ARCHIVES OF MECHANICS
Arch. Mech. **76** (6), 515–532, 2024, DOI: 10.24423/aom.4555

Effect of initial stress on the reflection and transmission of plane waves at an imperfect interface between two orthotropic elastic half-spaces

D. X. TUNG

Faculty of Civil Engineering, Hanoi Architectural University, Km 10 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam, e-mail address: tungdx2783@gmail.com

The effects of initial stress on the reflection and transmission waves at the imperfect interface between two orthotropic half spaces are studied in this paper. A linear spring model is used to describe the imperfection of bonding behavior at the interface. Reflection and transmission coefficients (RTCs) have been derived analytically when a quasi-longitudinal (qP) wave strikes for both the imperfect and perfect interface. Finally, numerical examples are provided to show the effect of the imperfect interface, initial stress and incident angle on the RTCs, energy ratios, reflection and transmission angles of waves.

Key words: reflection, transmission, orthotropic, initial stress, imperfect interface.

1. Introduction

The study of reflection and transmission phenomena of a plane wave at the separating interface of two different media is of core importance in many research areas such as non-destructive testing, reflection seismology, smart material technology, etc. Considering initial stresses in the study of reflection and transmission phenomena is of extreme importance as they impact the mechanical reaction of materials when acted upon by an incident wave. Pioneering work on developing the theory of waves in elastic media with initial stresses was done by Biot [3, 4], who developed the constitutive relations for a prestressed elastic medium and presented the equations of motion for elastic waves. However, Shams and Ogden [22] pointed out that the initial stress was generally not associated with finite deformation in Biot's works, although he did apply his theory to the case in which the initial stress is accompanied by a finite deformation. In [23], the effect of prestresses on the existence of longitudinal waves in an anisotropic elastic medium is taken into consideration. Due to prestresses, a medium of any type (isotropic/anisotropic) behaves anisotropic to wave propagation, and the equation of motion is significantly affected. Singh and Tomar [25] studied the problem of qP-waves at a corrugated interface between two dissimilar prestressed elastic half-spaces. Using Rayleigh's method of approximation, authors obtained the reflection and refraction coefficients corresponding to regular and irregular waves. Additionally, the problem of wave propagation in a prestressed piezoelectric, elastic half-space and the effect of initial stresses on the reflection coefficients of the reflected qP- and qSV-waves are discussed by many authors [5, 7, 9, 10, 17, 21, 24, 26, 27].

In past studies, many researchers have considered a perfectly bonded (welded contact) separating interface between two dissimilar media. In such a case, displacement and traction components are continuous at the separating interface. But, due to various reasons such as the aging of glue applied to two conjunct solids, micro-defects, diffusion impurities, and other forms of damage, discontinuities may occur at the separating interface between two dissimilar media. In the present investigation, imperfect bonding is defined as a condition where the stress components are continuous and a small displacement field is not. For the imperfect interface, the spring model of Hashin [13] is commonly used. In this model, the properties of the imperfect interface between the two solids can be characterized by the normal and tangential interfacial stiffnesses. Several authors have attempted to incorporate the effect of imperfect bonding on their problems, for example, Goyal *et al.* [8], Huang and Rokhlin [14], Kumar *et al.* [16], Pang and Liu [20], etc. Tung [28, 29] investigated the influence of boundary conditions on the reflection and transmission of qP-wave at an interface between two nonlocal transversely isotropic elastic, liquid-saturated porous half-spaces.

In light of the aforementioned literature, it is evident that no investigations have yet been conducted on the reflection and transmission of waves at the imperfect interface between two prestressed half-spaces. Recently, Guha and Singh [11] have investigated the plane wave reflection/transmission in imperfectly bonded initially stressed rotating piezothermoelastic fiber reinforced composite half-spaces. The medium that the authors consider is quite general. However, the authors obtained the following polynomial characteristic equation (40) of degree eight in q^m. It is an equation with complex coefficients. The choice of reflected and transmitted waves from this equation remains an unsatisfactory problem. In this medium, waves are inhomogeneous. Therefore, in this paper, the reflection and transmission problem at an imperfect interface between two orthotropic elastic half-spaces with prestressed effects taken into account are considered. The novelty of this paper is the comparison of reflection, transmission coefficients and reflection, transmission angles of waves with the presence or absence of initial stress as well as perfect or imperfect interface. These results are recorded and used as the input data of the inverse problem (nondestructive evaluation of materials). Such a study may also be helpful in predicting the anomalies and the actual cause of destruction. Moreover, this makes it an essential parameter that should be considered when designing devices.

2. Basic equations and formulation of the problem

We consider a plane strain problem in which the displacement fields u_1, u_3 are only the function of x_1 and x_3. The components of strains $\varepsilon_{11}, \varepsilon_{33}, \varepsilon_{13}$ are related to the displacement field u_1, u_3 are given by:

$$\varepsilon_{11} = u_{1,1}, \qquad \varepsilon_{33} = u_{3,3}, \qquad \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}(u_{1,3} + u_{3,1}). \tag{2.1}$$

The constitutive relations for a homogeneous orthotropic elastic medium can be written as [19]:

$$\begin{aligned} &\sigma_{11} = c_{11}\varepsilon_{11} + c_{13}\varepsilon_{33}, \qquad \sigma_{33} = c_{13}\varepsilon_{11} + c_{33}\varepsilon_{33}, \\ &\sigma_{13} = \sigma_{31} = 2c_{55}\varepsilon_{13}, \end{aligned} \tag{2.2}$$

where $\sigma_{11}, \sigma_{13}, \sigma_{33}$ are the components of stress and $c_{11}, c_{13}, c_{33}, c_{55}$ are characteristic constants of the orthotropic elastic material.

The mechanical governing equation with the initial stress (without body forces) considered can be expressed as [6, 18, 19]

$$\sigma_{ij,i} + (u_{j,k}\sigma^0_{ki})_{,i} = \rho\ddot{u}_j, \tag{2.3}$$

where ρ is the mass density, σ^0_{ki} are components of initial stresses, the superposed dot represents the temporal derivative and a comma in the subscript denotes the spatial derivative.

Substituting (2.2) into (2.3) and taking into account (2.1), we obtain the following field equations of the pre-stressed elastic solid, namely:

$$\begin{aligned} &(c_{11} + \sigma^0_{11})u_{1,11} + (c_{13} + c_{55})u_{3,13} + (c_{55} + \sigma^0_{33})u_{1,33} + 2\sigma^0_{13}u_{1,13} = \rho\ddot{u}_1, \\ &(c_{55} + \sigma^0_{11})u_{3,11} + (c_{13} + c_{55})u_{1,13} + (c_{33} + \sigma^0_{33})u_{3,33} + 2\sigma^0_{13}u_{3,13} = \rho\ddot{u}_3. \end{aligned} \tag{2.4}$$

Considering the problem shown in Fig. 1, let us assume a Cartesian coordinate system in such a way that x_1-axis lies along the separating interface of two halfspaces and the x_3-axis is vertically upwards. For the oblique incidence of the qP wave from the Ω^+ medium at the interface $x_3 = 0$, all kinds of scattered waves are depicted in Fig. 1. The transmitted wave fields consist of the transmitted qP wave and the quasi-transverse (qSV) wave, while the reflected wave fields comprise the reflected qP and qSV waves.

The plane wave solutions of (2.4) in x_1–x_3 plane are of the form [15, 28, 29],

$$\begin{cases} u_1 = a_1 e^{ik(x_1 + \xi x_3 - ct)}, \\ u_3 = a_3 e^{ik(x_1 + \xi x_3 - ct)}, \end{cases} \tag{2.5}$$

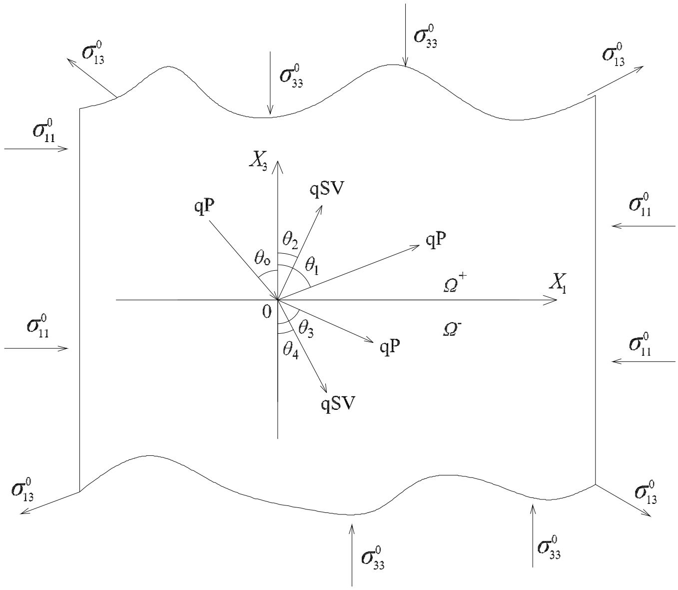


FIG. 1. Geometry of the problem.

where k is the x_1-component of the wavenumber, c is phase velocity along x_1, ξ is an unknown ratio of the wave vector components along the x_3- and x_1-directions, a_1, a_3 are unknown amplitudes of the displacement. The generalized Snell law has been taken into account in (2.5).

Now, with the aid of (2.5), Eqs.(2.4) lead to

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.6}$$

where

$$\begin{aligned} t_{11} &= (c_{55}+\sigma^0_{33})\xi^2 + 2\sigma^0_{13}\xi + c_{11} + \sigma^0_{11} - \rho c^2; \quad t_{12} = (c_{13}+c_{55})\xi, \\ t_{21} &= (c_{13}+c_{55})\xi, \quad t_{22} = (c_{33}+\sigma^0_{33})\xi^2 + 2\sigma^0_{13}\xi + c_{55} + \sigma^0_{11} - \rho c^2. \end{aligned} \tag{2.7}$$

The condition of existing non-trivial solution is $\det(t_{ij}) = 0$, we obtain a fourth-degree polynomial equation for ξ, namely

$$t_4\xi^4 + t_3\xi^3 + t_2\xi^2 + t_1\xi + t_0 = 0, \tag{2.8}$$

where the coefficients t_4, t_3, t_2, t_1, t_0 are provided in Appendix.

For the orthotropic solid considered, the value of ξ indicates that there are four possible part waves. These waves are qP and qSV propagating in an opposite direction in the medium. Moreover, for the propagation of plane waves with phase velocity v in the direction making an angle θ with the vertical axis, a plane wave is expressed by [28, 29]

$$\begin{cases} u_1 = a_1 e^{ik_0(p_1x_1+p_3x_3-vt)}, \\ u_3 = a_3 e^{ik_0(p_1x_1+p_3x_3-vt)}, \end{cases} \tag{2.9}$$

where $p_1 = \sin\theta$, $p_3 = -\cos\theta$ are components of propagation unit vector. It is noted that $k_0 = k/p_1$ and $v = p_1c$. Substituting (2.9) into (2.4) and obtaining the system of equations similar to Eq. (2.6), namely

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.10}$$

where

$$\begin{aligned} h_{11} &= (c_{55} + \sigma_{33}^0)p_3^2 + 2\sigma_{13}^0 p_1p_3 + (c_{11} + \sigma_{11}^0)p_1^2 - \rho v^2, \\ h_{12} &= (c_{13} + c_{55})p_1p_3, \\ h_{21} &= (c_{13} + c_{55})p_1p_3, \\ h_{22} &= (c_{33} + \sigma_{33}^0)p_3^2 + 2\sigma_{13}^0 p_1p_3 + (c_{55} + \sigma_{11}^0)p_1^2 - \rho v^2. \end{aligned} \tag{2.11}$$

By letting the determinant of this matrix equal zero, we have a quadratic equation in v^2:

$$(v^2)^2 - h_1v^2 - h_0 = 0, \tag{2.12}$$

where

$$\begin{aligned} h_1 &= (c_{55} + \sigma_{33}^0)p_3^2 + 2\sigma_{13}^0 p_1p_3 + (c_{11} + \sigma_{11}^0)p_1^2 \\ &\quad + (c_{33} + \sigma_{33}^0)p_3^2 + 2\sigma_{13}^0 p_1p_3 + (c_{55} + \sigma_{11}^0)p_1^2, \\ h_0 &= (c_{13} + c_{55})^2 p_1^2 p_3^2. \end{aligned}$$

Therefore, we obtain two real roots v_j $(j = 1, 2)$ corresponding to speeds of qP and qSV waves propagating in the medium.

3. Boundary conditions

Consider two bonded orthotropic half-spaces as shown in Fig. 1. If the bonding is imperfect then the spring model is introduced. At the imperfect interface the normal and tangential stiffness represented by k_N and k_T, respectively, characterize the properties of the interface between the two elastic half-spaces [13]. To investigate the influence of boundary conditions on the reflection and transmission coefficients, the different cases of imperfect interfaces namely imperfect interface, perfect interface and slip interface have been discussed.

3.1. Imperfect interface

For finite values of k_N and k_T the interface behaves as imperfect bonding or weak interface, which can be expressed by [8, 12, 13]:

$$\begin{aligned}
&\sigma_{j3}^{+} + \sigma_{13}^{0} u_{j,1}^{+} + \sigma_{33}^{0} u_{j,3}^{+} = \sigma_{j3}^{-} + \sigma_{13}^{0} u_{j,1}^{-} + \sigma_{33}^{0} u_{j,3}^{-} \quad (j = 1, 3),\\
&\sigma_{13}^{+} + \sigma_{13}^{0} u_{1,1}^{+} + \sigma_{33}^{0} u_{1,3}^{+} = k_T (u_1^{+} - u_1^{-}),\\
&\sigma_{33}^{+} + \sigma_{13}^{0} u_{3,1}^{+} + \sigma_{33}^{0} u_{3,3}^{+} = k_N (u_3^{+} - u_3^{-}),
\end{aligned} \tag{3.1}$$

where k_N and k_T are the normal and tangential spring constants, respectively.

3.2. Perfect interface

When the stiffness parameters $k_N, k_T \to \infty$, the interface between the two half-spaces is completely perfect [2, 8, 12]. In perfect bonding, the stresses and displacement are continuous at the interface:

$$u_j^{+} = u_j^{-}, \quad \sigma_{j3}^{+} + \sigma_{13}^{0} u_{j,1}^{+} + \sigma_{33}^{0} u_{j,3}^{+} = \sigma_{j3}^{-} + \sigma_{13}^{0} u_{j,1}^{-} + \sigma_{33}^{0} u_{j,3}^{-} \quad (j = 1, 3). \tag{3.2}$$

3.3. Slip interface

In the case of the slip interface, the tangential force along the interface is not supported by the interface. Displacements and normal stresses are continuous but tangential stresses vanish, i.e., $k_N \to \infty$ and $k_T \to 0$ [8].

4. The reflection, transmission coefficients

From the characteristic equation (2.8), the incident qP wave at the interface generates the reflected qP, qSV waves in halfspace Ω^{+} and the transmitted qP, qSV waves in the Ω^{-} as shown in Fig. 1.

i) Solution for incident medium Ω^{+}

A qP wave incident at the interface from the upper orthotropic half-space. Therefore, using (2.5) the displacement can be given as:

$$\begin{bmatrix} u_1^0 \\ u_3^0 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ w_0 \end{bmatrix} e^{ik(x_1 + \xi_0 x_3 - ct)}. \tag{4.1}$$

After reflecting at the interface $x_3 = 0$, the reflected waves qP and qSV in originating medium Ω^{+} may be written as:

$$\begin{bmatrix} u_1^1 \\ u_3^1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ w_1 \end{bmatrix} e^{ik(x_1 + \xi_1 x_3 - ct)}, \tag{4.2}$$

$$\begin{bmatrix} u_1^2 \\ u_3^2 \end{bmatrix} = a_2 \begin{bmatrix} 1 \\ w_2 \end{bmatrix} e^{ik(x_1 + \xi_2 x_3 - ct)}. \tag{4.3}$$

ii) Solution for transmitted medium Ω^-

The transmitted waves qP and qSV in the continuing medium Ω^- are shown as:

$$\begin{bmatrix} u_1^3 \\ u_3^3 \end{bmatrix} = a_3 \begin{bmatrix} 1 \\ w_3 \end{bmatrix} e^{ik(x_1+\xi_3 x_3-ct)}, \tag{4.4}$$

$$\begin{bmatrix} u_1^4 \\ u_3^4 \end{bmatrix} = a_4 \begin{bmatrix} 1 \\ w_4 \end{bmatrix} e^{ik(x_1+\xi_4 x_3-ct)}, \tag{4.5}$$

where a_i are the amplitudes of the displacement, $w_i = u_3^i/u_1^i$ are the wave amplitude ratios that determined from (2.6), namely

$$w_i = -\frac{(c_{13}+c_{55})\xi_i}{c_{11}+c_{55}\xi_i^2-\rho c^2+\sigma_{33}^0\xi_i^2+2\sigma_{13}^0\xi_i+\sigma_{11}^0}, \qquad i=0,1,2,3,4,$$

correspond to the incident, reflected, transmitted waves.

4.1. Imperfect interface

Substituting the quantities of the incident, reflected and transmitted wave field Eqs.(4.1)–(4.5) into the imperfect mechanical bonding (3.1), we obtain the following four linear equations about the amplitudes of the reflected and transmitted waves. These equations are expressed as:

$$\begin{cases}
-\big(c_{55}^+(\xi_1+w_1)+\sigma_{13}^0+\sigma_{33}^0\xi_1\big)a_1-\big(c_{55}^+(\xi_2+w_2)+\sigma_{13}^0+\sigma_{33}^0\xi_2\big)a_2 \\
+\big(c_{55}^-(\xi_3+w_3)+\sigma_{13}^0+\sigma_{33}^0\xi_3\big)a_3+\big(c_{55}^-(\xi_4+w_4)+\sigma_{13}^0+\sigma_{33}^0\xi_4\big)a_4 \\
\qquad\qquad = \big(c_{55}^+(\xi_0+w_0)+\sigma_{13}^0+\sigma_{33}^0\xi_0\big)a_0, \\[2mm]
-\big(c_{13}^++c_{33}^+w_1\xi_1+\sigma_{13}^0\xi_1+\sigma_{33}^0\xi_1w_1\big)a_1 \\
-\big(c_{13}^++c_{33}^+w_2\xi_2+\sigma_{13}^0\xi_2+\sigma_{33}^0\xi_2w_2\big)a_2 \\
+\big(c_{13}^-+c_{33}^-w_3\xi_3+\sigma_{13}^0\xi_3+\sigma_{33}^0\xi_3w_3\big)a_3 \\
+\big(c_{13}^-+c_{33}^-w_4\xi_4+\sigma_{13}^0\xi_4+\sigma_{33}^0\xi_4w_4\big)a_4 \\
\qquad\qquad = \big(c_{13}^++c_{33}^+w_0\xi_0+\sigma_{13}^0\xi_0+\sigma_{33}^0\xi_0w_0\big)a_0, \\[2mm]
\left(ic_{55}^+(w_1+\xi_1)+\sigma_{13}^0i+\sigma_{33}^0i\xi_1-\dfrac{k_T}{k}\right)a_1 \\
\qquad +\left(ic_{55}^+(w_2+\xi_2)+\sigma_{13}^0i+\sigma_{33}^0i\xi_2-\dfrac{k_T}{k}\right)a_2+\dfrac{k_T}{k}a_3+\dfrac{k_T}{k}a_4 \\
\qquad\qquad = \left(\dfrac{k_T}{k}-ic_{55}^+(w_0+\xi_0)-\sigma_{13}^0i-\sigma_{33}^0i\xi_0\right)a_0, \\[2mm]
\left((c_{13}^++c_{33}^+w_1\xi_1+\sigma_{13}^0w_1+\sigma_{33}^0\xi_1w_1)i-\dfrac{k_N}{k}w_1\right)a_1 \\
\quad +\left((c_{13}^++c_{33}^+w_2\xi_2+\sigma_{13}^0w_2+\sigma_{33}^0\xi_2w_2)i-\dfrac{k_N}{k}w_2\right)a_2+\dfrac{k_N}{k}w_3a_3+\dfrac{k_N}{k}w_4a_4 \\
\qquad\qquad = \left(\dfrac{k_N}{k}w_0-(c_{13}^++c_{33}^+w_0\xi_0+\sigma_{13}^0w_0+\sigma_{33}^0\xi_0w_0)i\right)a_0.
\end{cases} \tag{4.6}$$

4.2. Perfect interface

Similarly, using Eqs. (4.1) to (4.5) in the perfect boundary conditions (3.2) we get four equations:

$$
\begin{cases}
-a_1-a_2+a_3+a_4=a_0,\\
-w_1a_1-w_2a_2+w_3a_3+w_4a_4=w_0a_0,\\
-\big(c_{55}^{+}(\xi_1+w_1)+\sigma_{13}^0+\sigma_{33}^0\xi_1\big)a_1-\big(c_{55}^{+}(\xi_2+w_2)+\sigma_{13}^0+\sigma_{33}^0\xi_2\big)a_2\\
+\big(c_{55}^{-}(\xi_3+w_3)+\sigma_{13}^0+\sigma_{33}^0\xi_3\big)a_3+\big(c_{55}^{-}(\xi_4+w_4)+\sigma_{13}^0+\sigma_{33}^0\xi_4\big)a_4\\
\qquad=\big(c_{55}^{+}(\xi_0+w_0)+\sigma_{13}^0+\sigma_{33}^0\xi_0\big)a_0,\\
-\big(c_{13}^{+}+c_{33}^{+}w_1\xi_1+\sigma_{13}^0\xi_1+\sigma_{33}^0\xi_1w_1\big)a_1-\big(c_{13}^{+}+c_{33}^{+}w_2\xi_2+\sigma_{13}^0\xi_2+\sigma_{33}^0\xi_2w_2\big)a_2\\
+\big(c_{13}^{-}+c_{33}^{-}w_3\xi_3+\sigma_{13}^0\xi_3+\sigma_{33}^0\xi_3w_3\big)a_3+\big(c_{13}^{-}+c_{33}^{-}w_4\xi_4+\sigma_{13}^0\xi_4+\sigma_{33}^0\xi_4w_4\big)a_4\\
\qquad=\big(c_{13}^{+}+c_{33}^{+}w_0\xi_0+\sigma_{13}^0\xi_0+\sigma_{33}^0\xi_0w_0\big)a_0.
\end{cases}
\tag{4.7}
$$

Eqs. (4.6) and (4.7) give the amplitude ratios of reflected qP, reflected qSV, transmitted qP and transmitted qSV waves.

Denoting $A_1=a_1/a_0$, $A_2=a_2/a_0$, $A_3=a_3/a_0$, $A_4=a_4/a_0$. The reflection, transmission coefficients (RTCs) are defined by the ratio of the reflected/transmitted amplitudes to the incident amplitude:

$$
\begin{aligned}
R_1&=\frac{\sqrt{1+w_1^2}}{\sqrt{1+w_0^2}}A_1, \qquad R_2=\frac{\sqrt{1+w_2^2}}{\sqrt{1+w_0^2}}A_2,\\
T_1&=\frac{\sqrt{1+w_3^2}}{\sqrt{1+w_0^2}}A_3, \qquad T_2=\frac{\sqrt{1+w_4^2}}{\sqrt{1+w_0^2}}A_4.
\end{aligned}
\tag{4.8}
$$

5. Validation

To validate the established analytical results of the reflection and transmission coefficients (RTCs), we compute the distribution of energy among the reflected and transmitted waves due to the incidence of qP wave at its separating interface. They should be checked to ensure that the energy of incident waves is equal to the energy sum of reflection and transmission waves. The scalar product of the surface traction and particle velocity per unit area denoted by P_i represents the rate at which energy is communicated per unit area at the surface $x_3=0$. This is given by [1, 12, 19]

$$
P_i=-\dot{u}_j\sigma_{ji}-\sigma_{ik}^0u_{j,k}\dot{u}_j. \tag{5.1}
$$

Further, the mathematical expression for averaged energy flux in one period is

$$
P_i^*=\frac{1}{2}\,\mathrm{Re}(-\dot{u}_j^*\sigma_{ji}-\sigma_{ik}^0u_{j,k}\dot{u}_j^*), \tag{5.2}
$$

where asterisks $(*)$ appearing above the notations denote the conjugate complex of the derivative of the displacement components.

We may obtain the expressions of average energy flux E^I, E^{rqP}, E^{rqSV}, E^{tqP}, E^{tqSV} of the incident qP wave, reflected qP, reflected qSV, transmitted qP, and transmitted qSV waves, respectively, through the unit area perpendicular to the propagation direction of the said waves for the prestressed medium:

$$E^I = \frac{1}{2}k^2c \times \Big(c_{55}^{+}(w_0+\xi_0)+(c_{13}^{+}+c_{33}^{+}w_0\xi_0).w_0^*+(\sigma_{13}^0+\xi_0\sigma_{33}^0)(1+w_0.w_0^*)\Big)a_0.a_0^*, \tag{5.3}$$

$$\begin{aligned} E^{rqP} &= \frac{1}{2}k^2c \times \Big(c_{55}^{+}(w_1+\xi_1)+(c_{13}^{+}+c_{33}^{+}w_1\xi_1).w_1^*+(\sigma_{13}^0+\xi_1\sigma_{33}^0)(1+w_1.w_1^*)\Big)a_1.a_1^*, \\ E^{rqSV} &= \frac{1}{2}k^2c \times \Big(c_{55}^{+}(w_2+\xi_2)+(c_{13}^{+}+c_{33}^{+}w_2\xi_2).w_2^*+(\sigma_{13}^0+\xi_2\sigma_{33}^0)(1+w_2.w_2^*)\Big)a_2.a_2^*, \end{aligned} \tag{5.4}$$

$$\begin{aligned} E^{tqP} &= \frac{1}{2}k^2c \times \Big(c_{55}^{-}(w_3+\xi_3)+(c_{13}^{-}+c_{33}^{-}w_3\xi_3).w_3^*+(\sigma_{13}^0+\xi_3\sigma_{33}^0)(1+w_3.w_3^*)\Big)a_3.a_3^*, \\ E^{tqSV} &= \frac{1}{2}k^2c \times \Big(c_{55}^{-}(w_4+\xi_4)+(c_{13}^{-}+c_{33}^{-}w_4\xi_4).w_4^*+(\sigma_{13}^0+\xi_4\sigma_{33}^0)(1+w_4.w_4^*)\Big)a_4.a_4^*. \end{aligned} \tag{5.5}$$

Denoting $E_{r1} = E^{rqP}/E^I$, $E_{r2} = E^{rqSV}/E^I$, $E_{t1} = E^{tqP}/E^I$, $E_{t2} = E^{tqSV}/E^I$. From the energy balance at the surface $x_3 = 0$, we should have

$$E = E_{r1} + E_{r2} + E_{t1} + E_{t2} = 1, \tag{5.6}$$

which can be used as a check on the numerical computations.

6. Numerical calculation and discussion

In this section, numerical calculations were performed to illustrate the theoretical results obtained in the preceding sections. For convenience in the numerical analysis, we introduce the dimensionless quantities:

$$
\begin{aligned}
&f_1^+ = \frac{c_{11}^+}{c_{55}^+}, \quad f_2^+ = \frac{c_{13}^+}{c_{55}^+}, \quad f_3^+ = \frac{c_{33}^+}{c_{55}^+},\\
&f_1^- = \frac{c_{11}^-}{c_{55}^-}, \quad f_2^- = \frac{c_{13}^-}{c_{55}^-}, \quad f_3^- = \frac{c_{33}^-}{c_{55}^-},\\
&f_0 = \frac{c_{55}^-}{c_{55}^+}, \quad f_N = \frac{k_N}{kc_{55}^+}, \quad f_T = \frac{k_T}{kc_{55}^+},\\
&g_1 = \frac{\sigma_{11}^0}{c_{55}^+}, \quad g_2 = \frac{\sigma_{13}^0}{c_{55}^+}, \quad g_3 = \frac{\sigma_{33}^0}{c_{55}^+}, \quad r = \frac{\rho^-}{\rho^+}.
\end{aligned} \tag{6.1}
$$

The values of relevant elastic parameters for two halfspaces are taken following [8].

For halfspace Ω^+: material constants for AIN are as follows:

$$
\begin{aligned}
&c_{11}^+ = 3.45 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}, \quad c_{13}^+ = 1.20 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2},\\
&c_{33}^+ = 3.95 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}, \quad c_{55}^+ = 1.18 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2},\\
&\rho^+ = 3.62 \times 10^{3}\,\mathrm{kg}\cdot\mathrm{m}^{-3}.
\end{aligned}
$$

For halfspace Ω^-: material constants for $\mathrm{BaTiO_3}$ are given as:

$$
\begin{aligned}
&c_{11}^- = 1.66 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}, \quad c_{13}^- = 0.17 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2},\\
&c_{33}^- = 1.62 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}, \quad c_{55}^- = 0.453 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2},\\
&\rho^- = 5.8 \times 10^{3}\,\mathrm{kg}\cdot\mathrm{m}^{-3}.
\end{aligned}
$$

Besides the value of the initial stresses are given $\sigma_{11}^0 = 8 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}$, $\sigma_{13}^0 = -0.01 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}$, $\sigma_{33}^0 = 0.02 \times 10^{11}\,\mathrm{N}\cdot\mathrm{m}^{-2}$ and the x_1-component of the wavenumber k is fixed at 1. Moroever, to analyze separately the influence of each boundary condition, the tangential and normal spring constants are such that $k_T = 2.0$ and $k_N = 1.2$ in the first case (imperfect interface). In the second case (perfect interface), $k_T = 2^{100}$ and $k_N = 12^{100}$. For the slip interface, $k_T = 0$ and $k_N = 12^{100}$.

To facilitate comparison and evaluation, in Figs. 2–8, the graphic characteristics related to qP, qSV in halfspace Ω^+ and Ω^- are denoted by blue, red, yellow, and green, respectively. Additionally, the characteristics of waves with and without initial stress are illustrated by solid lines and dashed lines, respectively.

The effect of the initial stress on the dimensionless velocities of qP and qSV wave are plotted in Fig. 2. It can be seen from this figure that these velocities in the medium with the initial stress are greater than those in the medium without the initial stress. This makes it an essential parameter that should be considered when designing devices.

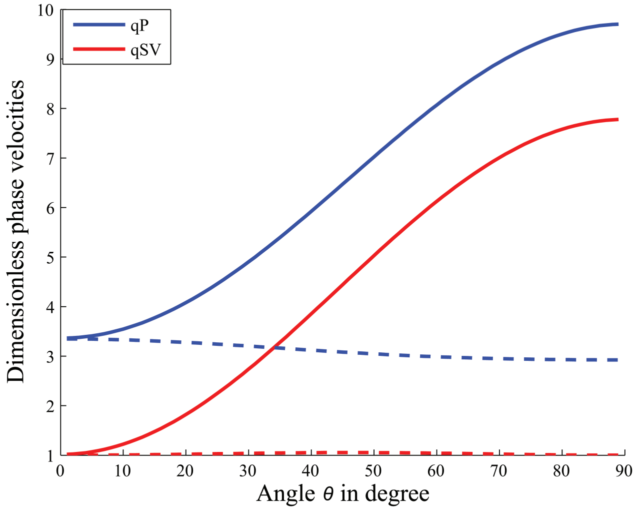


Fig. 2. Velocities of waves for presence and absence of the initial stress.

Figure 3 shows the effect of the initial stress on the RTCs in the case of the perfect interface. The RTCs of the reflected and transmitted waves are significantly affected by the initial stress. In contrast, for the imperfect interface, the RTCs are mildly unaffected by the initial stress (see Fig. 4). The present numerical results might provide more relevant information about the wave prop-

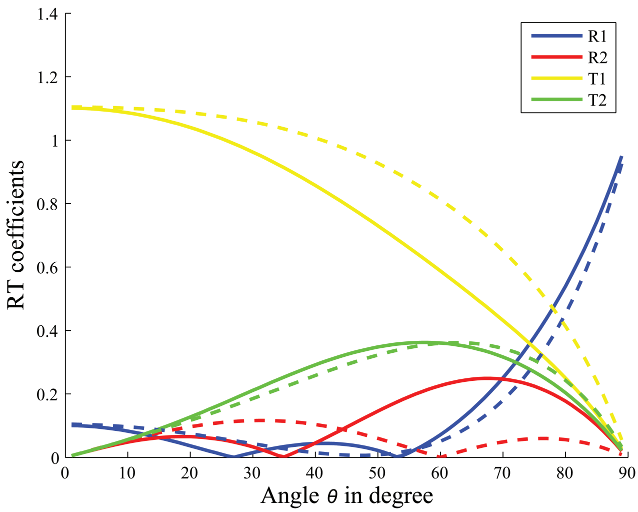


Fig. 3. The effect of the initial stress on the RTCs for the case of the perfect interface.

agation in an orthotropic medium with the initial stress. Moreover, our formulas can provide some fundamental insights for deriving RTCs formulas on the welded and non-welded interfaces between two more complex media. These results are recorded and used as the input data of the inverse problem (nondestructive evaluation of materials).

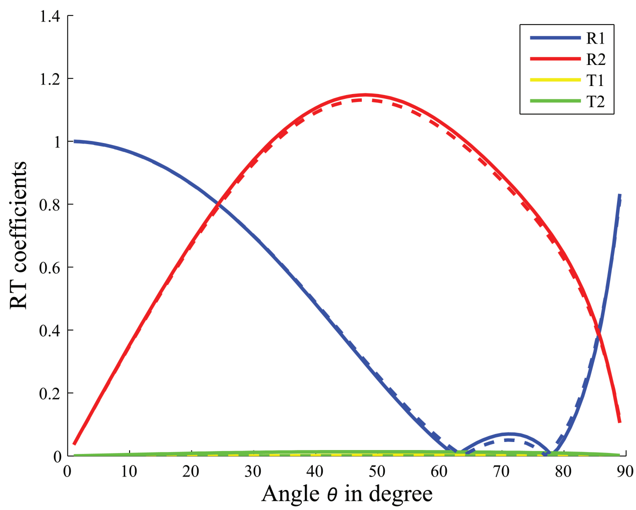
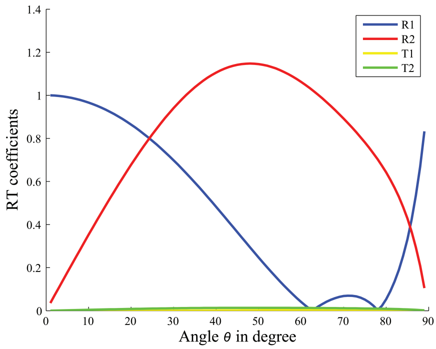


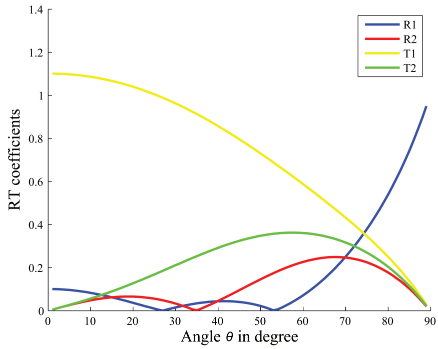
Fig. 4. The effect of the initial stress on the RTCs for the case of the imperfect interface.

Figure 5 is plotted to describe the influence of distinct common interfaces on RTCs. The comparative study of the RTCs curves suggests that the reflection coefficients are dominant for the imperfect interface ($k_N = 1.2$, $k_T = 2$), the transmission coefficients for this case are negligibly small in the whole range of an incident angle. While the transmission coefficient T_1 is quite large for the perfect interface ($k_N \to \infty$, $k_T \to \infty$) and the slip interface ($k_N \to \infty$, $k_T \to 0$). Moreover, the corresponding curves of RTC for imperfect (Fig. 5b) and perfect interface (Fig. 3) cases are coincident in the range $1°$–$89°$. This confirms the correctness of numerical calculations.

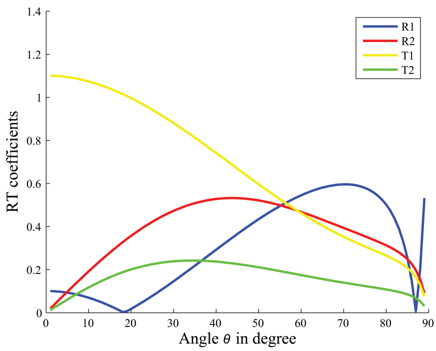
Figure 6 depicts the variation of the values of the energy ratios of reflected and transmitted waves with the angle of incidence for the presence and absence of initial stress. We also note that the ratio E_{t1} in the range $1°$–$80°$ (E_{r1} in the range $80°$–$89°$) of qP wave is highly dominated over all other energy ratios when the initial stress is present. The ratios E_{r1}, E_{r2}, E_{t2} (E_{t1}) in the medium with the initial stress are bigger (smaller) than the ones in the medium without the initial stress. Figure 6 once again confirms the computational results in this section are fully reliable.



$k_N = 1.2$, $k_T = 2$



$k_N = 12^{100}$, $k_T = 2^{100}$



$k_N = 12^{100}$, $k_T = 0$

FIG. 5. The RTCs of waves for distinct common interface.

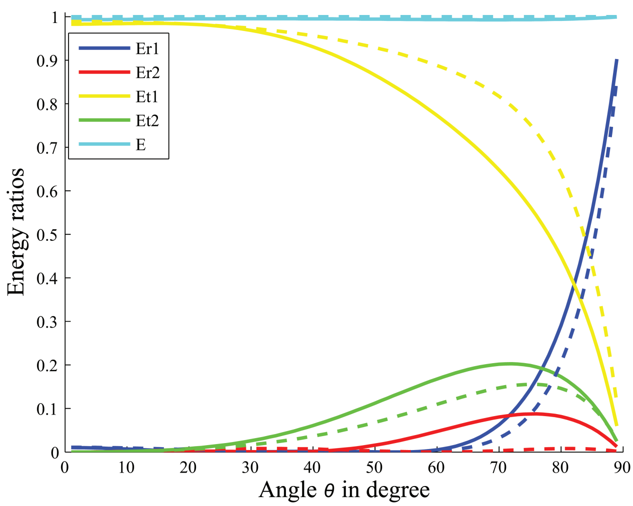


FIG. 6. Energy ratios.

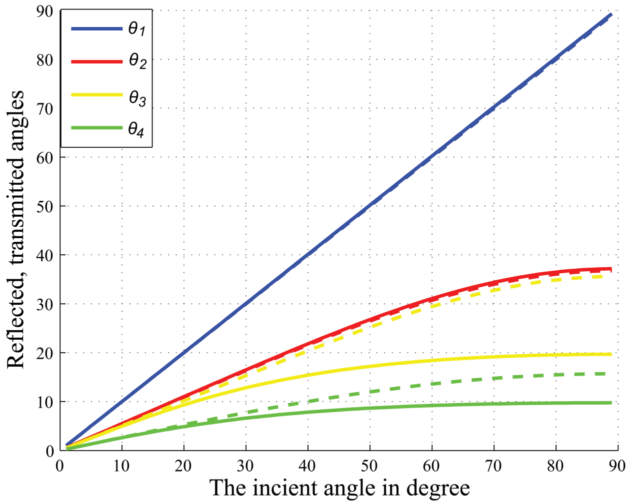


Fig. 7. Phase angles for presence and absence of the initial stress.

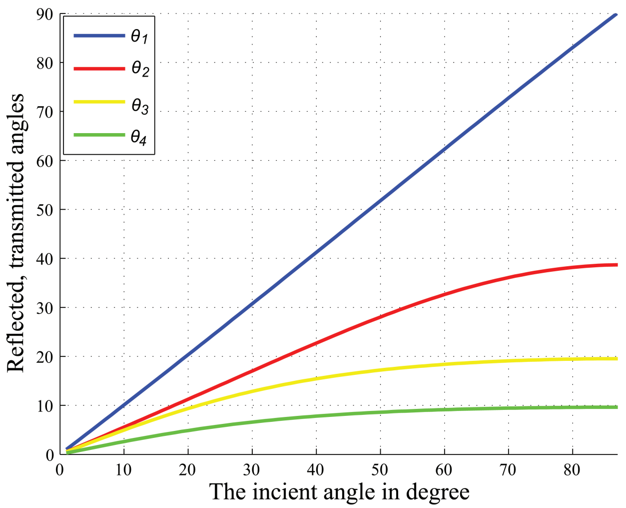


Fig. 8. Effect the initial stress on the phase angles of the reflected, transmitted waves.

If the wave propagation direction deviates the positive x_3 axis with angle θ, then $\mathrm{Real}(\xi) = \cot(\theta)$, which will be used to compute the reflection and transmission angles. Figure 7 shows the effects of the initial stress on the reflection and transmission angles. It is found that the existence of the initial stress makes the transmission angle decrease evidently but the reflection angle is unchanged.

It is an interesting thing that when the values of σ_{11}^0 and σ_{33}^0 remain unchanged while the value of σ_{13}^0 increases by 10 times, the reflection angle qP will no longer be equal to the incident angle qP in the range of the incident angle 30°–90° (see Fig. 8).

7. Conclusions

In conclusion, a mathematical study of the effects of the initial stress on the reflection and transmission coefficients at an imperfect interface separating two orthotropic elastic solid half spaces is made when the qP wave is incident. The key findings of the study of the study can be outlined as:

(i) The existence of the initial stress makes the reflection or the transmission coefficient of some waves increasing or decreasing at the total incident angle range, but, in general, the effects of the initial stress on the reflection and the transmission coefficient are dependent on the incident angle range. It is noticed that when the value of σ_{13}^0 gradually increases to 10 times then the reflection angle qP will no longer be equal to the incident angle qP in the range of the incident angle 30°–90°.

(ii) The reflection and transmission coefficients can be evidently affected by the distinct common interface. These results may also be helpful in predicting the anomalies and the actual cause of destruction.

(iii) It can be seen from this figure that these velocities in the medium with the initial stress are greater than those in the medium without the initial stress. This makes it an essential parameter that should be considered when designing devices.

It is anticipated that this work will be valuable for further theoretical and observational studies of wave reflection and transmission in more realistic models of prestressed solids present on Earth.

Appendix

The coefficients of the characteristic equation:

$$
\begin{aligned}
t_4 &= (c_{55} + \sigma_{33}^0)(c_{33} + \sigma_{33}^0),\\
t_3 &= 2\sigma_{13}^0(c_{33} + c_{55} + 2\sigma_{33}^0),\\
t_2 &= (c_{55} + \sigma_{33}^0)\big((c_{55} + \sigma_{11}^0) - \rho c^2\big) + (c_{33} + \sigma_{33}^0)\big((c_{11} + \sigma_{11}^0) - \rho c^2\big)\\
&\quad + 4(\sigma_{13}^0)^2 - (c_{13} + c_{55})^2,\\
t_1 &= 2\sigma_{13}^0(c_{11} + c_{55} + 2\sigma_{11}^0 - 2\rho c^2),\\
t_0 &= (c_{11} + \sigma_{11}^0 - \rho c^2)(c_{55} + \sigma_{11}^0 - \rho c^2).
\end{aligned}
$$

Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

1. J.D. ACHENBACH, *Wave Propagation in Elastic Solids*, Series in Applied Mechanics, **16**, North Holland, Amsterdam, 1973.
2. M.S. BARAK, V. KALIRAMAN, *Reflection and transmission of elastic waves from an imperfect boundary between micropolar elastic solid half space and fluid saturated porous solid half space*, Mechanics of Advanced Materials and Structures, **26**, 14, 1226–1233, 2018.
3. M.A. BIOT, *The influence of initial stress on elastic waves*, Journal of Applied Physics, **11**, 522–530, 1940.
4. M.A. BIOT, *Mechanics of Incremental Deformations*, John Wiley, New York, 1965.
5. A. CHATTOPADHYAY, S. BOSE, M. CHAKRABORTY, *Reflection of elastic waves under initial stress at a free surface: P and SV motion*, The Journal of the Acoustical Society of America, **72**, 255–263, 1982.
6. F. CHEN, Z. ZONG, A. STOVAS, X. YIN, *Wave reflection and transmission coefficients for layered transversely isotropic media with vertical symmetry axis under initial stress*, Geophysical Journal International, **233**, 1580–1595, 2023.
7. N. GARG, *Existence of longitudinal waves in pre-stressed anisotropic elastic medium*, Journal of Earth System Science, **118**, 6, 677–687, 2009.
8. S. GOYAL, S. SAHU, S. MONDAL, *Influence of imperfect bonding on the reflection and transmission of QP-wave at the interface of two functionally graded piezoelectric materials*, Wave Motion, **92**, 102431, 2019.
9. S. GUHA, A.K. SINGH, *Effects of initial stresses on reflection phenomenon of plane waves at the free surface of a rotating piezothermoelastic fiber-reinforced composite half-space*, International Journal of Mechanical Sciences, **181**, 105766, 2020.
10. A.K. SINGH, S. MAHTO, S. GUHA, *Analysis of plane wave reflection and transmission phenomenon at the interface of two distinct micro-mechanically modeled rotating initially stressed piezomagnetic fiber-reinforced half-spaces*, Mechanics of Advanced Materials and Structures, **29**, 28, 7623–7639, 2021.
11. S. GUHA, A.K. SINGH, *Plane wave reflection/transmission in imperfectly bonded initially stressed rotating piezothermoelastic fiber-reinforced composite half-spaces*, European Journal of Mechanics/A Solids, **88**, 104242, 2021.

12. X. Guo, P. Wei, *Effects of initial stress on the reflection and transmission waves at the interface between two piezoelectric half spaces*, International Journal of Solids and Structures, **51**, 3735–3751, 2014.

13. Z. Hashin, *The spherical inclusion with imperfect interface*, Journal of Applied Mechanics, **58**, 2, 444–449, 1991.

14. W. Huang, S.I. Rokhlin, *Interface waves along an anisotropic imperfect interface between anisotropic solids*, Journal of Nondestructive Evaluation, **11**, 185–198, 1992.

15. A. Khurana, S.K. Tomar, *Waves at interface of dissimilar nonlocal micropolar elastic half-spaces*, Mechanics of Advanced Materials and Structures, **26**, 10, 825–833, 2018.

16. R. Kumar, N. Sharma, P. Ram, *Interfacial imperfection on reflection and transmission of plane waves in anisotropic micropolar media*, Theoretical and Applied Fracture Mechanics, **49**, 305–312, 2008.

17. H. Liu, Z.K. Huang, T.J. Wang, *Effect of initial stress on the propagation behavior of Love waves in a layered piezoelectric structure*, International Journal of Solids and Structures, **38**, 37–51, 2001.

18. H. Liu, J.J. Lee, Z.M. Cai, *Analysis of nonlinear acoustoelastic effect of surface acoustic waves in laminated structures by transfer matrix method*, Mechanics Research Communications, **31**, 667–675, 2004.

19. M.K. Pal, A.K. Singh, *Analysis of reflection and transmission phenomenon at distinct bonding interfaces in a rotating pre-stressed functionally graded piezoelectric-orthotropic structure*, Applied Mathematics and Computation, **409**, 126398, 2021.

20. Y. Pang, J.X. Liu, *Reflection and transmission of plane waves at an imperfectly bonded interface between piezoelectric and piezomagnetic media*, European Journal of Mechanics A/Solids, **30**, 731–740, 2011.

21. Z. Qian, F. Jin, K. Kishimoto, Z. Wang, *Effect of initial stress on the propagation behavior of SH-waves in multilayered piezoelectric composite structures*, Sensors and Actuators A, **112**, 368–375, 2004.

22. M. Shams, R.W. Ogden, *On Rayleigh-type surface waves in an initially stressed incompressible elastic solid*, IMA Journal of Applied Mathematics, **79**, 360–376, 2014.

23. M.D. Sharma, N. Garg, *Wave velocities in a pre-stressed anisotropic elastic medium*, Journal of Earth System Science, **115**, 2, 257–265, 2006.

24. M.D. Sharma, *Wave propagation in a pre-stressed anisotropic generalized thermoelastic medium*, Earth Planets Space, **62**, 381–390, 2010.

25. S.S. Singh, S.K. Tomar, *qP-wave at a corrugated interface between two dissimilar pre-stressed elastic half-spaces*, Journal of Sound Vibration, **317**, 687–708, 2008.

26. B. Singh, *Wave propagation in a prestressed piezoelectric half-space*, Acta Mechanica, **211**, 337–344, 2010.

27. S.S. Singh, *Effect of initial stresses on incident qSV waves in pre-stressed elastic half-space*, The Australian & New Zeland Industrial and Applied Mathematics Journal, **52**, 4, 359–371, 2011.

28. D.X. Tung, *The reflection and transmission of waves at an imperfect interface between two nonlocal transversely isotropic liquid-saturated porous halfspaces*, Waves in Random and Complex Media, **34**, 3, 2251–2267, 2021.

29. D.X. Tung, *Influence of boundary conditions on the reflection and transmission of qP-wave at an interface between two nonlocal transversely isotropic elastic half-spaces*, International Journal for Computational Methods in Engineering Science and Mechanics, **24**, 2, 119–127, 2023.

Received July 17, 2024; revised version October 30, 2024.
Published online December 13, 2024.