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Motion of an aerosol sphere in hydrogel medium under thermal gradient

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In this work the thermophoretic migration of an aerosol sphere embedded in a hydrogel medium has been analytically investigated. The porous medium containing microstructure fluid of a micropolar type can be viewed as a hydrogel medium. The Reynolds and Péclet numbers are considered to be very small. We solve the governing equations of momentum and energy by applying a temperature jump, continuity of heat flux, and hydrodynamic boundary conditions such as viscous slip, thermal creep and thermal stress slip at the particle surface. Analytical expressions for thermophoretic velocity and thermophoretic force are obtained. The influence of the permeability, micropolarity, frictional slip, spin slip, thermal stress slip parameters, and thermal properties of particle and medium on thermophoretic velocity and force are discussed numerically. Our results show that the thermophoretic velocity and force are decreasing functions of micropolarity and microrotation thermal conductivity parameters, while the effect of thermal stress slip is to increase the thermophoretic velocity and force of the particle. The novelty of the research is the micropolarity parameter and permeability that characterizes the micropolar fluid flow through a porous medium. The results are also compared with previously published work. The study is applied to capture ash particles conducting thermophoresis in a porous filter formed by interconnected spherical pores in a hydrogel medium.

Key words: thermophoresis, aerosol sphere, porous medium, micropolar fluid.

1. Introduction

When a particle is suspended in a rarefied gaseous medium with a temperature gradient, it tends to move towards the direction of decreasing temperature. This movement is referred to as thermophoresis, and the force acting on the particle is known as the thermophoretic force. Thermophoresis has been the subject of extensive research in the past [1–3]. The Knudsen number is a crucial parameter in the study of thermophoresis. This quantity is defined as the ratio of the mean free path to the size of the particle. It is denoted by $K_n = l/a$. The narrow fluid layer, known as the Knudsen layer, flows along the direction of ambient temperature gradient. This phenomenon is important in various practical applications like microelectronic manufacturing, aerosol sampling, cleaning of air, and protecting nuclear reactors [4–7].

Many researchers have studied the thermophoresis of aerosol particles such as sphere and cylinder in the unbounded and bounded Newtonian fluid. Using the perturbation method, Brock [2] has calculated the thermophoretic velocity of a spherical particle at a constant temperature gradient ∇T_∞. He found that thermophoretic velocity depends upon certain thermal properties of the medium and hydrodynamic slip coefficients. Keh and Chang [8] examined the effects of the boundary on the Stokes flow and thermophoresis of spherical particle in a spherical cavity. Keh and Ho [9] discussed the effects of concentration on the thermophoresis of an aerosol sphere by using the cell model technique. Chang and Keh [10] examined the thermophoresis of an aerosol sphere with considering the influence of thermal stress slip, which shows increasing or decreasing effects on velocity and force according to the thermal characteristics of the particle and medium. Hsieh and Keh [11] considered the chemical reaction in the study of the thermophoretic motion of a spherical particle. Faltas and Ragab [12, 13] handled the thermophoresis of spherical and cylindrical particles in porous media governed by Darcy–Brinkman's equation. They have observed the thermophoretic velocity decreases with an increase of the permeability parameter. The impact of thermal stress slip on spherical particles within spherical cavities filled with viscous fluid has been discussed by Li and Keh [14]. El-Sapa [15] studied the thermophoresis of spherical particle in spherical cavity by considering the Brinkman model. Also, Li and Keh [16] analyzed the axisymmetric thermophoresis of a spherical particle positioned anywhere in a spherical cavity. Recently, Faltas and Saad [17] have investigated the thermophoresis of an aerosol particle within a spherical cavity filled with Brinkman's model.

Micropolar fluid is a non-Newtonian fluid proposed by Eringen [18], considers the effects of microstructure and the rotational motion of fluid particles. There are numerous applications in fields of science, engineering, and industries for this theory. Modelling the flows of biological fluids, suspensions, polymeric solutions, lubricants, etc. can be done with the micropolar fluid theory. By using the Eringen model [18] of micropolar fluid, Saad and Faltas [19] studied the thermophoresis of a spherical particle with consideration of photophoresis and chemical reaction in micropolar fluid and obtained the expressions for velocity and force in all the three cases. Also, Saad and Faltas [20] discussed the problem of a spherical particle being halfway embedded in micropolar fluid. In recent years, Faltas et al. [21] have studied the steady axisymmetric thermophoretic slow motion of a spherical particle located at an arbitrary position inside a microchannel with parallel permeable plane walls. When a small drop of one fluid is suspended in another immiscible fluid under a constant temperature gradient, the drop will move towards the direction of decreasing temperature, which is known as the thermocapillary effect. Faltas *et al.* [22] discussed the thermocapillary motion of a drop situated in the permeable cavity.

In recent years, studying non-Newtonian fluids in porous media has become increasingly significant for research, as it offers valuable insights into addressing various real-world fluid dynamics problems. This includes applications in the oil and petroleum industries, chemical filtration, and blood flow through porous vessels. Jaiswal and Yadav [23] discussed the Couette flow of a micropolar fluid through an annular porous region formed by two concentric rotating cylinders. They found that the velocity profile of the micropolar fluid decreases under slip conditions compared to no-slip conditions, while the microrotation velocity increases with the slip parameter. Kumar and Yadav [24] examined the peristaltic flow of immiscible non-Newtonian and micropolar fluids under the influence of magnetohydrodynamic effects. They derived analytical solutions for pressure, velocity, volume flow rate, concentration, and temperature distribution. Using the homotopy analysis method, Yadav and Yadav [25] studied the magnetohydrodynamic flow of a micropolar fluid in porous channels, considering suction and injection along with variable thermal conductivity and viscosity. Additionally, Yadav and Yadav [26] investigated the flow of immiscible couple stress fluids through curved porous channels under the impact of magnetohydrodynamics.

The previously discussed literature review highlights that many researchers have investigated the thermophoretic motion of particles suspended in Newtonian fluids and the Brinkman medium, while, there are only a few studies exploring thermophoresis in non-Newtonian fluids and non-Newtonian fluid saturated porous media. To the best of the authors' knowledge, no studies have investigated the thermophoretic phenomena of spherical particles in hydrogel media. The hydrogel medium is modeled as a porous medium saturated with a microstructure of the micropolar fluid. Understanding the thermophoretic migration of particles saturated in a porous medium of micropolar fluid as significant for industrial and environmental filtration applications. In industries such as metallurgy, chemical processing, and environmental monitoring, thermophoretic filtration plays a significant role in capturing and removing fine particulates, heavy metals, and pollutants from gases. Thermophoresis enhances the effectiveness of these filtration systems, particularly where other filtration methods may degrade or fail. One key application of thermophoresis is in pharmaceuticals, where it is used to purify nanoparticles by separating them based on size and properties [27]. By applying a temperature gradient, particles can be sorted according to their charge, size, and thermal characteristics, which aids in the purification of complex mixtures. Motivated by these real-world applications, we investigate the thermophoresis of spherical particles in a porous medium of micropolar fluid. The main objective of the study is to examine the influence of the permeability and micropolarity parameter, along with the thermal properties of the medium and particles on thermophoretic velocity and force. This research is an extension of the work by Faltas and Ragab [12] to the case of micropolar fluid.

2. Field equations

2.1. Fluid flow through porous medium

The fluid flow through porous media plays a significant role in various scientific, engineering, and technological applications. A porous medium is characterized by a structure composed of interconnected voids and solid spaces. This type of fluid flows are important in geothermal, fiber and granular insulation, sedimentation of particles, and nuclear reactor core applications [28, 29]. Neild and Bejan [30] proposed an equation that describes the fluid flow through the isotropic and homogenous porous medium of porosity φ. Suppose k_s, k_f, k_A are the thermal conductivity of the solid phase, fluid phase, and overall thermal conductivity of the porous medium, respectively. We make the assumptions of vanishing Péclet and Reynolds numbers, meaning that the effects of thermal convection and fluid inertia are neglected. So, the equation governing the thermal distributions for steady case, in the absence of heat source is [12, 30]

$$\nabla \cdot k_A \nabla T = 0, \tag{2.1}$$

where

$$k_A = (1 - \varphi)\, k_f + \varphi\, k_s. \tag{2.2}$$

2.2. Governing equations in micropolar fluid

The equations, which governs the flow of micropolar fluid in a porous medium under the Stokesian assumption are given by [31–33]:

$$\nabla \cdot \vec{u} = 0, \tag{2.3}$$

$$\nabla p + \frac{(\mu + \kappa_p)}{k_1}\, \vec{u} + (\mu + \kappa_p)\, \nabla \times \nabla \times \vec{u} - \kappa_p\, \nabla \times \vec{\nu} = 0, \tag{2.4}$$

$$\kappa_p \nabla \times \vec{u} - 2\, \kappa_p\, \vec{\nu} + (\alpha_0 + \beta_0 + \gamma_0)\, \nabla\nabla \cdot \vec{\nu} - \gamma_0\, \nabla \times \nabla \times\, \vec{\nu} = 0. \tag{2.5}$$

In Eq. (2.4), the second term denotes the Darcy term in porous medium and the third term is analogous to the Laplacian term that appears in the momentum equation of micropolar fluid. Where $\vec{u}$, p, μ, $\kappa_p, \vec{\nu}$, α_0, β_0, γ_0, k_1 denote the velocity vector, pressure, viscosity of classical fluid, rotational viscosity coefficient, microrotation vector and gyroviscosity coefficients of micropolar fluid, and permeability of porous medium, respectively.

The material coefficients $(\mu, \kappa_p, \alpha_0, \beta_0, \gamma_0)$ satisfy the inequalities:

$$\mu \geq 0, \qquad \kappa_p \geq 0, \qquad 2\mu + \kappa_p \geq 0,$$
$$\gamma_0 \geq 0, \quad -\gamma_0 \leq \beta \leq \gamma_0, \quad 3\alpha_0 + \beta_0 + \gamma_0 \geq 0.$$

The constitutive equations for stress tensor, couple stress tensor and heat flux vector of micropolar fluid are given by [18, 19]:

$$\Pi = -p\, I + \left(\mu + \frac{\kappa_p}{2}\right)(\nabla \vec{u} + (\nabla \vec{u})^t) + \kappa_p\, \varepsilon \cdot (\vec{\omega} - \vec{\nu}), \tag{2.6}$$

$$\mathbf{m} = -\frac{\delta}{\rho\, T_0}(\varepsilon \cdot \nabla T) + \alpha_0\, I \cdot \nabla \vec{\nu} + \beta_0\, \nabla\, \vec{\nu} + \gamma_0 (\nabla\, \vec{\nu})^t, \tag{2.7}$$

$$\vec{E} = k_A\, (\delta\, \rho\, T_0\, \nabla \times \vec{\nu} - \nabla T), \tag{2.8}$$

where $(\nabla\, \vec{u})^t$ is the transpose of $\nabla\, \vec{u}$, and $\vec{\omega}$, I, ε, ρ, k_A represent vorticity vector, unit dyadic, unit triadic, mass density, and overall thermal conductivity of the porous medium, respectively; δ indicates heat conduction due to microrotation, and T_0 denotes the temperature at the center of particle.

2.3. Boundary conditions for momentum equations

The first slip component, referred to as the frictional or viscous slip, and it can be expressed mathematically for micropolar fluid [19, 20]

$$\vec{u}_1 = \frac{2 C_m l}{2\mu + \kappa_p}(I - \vec{n}\vec{n}) \cdot (\vec{n} : \Pi), \tag{2.9}$$

where $\vec{n}$ denotes the unit normal vector at the solid surface, the quantity C_m is a dimensionless quantity known as the frictional or viscous slip coefficient. This coefficient is related to the momentum accommodation coefficient at the solid surface, can be considered in gas rarefaction.

The second slip component, referred to as a thermal creep. For micropolar fluid, it can be given as [19, 20]

$$\vec{u}_2 = -\frac{C_s(2\mu + \kappa_p)}{2\rho T_0 k_A}(I - \vec{n}\vec{n}) \cdot \vec{E}, \tag{2.10}$$

where C_s denotes the thermal creep coefficient. This velocity slip indicates thermal creep flow caused by the longitudinal temperature gradient over the particle's surface. The thermal creep effect can be considered in calculations of gas flows through microchannels, vacuum devices, and various other applications where temperature imbalance leads to the gas flow.

The third velocity slip component, referred to as a thermal stress slip; its general form in micropolar fluid is [19, 20]

$$\vec{u}_3 = \frac{C_m l C_h (2\mu + \kappa_p)}{2\rho T_0 k_A}(I - \vec{n}\vec{n}) \cdot (\vec{n} \cdot \nabla \vec{E}), \tag{2.11}$$

where C_h is the dimensionless thermal stress slip coefficient.

Additionally, the microrotation slip condition at the solid surface in micropolar fluid is [19, 20]

$$\vec{\nu} = \frac{C_n l}{2\gamma_0 + \beta_0}(I - \vec{n}\vec{n}) \cdot (\vec{n} : \mathbf{m}), \tag{2.12}$$

where C_n is a dimensionless spin slip coefficient. All C_m, C_s, C_h and C_n are of the order of unity and depend on the material of the particle and the fluid.

Further, velocity and microrotation vectors vanish far away from the particle as

$$\vec{u} \to 0, \quad \vec{\nu} \to 0 \quad \text{as } r \to \infty. \tag{2.13}$$

3. Mathematical formulation

Consider a small aerosol sphere with a radius a and thermal conductivity k_p that is surrounded by a hydrogel medium with an overall thermal conductivity k_A. We use a spherical coordinate system (r, θ, ϕ), with the origin at the center of the particle O. The unit vectors are represented by $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ along the r, θ, and ϕ directions, respectively. A uniform temperature gradient ∇T_∞ is maintained in the medium at a large distance from the particle, oriented

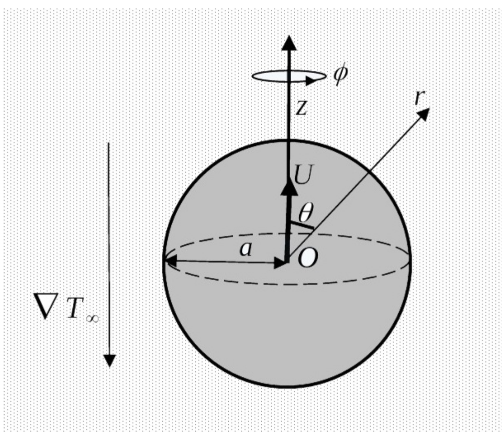

FIG. 1. Physical representation of thermophoresis motion of an aerosol sphere embedded in a porous medium saturated by micropolar fluid.

opposite to the positive z-axis, as shown in Fig. 1. In this analysis, we assume the thermal properties of both the porous medium and the particle as constants. Let U represent the thermophoretic velocity of the particle in the z-direction, which we aim to be determined. The micropolar fluid flow is assumed to be axisymmetric, so that the fluid velocity is independent of ϕ.

Then, we have supposed the components of the velocity and microrotation vectors are:

$$\vec{u} = u_r(r,\theta)\,\vec{e}_r + u_\theta(r,\theta)\,\vec{e}_\theta, \qquad \vec{\nu} = \nu_\phi(r,\theta)\,\vec{e}_\phi.$$

Under the conditions of incompressibility $\nabla\cdot\vec{u}=0$ and the uniform microrotation field $\nabla\cdot\vec{\nu}=0$, velocity components in terms of the stream function are defined as:

$$u_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{3.1}$$

By using the nondimensional parameters $r = a\,\tilde{r}$, $u = \tilde{u}\,U$, $p = (\mu\,U\,\tilde{p}/a)$, $\nabla = (\tilde{\nabla}/a)$, $\nu_\phi = (U\,\tilde{\nu}_\phi/a)$, we get the reduced form of Eqs. (2.3)–(2.5) as:

$$\nabla\cdot\vec{u} = 0, \tag{3.2}$$

$$\nabla p + (1+\chi)\,\vec{u}\,\eta^2 + (1+\chi)\,\nabla\times\nabla\times\vec{u} - \chi\,\nabla\times\vec{\nu} = 0, \tag{3.3}$$

$$\nabla\times\vec{u} - 2\,\vec{\nu} - m^{-2}\,\nabla\times\nabla\times\vec{\nu} = 0, \tag{3.4}$$

where $\eta = a/\sqrt{k_1}$ is the permeability parameter, $\chi = \kappa_p/\mu$ is the micropolarity parameter and $m^2 = \kappa_p\,a^2/\gamma_0$.

By eliminating the pressure term, we get:

$$E^2(r\sin\theta\nu_\phi) = \frac{(1+\chi)}{\chi}[E^4\psi - \eta^2E^2\psi], \tag{3.5}$$

and

$$\nu_\phi = \frac{1}{2r\sin\theta}[ME^4\psi + (1-M\eta^2)E^2\psi], \tag{3.6}$$

where

$$M = \frac{1+\chi}{\chi m^2} \quad\text{and}\quad E^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

From Eqs. (3.5) and (3.6), we get

$$E^2(E^2-\alpha^2)(E^2-\beta^2)\psi = 0, \tag{3.7}$$

where

$$\alpha^2 + \beta^2 = 2m^2 - M^{-1} + \eta^2, \quad \alpha^2\beta^2 = 2\eta^2m^2. \tag{3.8}$$

The solution of the sixth order partial differential Eq. (3.7) along with the boundary condition (2.13) is [33]:

$$\psi = \frac{1}{2}\left(\frac{A}{r} + B\sqrt{r}\,K_{3/2}(\alpha r) + C\sqrt{r}K_{3/2}(\beta r)\right)\sin^2\theta, \tag{3.9}$$

$$\nu_\phi = \frac{1}{4r}(B\xi_1\sqrt{r}\,K_{3/2}(\alpha r) + C\xi_2\,\sqrt{r}\,K_{3/2}(\beta r))\sin\theta, \tag{3.10}$$

where

$$\alpha = \sqrt{\frac{2m^2 - M^{-1} + \eta^2 + \sqrt{(2m^2 - M^{-1} + \eta^2)^2 - 8\eta^2 m^2}}{2}}, \qquad \beta = \sqrt{\frac{2m^2\eta^2}{\alpha^2}},$$

$$\xi_1 = (1 - M\eta^2)\alpha^2 + M\alpha^4, \qquad \xi_2 = (1 - M\eta^2)\beta^2 + M\beta^4,$$

and $K_{3/2}(*)$ represents the modified Bessel function of the second kind of the order $3/2$.

The expressions for velocity components, shear stress $t_{r\theta}$ and couple stress $m_{r\phi}$ are given as:

$$u_r = -\left[\frac{A}{r^3} + \frac{1}{r^2}(B\sqrt{r}\,K_{3/2}(\alpha r) + C\sqrt{r}\,K_{3/2}(\beta r))\right]\cos\theta, \tag{3.11}$$

$$\begin{aligned} u_\theta = -\frac{1}{2}\bigg[\frac{A}{r^3} &+ \frac{B}{r^{3/2}}(K_{3/2}(\alpha r) + \alpha r K_{1/2}(\alpha r)) \\ &+ \frac{C}{r^{3/2}}(K_{3/2}(\beta r) + \beta r K_{1/2}(\beta r))\bigg]\sin\theta, \end{aligned} \tag{3.12}$$

$$\begin{aligned} t_{r\theta} = \frac{(2\mu + \kappa_p)}{2}\bigg[\frac{3A}{r^4} &+ \frac{B}{r^{5/2}}(\sigma_1 K_{3/2}(\alpha r) + \alpha r K_{1/2}(\alpha\, r)) \\ &+ \frac{C}{r^{5/2}}(\sigma_2 K_{3/2}(\beta r) + \beta r K_{1/2}(\beta r)) \\ &+ \frac{\kappa_p}{(2\mu + \kappa_p)r^{1/2}}(BK_{3/2}(\alpha r)\sigma_3 + CK_{3/2}(\beta r)\sigma_4)\bigg]\sin\theta, \end{aligned} \tag{3.13}$$

$$\begin{aligned} m_{r\phi} = -\bigg[&\frac{(\beta_0 + 2\gamma_0)}{4r^{3/2}}(B\xi_1 K_{3/2}(\alpha r) + C\xi_2 K_{3/2}(\beta r)) \\ &+ \frac{\gamma_0}{4r^{1/2}}(B\xi_1\alpha K_{1/2}(\alpha r) + C\xi_2\beta K_{1/2}(\beta r)) \\ &- \frac{\delta E_\infty(1 - B_1 r^{-3})}{\rho T_0}\bigg]\sin\theta, \end{aligned} \tag{3.14}$$

where $K_{1/2}(*)$ represents the modified Bessel function of the second kind of the order $1/2$ and

$$\sigma_1 = \left(3 + \frac{\alpha^2 r^2}{2}\right), \quad \sigma_2 = \left(3 + \frac{\beta^2 r^2}{2}\right), \quad \sigma_3 = \left(\frac{\alpha^2}{2} - \frac{\xi_1}{2}\right), \quad \sigma_4 = \left(\frac{\beta^2}{2} - \frac{\xi_2}{2}\right).$$

The expression for pressure is

$$p = -\frac{(1 + \chi)\eta^2}{2r^2}A\cos\theta + \text{constant}. \tag{3.15}$$

3.1. Temperature distributions

The field equation which governs the thermal distribution for a micropolar fluid medium with constant thermal conductivities (k_A, δ) is given by the Laplace equation as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=0, \qquad r>a. \tag{3.16}$$

For the thermal field within the particle,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_p}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_p}{\partial\theta}\right)=0, \qquad r<a. \tag{3.17}$$

The components of the heat flux vector in the medium and the particle having the forms:

$$\vec{E} = E_r\,\vec{e}_r + E_\theta\,\vec{e}_\theta, \tag{3.18}$$

$$\vec{E}_p = E_{pr}\,\vec{e}_r + E_{p\theta}\,\vec{e}_\theta = -k_p\nabla T_p. \tag{3.19}$$

To obtain the thermal distributions at the particle surface $(r=1)$ and medium, we have used two types of boundary conditions: (i) temperature jump, (ii) continuity of heat flux for the present analysis.

The temperature jump is proportional to the normal temperature gradient. The temperature jump condition was suggested by Poisson in analogy with the slip boundary condition, which represents a discontinuity in the transport properties across the interface [20, 34, 36]

$$T - T_p = -\frac{C_t l}{k_A}(\vec{E}\cdot\vec{e}_r), \tag{3.20}$$

and the continuity of heat flux ensures that the amount of heat flowing into a boundary is equal to the amount flowing out, provided there is no heat generation or dissipation within at the boundary. Mathematically, it can be expressed as [20, 35]

$$\vec{E}\cdot\vec{e}_r = \vec{E}_p\cdot\vec{e}_r. \tag{3.21}$$

Also,

$$T \to T_\infty = T_0 - E_\infty r\cos\theta \qquad \text{as } r\to\infty, \tag{3.22}$$

where C_t denotes the temperature jump coefficient at the particle surface.

By solving Eqs. (3.16) and (3.17) with the boundary conditions Eqs. (3.20)–(3.22), we get the following system of linear equations:

$$A_1 k + 2B_1 + \frac{\delta\xi_1 S_1}{2V} B + \frac{\delta\xi_2 S_2}{2V} C + 1 = 0, \tag{3.23}$$

$$-A_1 + B_1(1 + 2\tilde{C}_t) + \frac{\tilde{C}_t \delta\xi_1 S_1}{2V} B + \frac{\tilde{C}_t \delta\xi_2 S_2}{2V} C + \tilde{C}_t - 1 = 0. \tag{3.24}$$

Here, $k = k_p/k_A$, $\tilde{C}_t = C_t l/a$, $V = E_\infty/\rho T_0$, $S_1 = K_{3/2}(\alpha)$, $S_2 = K_{3/2}(\beta)$.

Temperature fields for a fluid flow phase and inside the particle are given by:

$$T = T_0 - E_\infty r \cos\theta + E_\infty B_1 r^{-2} \cos\theta, \tag{3.25}$$

$$T_p = T_0 + E_\infty A_1 r \cos\theta. \tag{3.26}$$

The expressions of A_1 and B_1 are presented in Appendix.

3.2. Slip boundary conditions

To determine the closed-form solutions for the problem, it is essential to establish the interface conditions that are valid both physically and mathematically. For the current problem of thermophoretic motion of a spherical particle in micropolar fluid, we adopt the following slip conditions: frictional slip, thermal creep, thermal stress slip, and microrotation slip. Mathematically, at the particle surface, we establish the following boundary conditions [19, 20]:

$$u_r = U\cos\theta, \tag{3.27}$$

$$\begin{aligned} u_\theta = -U\sin\theta + \frac{2C_m l}{2\mu+\kappa_p} t_{r\theta} + \frac{C_s(2\mu+\kappa_p)}{2}\left(\frac{1}{\rho T_0 r}\frac{\partial T}{\partial \theta} + \frac{\delta}{r}\frac{\partial}{\partial r}(r\nu_\phi)\right) \\ - \frac{C_m l C_h(2\mu+\kappa_p)}{2}\left(\frac{1}{\rho T_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial T}{\partial \theta}\right) + \delta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\nu_\phi)\right)\right), \end{aligned} \tag{3.28}$$

$$\nu_\phi = \frac{C_n l}{2\gamma_0 + \beta_0} m_{r\phi}, \tag{3.29}$$

$$u_r = u_\theta = \nu_\phi = 0 \quad \text{as } r \to \infty. \tag{3.30}$$

Putting the expressions Eqs. (3.11)–(3.14) into the boundary conditions Eqs. (3.27)–(3.30), we found the system of linear equations for determining the unknowns A, B, and C:

$$A + BS_1 + CS_2 + U = 0, \tag{3.31}$$

$$\begin{aligned} -A(1+6\tilde{C}_m) - B\left(S_5 + \tilde{C}_m S_7 - \frac{C_s(2+\chi)\delta\xi_1}{4} S_5 - \frac{\tilde{C}_m C_h(2+\chi)\delta\xi_1}{4} S_9\right) \\ - C\left(S_6 + \tilde{C}_m S_8 - \frac{C_s(2+\chi)\delta\xi_2}{4} S_6 - \frac{\tilde{C}_m C_h(2+\chi)\delta\xi_2}{4} S_{10}\right) \\ + (C_s + 3\tilde{C}_m C_h)(2+\chi) V B_1 - C_s(2+\chi)V + 2U = 0, \end{aligned} \tag{3.32}$$

$$B(S_1+\tilde{C}_nS_{11})\xi_1+C(S_2+\tilde{C}_nS_{12})\xi_2+\frac{4\tilde{C}_n\tilde{\gamma_0}\delta V}{\gamma_0}B_1-\frac{4\tilde{C}_n\tilde{\gamma_0}\delta V}{\gamma_0}=0, \tag{3.33}$$

where

$$\tilde{C_m}=\frac{C_ml}{a},\quad \sigma=\frac{\chi}{2+\chi},\quad \tilde{C_n}=\frac{C_nl}{a},\quad \tilde{\gamma_0}=\frac{\gamma_0}{2\gamma_0+\beta_0},$$

S_i, $i=3,\dots,12$, and the expressions for A, B, and C are shown in Appendix.

3.3. Thermophoretic drag and velocity

The drag force exerted on the spherical particle embedded in micropolar fluid is given as:

$$F_T=2\pi a^2\int_0^\pi r^2[(t_{rr}\cos\theta-t_{r\theta}\sin\theta)]_{r=1}\sin\theta\,d\theta. \tag{3.34}$$

Substituting the expressions of t_{rr} and $t_{r\theta}$ in Eq. (3.34), we get the expression for the drag force:

$$F_T=\frac{2\pi}{3}\big[(1+\chi)A\eta^2-(BS_1(\alpha^2(2+2\chi)-\chi\xi_1)+CS_2(\beta^2(2+2\chi)-\chi\xi_2))\big]. \tag{3.35}$$

When the particle is suspended in a porous medium, the force F_T must be zero for the thermophoresis effect, therefore

$$(1+\chi)A\eta^2-(BS_1(\alpha^2(2+2\chi)-\chi\xi_1)+CS_2(\beta^2(2+2\chi)-\chi\xi_2))=0. \tag{3.36}$$

Using Eq. (3.36), we get thermophoretic velocity of the particle, which is given as:

$$U_T=\frac{12VT_6}{S_{18}}. \tag{3.37}$$

Also, by substituting $U_T=0$ in Eq. (3.35), we get the thermophoretic drag force given as:

$$F_T= 8\pi V\left(\frac{(1+\chi)W_1\eta^2-X_1S_1(\alpha^2(2+2\chi)-\chi\xi_1)-Y_1S_2(\beta^2(2+2\chi)-\chi\xi_2)}{S_{18}}\right), \tag{3.38}$$

where $T_6, S_{18}, W_1, X_1,$ and Y_1 are expressed in Appendix.

Additionally, the normalized thermophoretic velocity and force are, therefore:

$$U_T^*=\frac{U_T}{(1+\chi)C_sV}, \tag{3.39}$$

$$F_T^*=\frac{lF_T}{(1+\chi)^2a^2V}. \tag{3.40}$$

3.3.1. Special cases. The comparison of the current work with previously published results.

Case I: When $\chi = 0$, i.e., $(\kappa_p \to 0, \alpha \to \eta, \beta \to \infty)$, thermophoretic velocity and force are given as:

$$U_T = \frac{18\mu V(\eta+1)(C_s + C_s\tilde{C_t}k + \tilde{C_m}C_h(1-k+k\tilde{C_t}))}{(\eta^2+9\eta+9+\tilde{C_m}(\eta^3+3\eta^2+18\eta+18))(k+2+2k\tilde{C_t})}, \tag{3.41}$$

$$F_T = \frac{2}{3}\pi\mu U_T\left(\frac{\eta^2+9(\eta+1)+(\eta^3+3\eta^2+18(\eta+1))\tilde{C_m}}{1+(3+\eta)\tilde{C_m}}\right). \tag{3.42}$$

These results agree with the results of Faltas and Ragab [12].

Case II: When the particle embedded in Newtonian fluid, i.e., $k_1 \to \infty$, and $\chi \to 0$, Eqs. (3.41)–(3.42) reduce to:

$$U_T = 2\mu V\frac{C_s + C_s\tilde{C_t}k + \tilde{C_m}C_h(1-k+k\tilde{C_t})}{(1+2\tilde{C_m})(k+2+2k\tilde{C_t})}, \tag{3.43}$$

$$F_T = 6\pi\mu U_T\frac{1+2\tilde{C_m}}{1+3\tilde{C_m}}. \tag{3.44}$$

The above expressions are in agreement with the work of Chang and Keh [10].

4. Results and discussion

In this analysis, we followed an analytical methodology to obtain the expressions for temperature distributions, thermophoretic velocity, and force. The variation of the thermophoretic velocity and force for various parameters, including the thermal conductivity ratio, Knudsen number, viscous slip parameter, permeability, micropolarity parameter, and thermal stress slip parameter, are presented graphically and interpreted. By following [33, 37] the values of temperature jump coefficient, frictional slip coefficient, thermal creep coefficient, and microrotation slip coefficient have been established as: $C_t = 2.18$, $C_m = 1.14$, $C_s = 1.17$, $C_n = 1.2$, $\gamma_0/\mu a^2 = 0.3$, $\beta_0/\mu a^2 = 0.2$. In this work, we have studied the variation of the thermophoretic velocity and the force versus the various parameters:

1. Knudsen number $(0.01 \le K_n \le 1)$ [12, 13].
2. Thermal conductivity ratio $(0 < k < \infty)$ [12, 13].
3. Micropolarity parameter $(0 \le \chi < \infty)$ [19].
4. Permeability $(0 < k_1 < \infty)$ [12, 15].
5. Thermal stress slip parameter $(0 \le C_h < \infty)$ [13, 19].
6. Frictional slip parameter $(0 \le \tilde{C_m} < \infty)$ [15, 19].

7. Microrotation thermal conductivity parameter ($0 \leq \delta\mu/a^2 < \infty$) [19].
8. Temperature jump parameter ($0 < \tilde{C}_t < \infty$).
9. Spin slip parameter ($0 \leq \tilde{C}_n < \infty$).

The thermophoretic velocity represents the movement of a particle from hot regions to cold regions, suspended in a fluid under the influence of a temperature gradient.

Figure 2 shows the variation of the normalized thermophoretic velocity U_T^* versus the Knudsen number for different values of the thermal conductivity ratio and micropolarity parameters. In general, the plot indicates that U_T^* decreases monotonically as the Knudsen number increases. Furthermore, the velocity of the particle decreases with an increase in the thermal conductivity parameter. This consequence arises because a relatively high thermal conductivity of a particle reduces the local temperature gradient at the particle surface. For different values of K_n and k, the thermophoretic velocity decreases as the micropolarity parameter increases. The consequence is that the thickness of a micropolar fluid is higher than the Newtonian fluid. Therefore, it reduces the particle velocity.

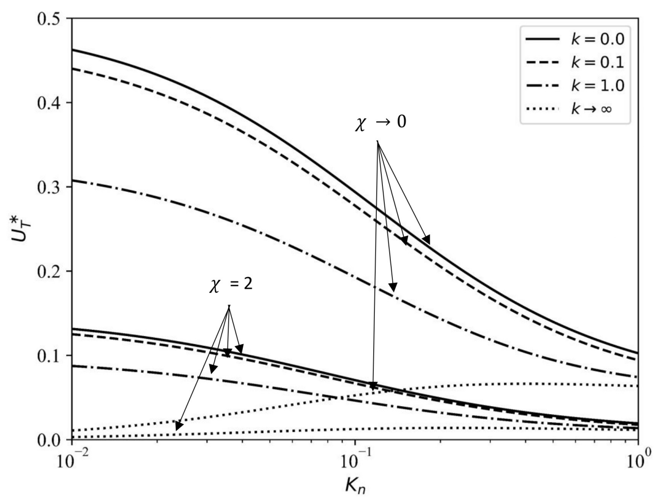

FIG. 2. Variation of U_T^* versus K_n when $C_h = 1$, $\delta\mu/a^2 = 0.2$, $k_1 = 0.01$.

Figure 3 depicts the impact of increasing frictional slip and thermal stress slip parameters on the thermophoretic velocity versus the Knudsen number. This plot shows that the thermophoretic velocity decreases when the frictional slip parameter rises, and increases as the thermal stress slip parameter increases. It is interesting to note that the effect of the thermal stress slip parameter on the velocity is the same when the frictional slip parameter is zero.

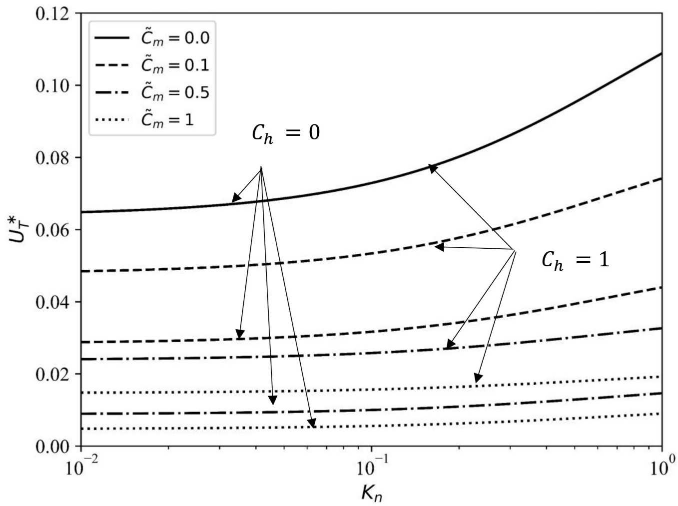

Fig. 3. Variation of U_T^* versus K_n when $k = 1$, $\chi = 3$, $\delta\,\mu/a^2 = 0.2$, $k_1 = 0.01$.

Figure 4 illustrates the plot of the thermophoretic velocity against micropolarity parameter for various values of the Knudsen number and the thermal stress slip parameter. As expected, it can be observed that the velocity decreases monotonically with an increase in the micropolarity parameter while keeping the other parameters unchanged. This causes due to a higher thickness in the medium that

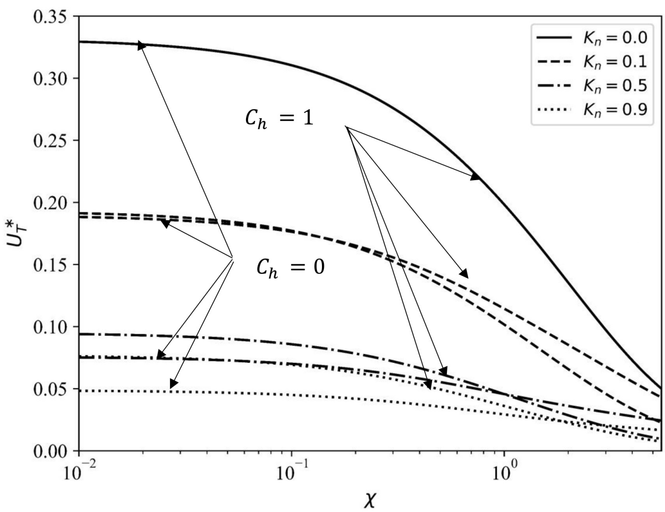

Fig. 4. Variation of U_T^* versus χ when $k = 1, \delta\,\mu/a^2 = 0.1$, $k_1 = 0.01$.

reduces the thermophoretic mobility. Further, the plots show that U_T^* decreases with an increase in the Knudsen number. The increasing effect of the thermal stress slip parameter on the thermophoretic velocity can be observed in this figure. This graph shows there is no effect of the thermal stress slip parameter on the velocity when $K_n \to 0$.

The plots of thermophoretic velocity versus the Knudsen number is visualized in Fig. 5. The normalized velocity decreases as the thermal conductivity of particle increases (thermal conductivity ratio increases) in the whole range of the Knudsen number. This happens due to a relatively high thermal conductivity of the particle, which reduces the local temperature gradient at the particle surface. However, for the case where the thermal conductivity ratio becomes very large, we see that the thermophoretic velocity increases with the Knudsen number. In this graph, it is also interesting to note that the thermophoretic velocity decreases for a higher microrotation thermal conductivity parameter. In all the above figures, the variation of thermophoretic velocity are the same as in the works of FALTAS and RAGAB [12] and SAAD and FALTAS [19].

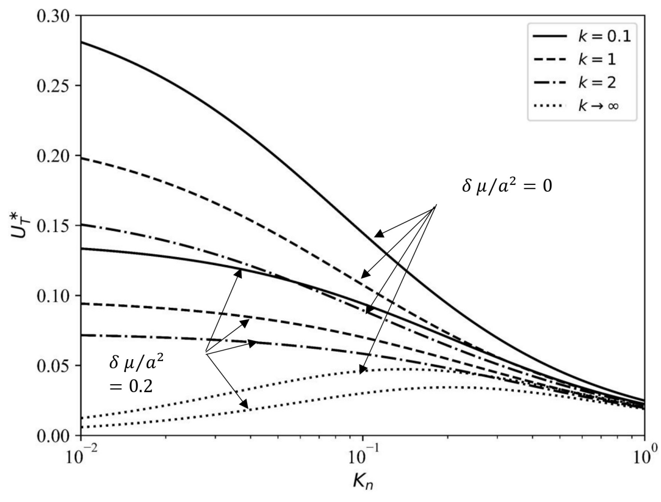

FIG. 5. Variation of U_T^* versus K_n when $C_h = 0, \chi = 2$, $k_1 = 0.01$.

Figures 6–10 illustrate the variation of the thermophoretic velocity against permeability for different parameters, including thermal conductivity ratio, frictional slip parameter, Knudsen number, microrotation thermal conductivity parameter, and thermal jump parameter. Figure 6 shows that the thermophoretic velocity increases as permeability increases. Lower permeability indicates that frictional forces dominate over inertial forces, which suggests that the medium is more porous. Conversely, higher permeability means that inertial

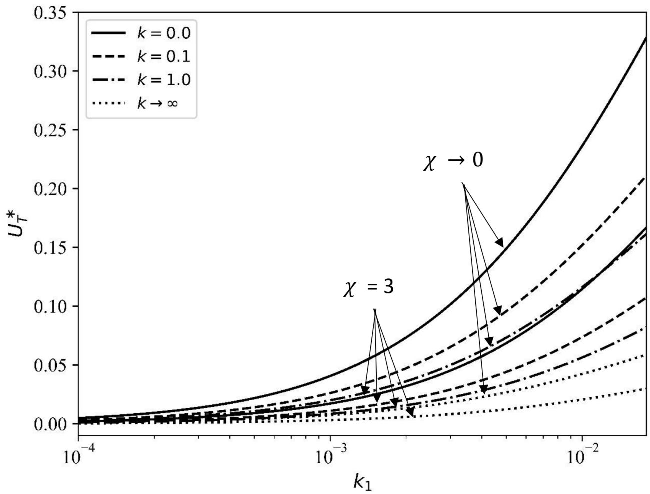

Fig. 6. Variation of U_T^* versus k_1 when $K_n = 0.2$, $C_h = 2$, $\delta\,\mu/a^2 = 0$.

forces dominate over frictional forces, indicating a clear fluid. Thus, as permeability increases, the thermophoretic velocity also rises. Additionally, it has been observed that as the thermal conductivity ratio (or the thermal conductivity of the particle) increases, the thermophoretic velocity decreases. This is because a higher thermal conductivity reduces the thermal gradient at the particle's surface. Figure 7 illustrates the effect of the frictional slip parameter on the

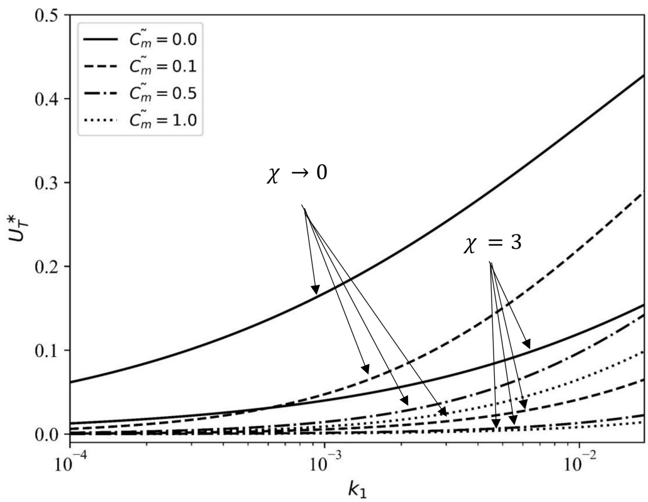

Fig. 7. Variation of U_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 2$, $\delta\,\mu/a^2 = 0.1$.

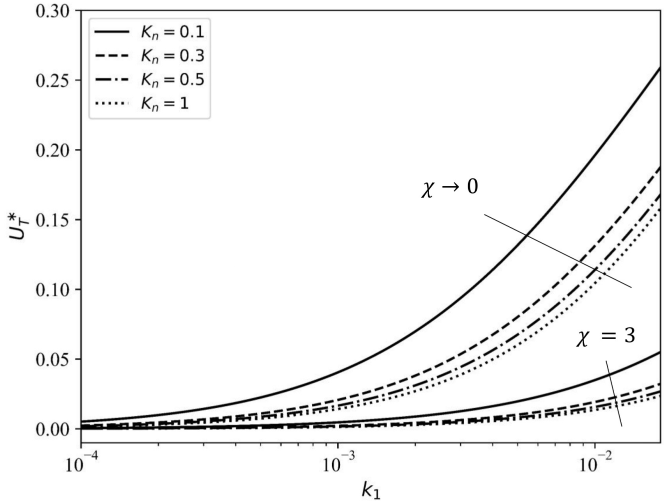

FIG. 8. Variation of U_T^* versus k_1 when $k = 1$, $C_h = 2$, $\delta\,\mu/a^2 = 0.1$.

thermophoretic velocity. The graph shows that the velocity U_T^* rises as the frictional slip parameter decreases. Consequently, this reduces the particle's mobility in the fluid medium. Figure 8 depicts the impact of the rising Knudsen number on thermophoretic velocity. The graph indicates that velocity decreases as the Knudsen number increases. Figure 9 presents the effect of the microrotational thermal conductivity parameter on thermophoretic velocity. Across the entire

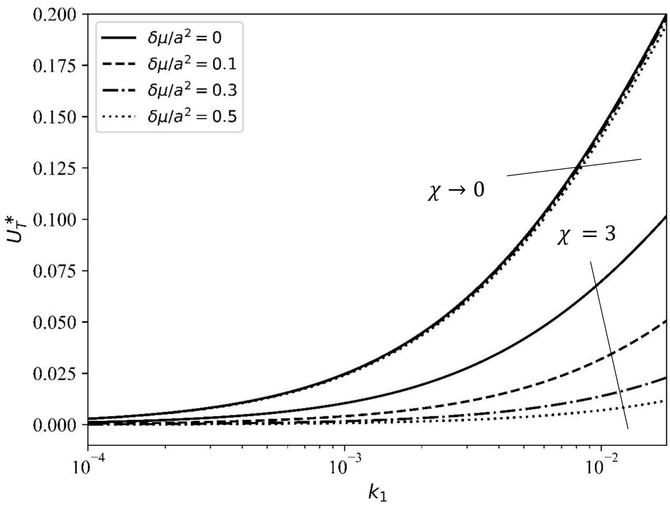

FIG. 9. Variation of U_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 1$.

range of permeability, thermophoretic velocity increases; however, it decreases with the value of the enhancing microrotation thermal conductivity parameter. The thermophoretic velocity for varying of the non-dimensional temperature jump parameter is shown in Fig. 10 which describes that the U_T^* increases monotonically with enhancing a temperature jump parameter. In all Figs. 6–10, we have observed that the velocity for the Newtonian fluid case ($\chi \to 0$) is higher than the micropolar fluid case. This happens because the thickness of micropolar fluid is greater than the Newtonian fluid.

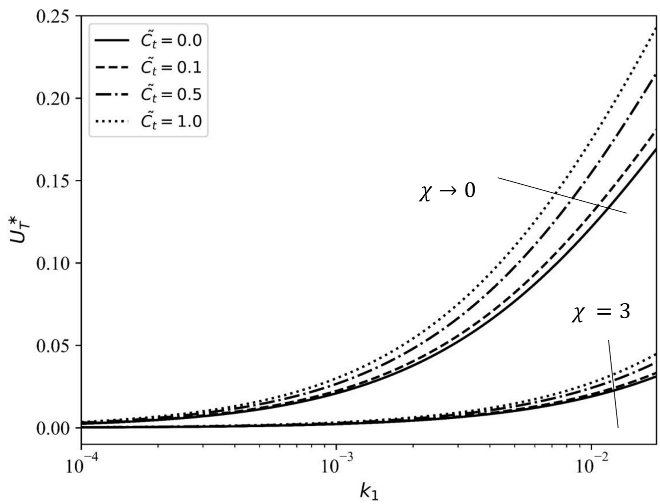

Fig. 10. Variation of U_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 2$, $\delta\,\mu/a^2 = 0.1$.

The thermophoretic force arises due to the motion of particles in a fluid caused by the temperature gradient. This force is proportional to the temperature gradient and depends on various parameters of the medium and the fluid. Figure 11 demonstrates the variation of normalized thermophoretic force versus the Knudsen number for different values of thermal stress slip and micropolarity parameters. In the whole range of the Knudsen number, the thermophoretic force increases. Further, the thermophoretic force rises with an increase in a thermal stress slip parameter. In this figure, we observe that variation in the thermophoretic force shows less effect for micropolar fluids than viscous fluids due to their thickness.

Figure 12 shows the thermophoretic force decreases as the thermal conductivity ratio and micropolarity parameters increase. This phenomenon may occur because the high thermal conductivity of the particle reduces the thermal gradient at the particle surface for various values of the Knudsen number. Further, we observe the thermophoretic force declines as the Knudsen number decreases.

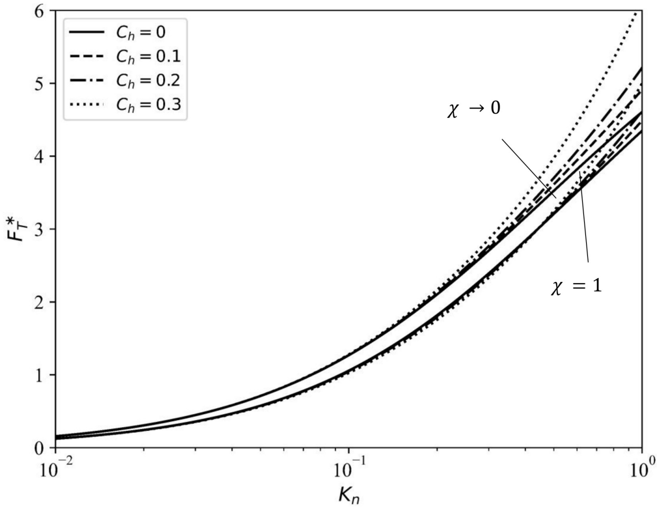

Fig. 11. Variation of F_T^* versus K_n when $k = 1$, $\delta\mu/a^2 = 0.1$, $k_1 = 1$.

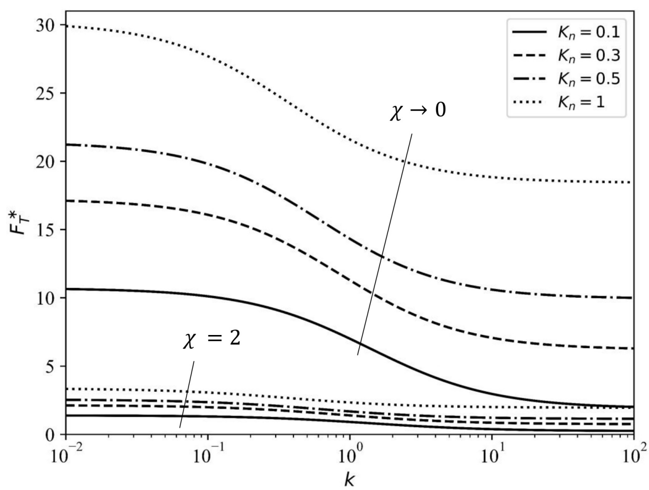

Fig. 12. Variation of F_T^* versus k when $C_h = 2$, $\delta\mu/a^2 = 0.1$, $k_1 = 0.01$.

In this graph, we conclude that the thermophoretic force is a decreasing function of the thermal conductivity ratio, and micropolarity parameters, and an increasing function of the Knudsen number.

Figure 13 portrays that the thermophoretic force decreases in the whole range of micropolarity parameter for various values of the frictional slip parameter and thermal conductivity ratio. An increased micropolarity parameter indicates

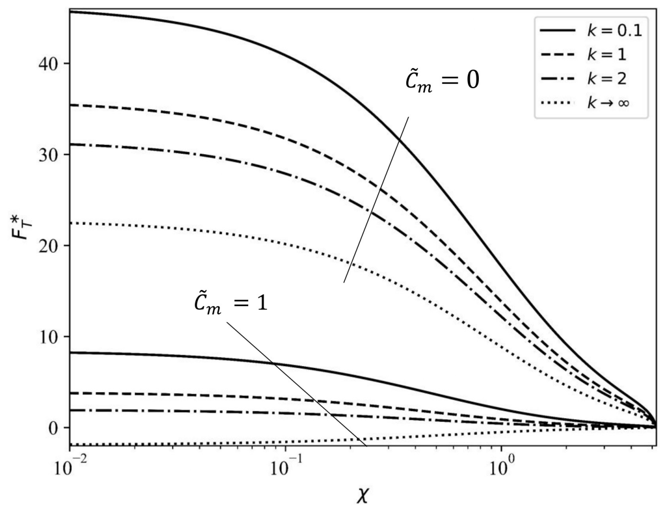

Fig. 13. Variation of F_T^* versus χ when $C_h = 0, K_n = 0.2$, $\delta\mu/a^2 = 0.2$, $k_1 = 0.01$.

higher microstructural effects than that of a Newtonian fluid and greater resistance for fluid flow caused by micro rotational viscosity. This causes a decreasing thermophoretic force. Also, we notice that the thermophoretic force is a decreasing function of the frictional slip parameter and thermal conductivity ratio. In all the above mentioned figures, the variation of the thermophoretic force is consistent with the findings of Faltas and Ragab [12] and Saad and Faltas [19].

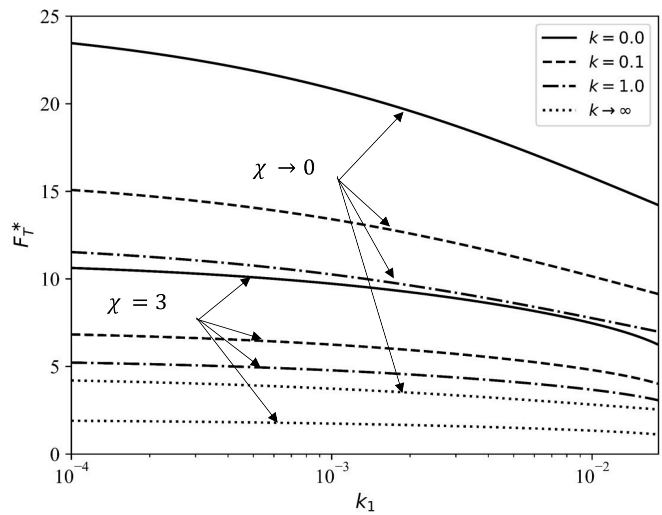

Fig. 14. Variation of F_T^* versus k_1 when $K_n = 0.2$, $C_h = 2$, $\delta\,\mu/a^2 = 0$.

The representation of the thermophoretic force of a particle versus the permeability for various parameters such as thermal conductivity ratio, frictional slip parameter, Knudsen number, microrotation thermal conductivity parameter, spin slip parameter, and temperature jump parameter is illustrated in Figs. 14–19. From these figures, it is evident that the thermophoretic force decreases as permeability increases. Figure 14 shows the impact of the thermal conductivity ratio on the thermophoretic force, indicating that the force declines

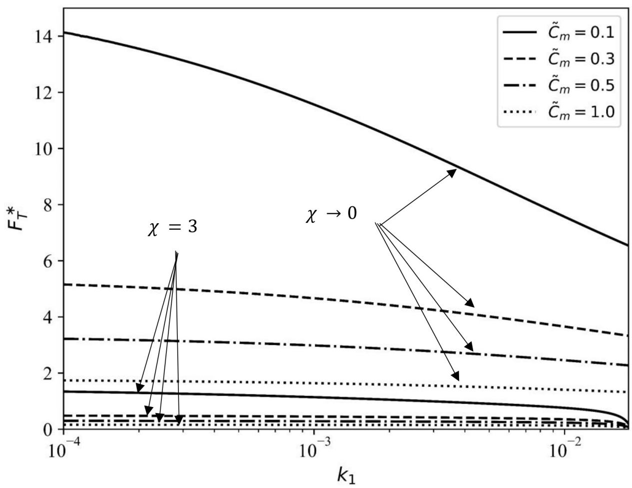

Fig. 15. Variation of F_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 2$, $\delta\,\mu/a^2 = 0.1$.

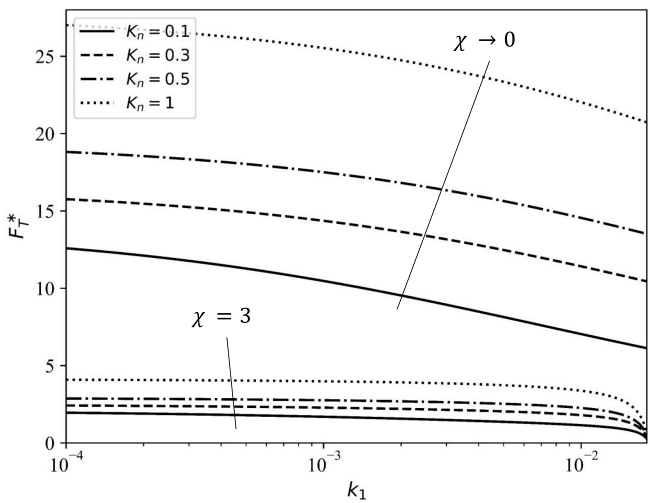

Fig. 16. Variation of F_T^* versus k_1 when $k = 1$, $C_h = 2$, $\delta\,\mu/a^2 = 0.1$.

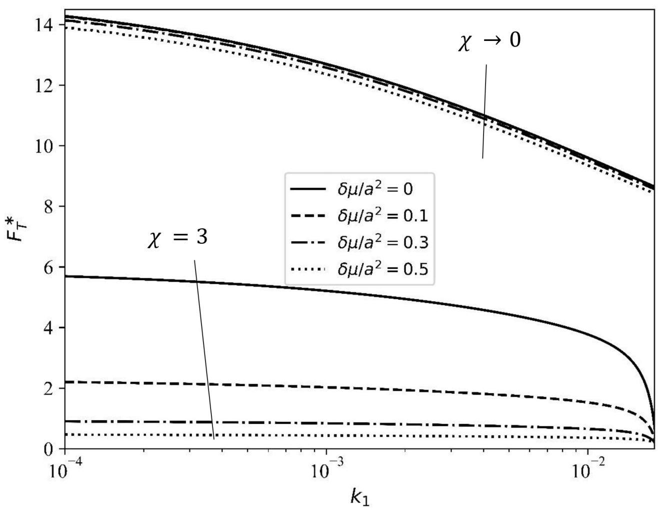

Fig. 17. Variation of F_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 1$.

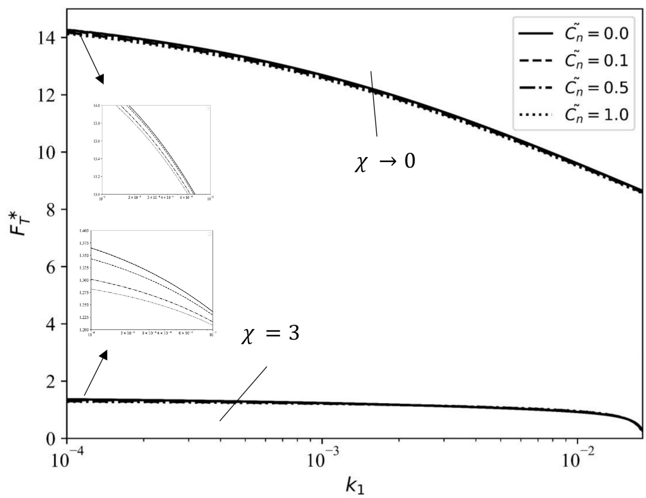

Fig. 18. Variation of F_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 1$, $\delta\,\mu/a^2 = 0.2$.

with higher values of the thermal conductivity ratio. In Fig. 15, the variation of the thermophoretic force with respect to the permeability is presented. The plot reveals that as the frictional slip parameter increases, the thermophoretic force decreases. Notably, the force is high when the frictional slip parameter is low. Graphical interpretation of the thermophoretic force of the particle with permeability corresponds to the Knudsen number and is pictured in Fig. 16.

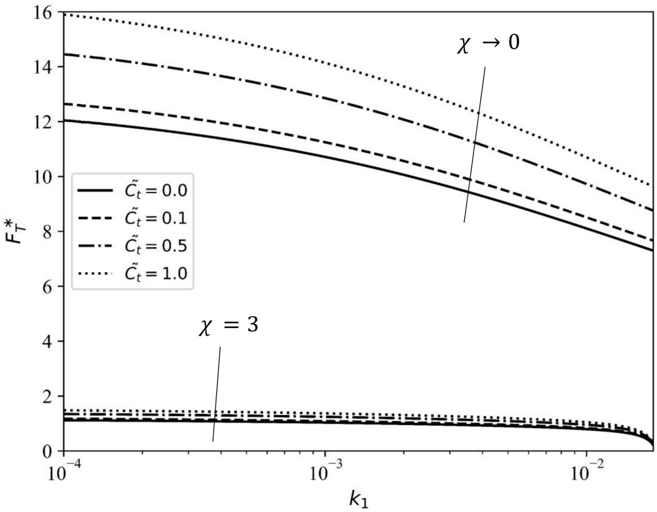

Fig. 19. Variation of F_T^* versus k_1 when $k = 1$, $K_n = 0.2$, $C_h = 2$, $\delta\,\mu/a^2 = 0.1$.

It demonstrates that the thermophoretic force F_T^* increases with a rise in the Knudsen number, while F_T^* decreases as the micropolarity parameter increases. In Fig. 17, we observe that the thermophoretic force is lower for higher values of the microrotation thermal conductivity parameter. Similarly, Fig. 18 shows that the thermophoretic force decreases with an increase in the spin slip parameter. The plot of F_T^* against permeability is visualized in Fig. 19. It can be seen that thermophoretic force increases with an increase in the thermal jump parameter. Throughout Figs. 14–19, we observe that the force in the case of a Newtonian fluid $\chi \to 0$ is higher than that of a micropolar fluid. The reason for this phenomenon has been explained earlier.

5. Conclusion

This work examines the thermophoretic motion of an aerosol sphere in an unbounded micropolar fluid saturated by the porous media. The temperature jump, continuity of heat flux and hydrodynamic slip conditions are assumed as the boundary conditions. Explicit expressions for the problem are reported. In this study, we have found that the thermophoretic velocity and thermophoretic force are functions of thermal conductivity ratio of particle and medium, Knudsen number, permeability, micropolarity parameter, microrotation thermal conductivity parameter, frictional slip parameter, and thermal stress slip parameter. The impact of these parameters on the thermophoretic velocity and force are shown graphically. From the figures, we concluded the following:

- Thermophoretis velocity shows an increasing effect with increasing permeability. However, the thermophoretic force decreases with enhancing permeability.
- As the micropolarity parameter increases, the thermophoretic velocity and force decline. Additionally, in the absence of frictional slip parameter, thermophoretic velocity and force rise.
- The thermophoretic velocity and force reduce for higher values of the microrotation thermal conductivity parameter.
- The increasing effect of thermal stress slip parameter on thermophoretic velocity and force is observed.
- Thermophoretic velocity decreases with increasing the Knudsen number, while the thermophoretic force shows increasing or decreasing effect with enhancing the Knudsen number.
- Thermophoresis velocity and thermophoretic force decrease as the thermal conductivity ratio increases.

Our results are also matched with previously published results available in the literature. The study of particle motion in micropolar fluid through a porous medium under the influence of a temperature field is very important from an industrial point of view to improve the filtration and separation process. Also, the analytical expressions and findings in this study are helpful for further understanding the thermophoretic motion of various shaped particles with strong interfacial interactions. For future research, it would be valuable to investigate the thermophoretic motion of different shaped particles, such as spheroids, cylinder and deformed spheres in non-Newtonian fluids and porous media saturated with non-Newtonian fluid.

6. Appendix

The expressions for S_3 to S_{12} appearing in Eqs.(3.31)–(3.33) are given as:

$$\begin{aligned}
S_3 &= K_{1/2}(\alpha), \qquad S_4 = K_{1/2}(\beta), \\
S_5 &= S_1 + \alpha S_3, \qquad S_6 = S_2 + \beta S_4, \\
S_7 &= S_1(\alpha^2 + 6 + \sigma(\alpha^2 - \xi_1)) + 2\alpha S_3, \\
S_8 &= S_2(\beta^2 + 6 + \sigma(\beta^2 - \xi_2)) + 2\beta S_4, \\
S_9 &= S_1(3 + \alpha^2) + \alpha S_3, \qquad S_{10} = S_2(3 + \beta^2) + \beta S_4, \\
S_{11} &= S_1 + \alpha S_3 \tilde{\gamma_0}, \qquad S_{12} = S_2 + \beta S_4 \tilde{\gamma_0}.
\end{aligned}$$

The expression for U_T is:

$$U_T = \frac{12VT_6}{S_{18}},$$

where the expressions for S_{18} and T_6 are:

$$S_{18} = \sum_{i=13}^{17} S_i, \quad T_6 = \sum_{j=1}^{5} T_j,$$

and

$$\begin{aligned}
S_{13} = {} & 2\tilde{C}_n\tilde{C}_t(C_h\tilde{C}_m S_2 S_9 - C_s S_1 S_6 + C_s S_2 S_5 - C_h\tilde{C}_m S_1 S_{10}) k\tilde{\gamma_0}\delta^3\eta^2\xi_1\xi_2\chi^2 \\
& + 2\tilde{C}_n(C_h\tilde{C}_m S_2 S_9 - C_s S_1 S_6 + C_s S_2 S_5 - C_h\tilde{C}_m S_1 S_{10})\tilde{\gamma_0}\delta^3\eta^2\xi_1\xi_2\chi^2 \\
& - (C_h\tilde{C}_m(2\tilde{C}_t + 1)(S_2 + \tilde{C}_n S_{12}) S_9 k\gamma_0\delta\eta^2\xi_1\xi_2\chi^2) \\
& + C_s(2\tilde{C}_t + 1)(\tilde{C}_n S_{11} + S_1) S_6 k\gamma_0\delta\eta^2\xi_1\xi_2\chi^2 \\
& - (C_s(2\tilde{C}_t + 1)(S_2 + \tilde{C}_n S_{12}) S_5 k\gamma_0\delta\eta^2\xi_1\xi_2\chi^2) \\
& - (2\tilde{C}_n(C_s + 3C_h\tilde{C}_m)\tilde{C}_t(S_{11}S_2 - S_1 S_{12}) k\gamma_0\delta\eta^2\xi_1\xi_2\chi^2) \\
& + C_h\tilde{C}_m(2\tilde{C}_t + 1) S_{10}(\tilde{C}_n S_{11} + S_1) k\gamma_0\delta\eta^2\xi_1\xi_2\chi^2 \\
& - (2(C_h\tilde{C}_m S_2 S_9 + C_h\tilde{C}_m\tilde{C}_n S_{12} S_9 - \tilde{C}_n C_s S_{11} S_6 - C_s S_1 S_6)\gamma_0\delta\eta^2\xi_1\xi_2\chi^2) \\
& - (2(C_s S_2 S_5 + \tilde{C}_n C_s S_{12} S_5 + \tilde{C}_n C_s S_{11} S_2 + 3C_h\tilde{C}_m\tilde{C}_n S_{11} S_2)\gamma_0\delta\eta^2\xi_1\xi_2\chi^2) \\
& + 2(\tilde{C}_n C_s S_1 S_{12} + 3C_h\tilde{C}_m\tilde{C}_n S_1 S_{12} + C_h\tilde{C}_m\tilde{C}_n S_{10} S_{11} \\
& + C_h\tilde{C}_m S_1 S_{10})\gamma_0\delta\eta^2\xi_1\xi_2\chi^2 - 6\tilde{C}_n\tilde{C}_t(C_h\tilde{C}_m S_2 S_9 \\
& - C_s S_1 S_6 + C_s S_2 S_5 - C_h\tilde{C}_m S_1 S_{10}) k\tilde{\gamma_0}\delta^3\eta^2\xi_1\xi_2\chi \\
& + 6\tilde{C}_n(C_h\tilde{C}_m S_2 S_9 - C_s S_1 S_6 + C_s S_2 S_5 - C_h\tilde{C}_m S_1 S_{10})\tilde{\gamma_0}\delta^3\eta^2\xi_1\xi_2\chi \\
& - (3C_h\tilde{C}_m(2\tilde{C}_t + 1)(S_2 + \tilde{C}_n S_{12}) S_9 k\gamma_0\delta\eta^2\xi_1\xi_2\chi) \\
& + 3C_s(2\tilde{C}_t + 1)(\tilde{C}_n S_{11} + S_1) S_6 k\gamma_0\delta\eta^2\xi_1\xi_2\chi,
\end{aligned}$$

$$\begin{aligned}
S_{14} = {} & -(3C_s(2\tilde{C}_t + 1)(S_2 + \tilde{C}_n S_{12}) S_5 k\gamma_0\delta\eta^2\xi_1\xi_2\chi) \\
& - (6\tilde{C}_n(C_s + 3C_h\tilde{C}_m)\tilde{C}_t(S_{11}S_2 - S_1 S_{12}) k\gamma_0\delta\eta^2\xi_1\xi_2\chi) \\
& + 3C_h\tilde{C}_m(2\tilde{C}_t + 1) S_{10}(\tilde{C}_n S_{11} + S_1) k\gamma_0\delta\eta^2\xi_1\xi_2\chi \\
& - (6(C_h\tilde{C}_m S_2 S_9 + C_h\tilde{C}_m\tilde{C}_n S_{12} S_9 - \tilde{C}_n C_s S_{11} S_6 - C_s S_1 S_6)\gamma_0\delta\eta^2\xi_1\xi_2\chi) \\
& - (6(C_s S_2 S_5 + \tilde{C}_n C_s S_{12} S_5 + \tilde{C}_n C_s S_{11} S_2 + 3C_h\tilde{C}_m\tilde{C}_n S_{11} S_2)\gamma_0\delta\eta^2\xi_1\xi_2\chi) \\
& + 6(\tilde{C}_n C_s S_1 S_{12} + 3C_h\tilde{C}_m\tilde{C}_n S_1 S_{12} + C_h\tilde{C}_m\tilde{C}_n S_{10} S_{11} \\
& + C_h\tilde{C}_m S_1 S_{10})\gamma_0\delta\eta^2\xi_1\xi_2\chi + 12(2\tilde{C}_m + 1)\tilde{C}_n(2\tilde{C}_t + 1) S_{11} S_2 k\gamma_0\xi_1\xi_2\chi \\
& - (12(2\tilde{C}_m + 1)\tilde{C}_n(2\tilde{C}_t + 1) S_1 S_{12} k\gamma_0\xi_1\xi_2\chi) \\
& + 24(2\tilde{C}_m + 1)\tilde{C}_n(S_{11}S_2 - S_1 S_{12})\gamma_0\xi_1\xi_2\chi \\
& - (8\tilde{C}_n S_2(\tilde{C}_m\tilde{C}_t S_7 k + \tilde{C}_t S_5 k + 2\tilde{C}_t S_1 k + \tilde{C}_m S_7)\tilde{\gamma_0}\delta^2\eta^2\xi_2\chi)
\end{aligned}$$

$$-(4S_2(2\tilde{C_n}S_5\tilde{\gamma_0}\delta^2+4\tilde{C_n}S_1\tilde{\gamma_0}\delta^2-2\tilde{C_m}\tilde{C_t}S_7k\gamma_0-\tilde{C_m}S_7k\gamma_0)\eta^2\xi_2\chi)$$
$$+4(2\tilde{C_t}+1)(\tilde{C_m}\tilde{C_n}S_{12}S_7+S_2S_5)k\gamma_0\eta^2\xi_2\chi,$$
$$S_{15}=4(2\tilde{C_t}+1)(\tilde{C_n}S_{12}S_5+2S_1S_2)k\gamma_0\eta^2\xi_2\chi+8(2\tilde{C_n}\tilde{C_t}S_1S_{12}k+\tilde{C_n}S_1S_{12}k$$
$$+\tilde{C_m}S_2S_7+\tilde{C_m}\tilde{C_n}S_{12}S_7)\gamma_0\eta^2\xi_2\chi+8(S_2+\tilde{C_n}S_{12})(S_5+2S_1)\gamma_0\eta^2\xi_2\chi$$
$$-(48(2\tilde{C_m}+1)\tilde{C_n}S_1S_2(\tilde{C_t}k+1)\alpha^2\tilde{\gamma_0}\delta^2\xi_2\chi)$$
$$+24(2\tilde{C_m}+1)(2\tilde{C_t}+1)S_1S_2k\alpha^2\gamma_0\xi_2\chi$$
$$+24(2\tilde{C_m}+1)\tilde{C_n}(2\tilde{C_t}+1)S_1S_{12}k\alpha^2\gamma_0\xi_2\chi$$
$$+48(2\tilde{C_m}+1)S_1(S_2+\tilde{C_n}S_{12})\alpha^2\gamma_0\xi_2\chi$$
$$+8\tilde{C_n}S_1(2\tilde{C_n}\tilde{C_t}S_1S_{12}k+\tilde{C_n}S_1S_{12}k+\tilde{C_m}S_2S_7$$
$$+\tilde{C_m}\tilde{C_n}S_{12}S_7)\tilde{\gamma_0}\delta^2\eta^2\xi_1\chi+4\tilde{C_n}(2S_1S_6\tilde{\gamma_0}\delta^2+4S_1S_2\tilde{\gamma_0}\delta^2$$
$$-2\tilde{C_m}\tilde{C_t}S_{11}S_8k\gamma_0-\tilde{C_m}S_{11}S_8k\gamma_0)\eta^2\xi_1\chi$$
$$-(4(2\tilde{C_t}+1)(\tilde{C_m}S_1S_8+\tilde{C_n}S_{11}S_6)k\gamma_0\eta^2\xi_1\chi)$$
$$-(4(2\tilde{C_t}+1)(S_1S_6+2\tilde{C_n}S_{11}S_2)k\gamma_0\eta^2\xi_1\chi)$$
$$-(8(2\tilde{C_n}\tilde{C_t}S_1S_{12}k+\tilde{C_n}S_1S_{12}k+\tilde{C_m}S_2S_7+\tilde{C_m}\tilde{C_n}S_{12}S_7)\gamma_0\eta^2\xi_1\chi)$$
$$-(8(\tilde{C_n}S_{11}+S_1)(S_6+2S_2)\gamma_0\eta^2\xi_1\chi)$$
$$+48(2\tilde{C_m}+1)\tilde{C_n}S_1S_2(\tilde{C_t}k+1)\beta^2\tilde{\gamma_0}\delta^2\xi_1\chi,$$
$$S_{16}=-(24(2\tilde{C_m}+1)\tilde{C_n}(2\tilde{C_t}+1)S_{11}S_2k\beta^2\gamma_0\xi_1\chi)$$
$$-(24(2\tilde{C_m}+1)(2\tilde{C_t}+1)S_1S_2k\beta^2\gamma_0\xi_1\chi)$$
$$-(48(2\tilde{C_m}+1)(\tilde{C_n}S_{11}+S_1)S_2\beta^2\gamma_0\xi_1\chi)+4\tilde{C_n}\tilde{C_t}(C_h\tilde{C_m}S_2S_9$$
$$-C_sS_1S_6+C_sS_2S_5-C_h\tilde{C_m}S_1S_{10})k\tilde{\gamma_0}\delta^3\eta^2\xi_1\xi_2$$
$$+4\tilde{C_n}(C_h\tilde{C_m}S_2S_9-C_sS_1S_6+C_sS_2S_5-C_h\tilde{C_m}S_1S_{10})\tilde{\gamma_0}\delta^3\eta^2\xi_1\xi_2$$
$$-(2C_h\tilde{C_m}(2\tilde{C_t}+1)(S_2+\tilde{C_n}S_{12})S_9k\gamma_0\delta\eta^2\xi_1\xi_2)$$
$$+2C_s(2\tilde{C_t}+1)(\tilde{C_n}S_{11}+S_1)S_6k\gamma_0\delta\eta^2\xi_1\xi_2$$
$$-(2C_s(2\tilde{C_t}+1)(S_2+\tilde{C_n}S_{12})S_5k\gamma_0\delta\eta^2\xi_1\xi_2)$$
$$-(4\tilde{C_n}(C_s+3C_h\tilde{C_m})\tilde{C_t}(S_{11}S_2-S_1S_{12})k\gamma_0\delta\eta^2\xi_1\xi_2)$$
$$+2C_h\tilde{C_m}(2\tilde{C_t}+1)S_{10}(\tilde{C_n}S_{11}+S_1)k\gamma_0\delta\eta^2\xi_1\xi_2$$
$$-(4(C_h\tilde{C_m}S_2S_9+C_h\tilde{C_m}\tilde{C_n}S_{12}S_9-\tilde{C_n}C_sS_{11}S_6-C_sS_1S_6)\gamma_0\delta\eta^2\xi_1\xi_2)$$
$$-(4(C_sS_2S_5+\tilde{C_n}C_sS_{12}S_5+\tilde{C_n}C_sS_{11}S_2+3C_h\tilde{C_m}\tilde{C_n}S_{11}S_2)\gamma_0\delta\eta^2\xi_1\xi_2)$$
$$+4(\tilde{C_n}C_sS_1S_{12}+3C_h\tilde{C_m}\tilde{C_n}S_1S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{10}S_{11}$$
$$+C_h\tilde{C_m}S_1S_{10})\gamma_0\delta\eta^2\xi_1\xi_2,$$

$$
\begin{aligned}
S_{17} = & -(8\tilde{C}_nS_2(\tilde{C}_m\tilde{C}_tS_7k+\tilde{C}_tS_5k+2\tilde{C}_tS_1k+\tilde{C}_mS_7)\tilde{\gamma_0}\delta^2\eta^2\xi_2)\\
& -(4S_2(2\tilde{C}_nS_5\tilde{\gamma_0}\delta^2+4\tilde{C}_nS_1\tilde{\gamma_0}\delta^2-2\tilde{C}_m\tilde{C}_tS_7k\gamma_0-\tilde{C}_mS_7k\gamma_0)\eta^2\xi_2)\\
& +4(2\tilde{C}_t+1)(\tilde{C}_m\tilde{C}_nS_{12}S_7+S_2S_5)k\gamma_0\eta^2\xi_2\\
& +4(2\tilde{C}_t+1)(\tilde{C}_nS_{12}S_5+2S_1S_2)k\gamma_0\eta^2\xi_2\\
& +8(2\tilde{C}_n\tilde{C}_tS_1S_{12}k+\tilde{C}_nS_1S_{12}k+\tilde{C}_mS_2S_7+\tilde{C}_m\tilde{C}_nS_{12}S_7)\gamma_0\eta^2\xi_2\\
& +8(S_2+\tilde{C}_nS_{12})(S_5+2S_1)\gamma_0\eta^2\xi_2\\
& -(48(2\tilde{C}_m+1)\tilde{C}_nS_1S_2(\tilde{C}_tk+1)\alpha^2\tilde{\gamma_0}\delta^2\xi_2)\\
& +24(2\tilde{C}_m+1)(2\tilde{C}_t+1)S_1S_2k\alpha^2\gamma_0\xi_2\\
& +24(2\tilde{C}_m+1)\tilde{C}_n(2\tilde{C}_t+1)S_1S_{12}k\alpha^2\gamma_0\xi_2\\
& +48(2\tilde{C}_m+1)S_1(S_2+\tilde{C}_nS_{12})\alpha^2\gamma_0\xi_2+8\tilde{C}_nS_1(\tilde{C}_m\tilde{C}_tS_8k\\
& +\tilde{C}_tS_6k+2\tilde{C}_tS_2k+\tilde{C}_mS_8)\tilde{\gamma_0}\delta^2\eta^2\xi_1\\
& +4\tilde{C}_n(2S_1S_6\tilde{\gamma_0}\delta^2+4S_1S_2\tilde{\gamma_0}\delta^2-2\tilde{C}_m\tilde{C}_tS_{11}S_8k\gamma_0-\tilde{C}_mS_{11}S_8k\gamma_0)\eta^2\xi_1\\
& -(4(2\tilde{C}_t+1)(\tilde{C}_mS_1S_8+\tilde{C}_nS_{11}S_6)k\gamma_0\eta^2\xi_1)\\
& -(4(2\tilde{C}_t+1)(S_1S_6+2\tilde{C}_nS_{11}S_2)k\gamma_0\eta^2\xi_1)\\
& -(8(2\tilde{C}_tS_1S_2k+S_1S_2k+\tilde{C}_m\tilde{C}_nS_{11}S_8+\tilde{C}_mS_1S_8)\gamma_0\eta^2\xi_1)\\
& -(8(\tilde{C}_nS_{11}+S_1)(S_6+2S_2)\gamma_0\eta^2\xi_1)\\
& +48(2\tilde{C}_m+1)\tilde{C}_nS_1S_2(\tilde{C}_tk+1)\beta^2\tilde{\gamma_0}\delta^2\xi_1\\
& -(24(2\tilde{C}_m+1)\tilde{C}_n(2\tilde{C}_t+1)S_{11}S_2k\beta^2\gamma_0\xi_1)\\
& -(24(2\tilde{C}_m+1)(2\tilde{C}_t+1)S_1S_2k\beta^2\gamma_0\xi_1)\\
& -(48(2\tilde{C}_m+1)(\tilde{C}_nS_{11}+S_1)S_2\beta^2\gamma_0\xi_1),\\
T_1 = & -\tilde{C}_n\tilde{C}_t(C_h\tilde{C}_mS_2S_9-C_sS_1S_6+C_sS_2S_5-C_h\tilde{C}_mS_1S_{10})k\tilde{\gamma_0}\delta^2\xi_1\xi_2\chi\\
& -\tilde{C}_n(C_h\tilde{C}_mS_2S_9-C_sS_1S_6+C_sS_2S_5-C_h\tilde{C}_mS_1S_{10})\tilde{\gamma_0}\delta^2\xi_1\xi_2\chi\\
& +(\tilde{C}_nk(C_s\tilde{C}_tS_{11}S_2+C_h\tilde{C}_m\tilde{C}_tS_{11}S_2-C_h\tilde{C}_mS_{11}S_2-C_s\tilde{C}_tS_1S_{12})\gamma_0\xi_1\xi_2\chi)\\
& -\tilde{C}_n(C_h\tilde{C}_m\tilde{C}_tS_1S_{12}k-C_h\tilde{C}_mS_1S_{12}k-C_sS_{11}S_2-C_h\tilde{C}_mS_{11}S_2)\gamma_0\xi_1\xi_2\chi\\
& -\tilde{C}_nS_1(C_sS_{12}\gamma_0\xi_1+C_h\tilde{C}_mS_{12}\gamma_0\xi_1+C_s\tilde{C}_tS_6k\tilde{\gamma_0}\delta^2\eta^2\\
& -2C_h\tilde{C}_m\tilde{C}_tS_2k\tilde{\gamma_0}\delta^2\eta^2)\xi_2\chi^2\\
& -\tilde{C}_nS_1(C_h\tilde{C}_m\tilde{C}_tS_{10}k+C_sS_6-2C_h\tilde{C}_mS_2+C_h\tilde{C}_mS_{10})\tilde{\gamma_0}\delta^2\eta^2\xi_2\chi\\
& +(S_1(C_s\tilde{C}_tS_2+C_h\tilde{C}_m\tilde{C}_tS_2-C_h\tilde{C}_mS_2+\tilde{C}_nC_s\tilde{C}_tS_{12})k\gamma_0\eta^2\xi_2\chi)\\
& +(S_1(C_h\tilde{C}_m\tilde{C}_n\tilde{C}_tS_{12}k-C_h\tilde{C}_m\tilde{C}_nS_{12}k+C_sS_2+C_h\tilde{C}_mS_2)\gamma_0\eta^2\xi_2\chi)\\
& +(\tilde{C}_nS_1(C_sS_{12}\gamma_0\eta^2+C_h\tilde{C}_mS_{12}\gamma_0\eta^2-2C_s\tilde{C}_tS_6k\alpha^2\tilde{\gamma_0}\delta^2
\end{aligned}
$$

$$+4C_h\tilde{C_m}\tilde{C_t}S_2k\alpha^2\tilde{\gamma_0}\delta^2)\xi_2\chi)$$
$$-2\tilde{C_n}S_1(C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6-2C_h\tilde{C_m}S_2+C_h\tilde{C_m}S_{10})\alpha^2\tilde{\gamma_0}\delta^2\xi_2\chi$$
$$+(2S_1(C_s\tilde{C_t}S_2+C_h\tilde{C_m}\tilde{C_t}S_2-C_h\tilde{C_m}S_2+\tilde{C_n}C_s\tilde{C_t}S_{12})k\alpha^2\gamma_0\xi_2\chi)$$
$$+(2S_1(C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}k-C_h\tilde{C_m}\tilde{C_n}S_{12}k+C_sS_2+C_h\tilde{C_m}S_2)\alpha^2\gamma_0\xi_2\chi)$$
$$+(\tilde{C_n}(2C_sS_1S_{12}\alpha^2\gamma_0\xi_2+C_s\tilde{C_t}S_2S_5k\tilde{\gamma_0}\delta^2\eta^2\xi_1+2C_h\tilde{C_m}S_1S_{12}\alpha^2\gamma_0\xi_2$$
$$+C_h\tilde{C_m}\tilde{C_t}S_2S_9k\tilde{\gamma_0}\delta^2\eta^2\xi_1)\chi)$$
$$-\tilde{C_n}S_2(2C_h\tilde{C_m}\tilde{C_t}S_1k-C_h\tilde{C_m}S_9-C_sS_5+2C_h\tilde{C_m}S_1)\tilde{\gamma_0}\delta^2\eta^2\xi_1\chi$$
$$-(\tilde{C_n}C_s\tilde{C_t}S_{11}+C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{11}-C_h\tilde{C_m}\tilde{C_n}S_{11}+C_s\tilde{C_t}S_1)S_2k\gamma_0\eta^2\xi_1\chi,$$
$$T_2=-S_2(C_h\tilde{C_m}\tilde{C_t}S_1k-C_h\tilde{C_m}S_1k+\tilde{C_n}C_sS_{11}+C_h\tilde{C_m}\tilde{C_n}S_{11})\gamma_0\eta^2\xi_1\chi$$
$$-S_2(C_sS_1\gamma_0\eta^2+C_h\tilde{C_m}S_1\gamma_0\eta^2-2C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_9k\beta^2\tilde{\gamma_0}\delta^2$$
$$-2\tilde{C_n}C_s\tilde{C_t}S_5k\beta^2\tilde{\gamma_0}\delta^2)\xi_1\chi$$
$$-2\tilde{C_n}S_2(2C_h\tilde{C_m}\tilde{C_t}S_1k-C_h\tilde{C_m}S_9-C_sS_5+2C_h\tilde{C_m}S_1)\beta^2\tilde{\gamma_0}\delta^2\xi_1\chi$$
$$-2(\tilde{C_n}C_s\tilde{C_t}S_{11}+C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{11}-C_h\tilde{C_m}\tilde{C_n}S_{11}+C_s\tilde{C_t}S_1)S_2k\beta^2\gamma_0\xi_1\chi$$
$$-2S_2(C_h\tilde{C_m}\tilde{C_t}S_1k-C_h\tilde{C_m}S_1k+\tilde{C_n}C_sS_{11}+C_h\tilde{C_m}\tilde{C_n}S_{11})\beta^2\gamma_0\xi_1\chi$$
$$-2\xi_1\chi(C_sS_1S_2\beta^2\gamma_0\chi+C_h\tilde{C_m}S_1S_2\beta^2\gamma_0\chi$$
$$+C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_2S_9k\tilde{\gamma_0}\delta^2\xi_2-\tilde{C_n}C_s\tilde{C_t}S_1S_6k\tilde{\gamma_0}\delta^2\xi_2)$$
$$-2\tilde{C_n}(C_s\tilde{C_t}S_2S_5k-C_h\tilde{C_m}\tilde{C_t}S_1S_{10}k+C_h\tilde{C_m}S_2S_9-C_sS_1S_6)\tilde{\gamma_0}\delta^2\xi_1\xi_2\chi$$
$$-2\tilde{C_n}(C_sS_2S_5\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}S_1S_{10}\tilde{\gamma_0}\delta^2$$
$$-C_s\tilde{C_t}S_{11}S_2k\gamma_0-C_h\tilde{C_m}\tilde{C_t}S_{11}S_2k\gamma_0)\xi_1\xi_2\chi$$
$$-2\tilde{C_n}(C_h\tilde{C_m}S_{11}S_2+C_s\tilde{C_t}S_1S_{12}+C_h\tilde{C_m}\tilde{C_t}S_1S_{12}-C_h\tilde{C_m}S_1S_{12})k\gamma_0\xi_1\xi_2\chi$$
$$+(2\tilde{C_n}(C_s+C_h\tilde{C_m})(S_{11}S_2-S_1S_{12})\gamma_0\xi_1\xi_2\chi)$$
$$-3\tilde{C_n}S_1(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\tilde{\gamma_0}\delta^2\eta^2\xi_2\chi$$
$$+(3S_1(2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0$$
$$+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\eta^2\xi_2\chi)$$
$$-3S_1(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\gamma_0\eta^2\xi_2\chi$$
$$+(3(C_s+C_h\tilde{C_m})S_1(S_2+\tilde{C_n}S_{12})\gamma_0\eta^2\xi_2\chi)$$
$$-6\tilde{C_n}S_1(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\alpha^2\tilde{\gamma_0}\delta^2\xi_2\chi,$$
$$T_3=(2\tilde{C_n}\tilde{\gamma_0}\delta(6C_h\tilde{C_m}S_1S_2\alpha^2\delta-3C_h\tilde{C_m}S_1S_{10}\alpha^2\delta+2\tilde{C_m}\tilde{C_t}S_2S_7k$$
$$+2\tilde{C_t}S_2S_5k)\xi_2\chi)$$
$$-4\tilde{C_n}S_2(6\tilde{C_m}\tilde{C_t}S_1k+\tilde{C_t}S_1k-\tilde{C_m}S_7-S_5)\tilde{\gamma_0}\delta\xi_2\chi$$

$$
\begin{aligned}
&-2S_1S_2(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\xi_2\chi\\
&-6S_1(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\alpha^2\gamma_0\xi_2\chi\\
&+(6(C_s+C_h\tilde{C_m})S_1(S_2+\tilde{C_n}S_{12})\alpha^2\gamma_0\xi_2\chi)\\
&+(3\tilde{C_n}S_2(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k\\
&+C_h\tilde{C_m}S_9)\tilde{\gamma_0}\delta^2\eta^2\xi_1\chi)\\
&+(3\tilde{C_n}S_2(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0\\
&-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\eta^2\xi_1\chi)\\
&+(3(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)S_2k\gamma_0\eta^2\xi_1\chi)\\
&-3(C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1)S_2\gamma_0\eta^2\xi_1\chi\\
&+(6\tilde{C_n}S_2(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\beta^2\tilde{\gamma_0}\delta^2\xi_1\chi)\\
&+(2\tilde{C_n}\tilde{\gamma_0}\delta(3C_sS_2S_5\beta^2\delta-6C_h\tilde{C_m}S_1S_2\beta^2\delta-2\tilde{C_m}\tilde{C_t}S_1S_8k\\
&-2\tilde{C_t}S_1S_6k)\xi_1\chi)\\
&+(4\tilde{C_n}S_1(6\tilde{C_m}\tilde{C_t}S_2k+\tilde{C_t}S_2k-\tilde{C_m}S_8-S_6)\tilde{\gamma_0}\delta\xi_1\chi)\\
&+(2\tilde{C_n}S_2(12\tilde{C_m}S_1\tilde{\gamma_0}\delta+2S_1\tilde{\gamma_0}\delta-3C_s\tilde{C_t}S_{11}k\beta^2\gamma_0\\
&-3C_h\tilde{C_m}\tilde{C_t}S_{11}k\beta^2\gamma_0)\xi_1\chi)\\
&+(6(C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1)),
\end{aligned}
$$

$$
\begin{aligned}
T_4 = {}&(4\tilde{C_n}\tilde{C_t}(\tilde{C_m}S_1S_8-\tilde{C_m}S_2S_7+S_1S_6-S_2S_5)k\tilde{\gamma_0}\delta\eta^2\chi)\\
&+(4\tilde{C_n}(\tilde{C_m}S_1S_8-\tilde{C_m}S_2S_7+S_1S_6-S_2S_5)\tilde{\gamma_0}\delta\eta^2\chi)\\
&-8\tilde{C_n}\tilde{C_t}S_2(\tilde{C_m}S_7+S_5-6\tilde{C_m}S_1-S_1)k\beta^2\tilde{\gamma_0}\delta\chi\\
&-8\tilde{C_n}S_2(\tilde{C_m}S_7+S_5-6\tilde{C_m}S_1-S_1)\beta^2\tilde{\gamma_0}\delta\chi\\
&+(8\tilde{C_n}\tilde{C_t}S_1(\tilde{C_m}S_8+S_6-6\tilde{C_m}S_2-S_2)k\alpha^2\tilde{\gamma_0}\delta\chi)\\
&+(8\tilde{C_n}S_1(\tilde{C_m}S_8+S_6-6\tilde{C_m}S_2-S_2)\alpha^2\tilde{\gamma_0}\delta\chi)\\
&-2\tilde{C_n}S_1(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\tilde{\gamma_0}\delta^2\eta^2\xi_2\\
&+(2S_1(2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0\\
&+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\eta^2\xi_2)\\
&-2S_1(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}\\
&+C_h\tilde{C_m}\tilde{C_n}S_{12})k\gamma_0\eta^2\xi_2\\
&+(2(C_s+C_h\tilde{C_m})S_1(S_2+\tilde{C_n}S_{12})\gamma_0\eta^2\xi_2)\\
&-4\tilde{C_n}S_1(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\alpha^2\tilde{\gamma_0}\delta^2\xi_2\\
&+(4S_1\alpha^2(2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0
\end{aligned}
$$

$$
\begin{aligned}
&+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\xi_2)\\
&-4S_1(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\alpha^2\gamma_0\xi_2\\
&+(4(C_s+C_h\tilde{C_m})S_1(S_2+\tilde{C_n}S_{12})\alpha^2\gamma_0\xi_2)\\
&-(2\tilde{C_n}S_2(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\tilde{\gamma_0}\delta^2\eta^2\xi_1),\\
T_5 =&-(2\tilde{C_n}S_2(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0\\
&-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\eta^2\xi_1)\\
&-(2(C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1)S_2k\gamma_0\eta^2\xi_1)\\
&-2(C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1)S_2\gamma_0\eta^2\xi_1\\
&+(4\tilde{C_n}S_2(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\beta^2\tilde{\gamma_0}\delta^2\xi_1)\\
&-(4\tilde{C_n}S_2\beta^2(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0\\
&-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\xi_1)\\
&+(4(C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1)\\
&-4(C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1))S_2\beta^2\gamma_0\xi_1\\
&+(4\tilde{C_n}\tilde{C_t}(\tilde{C_m}S_1S_8-\tilde{C_m}S_2S_7+S_1S_6-S_2S_5)k\tilde{\gamma_0}\delta\eta^2)\\
&+(4\tilde{C_n}(\tilde{C_m}S_1S_8-\tilde{C_m}S_2S_7+S_1S_6-S_2S_5)\tilde{\gamma_0}\delta\eta^2)\\
&-8\tilde{C_n}\tilde{C_t}S_2(\tilde{C_m}S_7+S_5-6\tilde{C_m}S_1-S_1)k\beta^2\tilde{\gamma_0}\delta\\
&+8\tilde{C_n}S_2(\tilde{C_m}S_7+S_5-6\tilde{C_m}S_1-S_1)\beta^2\tilde{\gamma_0}\delta\\
&-(8\tilde{C_n}\tilde{C_t}S_1(\tilde{C_m}S_8+S_6-6\tilde{C_m}S_2-S_2)k\alpha^2\tilde{\gamma_0}\delta)\\
&-(8\tilde{C_n}S_1(\tilde{C_m}S_8+S_6-6\tilde{C_m}S_2-S_2)\alpha^2\tilde{\gamma_0}\delta).
\end{aligned}
$$

The expression for A is:

$$A = \frac{W_1}{S_{18}},$$

where

$$
\begin{aligned}
W_1 =&\ \tilde{C_n}S_1(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\tilde{\gamma_0}\delta^2\xi_2\chi\\
&-(S_1(2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0\\
&+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\xi_2\chi)\\
&+S_1(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\gamma_0\xi_2\chi\\
&-((C_s+C_h\tilde{C_m})S_1(S_2+\tilde{C_n}S_{12})\gamma_0\xi_2\chi)\\
&-(\tilde{C_n}S_2(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\tilde{\gamma_0}\delta^2\xi_1\chi)\\
&-(\tilde{C_n}S_2(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\xi_1\chi)\\
&-((C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1)S_2k\gamma_0\xi_1\chi)
\end{aligned}
$$

$$
\begin{aligned}
&+(C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1)S_2\gamma_0\xi_1\chi\\
&+2\tilde{C_n}S_1(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\tilde{\gamma_0}\delta^2\xi_2\\
&-(2S_1(2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0\\
&+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\xi_2)\\
&+2S_1(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\gamma_0\xi_2\\
&-(2(C_s+C_h\tilde{C_m})S_1(S_2+\tilde{C_n}S_{12})\gamma_0\xi_2)\\
&-(2\tilde{C_n}S_2(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\tilde{\gamma_0}\\
&-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\tilde{\gamma_0}\delta^2\xi_1)\\
&-(2\tilde{C_n}S_2(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\xi_1)\\
&-(2(C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1)S_2k\gamma_0\xi_1)\\
&+2(C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1)S_2\gamma_0\xi_1\\
&-(4\tilde{C_n}\tilde{C_t}(\tilde{C_m}S_1S_8-\tilde{C_m}S_2S_7+S_1S_6-S_2S_5)k\tilde{\gamma_0}\delta)\\
&-(4\tilde{C_n}(\tilde{C_m}S_1S_8-\tilde{C_m}S_2S_7+S_1S_6-S_2S_5)\tilde{\gamma_0}\delta).
\end{aligned}
$$

The expression for B is:

$$B=\frac{X_1}{S_{18}},$$

where

$$
\begin{aligned}
X_1=&-\tilde{C_n}(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\tilde{\gamma_0}\delta^2\xi_2\chi\\
&+((2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0\\
&+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\xi_2\chi)\\
&-(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\gamma_0\xi_2\chi\\
&+((C_s+C_h\tilde{C_m})(S_2+\tilde{C_n}S_{12})\gamma_0\xi_2\chi)\\
&-2\tilde{C_n}(C_s\tilde{C_t}S_6k-2C_h\tilde{C_m}\tilde{C_t}S_2k+C_h\tilde{C_m}\tilde{C_t}S_{10}k+C_sS_6)\tilde{\gamma_0}\delta^2\xi_2\\
&+(2(2C_h\tilde{C_m}\tilde{C_n}S_2\tilde{\gamma_0}\delta^2-C_h\tilde{C_m}\tilde{C_n}S_{10}\tilde{\gamma_0}\delta^2+C_s\tilde{C_t}S_2k\gamma_0\\
&+C_h\tilde{C_m}\tilde{C_t}S_2k\gamma_0)\xi_2)\\
&-2(C_h\tilde{C_m}S_2-\tilde{C_n}C_s\tilde{C_t}S_{12}-C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_{12}+C_h\tilde{C_m}\tilde{C_n}S_{12})k\gamma_0\xi_2\\
&+(2(C_s+C_h\tilde{C_m})(S_2+\tilde{C_n}S_{12})\gamma_0\xi_2)\\
&+(4\tilde{C_n}\tilde{C_t}(\tilde{C_m}S_8+S_6-6\tilde{C_m}S_2-S_2)k\tilde{\gamma_0}\delta)\\
&+(4\tilde{C_n}(\tilde{C_m}S_8+S_6-6\tilde{C_m}S_2-S_2)\tilde{\gamma_0}\delta).
\end{aligned}
$$

The expression for C is:

$$C=\frac{Y_1}{S_{18}},$$

where

$$\begin{aligned}
Y_1 = {} & \tilde{C_n}(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\tilde{\gamma_0}\delta^2\xi_1\chi \\
& +\tilde{C_n}(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\xi_1\chi \\
& +(C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1)k\gamma_0\xi_1\chi \\
& -((C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1)\gamma_0\xi_1\chi) \\
& +2\tilde{C_n}(C_h\tilde{C_m}\tilde{C_t}S_9k+C_s\tilde{C_t}S_5k-2C_h\tilde{C_m}\tilde{C_t}S_1k+C_h\tilde{C_m}S_9)\tilde{\gamma_0}\delta^2\xi_1 \\
& +2\tilde{C_n}(C_sS_5\tilde{\gamma_0}\delta^2-2C_h\tilde{C_m}S_1\tilde{\gamma_0}\delta^2-C_s\tilde{C_t}S_{11}k\gamma_0-C_h\tilde{C_m}\tilde{C_t}S_{11}k\gamma_0)\xi_1 \\
& +2(C_h\tilde{C_m}\tilde{C_n}S_{11}-C_s\tilde{C_t}S_1-C_h\tilde{C_m}\tilde{C_t}S_1+C_h\tilde{C_m}S_1) \\
& -(2(C_s+C_h\tilde{C_m})(\tilde{C_n}S_{11}+S_1)\gamma_0\xi_1) \\
& -(4\tilde{C_n}\tilde{C_t}(\tilde{C_m}S_7+S_5-6\tilde{C_m}S_1-S_1)k\tilde{\gamma_0}\delta) \\
& -(4\tilde{C_n}(\tilde{C_m}S_7+S_5-6\tilde{C_m}S_1-S_1)\tilde{\gamma_0}\delta).
\end{aligned}$$

The expression for A_1 is:

$$A_1 = \frac{A_0}{S_{18}},$$

where

$$\begin{aligned}
A_0 = {} & -(3(C_h\tilde{C_m}S_2S_9\delta\xi_1\xi_2\chi+C_h\tilde{C_m}\tilde{C_n}S_{12}S_9\delta\xi_1\xi_2\chi-\tilde{C_n}C_sS_{11}S_6\delta\xi_1\xi_2\chi \\
& -C_sS_1S_6\delta\xi_1\xi_2\chi+C_sS_2S_5\delta\xi_1\xi_2\chi+\tilde{C_n}C_sS_{12}S_5\delta\xi_1\xi_2\chi \\
& +2C_h\tilde{C_m}\tilde{C_n}S_{11}S_2\delta\xi_1\xi_2\chi-2C_h\tilde{C_m}\tilde{C_n}S_1S_{12}\delta\xi_1\xi_2\chi-C_h\tilde{C_m}\tilde{C_n}S_{10}S_{11}\delta\xi_1\xi_2\chi \\
& -C_h\tilde{C_m}S_1S_{10}\delta\xi_1\xi_2\chi+2C_h\tilde{C_m}S_2S_9\delta\xi_1\xi_2+2C_h\tilde{C_m}\tilde{C_n}S_{12}S_9\delta\xi_1\xi_2 \\
& -2\tilde{C_n}C_sS_{11}S_6\delta\xi_1\xi_2-2C_sS_1S_6\delta\xi_1\xi_2+2C_sS_2S_5\delta\xi_1\xi_2+2\tilde{C_n}C_sS_{12}S_5\delta\xi_1\xi_2 \\
& +4C_h\tilde{C_m}\tilde{C_n}S_{11}S_2\delta\xi_1\xi_2-4C_h\tilde{C_m}\tilde{C_n}S_1S_{12}\delta\xi_1\xi_2-2C_h\tilde{C_m}\tilde{C_n}S_{10}S_{11}\delta\xi_1\xi_2 \\
& -2C_h\tilde{C_m}S_1S_{10}\delta\xi_1\xi_2-4\tilde{C_m}S_2S_7\xi_2-4\tilde{C_m}\tilde{C_n}S_{12}S_7\xi_2 \\
& -4S_2S_5\xi_2-4\tilde{C_n}S_{12}S_5\xi_2+24\tilde{C_m}S_1S_2\xi_2+4S_1S_2\xi_2+24\tilde{C_m}\tilde{C_n}S_1S_{12}\xi_2 \\
& +4\tilde{C_n}S_1S_{12}\xi_2+4\tilde{C_m}\tilde{C_n}S_{11}S_8\xi_1+4\tilde{C_m}S_1S_8\xi_1+4\tilde{C_n}S_{11}S_6\xi_1+4S_1S_6\xi_1 \\
& -24\tilde{C_m}\tilde{C_n}S_{11}S_2\xi_1-4\tilde{C_n}S_{11}S_2\xi_1-24\tilde{C_m}S_1S_2\xi_1-4S_1S_2\xi_1))/4.
\end{aligned}$$

The expression for B_1 is:

$$B_1 = \frac{B_0}{S_{18}},$$

where

$$\begin{aligned}
B_0 = {} & -(2C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_2S_9k\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi-2\tilde{C_n}C_s\tilde{C_t}S_1S_6k\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi \\
& +2\tilde{C_n}C_s\tilde{C_t}S_2S_5k\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi-2C_h\tilde{C_m}\tilde{C_n}\tilde{C_t}S_1S_{10}k\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi \\
& +2C_h\tilde{C_m}\tilde{C_n}S_2S_9\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi-2\tilde{C_n}C_sS_1S_6\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi
\end{aligned}$$

$$
\begin{aligned}
&+2\tilde{C}_n C_s S_2 S_5 \tilde{\gamma_0}\delta^3\xi_1\xi_2\chi - 2C_h\tilde{C}_m\tilde{C}_n S_1 S_{10}\tilde{\gamma_0}\delta^3\xi_1\xi_2\chi \\
&+C_h\tilde{C}_m\tilde{C}_t S_2 S_9 k\gamma_0\delta\xi_1\xi_2\chi - C_h\tilde{C}_m S_2 S_9 k\gamma_0\delta\xi_1\xi_2\chi \\
&+C_h\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{12} S_9 k\gamma_0\delta\xi_1\xi_2\chi - C_h\tilde{C}_m\tilde{C}_n S_{12} S_9 k\gamma_0\delta\xi_1\xi_2\chi \\
&-\tilde{C}_n C_s\tilde{C}_t S_{11} S_6 k\gamma_0\delta\xi_1\xi_2\chi + \tilde{C}_n C_s S_{11} S_6 k\gamma_0\delta\xi_1\xi_2\chi \\
&-C_s\tilde{C}_t S_1 S_6 k\gamma_0\delta\xi_1\xi_2\chi + C_s S_1 S_6 k\gamma_0\delta\xi_1\xi_2\chi + C_s\tilde{C}_t S_2 S_5 k\gamma_0\delta\xi_1\xi_2\chi \\
&-C_s S_2 S_5 k\gamma_0\delta\xi_1\xi_2\chi + \tilde{C}_n C_s\tilde{C}_t S_{12} S_5 k\gamma_0\delta\xi_1\xi_2\chi - \tilde{C}_n C_s S_{12} S_5 k\gamma_0\delta\xi_1\xi_2\chi \\
&-2\tilde{C}_n C_s\tilde{C}_t S_{11} S_2 k\gamma_0\delta\xi_1\xi_2\chi + 2\tilde{C}_n C_s\tilde{C}_t S_1 S_{12} k\gamma_0\delta\xi_1\xi_2\chi \\
&-C_h\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{10} S_{11} k\gamma_0\delta\xi_1\xi_2\chi + C_h\tilde{C}_m\tilde{C}_n S_{10} S_{11} k\gamma_0\delta\xi_1\xi_2\chi \\
&-C_h\tilde{C}_m\tilde{C}_t S_1 S_{10} k\gamma_0\delta\xi_1\xi_2\chi + C_h\tilde{C}_m S_1 S_{10} k\gamma_0\delta\xi_1\xi_2\chi + C_h\tilde{C}_m S_2 S_9\gamma_0\delta\xi_1\xi_2\chi \\
&+C_h\tilde{C}_m\tilde{C}_n S_{12} S_9\gamma_0\delta\xi_1\xi_2\chi - \tilde{C}_n C_s S_{11} S_6\gamma_0\delta\xi_1\xi_2\chi - C_s S_1 S_6\gamma_0\delta\xi_1\xi_2\chi \\
&+C_s S_2 S_5\gamma_0\delta\xi_1\xi_2\chi + \tilde{C}_n C_s S_{12} S_5\gamma_0\delta\xi_1\xi_2\chi - 2\tilde{C}_n C_s S_{11} S_2\gamma_0\delta\xi_1\xi_2\chi \\
&+2\tilde{C}_n C_s S_1 S_{12}\gamma_0\delta\xi_1\xi_2\chi - C_h\tilde{C}_m\tilde{C}_n S_{10} S_{11}\gamma_0\delta\xi_1\xi_2\chi \\
&-C_h\tilde{C}_m S_1 S_{10}\gamma_0\delta\xi_1\xi_2\chi + 4C_h\tilde{C}_m\tilde{C}_n\tilde{C}_t S_2 S_9 k\tilde{\gamma_0}\delta^3\xi_1\xi_2 \\
&-4\tilde{C}_n C_s\tilde{C}_t S_1 S_6 k\tilde{\gamma_0}\delta^3\xi_1\xi_2 + 4\tilde{C}_n C_s\tilde{C}_t S_2 S_5 k\tilde{\gamma_0}\delta^3\xi_1\xi_2 \\
&-4C_h\tilde{C}_m\tilde{C}_n\tilde{C}_t S_1 S_{10} k\tilde{\gamma_0}\delta^3\xi_1\xi_2 + 4C_h\tilde{C}_m\tilde{C}_n S_2 S_9\tilde{\gamma_0}\delta^3\xi_1\xi_2 \\
&-4\tilde{C}_n C_s S_1 S_6\tilde{\gamma_0}\delta^3\xi_1\xi_2 + 4\tilde{C}_n C_s S_2 S_5\tilde{\gamma_0}\delta^3\xi_1\xi_2 \\
&-4C_h\tilde{C}_m\tilde{C}_n S_1 S_{10}\tilde{\gamma_0}\delta^3\xi_1\xi_2 + 2C_h\tilde{C}_m\tilde{C}_t S_2 S_9 k\gamma_0\delta\xi_1\xi_2 \\
&-2C_h\tilde{C}_m S_2 S_9 k\gamma_0\delta\xi_1\xi_2 + 2C_h\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{12} S_9 k\gamma_0\delta\xi_1\xi_2 \\
&-2C_h\tilde{C}_m\tilde{C}_n S_{12} S_9 k\gamma_0\delta\xi_1\xi_2 - 2\tilde{C}_n C_s\tilde{C}_t S_{11} S_6 k\gamma_0\delta\xi_1\xi_2 \\
&+2\tilde{C}_n C_s S_{11} S_6 k\gamma_0\delta\xi_1\xi_2 - 2C_s\tilde{C}_t S_1 S_6 k\gamma_0\delta\xi_1\xi_2 + 2C_s S_1 S_6 k\gamma_0\delta\xi_1\xi_2 \\
&+2C_s\tilde{C}_t S_2 S_5 k\gamma_0\delta\xi_1\xi_2 - 2C_s S_2 S_5 k\gamma_0\delta\xi_1\xi_2 + 2\tilde{C}_n C_s\tilde{C}_t S_{12} S_5 k\gamma_0\delta\xi_1\xi_2 \\
&-2\tilde{C}_n C_s S_{12} S_5 k\gamma_0\delta\xi_1\xi_2 - 4\tilde{C}_n C_s\tilde{C}_t S_{11} S_2 k\gamma_0\delta\xi_1\xi_2 \\
&+4\tilde{C}_n C_s\tilde{C}_t S_1 S_{12} k\gamma_0\delta\xi_1\xi_2 - 2C_h\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{10} S_{11} k\gamma_0\delta\xi_1\xi_2 \\
&+2C_h\tilde{C}_m\tilde{C}_n S_{10} S_{11} k\gamma_0\delta\xi_1\xi_2 - 2C_h\tilde{C}_m\tilde{C}_t S_1 S_{10} k\gamma_0\delta\xi_1\xi_2 \\
&+2C_h\tilde{C}_m S_1 S_{10} k\gamma_0\delta\xi_1\xi_2 + 2C_h\tilde{C}_m S_2 S_9\gamma_0\delta\xi_1\xi_2 + 2C_h\tilde{C}_m\tilde{C}_n S_{12} S_9\gamma_0\delta\xi_1\xi_2 \\
&-2\tilde{C}_n C_s S_{11} S_6\gamma_0\delta\xi_1\xi_2 - 2C_s S_1 S_6\gamma_0\delta\xi_1\xi_2 + 2C_s S_2 S_5\gamma_0\delta\xi_1\xi_2 \\
&+2\tilde{C}_n C_s S_{12} S_5\gamma_0\delta\xi_1\xi_2 - 4\tilde{C}_n C_s S_{11} S_2\gamma_0\delta\xi_1\xi_2 + 4\tilde{C}_n C_s S_1 S_{12}\gamma_0\delta\xi_1\xi_2 \\
&-2C_h\tilde{C}_m\tilde{C}_n S_{10} S_{11}\gamma_0\delta\xi_1\xi_2 - 2C_h\tilde{C}_m S_1 S_{10}\gamma_0\delta\xi_1\xi_2 - 8\tilde{C}_m\tilde{C}_n\tilde{C}_t S_2 S_7 k\tilde{\gamma_0}\delta^2\xi_2 \\
&-8\tilde{C}_n\tilde{C}_t S_2 S_5 k\tilde{\gamma_0}\delta^2\xi_2 + 48\tilde{C}_m\tilde{C}_n\tilde{C}_t S_1 S_2 k\tilde{\gamma_0}\delta^2\xi_2 + 8\tilde{C}_n\tilde{C}_t S_1 S_2 k\tilde{\gamma_0}\delta^2\xi_2 \\
&-8\tilde{C}_m\tilde{C}_n S_2 S_7\tilde{\gamma_0}\delta^2\xi_2 - 8\tilde{C}_n S_2 S_5\tilde{\gamma_0}\delta^2\xi_2 + 48\tilde{C}_m\tilde{C}_n S_1 S_2\tilde{\gamma_0}\delta^2\xi_2
\end{aligned}
$$

$$
\begin{aligned}
&+8\tilde{C}_n S_1 S_2 \tilde{\gamma_0}\delta^2\xi_2 - 4\tilde{C}_m\tilde{C}_t S_2 S_7 k\gamma_0\xi_2 + 4\tilde{C}_m S_2 S_7 k\gamma_0\xi_2\\
&-4\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{12} S_7 k\gamma_0\xi_2 + 4\tilde{C}_m\tilde{C}_n S_{12} S_7 k\gamma_0\xi_2 - 4\tilde{C}_t S_2 S_5 k\gamma_0\xi_2\\
&+4S_2 S_5 k\gamma_0\xi_2 - 4\tilde{C}_n\tilde{C}_t S_{12} S_5 k\gamma_0\xi_2 + 4\tilde{C}_n S_{12} S_5 k\gamma_0\xi_2\\
&+24\tilde{C}_m\tilde{C}_t S_1 S_2 k\gamma_0\xi_2 + 4\tilde{C}_t S_1 S_2 k\gamma_0\xi_2 - 24\tilde{C}_m S_1 S_2 k\gamma_0\xi_1\\
&-(4S_1(S_2 - 6\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{12} - \tilde{C}_n\tilde{C}_t S_{12} + 6\tilde{C}_m\tilde{C}_n S_{12})k\gamma_0\xi_2)\\
&-(4(\tilde{C}_n S_1 S_{12} k + \tilde{C}_m S_2 S_7 + \tilde{C}_m\tilde{C}_n S_{12} S_7 + S_2 S_5)\gamma_0\xi_2)\\
&-(4(\tilde{C}_n S_{12} S_5 - 6\tilde{C}_m S_1 S_2 - S_1 S_2 - 6\tilde{C}_m\tilde{C}_n S_1 S_{12})\gamma_0\xi_2)\\
&+4\tilde{C}_n S_1(S_{12}\gamma_0\xi_2 + 2\tilde{C}_m\tilde{C}_t S_8 k\tilde{\gamma_0}\delta^2\xi_1 + 2\tilde{C}_t S_6 k\tilde{\gamma_0}\delta^2\xi_1 - 12\tilde{C}_m\tilde{C}_t S_2 k\tilde{\gamma_0}\delta^2\xi_1)\\
&-(8\tilde{C}_n S_1(\tilde{C}_t S_2 k - \tilde{C}_m S_8 - S_6 + 6\tilde{C}_m S_2)\tilde{\gamma_0}\delta^2\xi_1) - (4(2\tilde{C}_n S_1 S_2\tilde{\gamma_0}\delta^2\\
&-\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{11} S_8 k\gamma_0 + \tilde{C}_m\tilde{C}_n S_{11} S_8 k\gamma_0 - \tilde{C}_m\tilde{C}_t S_1 S_8 k\gamma_0)\xi_1)\\
&-(4(\tilde{C}_m S_1 S_8 - \tilde{C}_n\tilde{C}_t S_{11} S_6 + \tilde{C}_n S_{11} S_6 - \tilde{C}_t S_1 S_6)k\gamma_0\xi_1)\\
&-(4(S_1 S_6 + 6\tilde{C}_m\tilde{C}_n\tilde{C}_t S_{11} S_2 + \tilde{C}_n\tilde{C}_t S_{11} S_2 - 6\tilde{C}_m\tilde{C}_n S_{11} S_2)k\gamma_0\xi_1)\\
&+4(\tilde{C}_n S_{11} - 6\tilde{C}_m\tilde{C}_t S_1 - \tilde{C}_t S_1 + 6\tilde{C}_m S_1)S_2 k\gamma_0\xi_1\\
&+4(S_1 S_2 k + \tilde{C}_m\tilde{C}_n S_{11} S_8 + \tilde{C}_m S_1 S_8 + \tilde{C}_n S_{11} S_6)\gamma_0\xi_1 + 4(S_1 S_6 - 6\tilde{C}_m\tilde{C}_n S_{11} S_2\\
&-\tilde{C}_n S_{11} S_2 - 6\tilde{C}_m S_1 S_2 - S_1 S_2)\gamma_0\xi_1)/(4\gamma_0).
\end{aligned}
$$

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