson–Thompson thermoporoelasticity. In Section 3, the fundamental solution of the system of steady vibration equations is constructed explicitly by means of elementary functions and its basic properties are established. In Sections 4 and 5, the basic internal and external BVPs are formulated and the formula of integral representation of regular vectors is obtained. In Section 6, the surface and volume potentials are introduced and their basic properties are given. Then, some helpful singular integral operators are defined for which the symbolic determinants and indexes are calculated. In Section 7, the BVPs of steady vibrations are reduced to the equivalent singular integral equations and the existence theorems for classical solutions of these BVPs in the foregoing 5 theories of thermoporoelasticity are proved with the help of the potential method and the theory of singular integral equations. The paper ends with a conclusion section. In addition, since the proof of existence theorems requires uniqueness theorems, the paper is accompanied by Appendix, where these theorems are given.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$. In what follows we consider a porous material that occupies the region Ω of $\mathbb{R}^3$, the skeleton of which is an isotropic and homogeneous elastic solid, and the pores are filled with a fluid.

We assume that repeated indices are summed over the range $(1, 2, 3)$ and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. Subsequently, vectors and matrices are marked with bold letters.

Let $\mathbf{u}(\mathbf{x})$ be the displacement vector in solid, $\mathbf{u} = (u_1, u_2, u_3)$, $p(\mathbf{x})$ is the change of the fluid pressure from the reference configuration, $\theta(\mathbf{x})$ is the temperature measured from some constant absolute temperature T_0 (> 0), and $\vartheta(\mathbf{x})$ is the thermal displacement variable.

Following Svanadze [35], the governing system of field equations of steady vibrations in the theory of MGT thermoporoelasticity is composed of the next six sets of equations:

1. *Constitutive equations*:

$$
\begin{aligned}
t_{lj} &= 2\mu e_{lj} + \lambda e_{rr}\delta_{lj} - (\beta p + \varepsilon\theta)\delta_{lj},\\
\rho\eta &= c\theta + \varepsilon e_{rr} + \gamma p, \qquad l,j = 1,2,3,
\end{aligned}
\tag{2.1}
$$

where t_{lj} is the component of the total stress tensor, ρ (>0) is the reference mass density of the skeleton, η is the entropy per unit mass, λ and μ are the Lamé constants, β is the effective stress parameter, ε is the thermal expansion coefficient, c (>0) is the thermal capacity, γ is the constitutive thermal constant of the porous material, δ_{lj} is the Kronecker delta, e_{lj} is the component of the strain tensor and is defined by

$$
e_{lj} = \tfrac{1}{2}(u_{l,j} + u_{j,l}). \tag{2.2}
$$

2. *Equation of steady vibrations*:

$$
t_{lj,j} + \rho\omega^2 u_l = -\rho\mathcal{F}_l, \qquad l = 1,2,3, \tag{2.3}
$$

where $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is the body force per unit mass and $\omega(>0)$ is the oscillation frequency.

3. *Equation of fluid mass conservation*:

$$
v_{j,j} - i\omega(\alpha p + \beta e_{jj} + \gamma\theta) = 0, \tag{2.4}
$$

where i is the imaginary unit, $\mathbf{v} = (v_1, v_2, v_3)$ and α are the fluid flux vector and the compressibility of the pore network, respectively.

4. *Darcy's law*:

$$
\mathbf{v} = -\frac{\kappa_0}{\mu'}\nabla p - \rho_1\mathbf{s}', \tag{2.5}
$$

where μ' is the fluid viscosity, κ_0 is the macroscopic intrinsic permeability associated with the pore network, ρ_1 (>0) is the density of fluid, $\mathbf{s}' = (s_1', s_2', s_3')$ is the external force (such as gravity) for the pore network, ∇ is the gradient operator.

5. *Heat transfer equation*:

$$
\operatorname{div}\mathbf{q} = i\omega T_0\rho\eta - \rho s, \tag{2.6}
$$

where $\mathbf{q} = (q_1, q_2, q_3)$ is the heat flux vector and s is the heat source.

6. *MGT law of heat conduction*:

In the steady vibrations case, we can write this law in the following form:

$$
(1 - i\omega\tau)q_l = -(k^*\vartheta_{,l} + k\theta_{,l}), \qquad l = 1,2,3, \tag{2.7}
$$

where k^* (≥ 0) is the conductivity rate parameter, k (≥ 0) is the thermal conductivity, and τ (≥ 0) is the relaxation parameter. Moreover, if we take into account the following relationship between heat displacement and temperature change

$$-i\omega\vartheta = \theta,$$

we get from (2.7)

$$(\omega^2\tau + i\omega)q_l = k_0\theta_{,l}, \tag{2.8}$$

where $k_0 = k^* - i\omega k$.

Substituting Eqs. (2.1), (2.2), (2.5) and (2.8) into (2.3), (2.4) and (2.6), we obtain the following system of equations of steady vibrations in the linear theory of MGT thermoporoelasticity expressed in terms of the displacement vector field $\mathbf{u}$, the change of pressure p of fluid in the pore network and the change of temperature θ of the porous material [35]:

$$\begin{aligned}
&(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} - \beta\nabla p - \varepsilon\nabla\theta = -\rho\boldsymbol{\mathcal{F}},\\
&(\kappa\Delta + \alpha')p + \beta'\operatorname{div}\mathbf{u} + \gamma'\theta = -\rho_1\operatorname{div}\mathbf{s}',\\
&(k_0\Delta + mc)\theta + m\varepsilon\operatorname{div}\mathbf{u} + m\gamma p = (\omega^2\tau + i\omega)\rho s,
\end{aligned} \tag{2.9}$$

where Δ is the Laplacian operator, $\kappa = \frac{\kappa_0}{\mu'}(>0)$, the physical constant κ is called the coefficient of permeability of the porous material (see Biot [26]), and

$$\alpha' = i\omega\alpha, \qquad \beta' = i\omega\beta, \qquad \gamma' = i\omega\gamma, \qquad m = T_0\omega^2(1 - i\omega\tau).$$

Now let us consider those special cases of parameters k, k^* and τ that give us different thermoporoelasticity theories. Obviously, by changing these parameters, only the last equation of the system (2.9) will change. We have the following 5 cases:

1. If $k^* = \tau = 0$ and $k > 0$, then from the last equation of (2.9) it follows that

$$k\Delta\theta + i\omega T_0(c\theta + \varepsilon\operatorname{div}\mathbf{u} + \gamma p) = -\rho s.$$

Consequently, in this case from (2.9) we get the system of equations of thermoporoelasticity based on the Fourier classical law of heat conduction. That is why we call this theory as *the classical thermoporoelasticity*.

2. If $k^* = 0$, $k > 0$ and $\tau > 0$, then the last equation of (2.9) is replaced by

$$k\Delta\theta + i\omega T_0(1 - i\omega\tau)(c\theta + \varepsilon\operatorname{div}\mathbf{u} + \gamma p) = (i\omega\tau - 1)\rho s.$$

Clearly, we get the system of equations of thermoporoelasticity based on the Cattaneo–Vernotte law of heat conduction (see [1–3]). Obviously, this system of equations is the extension of Lord–Shulman [4] equations of thermoelasticity for porous materials. Because of this we call this theory as *the LS thermoporoelasticity*.

3. If $k^* > 0$ and $k = \tau k^*$, then the last equation of (2.9) now reduces to

$$k^*\Delta\theta + m_1(c\theta + \varepsilon \operatorname{div} \mathbf{u} + \gamma p) = -i\omega\rho s,$$

where $m_1 = T_0\omega^2$. In this case from (2.9) we obtain the system of equations of thermoporoelasticity based on the Green–Naghdi type II heat conduction. We call this theory as *the GN type II thermoporoelasticity.*

4. If $\tau = 0$, $k > 0$ and $k^* > 0$, then the last equation of (2.9) can be expressed as

$$k_0\Delta\theta + m_1(c\theta + \varepsilon \operatorname{div} \mathbf{u} + \gamma p) = i\omega\rho s.$$

In this case, we have the system of equations of thermoporoelasticity based on the Green–Naghdi type III heat conduction. That's why we call this theory as *the GN type III thermoporoelasticity.*

5. If $k^* > 0$, $\tau > 0$, and $k - \tau k^* > 0$, then we have the system (2.9) of *the MGT thermoporoelasticity.*

The purpose of this article is to prove the existence theorems of the classical solutions of the basic BVPs of steady vibrations for the foregoing 5 theories of thermoporoelasticity using the potentials method. For this, it is necessary: (a) to construct the fundamental solution of the system (2.9) and the potentials, and to determine their properties; (b) to reduce the BVPs to the corresponding singular integral equations, and (c) to establish the solvability of these equations. These issues are studied in the next sections.

3. Fundamental solution

In this section the fundamental solution of the system (2.9) is constructed explicitly by means of elementary functions and its basic properties are established.

In what follows we assume that $\mu > 0$, $3\lambda + 2\mu > 0$, $k_0 \neq 0$. We introduce the matrix differential operator $\mathbf{A}(\mathbf{D}_\mathbf{x}) = (A_{lj}(\mathbf{D}_\mathbf{x}))_{5\times 5}$, where:

$$\mathbf{A}(\mathbf{D}_\mathbf{x}) = (A_{lj}(\mathbf{D}_\mathbf{x}))_{5\times 5}, \qquad A_{lj}(\mathbf{D}_\mathbf{x}) = (\mu\Delta + \rho\omega^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l4}(\mathbf{D}_\mathbf{x}) = -\beta\frac{\partial}{\partial x_l}, \qquad A_{l5}(\mathbf{D}_\mathbf{x}) = -\varepsilon\frac{\partial}{\partial x_l}, \qquad A_{4l}(\mathbf{D}_\mathbf{x}) = \beta'\frac{\partial}{\partial x_l},$$

$$A_{44}(\mathbf{D}_\mathbf{x}) = \kappa\Delta + \alpha', \quad A_{45}(\mathbf{D}_\mathbf{x}) = \gamma', \quad A_{5l}(\mathbf{D}_\mathbf{x}) = m\varepsilon\frac{\partial}{\partial x_l}, \quad A_{54}(\mathbf{D}_\mathbf{x}) = m\gamma,$$

$$A_{55}(\mathbf{D}_\mathbf{x}) = k_0\Delta + mc, \qquad \mathbf{D}_\mathbf{x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad l, j = 1, 2, 3.$$

Obviously, we can rewrite the system (2.9) in the form

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\,\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{3.1}$$

where $\mathbf{U} = (\mathbf{u}, p, \theta)$ and $\mathbf{F} = (-\rho\boldsymbol{\mathcal{F}}, -\rho_1 \operatorname{div} \mathbf{s}', (\omega^2\tau + i\omega)\rho s)$ are five-component vector functions, and $\mathbf{x} \in \Omega$.

DEFINITION 1. The fundamental solution of the system (2.9) (the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$) is the matrix $\boldsymbol{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{5\times5}$ satisfying the equation

$$\mathbf{A}(\mathbf{D_x})\boldsymbol{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}$$

in the class of generalized functions, where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{5\times5}$ is the unit matrix, and $\mathbf{x} \in \mathbb{R}^3$.

Let $\mathbf{B}(\mathbf{D_x})$ be the following matrix differential operator

$$\mathbf{B}(\mathbf{D_x}) = (B_{lj}(\mathbf{D_x}))_{3\times3} = \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & -\beta\Delta & -\varepsilon\Delta \\ \beta' & \kappa\Delta + \alpha' & \gamma' \\ m\varepsilon & m\gamma & k_0\Delta + mc \end{pmatrix}_{3\times3},$$

where $\mu_0 = \lambda + 2\mu$. We introduce the notation

$$\Lambda_1(\Delta) = \frac{1}{\mu_0\kappa k_0} \det \mathbf{B}(\Delta) = \prod_{j=1}^{3}(\Delta + \zeta_j^2),$$

where ζ_1^2, ζ_2^2 and ζ_3^2 are the roots of the equation $\Lambda_1(-\xi) = 0$ (with respect to ξ).

We assume that the values ζ_1, ζ_2, ζ_3 and ζ_4 are distinct, $\mathrm{Im}\zeta_l > 0$ for negative or complex number ζ_l^2 and $\zeta_l > 0$ for $\zeta_l^2 > 0$ $(l = 1, 2, 3)$. Here $\zeta_4 = \sqrt{\rho\omega^2\mu^{-1}}$.

Now we introduce the matrix differential operator $\mathbf{L}(\mathbf{D_x}) = (L_{lj}(\mathbf{D_x}))_{5\times5}$ as follows:

$$\begin{aligned} &L_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\Lambda_1(\Delta)\,\delta_{lj} + n_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j}, \quad L_{lr}(\mathbf{D_x}) = n_{1;r-2}(\Delta)\frac{\partial}{\partial x_l}, \\ &L_{rl}(\mathbf{D_x}) = n_{r-2;1}(\Delta)\frac{\partial}{\partial x_l}, \qquad L_{r4}(\mathbf{D_x}) = n_{r-2;2}(\Delta), \\ &L_{r5}(\mathbf{D_x}) = n_{r-2;3}(\Delta), \qquad l, j = 1, 2, 3,\ r = 4, 5, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} n_{l1}(\Delta) &= -\frac{1}{\mu\mu_0\kappa k_0}[(\lambda + \mu)B_{l1}^*(\Delta) + \beta' B_{l2}^*(\Delta) + m\varepsilon B_{l3}^*(\Delta)], \\ n_{lj}(\Delta) &= \frac{1}{\mu_0\kappa k_0} B_{lj}^*(\Delta), \qquad l = 1, 2, 3,\ j = 2, 3, \end{aligned}$$

and B_{lj}^* is the cofactor of the element B_{lj} of the matrix $\mathbf{B}$.

It is not difficult to prove the following identity by direct calculations

$$\mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x}) = \boldsymbol{\Lambda}(\Delta), \tag{3.3}$$

where:

$$\begin{aligned}\mathbf{\Lambda}(\Delta) &= (\Lambda_{lj}(\Delta))_{5\times 5},\\ \Lambda_{11}(\Delta) = \Lambda_{22}(\Delta) &= \Lambda_{33}(\Delta) = \Lambda_1(\Delta)(\Delta+\zeta_4^2),\\ \Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) &= \Lambda_1(\Delta),\\ \Lambda_{lj}(\Delta) &= 0, \qquad l\neq j,\ l,j=1,2,\dots,5.\end{aligned}$$

In our further analysis we need the matrix $\mathbf{Y}(\mathbf{x}) = (Y_{lr}(\mathbf{x}))_{5\times 5}$, where:

$$\begin{aligned}Y_{11}(\mathbf{x}) = Y_{22}(\mathbf{x}) = Y_{33}(\mathbf{x}) &= \sum_{j=1}^{4}\eta_{2j}\gamma^{(j)}(\mathbf{x}),\\ Y_{44}(\mathbf{x}) = Y_{55}(\mathbf{x}) &= \sum_{j=1}^{3}\eta_{1j}\gamma^{(j)}(\mathbf{x}),\\ Y_{lr}(\mathbf{x}) &= 0, \qquad l\neq r,\ l,r=1,2,\dots,5.\end{aligned}\tag{3.4}$$

Here we used the following notation

$$\gamma^{(r)}(\mathbf{x}) = -\frac{e^{i\zeta_r|\mathbf{x}|}}{4\pi|\mathbf{x}|}\tag{3.5}$$

and

$$\eta_{1j} = \prod_{l=1,\, l\neq j}^{3}(\zeta_l^2-\zeta_j^2)^{-1}, \quad \eta_{2r} = \prod_{l=1,\, l\neq r}^{4}(\zeta_l^2-\zeta_r^2)^{-1}, \quad j=1,2,3,\ r=1,2,3,4.$$

Obviously, $\mathbf{Y}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{\Lambda}(\Delta)$, i.e.,

$$\mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.\tag{3.6}$$

Let us introduce the notation

$$\mathbf{\Gamma}(\mathbf{x}) = \mathbf{L}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}).\tag{3.7}$$

On the basis of the relations (3.3) and (3.6) we get

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) = \mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Hence, $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$. Consequently, we have the following theorem.

Theorem 1. *The matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is the fundamental solution of the system* (2.9), *where the matrices* $\mathbf{L}(\mathbf{D_x})$ *and* $\mathbf{Y}(\mathbf{x})$ *are given by* (3.2) *and* (3.4), *respectively.*

Clearly, the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed explicitly by 4 metaharmonic functions $\gamma^{(r)}$ $(r=1,2,3,4)$ (see (3.5)).

Theorem 1 leads to the following basic properties of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$.

Theorem 2. *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of the homogeneous equation* $\mathbf{A}(\mathbf{D}_\mathbf{x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0}$ *at every point except the origin of* $\mathbb{R}^3$.

Theorem 3. *The fundamental solution of the system*

$$\begin{aligned}&\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\nabla\operatorname{div}\mathbf{u} = \mathbf{0},\\ &\kappa\Delta p = 0, \qquad k_0\Delta\theta = 0\end{aligned}$$

is the matrix $\mathbf{\Psi}(\mathbf{x}) = (\Psi_{lj}(\mathbf{x}))_{5\times 5}$, *where:*

$$\begin{aligned}&\Psi_{lj}(\mathbf{x}) = \lambda'\,\frac{\delta_{lj}}{|\mathbf{x}|} + \mu'\,\frac{x_l x_j}{|\mathbf{x}|^3}, \quad \Psi_{44}(\mathbf{x}) = \frac{1}{\kappa}\gamma^{(0)}(\mathbf{x}), \quad \Psi_{55}(\mathbf{x}) = \frac{1}{k_0}\gamma^{(0)}(\mathbf{x}),\\ &\Psi_{lr}(\mathbf{x}) = \Psi_{rl}(\mathbf{x}) = \Psi_{45}(\mathbf{x}) = \Psi_{54}(\mathbf{x}) = 0,\\ &\gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \quad \lambda' = -\frac{\lambda+3\mu}{8\pi\mu\mu_0}, \quad \mu' = -\frac{\lambda+\mu}{8\pi\mu\mu_0}, \qquad l,j = 1,2,3,\ r = 4,5.\end{aligned}$$

Obviously, Theorems 1 and 3 imply the following results.

Theorem 4. *The relations*

$$\Psi_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \Psi_{44}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \Psi_{55}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \tag{3.8}$$

and

$$\begin{aligned}&\Gamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \Gamma_{44}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \Gamma_{55}(\mathbf{x}) = O(|\mathbf{x}|^{-1}),\\ &\Gamma_{lr}(\mathbf{x}) = O(1), \quad \Gamma_{rl}(\mathbf{x}) = O(1), \quad \Gamma_{45}(\mathbf{x}) = O(1), \quad \Gamma_{54}(\mathbf{x}) = O(1)\end{aligned}$$

hold in the neighborhood of the origin of $\mathbb{R}^3$, *where* $l,j = 1,2,3$, $r = 4,5$.

Theorem 5. *The relations*

$$\Gamma_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|) \tag{3.9}$$

hold in the neighborhood of the origin of $\mathbb{R}^3$, *where* $l,j = 1,2,\ldots,5$.

Thus, employing (3.8) and (3.9), the matrix $\mathbf{\Psi}\,(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}\,(\mathbf{x})$ in the neighborhood of the origin.

4. Boundary value problems

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{2,\nu}$, $0 < \nu \le 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$; $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to S at $\mathbf{z}$.

DEFINITION 2 (see [35]). A vector function $\mathbf{U} = (\mathbf{u}, p, \theta) = (U_1, U_2, \cdots, U_5)$ is called *regular* in Ω^- (or Ω^+) if

(i) $U_l \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ (or $U_l \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$),

(ii) $U_l = \sum_{j=1}^4 U_l^{(j)}, \quad U_l^{(j)} \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$,

(iii) $(\Delta + \zeta_j^2) U_l^{(j)}(\mathbf{x}) = 0$ and

$$\left(\frac{\partial}{\partial |\mathbf{x}|} - i\zeta_j \right) U_l^{(j)}(\mathbf{x}) = e^{i\zeta_j |\mathbf{x}|} o(|\mathbf{x}|^{-1}) \quad \text{for } |\mathbf{x}| \gg 1, \tag{4.1}$$

where $U_r^{(4)} = 0$, $j = 1, 2, 3, 4$, $l = 1, 2, \ldots, 5$, $r = 4, 5$.

Obviously, the relation (4.1) implies (for details seeVEKUA [36])

$$U_l^{(j)}(\mathbf{x}) = e^{i\zeta_j |\mathbf{x}|} O(|\mathbf{x}|^{-1}) \quad \text{for } |\mathbf{x}| \gg 1, \tag{4.2}$$

where $j = 1, 2, 3, 4$, $l = 1, 2, \ldots, 5$.

Relations (4.1) and (4.2) are the radiation conditions in the theory of MGT thermoporoelasticity.

In the sequel, we use the matrix differential operator

$$\mathbf{R}(\mathbf{D_x}, \mathbf{n}) = (R_{lj}(\mathbf{D_x}, \mathbf{n}))_{5\times 5},$$

where

$$\begin{aligned}
&R_{lj}(\mathbf{D_x}, \mathbf{n}) = \mu \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}, \qquad R_{l4}(\mathbf{D_x}, \mathbf{n}) = -\beta n_l, \\
&R_{l5}(\mathbf{D_x}, \mathbf{n}) = -\varepsilon n_l, \qquad R_{44}(\mathbf{D_x}, \mathbf{n}) = \kappa \frac{\partial}{\partial \mathbf{n}}, \qquad R_{55}(\mathbf{D_x}, \mathbf{n}) = k_0 \frac{\partial}{\partial \mathbf{n}}, \\
&R_{4j}(\mathbf{D_x}, \mathbf{n}) = R_{45}(\mathbf{D_x}, \mathbf{n}) = R_{5j}(\mathbf{D_x}, \mathbf{n}) = R_{54}(\mathbf{D_x}, \mathbf{n}) = 0, \quad l, j = 1, 2, 3,
\end{aligned} \tag{4.3}$$

and $\partial/\partial\mathbf{n}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the linear theory of MGT thermoporoelasticity are formulated as follows.

Find a regular (classical) solution to (3.1) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{4.4}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{4.5}$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed five-component vector functions.

Find a regular (classical) solution to (3.1) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{4.6}$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{4.7}$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed five-component vector functions, $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

The uniqueness theorems for classical solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ in the aforementioned 5 theories of thermoporoelasticity are proved by SVANADZE [35]. We need these theorems in the proof of existence theorems (see Section 6), which is why they are given in Appendix.

5. Integral representation of regular vectors

In this section the formula of the integral representation of regular five-component vector functions is established which help us to determine the structure of surface and volume potentials.

Let $\tilde{\mathbf{A}}(\mathbf{D_x}) = (\tilde{A}_{lj}(\mathbf{D_x}))_{5\times 5}$ be the associate operator of $\mathbf{A}(\mathbf{D_x})$, i.e. $\tilde{\mathbf{A}}(\mathbf{D_x}) = \mathbf{A}^\top(-\mathbf{D_x})$, where $\mathbf{A}^\top$ is the transpose of the matrix $\mathbf{A}$. Let $\mathbf{U} = (\mathbf{u}, p, \theta) = (U_1, U_2, \dots, U_5)$, the vector $\tilde{\mathbf{U}}_j$ is the j-th column of the matrix $\tilde{\mathbf{U}} = (\tilde{U}_{lj})_{5\times 5}$.

It is not difficult to prove the following theorem by direct calculations.

THEOREM 6. *If* $\mathbf{U}$ *and* $\tilde{\mathbf{U}}_j$ $(j = 1, 2, \cdots, 5)$ *are regular vectors in* Ω^+, *then*

$$\begin{aligned} &\int_{\Omega^+} \left\{[\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\right\} d\mathbf{y} \\ &\qquad = \int_S \left\{[\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_{\mathbf{z}}S, \end{aligned} \tag{5.1}$$

where the operator $\mathbf{R}(\mathbf{D_z}, \mathbf{n})$ *is given by (4.3) and the operator* $\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})$ *is defined as*

$$\begin{aligned} &\tilde{\mathbf{R}}(\mathbf{D_x}, \mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D_x}, \mathbf{n}))_{5\times 5}, \quad \tilde{R}_{lj}(\mathbf{D_x}, \mathbf{n}) = R_{lj}(\mathbf{D_x}, \mathbf{n}), \\ &\tilde{R}_{l4}(\mathbf{D_x}, \mathbf{n}) = -\beta' n_l, \quad \tilde{R}_{l5}(\mathbf{D_x}, \mathbf{n}) = -m\varepsilon n_l, \quad \tilde{R}_{4r}(\mathbf{D_x}, \mathbf{n}) = R_{4r}(\mathbf{D_x}, \mathbf{n}), \\ &\tilde{R}_{5r}(\mathbf{D_x}, \mathbf{n}) = R_{5r}(\mathbf{D_x}, \mathbf{n}), \quad l, j = 1, 2, 3, \ r = 1, 2, \dots, 5. \end{aligned} \tag{5.2}$$

Theorem 6 and the radiation conditions (4.1) and (4.2) lead to the following result.

THEOREM 7. *If* $\mathbf{U}$ *and* $\tilde{\mathbf{U}}_j$ $(j = 1, 2, \ldots, 5)$ *are regular vectors in* Ω^-, *then*

$$\int\limits_{\Omega^-} \left\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \right\} d\mathbf{y} \\ = -\int\limits_S \left\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \right\} d_\mathbf{z} S. \tag{5.3}$$

The formulas (5.1) and (5.3) are Green's second identities in the theory of MGT thermoporoelasticity for domains Ω^+ and Ω^-, respectively.

Let $\tilde{\mathbf{\Gamma}}(\mathbf{x})$ be the fundamental martix of the operator $\tilde{\mathbf{A}}(\mathbf{D_x})$. Obviously, the matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x})$ satisfies the following condition

$$\tilde{\mathbf{\Gamma}}(\mathbf{x}) = \mathbf{\Gamma}^\top(-\mathbf{x}), \tag{5.4}$$

where the matrix $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$ and defined by (3.7).

THEOREM 8. *If* $\mathbf{U}$ *is a regular vector in* Ω^+, *then*

$$\mathbf{U}(\mathbf{x}) = \int\limits_S \left\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x} - \mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x} - \mathbf{z})\, \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \right\} d_\mathbf{z} S \\ + \int\limits_{\Omega^+} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) d\mathbf{y} \quad \textit{for } \mathbf{x} \in \Omega^+. \tag{5.5}$$

Proof. Let $\mathcal{B}(\mathbf{x}, \varsigma)$ and $S(\mathbf{x}, \varsigma)$ be the open ball and sphere of radius ς and center $\mathbf{x}$, respectively, where $\mathbf{x} \in \Omega^+$. Pick $\varsigma > 0$ such that $\overline{\mathcal{B}(\mathbf{x}, \varsigma)} \subset \Omega^+$. Applying the formula (5.1) in the domain $\Omega^+ \backslash \overline{\mathcal{B}(\mathbf{x}, \varsigma)}$ with $\tilde{\mathbf{U}}(\mathbf{y}) = \tilde{\mathbf{\Gamma}}(\mathbf{y} - \mathbf{x})$ we get

$$\int\limits_{\Omega^+ \backslash \overline{\mathcal{B}(\mathbf{x}, \varsigma)}} \left\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{\Gamma}}(\mathbf{y} - \mathbf{x})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{\Gamma}}(\mathbf{y} - \mathbf{x})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \right\} d\mathbf{y} \\ = \int\limits_S \left\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \right\} d_\mathbf{z} S \\ - \int\limits_{S(\mathbf{x}, \varsigma)} \left\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \right\} d_\mathbf{z} S. \tag{5.6}$$

Here, in the third integral of (5.6), $\mathbf{n}(\mathbf{z})$ is the external (with respect to $S(\mathbf{x},\varsigma)$) unit normal vector to $S(\mathbf{x},\varsigma)$ for $\mathbf{z}\in S(\mathbf{x},\varsigma)$.

Letting $\varsigma\to 0$, from (5.6) we obtain the relation (5.5) by using (5.4) and the basic properties of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$. □

In a similar way, by virtue of the radiation conditions (4.1) and (4.2), from (5.3) we obtain the following result.

Theorem 9. *If* $\mathbf{U}$ *is a regular vector in* Ω^-, *then*

$$\begin{aligned}(5.7)\qquad \mathbf{U}(\mathbf{x}) = &\\ &-\int_S \big\{[\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{z})]^\top\mathbf{U}(\mathbf{z})-\mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\,\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{U}(\mathbf{z})\big\}\,d_{\mathbf{z}}S\\ &+\int_{\Omega^-}\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\,\mathbf{A}(\mathbf{D}_{\mathbf{y}})\mathbf{U}(\mathbf{y})\,d\mathbf{y}\qquad \textit{for } \mathbf{x}\in\Omega^-.\end{aligned}$$

The formulas (5.5) and (5.7) are the integral representations of regular vectors (Green's third identities) in the theory of MGT thermoporoelasticity for domains Ω^+ and Ω^-, respectively.

6. Potentials and singular integral operators

In this section, the surface (single-layer and double-layer) and volume potentials are introduced and their basic properties are established. Then, some useful singular integral operators are studied.

The basic definitions in the theory of singular integral equations (a normal type singular integral operator, the symbol and the index of operator, Noether's theorems for the singular integral equations, etc.) are given in the books by Kupradze *et al.* [37] and Mikhlin [38].

Let us introduce the notation:

(i) $\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g})=\int_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the single-layer potential,

(ii) $\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g})=\int_S [\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the double-layer potential, and

(iii) $\mathbf{Z}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^\pm)=\int_{\Omega^\pm}\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y})d\mathbf{y}$ is the volume potential,

where $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D}_{\mathbf{x}})$, $\mathbf{g}$ and $\boldsymbol{\phi}$ are five-component vector functions, the matrix differential operator $\tilde{\mathbf{R}}$ is defined by (5.2).

It is worth noting that, in view of Theorems 8 and 9, a regular vector is represented by the foregoing three potentials as:

$$\begin{aligned}&\mathbf{U}(\mathbf{x}) = -\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{RU}) + \mathbf{Z}^{(2)}(\mathbf{x},\mathbf{U}) + \mathbf{Z}^{(3)}(\mathbf{x},\mathbf{AU},\Omega^+) \qquad \text{for } \mathbf{x}\in\Omega^+,\\ &\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x},\mathbf{RU}) - \mathbf{Z}^{(2)}(\mathbf{x},\mathbf{U}) + \mathbf{Z}^{(3)}(\mathbf{x},\mathbf{AU},\Omega^-) \qquad \text{for } \mathbf{x}\in\Omega^-.\end{aligned}$$

On the basis of the properties of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ it is not very difficult to prove the following theorems.

THEOREM 10. *If $S\in C^{r+1,\nu}$, $\mathbf{g}\in C^{r,\nu'}(S)$, $0<\nu'<\nu\le 1$, and r is a non-negative integer, then:*
(a) $\mathbf{Z}^{(1)}(\cdot,\mathbf{g})\in C^{0,\nu'}(\mathbb{R}^3)\cap C^{r+1,\nu'}(\overline{\Omega^\pm})\cap C^\infty(\Omega^\pm)$,
(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g})=\mathbf{0}$,
(c)

$$\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}), \tag{6.1}$$

(d) $\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})$
is a singular integral, where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^\pm$.

THEOREM 11. *If $S\in C^{r+1,\nu}$, $\mathbf{g}\in C^{r,\nu'}(S)$, $0<\nu'<\nu\le 1$, then:*
(a) $\mathbf{Z}^{(2)}(\cdot,\mathbf{g})\in C^{r,\nu'}(\overline{\Omega^\pm})\cap C^\infty(\Omega^\pm)$,
(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g})=\mathbf{0}$,
(c)

$$\{\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\}^\pm = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}) \tag{6.2}$$

for the non-negative integer r,
(d) $\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral, where $\mathbf{z}\in S$,*
(e) $\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\}^+ = \{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\}^-$,
for the natural number r, where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^\pm$.

THEOREM 12. *If $S\in C^{1,\nu}$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^+)$, $0<\nu'<\nu\le 1$, then:*
(a) $\mathbf{Z}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^+)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^+)\cap C^{2,\nu'}(\overline{\Omega_0^+})$,
(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^+)=\boldsymbol{\phi}(\mathbf{x})$,
where $\mathbf{x}\in\Omega^+$, Ω_0^+ is a domain in $\mathbb{R}^3$ and $\overline{\Omega_0^+}\subset\Omega^+$.

THEOREM 13. *If $S\in C^{1,\nu}$,* $\operatorname{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^-)$, $0<\nu'<\nu\le 1$, *then:*

(a) $\mathbf{Z}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^-)\cap C^{2,\nu'}(\overline{\Omega_0^-})$,

(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-)=\boldsymbol{\phi}(\mathbf{x})$,

where $\mathbf{x}\in\Omega^-$, Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^-}\subset\Omega^-$.

Let us introduce the following matrix singular integral operators:

$$
\begin{aligned}
\mathcal{K}^{(1)}\,\mathbf{g}(\mathbf{z})&\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(2)}\,\mathbf{g}(\mathbf{z})&\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z})&\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(4)}\mathbf{g}(\mathbf{z})&\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}_\chi\mathbf{g}(\mathbf{z})&\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\chi\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})
\end{aligned}
\tag{6.3}
$$

for $\mathbf{z}\in S$, where χ is a complex number. The symbol of the operator $\mathcal{K}^{(j)}$ ($j=1,2,3,4$) we denote by $\boldsymbol{\Upsilon}^{(j)}=(\Upsilon^{(j)}_{lm})_{5\times5}$. It is not difficult to obtain the following result from (6.3) (for details, see, KUPRADZE *et al.* [37], p. 357)

$$
\begin{aligned}
\det\boldsymbol{\Upsilon}^{(1)}&=-\det\boldsymbol{\Upsilon}^{(2)}=-\det\boldsymbol{\Upsilon}^{(3)}=\det\boldsymbol{\Upsilon}^{(4)}\\
&=\Big(-\frac{1}{2}\Big)^5\Big[\frac{\mu^2}{(\lambda+2\mu)^2}-1\Big]=\frac{(\lambda+\mu)(\lambda+3\mu)}{32(\lambda+2\mu)^2}>0.
\end{aligned}
\tag{6.4}
$$

Therefore, the operator $\mathcal{K}^{(j)}$ is of the normal type, where $j=1,2,3,4$.

Let $\boldsymbol{\Upsilon}_\chi$ and $\operatorname{ind}\mathcal{K}_\chi$ be the symbol and the index of the integral operator $\mathcal{K}_\chi$, respectively. It is easily verified that

$$
\det\boldsymbol{\Upsilon}_\chi=\Big(-\frac{1}{2}\Big)^5\Big[\frac{\mu^2\chi^2}{(\lambda+2\mu)^2}-1\Big]=\frac{(\lambda+2\mu)^2-\mu^2\chi^2}{32(\lambda+2\mu)^2}
$$

and $\det\boldsymbol{\Upsilon}_\chi=0$ only at two points χ_1 and χ_2 of the complex plane. In view of the relation (6.4) and $\det\boldsymbol{\Upsilon}_1=\det\boldsymbol{\Upsilon}^{(1)}$ we can write $\chi_j\neq1$ ($j=1,2$) and consequently,

$$
\operatorname{ind}\mathcal{K}_1=\operatorname{ind}\mathcal{K}^{(1)}=\operatorname{ind}\mathcal{K}_0=0.
$$

In a similar way we obtain $\operatorname{ind}\mathcal{K}^{(2)}=-\operatorname{ind}\mathcal{K}^{(3)}=0$ and $\operatorname{ind}\mathcal{K}^{(4)}=-\operatorname{ind}\mathcal{K}^{(1)}=0$.

Therefore, the operator $\mathcal{K}^{(j)}$ ($j=1,2,3,4$) is of the normal type with an index equal to zero, and consequently, we have the following result.

THEOREM 14. *Noether's theorems are valid for the singular integral operator* $\mathcal{K}^{(j)}$, *where* $j=1,2,3,4$.

7. Existence theorems

Now we are ready to prove the existence theorems of classical solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ using the potential method. Note first that by Theorems 12 and 13 the volume potential $\mathbf{Z}^{(3)}(\mathbf{x},\mathbf{F},\Omega^{\pm})$ is a regular particular solution of (3.1), where $\mathbf{F}\in C^{0,\nu'}(\Omega^{\pm})$, $0<\nu'\le 1$ and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^{-}. Therefore, further we consider problems $(I)^{\pm}_{\mathbf{0},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{0},\mathbf{f}}$, and we prove the existence theorems of a regular (classical) solution of these BVPs.

In addition, as we know (see Appendix), the uniqueness theorems in the GN type II thermoporoelasticity and the other 4 theories of thermoporoelasticity are essentially different, so we consider the following cases separately: 1) $k-\tau k^{*}>0$ and 2) $k-\tau k^{*}=0$, $k^{*}>0$.

1) Let

$$k-\tau k^{*}>0. \tag{7.1}$$

Problem $(I)^{+}_{\mathbf{0},\mathbf{f}}$. We assume that ω is not an eigenfrequency of the internal BVP $(I)^{+}_{\mathbf{0},\mathbf{0}}$ (see Theorem A1). We seek a regular solution to this problem in the form of the double-layer potential

$$\mathbf{U}(\mathbf{x})=\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g}) \quad \text{for } \mathbf{x}\in\Omega^{+}, \tag{7.2}$$

where $\mathbf{g}$ is the required five-component vector function.

Obviously, by Theorem 11 the vector function $\mathbf{U}$ is a solution of the homogeneous equation

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{U}(\mathbf{x})=\mathbf{0} \tag{7.3}$$

for $\mathbf{x}\in\Omega^{+}$. Keeping in mind the boundary condition (4.4) and using (6.2), from (7.2) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(1)}\,\mathbf{g}(\mathbf{z})=\mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z}\in S. \tag{7.4}$$

We prove that Eq. (7.4) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the associate homogeneous equation

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z})=\mathbf{0} \quad \text{for } \mathbf{z}\in S, \tag{7.5}$$

where $\mathbf{h}$ is the required five-component vector function. Now we prove that (7.5) has only the trivial solution.

Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (7.5). On the basis of Theorem 10 and Eq. (7.5) the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of the external homogeneous BVP $(II)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem A3, the problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^-. \tag{7.6}$$

On the other hand, by Theorem 10 and (7.6) we get

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \quad \text{for } \mathbf{z} \in S, \tag{7.7}$$

i.e., on the basis of Theorem 10 the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem A1 and the assumption that ω is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$, the problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^+. \tag{7.8}$$

By virtue of (7.6), (7.8) and the identity (6.1) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for } \mathbf{z} \in S.$$

Thus, the homogeneous equation (7.5) has only the trivial solution and therefore on the basis of Noether's theorem the integral equation (7.4) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following theorem.

THEOREM 15. *If the condition* (7.1) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *and* ω *is not an eigenfrequency of the BVP* $(I)^+_{\mathbf{0},\mathbf{0}}$, *then a regular solution of the internal BVP* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a double-layer potential* (7.2), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.4) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$. Let us assume that ω is not an eigenfrequency of the internal BVP $(II)^+_{\mathbf{0},\mathbf{0}}$ (see Theorem A2). We seek a regular solution to this problem in the form of the single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) \quad \text{for } \mathbf{x} \in \Omega^+, \tag{7.9}$$

where $\mathbf{g}$ is the required five-component vector function.

Obviously, by Theorem 10 the vector function $\mathbf{U}$ is a solution of the homogeneous equation (7.3) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition (4.5) and using (6.1), from (7.9) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(2)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S. \tag{7.10}$$

We prove that Eq. (7.10) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the homogeneous equation

$$-\frac{1}{2}\,\mathbf{g}_0(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}_0)=\mathbf{0}\quad \text{for } \mathbf{z}\in S, \tag{7.11}$$

where $\mathbf{g}_0$ is the required five-component vector function. Now we prove that (7.11) has only the trivial solution. On the basis of Theorem 10 and Eq. (7.11) the vector function $\mathbf{V}(\mathbf{x})=\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}_0)$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem A2 and the assumption that ω is not an eigenfrequency of the problem $(II)^+_{\mathbf{0},\mathbf{0}}$, this problem has only the trivial solution and we have the relation (7.8).

On the other hand, by Theorem 10 and (7.8) we have (7.7), i.e., on the basis of Theorem 10 the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem A3 the problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution and we get the relation (7.6). By virtue of (7.6), (7.8) and the identity (6.1) we obtain

$$\mathbf{g}_0(\mathbf{z})=\{\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^+=\mathbf{0}\quad \text{for } \mathbf{z}\in S.$$

Thus, the homogeneous equation (7.11) has only the trivial solution and therefore, on the basis of Noether's theorem the integral equation (7.10) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved the following result.

THEOREM 16. *If the condition* (7.1) *is fulfilled,* $S\in C^{2,\nu}$*,* $\mathbf{f}\in C^{0,\nu'}(S)$*,* $0<\nu'<\nu\le 1$*, and* ω *is not an eigenfrequency of the BVP* $(II)^+_{\mathbf{0},\mathbf{0}}$*, then a regular solution of the internal BVP* $(II)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a single-layer potential* (7.9)*, where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.10) *which is always solvable for an arbitrary vector* $\mathbf{f}$*.*

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to this problem in the sum of double-layer and single-layer potentials

$$\mathbf{U}(\mathbf{x})=\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g})+(1-i)\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g})\quad \text{for } \mathbf{x}\in\Omega^-, \tag{7.12}$$

where $\mathbf{g}$ is the required five-component vector function.

Obviously, by Theorems 10 and 11 the vector function $\mathbf{U}$ is a solution of the homogeneous equation (7.3) for $\mathbf{x}\in\Omega^-$. Keeping in mind the boundary condition (4.6) and using (6.2), from (7.12) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(5)}\,\mathbf{g}(\mathbf{z})\equiv\mathcal{K}^{(3)}\,\mathbf{g}(\mathbf{z})+(1-i)\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})=\mathbf{f}(\mathbf{z})\quad \text{for } \mathbf{z}\in S. \tag{7.13}$$

We prove that Eq. (7.13) is always solvable for an arbitrary vector $\mathbf{f}$.

Clearly, the singular integral operator $\mathcal{K}^{(5)}$ is of the normal type and $\operatorname{ind}\mathcal{K}^{(5)} = \operatorname{ind}\mathcal{K}^{(3)} = 0$. Consequently, Noether's theorems are valid for the singular integral operator $\mathcal{K}^{(5)}$ and it is sufficient to show that the homogeneous equation

$$\mathcal{K}^{(5)}\,\mathbf{g}_0(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S \tag{7.14}$$

has only a trivial solution. Indeed, let $\mathbf{g}_0$ be a solution of the homogeneous equation (7.14). Then the vector

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}_0) + (1-i)\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}_0) \quad \text{for } \mathbf{x} \in \Omega^- \tag{7.15}$$

is a regular solution of the problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem A3 we have the relation (7.6).

On the other hand, by Theorems 10 and 11 from (7.15) we get:

$$\begin{aligned} &\{\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{V}(\mathbf{z})\}^+ = -\mathbf{g}_0(\mathbf{z}),\\ &\{\mathbf{R}(\mathbf{D}_{\mathbf{z}}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_{\mathbf{z}}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = (1-i)\mathbf{g}_0(\mathbf{z}) \end{aligned} \tag{7.16}$$

for $\mathbf{z} \in S$. On the basis of (7.6) from (7.16) it follows that

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}}, \mathbf{n})\mathbf{V}(\mathbf{z}) + (1-i)\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.17}$$

Obviously, the vector $\mathbf{V}$ is a solution of the homogeneous equation (7.3) satisfying the boundary condition (7.17). It is not difficult to prove that from (7.3) and (7.17) it follows the relation

$$\{\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.18}$$

Clearly, by virtue of (7.6) and (7.18) from the first equation of (7.16) we get $\mathbf{g}_0(\mathbf{z}) \equiv \mathbf{0}$ for $\mathbf{z} \in S$. Thus, the homogeneous equation (7.14) has only the trivial solution and therefore, on the basis of Noether's theorem the integral equation (7.13) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following result.

Theorem 17. *If the condition* (7.1) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *then a regular solution* $\mathbf{U}$ *of the external BVP* $(I)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by the sum of double-layer and single-layer potentials* (7.12), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.13) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$. Foremost, the scalar product of two vectors

$$\mathbf{U} = (u_1, u_2, \dots, u_5) \quad \text{and} \quad \mathbf{V} = (v_1, v_2, \dots, v_5)$$

is denoted by $\mathbf{U}\cdot\mathbf{V} = \sum_{j=1}^{5} u_j \bar{v}_j$, where $\bar{v}_j$ is the complex conjugate of v_j.

We seek a regular solution to this problem in the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h}) + \mathbf{U}^*(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega^-, \tag{7.19}$$

where $\mathbf{h}$ is the required five-component vector function and the vector function $\mathbf{U}^*$ is a regular solution of the equation

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}^*(\mathbf{x}) = \mathbf{0} \qquad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{7.20}$$

Keeping in mind the boundary condition (4.7) and using (6.1), from (7.19) we obtain the following singular integral equation for determining the unknown vector $\mathbf{h}$

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{f}^*(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S, \tag{7.21}$$

where

$$\mathbf{f}^*(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \{\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}^*(\mathbf{z})\}^-. \tag{7.22}$$

Now we prove that Eq. (7.21) is always solvable for an arbitrary vector $\mathbf{f}$. We assume that the homogeneous singular integral equation

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \tag{7.23}$$

has r linearly independent solutions $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^r$ that are assumed to be orthonormal. By Noether's theorem the solvability condition of Eq. (7.21) can be written as

$$\int_S \{\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}^*(\mathbf{z})\}^- \cdot \boldsymbol{\psi}^{(l)}(\mathbf{z}) d_{\mathbf{z}}S = M_l, \tag{7.24}$$

where

$$M_l = \int_S \mathbf{f}(\mathbf{z}) \cdot \boldsymbol{\psi}^{(l)}(\mathbf{z})\, d_{\mathbf{z}}S$$

and $\{\boldsymbol{\psi}^{(l)}(\mathbf{z})\}_{l=1}^r$ is a complete system of solutions of the homogeneous associated equation of (7.23), i.e.,

$$\mathcal{K}^{(1)}\,\boldsymbol{\psi}^{(l)} = \mathbf{0}, \qquad l = 1,2,\ldots,r.$$

It is easy to see that condition (7.24) takes the form (for details, see Kupradze *et al.* [37])

$$\int_S \mathbf{h}^{(l)}(\mathbf{z}) \cdot \{\mathbf{U}^*(\mathbf{z})\}^- d_{\mathbf{z}}S = -N_l, \qquad l = 1,2,\ldots,r. \tag{7.25}$$

Let the vector $\mathbf{U}^*$ be a solution of (7.20) and satisfies the boundary condition

$$\{\mathbf{U}^*(\mathbf{z})\}^- = \hat{\mathbf{f}}(\mathbf{z}), \tag{7.26}$$

where

$$\hat{\mathbf{f}}(\mathbf{z}) = \sum_{l=1}^{r} M_l \mathbf{h}^{(l)}(\mathbf{z}). \tag{7.27}$$

By virtue of Theorem 17 the BVP (7.20), (7.26) is always solvable. Because of the orthonormalization of $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^{r}$, the condition (7.25) is fulfilled automatically and the solvability of (7.21) is proved. Consequently, the existence of regular solution of the problem $(II)^-_{\mathbf{0},\mathbf{f}}$ is proved too. Thus, the following theorem has been proved.

THEOREM 18. *If the condition* (7.1) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *then a regular solution* $\mathbf{U}$ *of the external BVP* $(II)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a sum* (7.19), *where* $\mathbf{h}$ *is a solution of the singular integral equation* (7.21) *which is always solvable,* $\mathbf{U}^*$ *is the solution of BVP* (7.20), (7.26) *which is always solvable; and the vector functions* $\mathbf{f}^*$ *and* $\hat{\mathbf{f}}$ *are defined by* (7.22) *and* (7.27), *respectively.*

2) Let us now consider the case

$$k - \tau k^* = 0, \qquad k^* > 0. \tag{7.28}$$

Similarly, as in Theorems 15 to 18, it is not very difficult to prove the following theorems.

THEOREM 19. *If the condition* (7.28) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *and* ω *is not an eigenfrequency of the BVP* (A.6), (A.7), *then a regular solution of the internal BVP* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a double-layer potential* (7.2), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.4) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

THEOREM 20. *If the condition* (7.28) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *and* ω *is not an eigenfrequency of the BVP* (A.6), (A.8), *then a regular solution of the internal BVP* $(II)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a single-layer potential* (7.9), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.10) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

THEOREM 21. *If the condition* (7.28) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *then a regular solution* $\mathbf{U}$ *of the external BVP* $(I)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a sum of double-layer and single-layer potentials* (7.12), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.13) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Theorem 22. *If the condition* (7.28) *is fulfilled,* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *then a regular solution* $\mathbf{U}$ *of the external BVP* $(II)^{-}_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a sum* (7.19), *where* $\mathbf{h}$ *is a solution of the singular integral equation* (7.21) *which is always solvable,* $\mathbf{U}^*$ *is the solution of BVP* (7.20), (7.26) *which is always solvable; and the vector functions* $\mathbf{f}^*$ *and* $\hat{\mathbf{f}}$ *are defined by* (7.22) *and* (7.27), *respectively.*

8. Concluding remarks

1. In the present paper the linear theory of MGT thermoporoelasticity is considered and the following results are obtained:

(i) The fundamental solution of the system of equations of steady vibrations is constructed explicitly by means of elementary functions and its basic properties are established;

(ii) The formula of integral representation of regular vectors is obtained;

(iii) The surface and volume potentials are introduced and their basic properties are given;

(iv) Some useful singular integral operators are defined for which the symbolic determinants and indexes are calculated;

(v) The basic BVPs of steady vibrations are reduced to the equivalent singular integral equations;

(vi) The existence theorems for classical solutions of these BVPs in the foregoing 5 theories of thermoporoelasticity are proved with the help of the potential method and the theory of singular integral equations.

2. On the basis of results of this paper it is possible to investigate the nonclassical BVPs in the linear theories of thermoporoelasticity for materials with a multiple porosity by using the potential method.

3. As is well known, obtaining numerical solutions of the BVPs using the boundary element method consists of the following three stages of research: (i) reduction of the BVP to an equivalent always solvable integral equation using the potential method (the boundary integral equation method), (ii) obtaining the numerical solution of the integral equation using the boundary element method, and (iii) obtaining the numerical solution of the original BVP through the obtained numerical solution. In this paper, the first stage has been successfully solved.

Appendix

We have the following uniqueness theorems (for details, see [35]).

THEOREM A1. *If the condition* (7.1) *is fulfilled, then two regular solutions of the internal BVP* $(I)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, p, \theta)$, *where*

$$p(\mathbf{x}) = 0, \qquad \theta(\mathbf{x}) = 0 \tag{A.1}$$

and the vector $\mathbf{u}$ *is a regular solution of the following system of homogeneous equations*

$$(\mu\Delta + \rho\omega^2)\mathbf{u}(\mathbf{x}) = \mathbf{0}, \qquad \operatorname{div}\mathbf{u}(\mathbf{x}) = 0 \qquad for\ \mathbf{x} \in \Omega^+ \tag{A.2}$$

satisfying the homogeneous boundary condition

$$\{\mathbf{u}(\mathbf{z})\}^+ = \mathbf{0} \qquad for\ \mathbf{z} \in S. \tag{A.3}$$

Moreover, the homogeneous BVPs $(I)^+_{\mathbf{0},\mathbf{0}}$ *and* (A.2), (A.3) *have the same eigenfrequencies.*

THEOREM A2. *If the condition* (7.1) *is fulfilled, then two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, p, \theta)$, *where* p *and* θ *satisfy the condition* (A.1), *the vector* $\mathbf{u}$ *is a regular solution of the system of homogeneous equations* (A.2) *for* $\mathbf{x} \in \Omega^+$ *satisfying the homogeneous boundary condition*

$$\{\mathbf{R}^{(0)}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\mathbf{u}(\mathbf{z})\}^+ = \mathbf{0} \qquad for\ \mathbf{z} \in S, \tag{A.4}$$

where the matrix differential operator $\mathbf{R}^{(0)}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))$ *is defined by*

$$\mathbf{R}^{(0)}(\mathbf{D}_{\mathbf{x}}, \mathbf{n}) = (R^{(0)}_{lj}(\mathbf{D}_{\mathbf{x}}, \mathbf{n}))_{3\times 3}, \qquad R^{(0)}_{lj}(\mathbf{D}_{\mathbf{x}}, \mathbf{n}) = R_{lj}(\mathbf{D}_{\mathbf{x}}, \mathbf{n}), \qquad l, j = 1, 2, 3.$$

In addition, the homogeneous BVPs $(II)^+_{\mathbf{0},\mathbf{0}}$ *and* (A.2), (A.4) *have the same eigenfrequencies.*

THEOREM A3. *If the condition* (7.1) *is fulfilled, then the external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Consequently, in the classical thermoporoelasticity, LS thermoporoelasticity, GN type III thermoporoelasticity, and MGT thermoporoelasticity we have the same uniqueness theorems. Remarkably, in these 4 theories and in the classical thermoelasticity the corresponding internal BVPs have the same eigenfrequencies.

The following uniqueness theorems hold in the theory of GN type II thermoporoelasticity.

THEOREM A4. *If the condition* (7.27) *is fulfilled, then two regular solutions of the internal BVP* $(I)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, p, \theta)$, *where* p *satisfies the condition*

$$p(\mathbf{x}) = 0, \tag{A.5}$$

and the four-component vector $\mathbf{v} = (\mathbf{u}, \theta)$ *is a regular solution of the system of homogeneous equations*

$$\begin{aligned}
&(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} - \varepsilon\nabla\theta = \mathbf{0},\\
&(k^*\Delta + m_1 c)\theta + m_1\varepsilon \operatorname{div} \mathbf{u} = 0,\\
&\beta \operatorname{div} \mathbf{u} + \gamma\theta = 0
\end{aligned} \tag{A.6}$$

for $\mathbf{x} \in \Omega^+$ *satisfying the homogeneous boundary condition*

$$\{\mathbf{v}(\mathbf{z})\}^+ = \mathbf{0} \tag{A.7}$$

for $\mathbf{z} \in S$. *Moreover, the homogeneous BVPs* $(I)^+_{\mathbf{0},\mathbf{0}}$ *and* (A.6), (A.7) *have the same eigenfrequencies.*

THEOREM A5. *If the condition* (7.27) *is fulfilled, then two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, p, \theta)$, *where* p *satisfies the relation* (A.5) *and the four-component vector* $\mathbf{v} = (\mathbf{u}, \theta)$ *is a regular solution of the system of homogeneous equations* (A.6) *for* $\mathbf{x} \in \Omega^+$ *satisfying the homogeneous boundary condition*

$$\{\mathbf{R}^{(1)}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{v}(\mathbf{z})\}^+ = \mathbf{0} \tag{A.8}$$

for $\mathbf{z} \in S$. *Moreover, the homogeneous BVPs* $(II)^+_{\mathbf{0},\mathbf{0}}$ *and* (A.6), (A.8) *have the same eigenfrequencies. Here*

$$\begin{aligned}
&\mathbf{R}^{(1)}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z})) = (R^{(1)}_{lj}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z})))_{4\times 4},\\
&R^{(1)}_{lj} = \mu\delta_{lj}\frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}, \qquad R^{(1)}_{l4} = -\varepsilon n_l,\\
&R^{(1)}_{4j} = 0, \qquad R^{(1)}_{44} = k^*\frac{\partial}{\partial \mathbf{n}}, \qquad l, j = 1, 2, 3.
\end{aligned}$$

THEOREM A6. *If the condition* (7.27) *is fulfilled, then the external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Acknowledgments

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [Project # FR-23-4905].

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Received June 30, 2024; revised version November 7, 2024.
Published online January 27, 2025.