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On the deformation of elastic rods in a symmetric micromorphic theory

S. DE CICCO

Department of Structures for Engineering and Architecture,
University of Naples Federico II, Via Forno Vecchio 80131, Naples, Italy,
e-mail: simona.decicco@unina.it

We investigate the deformation of an elastic isotropic rod in the framework of a simplified micromorphic theory introduced by Forest and Sievert. In contrast with the classical micromorphic model, which includes 18 elastic constants, this theory is characterized by constitutive equations which involve 6 constants and a material length scale parameter to describe microstructure-dependent size effects. First, we formulate the equilibrium problem of a rod subjected to a resultant force and resultant moment acting on its plane ends. Then, we generalize the method of construction of the solution avoiding a priori assumptions proposed by Iesan in classical elasticity. The method leads to the decomposition of the general problem into the basic problems of extension, bending, torsion and flexure. The analytical solutions are obtained in a closed form and reduced to their classical elasticity counterparts when the microstructure effects are suppressed. The results are useful to obtain explicit solutions when the shape of the cross section is assigned and are preliminary to the solution of the problem of cylinders loaded on a lateral surface such as the Almansi–Michel problem.

Key words: symmetric micromorphic theory, elastic rods, Saint-Venant problem, extension, bending, torsion, flexure.

1. Introduction

The paper deals with a simplified micromorphic theory proposed by Forest and Sievert [1]. The micromorphic theory was introduced by Eringen and Suhubi [2] and by Eringen [3] as an extension of classical elasticity and viscolelasticity to microscopic length and time scales. The material bodies are regarded as a collection of deformable particles suitable for modeling materials with an inner structure, such as polymers with deformable molecules, granular and porous solids, geomaterials, biological tissues and so on. Among the micro-continuum field theories, because of its complexity, the micromorphic elasticity is the least-developed and contributions in this area have been rare. Theoretical difficulties depend on the number of extra kinematical variables, non symmetric stress and strain tensors, unavailability of material constants trough atomistic calculations or experimental measurements (see e.g. [4, 5]). Recently, in order to remove these barriers and to get more tractable field equations, various simplified micromorphic theories have been introduced. Teisseyre [6, 7] presented a symmetric micromorphic continuum suitable to describe earthquake processes. In this model the stress tensor and the body couples are symmetric, and the third order stress moment tensor is symmetric with respect to the last two indices. The relaxed micromorphic model presented by Neff *et al.* [8, 9] is characterized by symmetric stress and strain tensors and by a number of constitutive coefficients drastically reduced. In the Forest–Sievert approach [1], the micro-rotation and the macro-rotation are taken to be the same for infinitesimal deformation. This assumption leads to a reduction of the field equations (from 12 to 6) and the elastic moduli (from 18 to 7). For an in-depth discussion of various simplified micromorphic theories, the reader is referred to the paper by Neff *et al.* [8] and the references therein.

The effects of microstructure on the behaviour of beams and other structural members, such as shells and plates, have been intensively studied (see e.g. [10]). In the literature solutions are provided for elastic beams made of Cosserat, microstretch, porous and other generalized media. Iesan and Nappa [11] solved Saint-Venant's problem for elastic microstretch cylinders and established explicit solutions for the extension and bending in the case of a circular cross section [12]. The torsion and flexure of a circular microstretch rod was studied by De Cicco and Nappa [13], whereas the thermoelastic deformation of a microstretch beam was investigated by Nappa and Pesce [14]. The model of elastic micropolar beam has been considered by several authors. Special cases are discussed by Lakes and Drugan [15], Taliercio and Veber [16], and Taliercio [17]. Recent contributions to the theory of porous beams are due to De Cicco [18, 19] and De Cicco and Iesan [20] who considered materials with double porosity structure and chiral porous materials within the context of strain gradient elasticity. There are not many studies on elastic micromorphic beams. An analytical solution for anisotropic micromorphic solids was established by Iesan [21]. Particular cases have been investigated by Norouzzadeh *et al.* [22] and by Shaat *et al.* [23]. Experimental evaluation of micromorphic elastic constants were performed by Lakes [24]. Most usefully for practical applications are the results established in the context of simplified micromorphic theories. Rizzi *et al.* [25, 26] investigated the bending and torsion problem for micromorphic relaxed continuum. A technical solution for the Bernoulli–Euler beam model in the context of Forest–Sievert [1] theory has been presented by Zhang *et al.* [27].

In this paper we study the problem of extension, bending, torsion and flexure of elastic isotropic rods in the framework of a symmetric micromorphic theory. The analytical solution is given in a closed form and obtained avoiding a priori assumptions. The paper is organized as follows. In the Section 2, the basic equations of a simplified micromorphic theory are presented. In Section 3, the problem of a rod loaded with a resultant force and a resultant moment acting on its ends is formulated. In the Section 4, we generalize the method of construction of the solution proposed by IESAN [28, 29] in classical elasticity. The method can be useful for different constitutive equations characterizing various simplified micromorphic theories. In Section 5, the problem is reduced to the solution of four plain strain problems. The flexure of elastic rods is studied in Section 6. The paper concludes in Section 7 with a summary.

2. Basic equations

In this section, we present the field equations of a symmetric micromorphic theory for elastic solids introduced by FOREST and SIEVERT in [1]. In the classical Eringen micromorphic theory [3] the independent kinematic variables are the components of the 12-dimensional vector field $v = (\boldsymbol{u}, P)$, where $\boldsymbol{u}$ is the displacement vector and P is the micro-distorsion second order tensor. The linear strain measures associated with v are the relative deformation Γ, the micro-strain ε and the micro-curvature K, defined by:

$$\Gamma(v) = (\nabla \boldsymbol{u} - P)^T, \quad \varepsilon(v) = \operatorname{sym} P, \quad K(v) = \nabla P, \tag{2.1}$$

where $\nabla \boldsymbol{a}$ is the gradient of $\boldsymbol{a}$ and A^T is the transpose of A.

Following FOREST and SIEVERT [1] for infinitesimal deformations the microrotation and the macro-rotation are taken to be the same. This hypothesis leads to the modified strain measures:

$$e(v) = \operatorname{sym}(\nabla \boldsymbol{u} - P), \quad \varepsilon(v) = \operatorname{sym} P, \quad \kappa(v) = \nabla(\operatorname{sym} P) \tag{2.2}$$

or in components:

$$\begin{aligned} &e_{ij}(v) = \tfrac{1}{2}(\Gamma_{ij} + \Gamma_{ji}), \quad \varepsilon_{ij}(v) = \tfrac{1}{2}(P_{ij} + P_{ji}), \\ &\kappa_{kij} = \tfrac{1}{2}(P_{ij} + P_{ji})_{,k}, \end{aligned} \tag{2.3}$$

where we have used the notation $\frac{\partial g}{\partial x_k} = g_{,k}$. In Eq. (2.3) and in what follows Latin subscripts take the values $k = 1, 2, 3$ and Greek subscripts take the values 1 and 2. We denote by τ, τ^m and m, the stress tensor, the micro-stress tensor and the micro-moment tensor, respectively. The constitutive equations for isotropic elastic solids in the symmetric micromorphic theory are:

$$\begin{aligned} \tau_{ij}(v) &= 2\mu_e e_{ij}(v) + \lambda_e e_{rr}(v)\delta_{ij} + 2\mu_c \varepsilon_{ij}(v) + \lambda_c \varepsilon_{kk}(v)\delta_{ij}, \\ \tau^m_{ij}(v) &= 2\mu_c e_{ij}(v) + \lambda_c e_{rr}(v)\delta_{ij} + 2\mu_m \varepsilon_{ij}(v) + \lambda_m \varepsilon_{kk}(v)\delta_{ij}, \\ m_{kij}(v) &= l^2(2\mu_m \kappa_{kij}(v) + \lambda_m \kappa_{krr}(v)\delta_{ij}), \end{aligned} \tag{2.4}$$

where μ_e, μ_m, μ_c, λ_e, λ_m and λ_c are constitutive coefficients and l is a scale length parameter. We note that, in contrast with the micromorphic theory where the stress-strain relations involve 18 material constants, in this symmetric micromorphic theory the material constants are 6. The strain energy density function reads

$$\begin{aligned}(2.5)\qquad & w(e_{ij}(v), \varepsilon_{ij}(v), \kappa_{kij}(v)) \\ &= \mu_e e_{ij}(v)e_{ij}(v) + \tfrac{1}{2}\lambda_e e_{hh}(v)e_{kk}(v) \\ &\quad + 2\mu_c e_{ij}(v)\varepsilon_{ij}(v) + \lambda_c e_{jj}(v)\varepsilon_{ii}(v) + \mu_m \varepsilon_{ij}(v)\varepsilon_{ij}(v) \\ &\quad + \tfrac{1}{2}\lambda_m \varepsilon_{rr}(v)\varepsilon_{ss}(v) + l^2[(\mu_m \kappa kij(v)\kappa_{kij}(v)) + \tfrac{1}{2}\lambda_m(\kappa_{krr}(v)\kappa_{kss}(v))].\end{aligned}$$

The condition of strict positive definiteness of the energy density function implies the following relations:

$$(2.6)\quad \mu_e > 0, \quad \mu_c > 0, \quad \mu_m > 0, \quad 2\mu_e + 3\lambda_e > 0, \quad 2\mu_m + 3\lambda_m > 0, \quad l > 0.$$

The equilibrium equations are given by:

$$(2.7)\qquad \begin{aligned} &\tau_{ji,j}(v) + f_i = 0, \\ &m_{kij,k}(v) + \tau_{ji}(v) - \tau^m_{ji}(v) + G_{ij} = 0, \end{aligned}$$

where $\boldsymbol{f}$ is the body force vector and G is the body volume moment tensor. We denote by $\boldsymbol{n}$ the outward unit normal of the boundary ∂B and by $\tilde{\tau}$ and $\tilde{m}$ the tractions acting on ∂B. We have the boundary conditions:

$$(2.8)\qquad \tau_{ji}(v)n_j = \tilde{\tau}_i, \quad m_{kij}(v)n_k = \tilde{m}_{ij}.$$

The equilibrium problem of the body B in the symmetric micromorphic theory consists in finding a vector field v that satisfies Eqs. (2.3), (2.4), (2.7) on B and the boundary conditions (2.8) on ∂B. The necessary and sufficient conditions for the existence of a solution to the above formulated problem are:

$$(2.9)\qquad \begin{aligned} &\int_B f_i\,dv + \int_{\partial B} \tilde{\tau}_i\,da = 0, \\ &\int_B \epsilon_{kji}(x_j f_i + G_{ij})\,dv + \int_{\partial B} \epsilon_{kji}(x_j\tilde{\tau}_i + \tilde{m}_{ij})\,da = 0, \end{aligned}$$

where ϵ is the Levi–Civita tensor. The existence and uniqueness of the solution of the equilibrium problem in the micromorphic theory have been established by IESAN and NAPPA in [30]. Analogous results in the linear theory of elasticity with couple stresses were obtained by HLAVACEK and HLAVACECK [31]. Existence theorems for the dynamic theory of microstretch elastic solids were presented by IESAN and QUINTANILLA [32]. All these results generalize in the context of microcontinuum theories the existence theorems presented by FICHERA [33] in classical elasticity.

3. Formulation of the problem

In this section, we study the equilibrium problem of an elastic homogeneous isotropic cylinder in the symmetric micromorphic theory. The system of rectangular axes is chosen such that the Ox_3-axis is parallel to the generator of the cylinder and the origin O is the centre of one of its ends. We denote by h the length of the cylinder, by Π the lateral surface and by Σ_1 and Σ_2, respectively, the cross-section located at $x_3 = 0$ and $x_3 = h$. We suppose that the body loads and the tractions on the lateral surface are absent. The equilibrium equations (2.7) become:

$$\tau_{ji,j}(v) = 0, \quad m_{kij,k}(v) + \tau_{ji}(v) - \tau^m_{ji}(v) = 0. \tag{3.1}$$

The boundary conditions are given by:

$$\tau_{\beta i}(v) n_\beta = 0, \qquad m_{\beta ij}(v) n_\beta = 0 \quad (\beta = 1, 2) \quad \text{on } \Pi, \tag{3.2}$$

$$\tau_{3i}(v) = \tilde{\tau}_i^{(\alpha)}, \quad m_{3ij}(v) = \tilde{m}_{ij}^{(\alpha)} \quad (\alpha = 1, 2) \quad \text{on } \Sigma_\alpha. \tag{3.3}$$

We denote by $\boldsymbol{F}$ and $\boldsymbol{M}$ the resultant force and the resultant moment of the tractions acting on Σ_1:

$$\int\limits_{\Sigma_1} \tilde{\tau}_i^{(1)}\, da = F_i, \qquad \int\limits_{\Sigma_1} \epsilon_{kji}(x_j \tilde{\tau}_i^{(1)} + \tilde{m}_{ij}^{(1)})\, da = M_k \tag{3.4}$$

and by $R_i(v)$ and $\mathcal{M}_i(v)$ the following integrals:

$$\begin{aligned}
R_i(v) &= -\int\limits_{\Sigma_1} \tau_{3i}(v)\, da, \\
\mathcal{M}_\alpha(v) &= -\int\limits_{\Sigma_1} \epsilon_{\alpha\beta 3}[x_\beta \tau_{33}(v) + m_{33\beta}(v) - m_{3\beta 3}(v)]\, da, \\
\mathcal{M}_3(v) &= -\int\limits_{\Sigma_1} \epsilon_{3\alpha\beta}[x_\alpha \tau_{3\beta}(v) + m_{3\beta\alpha}(v)]\, da.
\end{aligned} \tag{3.5}$$

In the Saint-Venant problem the conditions (3.3) are replaced by:

$$R_i(v) = F_i, \quad \mathcal{M}_i(v) = M_i \quad \text{on } \Sigma_1. \tag{3.6}$$

In what follows the equilibrium problem of the cylinder is decomposed into problems P_1 and P_2 defined by:

• *extension, bending, torsion*:

$$F_\alpha = 0, \qquad P_1 = (F_3, M_1, M_2, M_3); \tag{3.7}$$

• *flexure*:

$$F_3 = 0, \qquad M_i = 0, \qquad P_2 = (F_1, F_2). \tag{3.8}$$

In the next section we present a method to solve the problem P_1 and then we use the solution of the problem P_1 to solve the problem P_2.

4. A method of constructing the solution of the problem P_1

In [28, 29], Iesan presented a rational method of deriving the Saint-Venant solution in classical linear elasticity. The method has been proven to be effective for other kinds of constitutive assumptions and is based on the following propositions:

1. *Let v be a solution of the problem P_1, then*
(i) $R_i(v_{,3}) = 0$, $\mathcal{M}_i(v_{,3}) = 0$,
(ii) *the vector $v_{,3}$ is a rigid displacement field.*

2. *Let v^0 be a rigid displacement field, integrating v^0 respect to the axial coordinate x_3, we obtain a solution of the problem P_1.*

As an immediate consequence, we define a rigid displacement field for micromorphic bodies and then we integrate it with respect to x_3. In the micromorphic theory the rigid deformation has the form:

$$u_i^0 = \alpha_i + \epsilon_{ijk}\beta_i x_k, \qquad P_{ij}^0 = \epsilon_{ikj}\beta_k, \tag{4.1}$$

where α_i and β_i are arbitrary constants. We get:

$$u_{i,3} = u_i^0, \qquad P_{ij,3} = P_{ij}^0. \tag{4.2}$$

From (4.2), except for an additive rigid deformation, we obtain:

$$\begin{aligned}
u_\alpha &= -\tfrac{1}{2}a_\alpha x_3^2 - a_4\epsilon_{\alpha\beta 3}x_\beta x_3 + \omega_\alpha(x_1, x_2),\\
u_3 &= (a_1x_1 + a_2x_2 + a_3)x_3 + \omega_3(x_1, x_2),\\
P_{\alpha\beta} &= \epsilon_{3\beta\alpha}a_4x_3 + Q_{\alpha\beta}(x_1, x_2),\\
P_{\alpha 3} &= -a_4x_3 + Q_{\alpha 3}(x_1, x_2),\\
P_{3\alpha} &= a_4x_3 + Q_{3\alpha}(x_1, x_2),\\
P_{33} &= Q_{33}(x_1, x_2).
\end{aligned} \tag{4.3}$$

Here, $V = (\omega, Q)$ is an arbitrary 12-dimensional vector field independent of x_3 and $a_s, (s = 1, 2, 3, 4)$, are arbitrary constants that are related to α_i and β_i by the following relations:

$$a_\alpha = \epsilon_{\rho\alpha 3}\beta_\rho, \quad a_3 = \alpha_3, \quad a_4 = \beta_3. \tag{4.4}$$

We introduce the notations:

$$\begin{aligned} \eta_{\alpha\beta}(V) &= \tfrac{1}{2}(\omega_{\alpha,\beta} + \omega_{\beta,\alpha}), \quad \eta_{\alpha 3}(V) = \eta_{3\alpha}(V) = \tfrac{1}{2}\omega_{3,\alpha}, \\ \varepsilon_{ij}(V) &= \tfrac{1}{2}(Q_{ij} + Q_{ji}), \quad \kappa_{\alpha ij}(V) = \tfrac{1}{2}(Q_{ij} + Q_{ji})_{,\alpha}. \end{aligned} \tag{4.5}$$

From (2.3), (4.3) and (4.5), we obtain:

$$\begin{aligned} & e_{\alpha\beta}(v) = e_{\alpha\beta}(V) = \eta_{\alpha\beta}(V) - \varepsilon_{\alpha\beta}(V), \\ & e_{\alpha 3}(v) = e_{\alpha 3}(V) + \tfrac{1}{2}\epsilon_{\beta\alpha 3}a_4 x_\beta, \quad e_{\alpha 3}(V) = \eta_{\alpha 3}(V) - \varepsilon_{\alpha 3}(V), \\ & e_{33}(v) = a_1 x_1 + a_2 x_2 + a_3 - \varepsilon_{33}(V), \\ & \varepsilon_{ij}(v) = \varepsilon_{ij}(V), \quad \kappa_{\alpha ij}(v) = \kappa_{\alpha ij}(V), \quad \kappa_{3ij}(v) = 0. \end{aligned} \tag{4.6}$$

Equation (2.4) becomes:

$$\begin{aligned} \tau_{\alpha\beta}(v) &= \tau_{\alpha\beta}(V) + \lambda_e(a_1 x_1 + a_2 x_2 + a_3)\delta_{\alpha\beta}, \\ \tau_{\alpha 3}(v) &= \tau_{\alpha 3}(V) + \epsilon_{\beta\alpha 3}\mu_e a_4 x_\beta, \\ \tau_{33}(v) &= \tau_{33}(V) + (2\mu_e + \lambda_e)(a_1 x_1 + a_2 x_2 + a_3), \\ \overset{m}{\tau}_{\alpha\beta}(v) &= \overset{m}{\tau}_{\alpha\beta}(V) + \lambda_c(a_1 x_1 + a_2 x_2 + a_3)\delta_{\alpha\beta}, \\ \overset{m}{\tau}_{\alpha 3}(v) &= \overset{m}{\tau}_{\alpha 3}(V) + \epsilon_{\beta\alpha 3}\mu_c a_4 x_\beta, \\ \overset{m}{\tau}_{33}(v) &= \overset{m}{\tau}_{33}(V) + (2\mu_c + \lambda_c)(a_1 x_1 + a_2 x_2 + a_3), \\ m_{\alpha ij}(v) &= m_{\alpha ij}(V), \quad m_{3ij}(v) = 0, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \tau_{\alpha\beta}(V) &= 2\mu_e e_{\alpha\beta}(V) + \lambda_e e_{\rho\rho}(V)\delta_{\alpha\beta} + 2\mu_c \varepsilon_{\alpha\beta}(V) \\ &\quad + \lambda_c \varepsilon_{\gamma\gamma}(V)\delta_{\alpha\beta} + (\lambda_c - \lambda_e)\varepsilon_{33}\delta_{\alpha\beta}, \\ \tau_{\alpha 3}(V) &= 2\mu_e e_{\alpha 3}(V) + 2\mu_c \varepsilon_{\alpha 3}(V), \\ \tau_{33}(V) &= \lambda_e e_{\rho\rho}(V) + \lambda_c \varepsilon_{\beta\beta}(V) - (2\mu_e + \lambda_e - 2\mu_c - \lambda_c)\varepsilon_{33}(V), \\ \overset{m}{\tau}_{\alpha\beta}(V) &= 2\mu_c e_{\alpha\beta}(V) + \lambda_c \varepsilon_{\gamma\gamma}(V)\delta_{\alpha\beta} + 2\mu_m \varepsilon_{\alpha\beta}(V) \\ &\quad + \lambda_m \varepsilon_{\rho\rho}(V)\delta_{\alpha\beta} + (\lambda_m - \lambda_c)\varepsilon_{33}(V)\delta_{\alpha\beta}, \\ \overset{m}{\tau}_{\alpha 3}(V) &= 2\mu_c e_{\alpha 3}(V) + 2\mu_m \varepsilon_{\alpha 3}(V), \\ \overset{m}{\tau}_{33}(V) &= \lambda_c e_{\rho\rho}(V) + \lambda_m \varepsilon_{\beta\beta}(V) - (2\mu_c + \lambda_c - 2\mu_m + \lambda_m)\varepsilon_{33}(V), \\ m_{\alpha ij}(V) &= l^2(2\mu_m \kappa_{\alpha ij}(V) + \lambda_m \kappa_{\alpha rr}(V)\delta_{ij}). \end{aligned} \tag{4.8}$$

The equilibrium equations (3.1) reduce to:

$$
\begin{aligned}
&\tau_{\beta\alpha,\beta}(V) + \lambda_e a_\alpha = 0, \qquad \tau_{\alpha 3,\alpha}(V) = 0,\\
&m_{\rho\alpha\beta,\rho}(V) + \tau_{\alpha\beta}(V) - \tau^m_{\alpha\beta}(V) + (\lambda_e - \lambda_c)(a_1x_1 + a_2x_2 + a_3)\delta_{\alpha\beta} = 0,\\
&m_{\rho\alpha 3,\rho}(V) + \tau_{\alpha 3}(V) - \tau^m_{\alpha 3}(V) + \epsilon_{\beta\alpha 3}(\mu_e - \mu_c)a_4x_3 = 0,\\
&m_{\rho 33,\rho}(V) + \tau_{33}(V) - \tau^m_{33}(V)\\
&\qquad\qquad + (2\mu_e - 2\mu_c + \lambda_e - \lambda_c)(a_1x_1 + a_2x_2 + a_3) = 0.
\end{aligned}
\tag{4.9}
$$

The boundary conditions (3.2) become:

$$
\begin{aligned}
\tau_{\beta\alpha}(V)n_\beta &= -\lambda_e(a_1x_1 + a_2x_2 + a_3)n_\alpha,\\
\tau_{\beta 3}(V)n_\beta &= \mu_e a_4(x_2n_1 - x_1n_2),\\
m_{\beta ij}(V)n_\beta &= 0 \quad \text{on } \Pi.
\end{aligned}
\tag{4.10}
$$

Equations (4.9) and (4.10) can be rewritten in the form:

$$
\begin{aligned}
&\tau_{\alpha i,\alpha}(V) + f_i = 0,\\
&m_{\rho ij,\rho}(V) + \tau_{ij}(V) - \tau^m_{ij}(V) + G_{ij} = 0 \quad \text{on } \Sigma,
\end{aligned}
\tag{4.11}
$$

and

$$
\tau_{\beta i}(V)n_\beta = \tilde{\tau}_i, \qquad m_{\beta ij}(V)n_\beta = \tilde{m_{ij}} \quad \text{on } L,
\tag{4.12}
$$

where L is the boundary of Σ and:

$$
\begin{aligned}
f_\alpha &= \lambda_e a_\alpha, \quad f_3 = 0,\\
G_{\alpha\beta} &= (\lambda_e - \lambda_c)(a_1x_1 + a_2x_2 + a_3)\delta_{\alpha\beta},\\
G_{\alpha 3} &= \epsilon_{\beta\alpha 3}(\mu_e - \mu_c)a_4x_3,\\
G_{33} &= (2\mu_e - 2\mu_c + \lambda_e - \lambda_c)(a_1x_1 + a_2x_2 + a_3),\\
\tilde{\tau_\alpha} &= -\lambda_e(a_1x_1 + a_2x_2 + a_3)n_\alpha,\\
\tilde{\tau_3} &= \mu_e a_4(x_2n_1 - x_1n_2), \quad \tilde{m_{ij}} = 0.
\end{aligned}
\tag{4.13}
$$

The vector $V = (\omega, Q)$ is the solution of the plane strain problem corresponding to the body loads f_i and G_{ij} and to the boundary data $\tilde{\tau}_i$ and $\tilde{m_{ij}}$. The necessary and sufficient conditions for the existence of a solution to the plane problem are [31, 33]:

$$
\begin{aligned}
&\int_{\Sigma_1} f_i da + \int_L \tilde{\tau}_i\, ds = 0,\\
&\int_{\Sigma_1} \varepsilon_{3\beta\alpha}(x_\beta f_\alpha + G_{\alpha\beta})\, da + \int_L \varepsilon_{3\beta\alpha}(x_\beta\tilde{\tau_\alpha} + \tilde{m_{\alpha\beta}})\, ds = 0.
\end{aligned}
\tag{4.14}
$$

Taking into account the relations (4.13) and by using the divergence theorem, the first condition of (4.14) is verified. The second condition of (4.14) becomes

$$
\begin{aligned}
(4.15)\qquad & \int_{\Sigma_1} \epsilon_{3\beta\alpha}[\lambda_e x_\beta a_\alpha + (l_e - l_c)(a_1 x_1 + a_2 x_2 + a_3)\delta_{\alpha\beta}]\, da \\
& \qquad - \int_L \epsilon_{3\beta\alpha}\lambda_e x_\beta (a_1 x_1 + a_2 x_2 + a_3) n_\alpha \, ds = 0.
\end{aligned}
$$

Since $\varepsilon_{3\beta\alpha}\delta_{\alpha\beta} = 0$, also the second condition of (4.14) is verified. We conclude that the necessary and sufficient conditions for the existence of a solution of the problem P_1 are satisfied for any constants a_k $(k = 1, 2, 3, 4)$. In the following we show that the plane strain problem (4.5)–(4.12) can be decomposed in four plane strain problems.

5. Decomposition of the problem P_1

We introduce the notations:

$$
(5.1)\qquad \omega_i = \sum_{k=1}^{4} a_k \omega_i^{(k)}(x_1, x_2), \qquad Q_{ij} = \sum_{k=1}^{4} a_k Q_{ij}^{(k)}(x_1, x_2).
$$

Equations (4.3) become:

$$
\begin{aligned}
u_\alpha &= -\frac{1}{2} a_\alpha x_3^2 - a_4 \epsilon_{\alpha\beta 3} x_\beta x_3 + \sum_{k=1}^{4} a_k \omega_\alpha^{(k)}(x_1, x_2), \\
u_3 &= (a_1 x_1 + a_2 x_2 + a_3) x_3 + \sum_{k=1}^{4} a_k \omega_3^{(k)}(x_1, x_2),
\end{aligned}
$$

$$
(5.2)\qquad
\begin{aligned}
P_{\alpha\beta} &= \epsilon_{3\beta\alpha} a_4 x_3 + \sum_{k=1}^{4} a_k Q_{\alpha\beta}^{(k)}(x_1, x_2), \\
P_{\alpha 3} &= -a_4 x_3 + \sum_{k=1}^{4} a_k Q_{\alpha 3}^{(k)}(x_1, x_2), \\
P_{3\alpha} &= a_4 x_3 + \sum_{k=1}^{4} a_k Q_{3\alpha}^{(k)}(x_1, x_2), \\
P_{33} &= \sum_{k=1}^{4} a_k Q_{33}^{(k)}(x_1, x_2).
\end{aligned}
$$

From (2.3) and (5.2), we obtain:

$$
\begin{aligned}
e_{\alpha\beta} &= \sum_{k=1}^{4} a_k e_{\alpha\beta}^{(k)},\\
e_{3\alpha} &= -\frac{1}{2} a_4 \epsilon_{\alpha\beta 3} x_\beta + \sum_{k=1}^{4} a_k e_{3\alpha}^{(k)},\\
e_{33} &= a_1 x_1 + a_2 x_2 + a_3 + \sum_{k=1}^{4} a_k e_{33}^{(k)},\\
\varepsilon_{ij} &= \Sigma a_k \varepsilon_{ij}^{(k)}, \qquad \kappa_{\rho ij} = \Sigma a_k \kappa_{\rho ij}^{(k)},
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
e_{\alpha\beta}^{(k)} &= \tfrac{1}{2}(\omega_{\alpha,\beta}^{(k)} + \omega_{\beta,\alpha}^{(k)} - Q_{\alpha\beta}^{(k)} - Q_{\beta\alpha}^{(k)}),\\
e_{3\alpha}^{(k)} &= \tfrac{1}{2}(\omega_{3,\alpha}^{(k)} - Q_{\alpha 3}^{(k)} - Q_{3\alpha}^{(k)}),\\
e_{33}^{(k)} &= -Q_{33}^{(k)}, \qquad \varepsilon_{ij}^{(k)} = \tfrac{1}{2}(Q_{ij}^{(k)} + Q_{ji}^{(k)}),\\
\kappa_{\rho ij}^{(k)} &= \tfrac{1}{2}(Q_{ij}^{(k)} + Q_{ji}^{(k)})_{,\rho}, \qquad \kappa_{3ij}^{(k)} = 0.
\end{aligned}
\tag{5.4}
$$

The stress tensors are given by:

$$
\begin{aligned}
\tau_{\alpha\beta} &= \sum_{k=1}^{4} a_k \tau_{\alpha\beta}^{(k)} + \lambda_e (a_1 x_1 + a_2 x_2 + a_3)\delta_{\alpha\beta},\\
\tau_{3\alpha} &= \sum_{k=1}^{4} a_k \tau_{3\alpha}^{(k)} - \epsilon_{\alpha\beta 3} a_4 \mu_e x_\beta,\\
\tau_{33} &= \sum_{k=1}^{4} a_k \tau_{33}^{(k)} + (2\mu_e + \lambda_e)(a_1 x_1 + a_2 x_2 + a_3),\\
\tau_{\alpha\beta}^{m} &= \sum_{k=1}^{4} a_k \tau_{\alpha\beta}^{(k)m} + \lambda_c (a_1 x_1 + a_2 x_2 + a_3)\delta_{\alpha\beta},\\
\tau_{3\alpha}^{m} &= \sum_{k=1}^{4} a_k \tau_{3\alpha}^{(k)m} - \epsilon_{\alpha\beta 3} a_4 \mu_c x_\beta,\\
\tau_{33}^{m} &= \sum_{k=1}^{4} a_k \tau_{33}^{(k)m} + (2\mu_c + \lambda_c)(a_1 x_1 + a_2 x_2 + a_3),\\
m_{\rho ij} &= \sum_{k=1}^{4} a_k m_{\rho ij}^{(k)},
\end{aligned}
\tag{5.5}
$$

where

$$
\begin{aligned}
\tau_{\alpha\beta}^{(k)} &= 2\mu_e e_{\alpha\beta}^{(k)} + \lambda_e e_{rr}^{(k)}\delta_{\alpha\beta} + 2\mu_c\varepsilon_{\alpha\beta}^{(k)} + \lambda_c\varepsilon_{rr}^{(k)}\delta_{\alpha\beta},\\
\tau_{3\alpha}^{(k)} &= 2\mu_e e_{3\alpha}^{(k)} + 2\mu_c\varepsilon_{3\alpha}^{(k)},\\
\tau_{33}^{(k)} &= 2\mu_e e_{33}^{(k)} + \lambda_e e_{rr}^{(k)} + 2\mu_c\varepsilon_{33}^{(k)} + \lambda_c\varepsilon_{rr}^{(k)},\\
\tau_{\alpha\beta}^{(k)m} &= 2\mu_c e_{\alpha\beta}^{(k)} + \lambda_c e_{rr}^{(k)}\delta_{\alpha\beta} + 2\mu_m\varepsilon_{\alpha\beta}^{(k)} + \lambda_m\varepsilon_{rr}^{(k)}\delta_{\alpha\beta},\\
\tau_{\alpha 3}^{(k)m} &= 2\mu_c e_{3\alpha}^{(k)} + 2\mu_m\varepsilon_{3\alpha}^{(k)},\\
\tau_{33}^{(k)m} &= 2\mu_c e_{33}^{(k)} + \lambda_c e_{rr}^{(k)} + 2\mu_m\varepsilon_{33}^{(k)} + \lambda_m\varepsilon_{rr}^{(k)},\\
m_{\rho ij}^{(k)} &= l^2(2\mu_m\kappa_{\rho ij}^{(k)} + \lambda_m\kappa_{\rho rr}^{(k)}\delta_{ij}).
\end{aligned}
\tag{5.6}
$$

It follows from (4.9), (4.13) and (5.5) that the equilibrium equations can be expressed in the form:

$$
\begin{aligned}
&\sum_{k=1}^{4} a_k\tau_{\beta\alpha,\beta}^{(k)} + f_\alpha = 0, \quad \sum_{k=1}^{4} a_k\tau_{\beta 3,\beta}^{(k)} = 0,\\
&\sum_{k=1}^{4} a_k(m_{\rho\alpha\beta,\rho}^{(k)} + \tau_{\alpha\beta}^{(k)} - \tau_{\alpha\beta}^{(k)m}) + G_{\alpha\beta} = 0,\\
&\sum_{k=1}^{4} a_k(m_{\rho\alpha 3,\rho}^{(k)} + \tau_{\alpha 3}^{(k)} - \tau_{\alpha 3}^{(k)m}) + G_{\alpha 3} = 0,\\
&\sum_{k=1}^{4} a_k(m_{\rho 33,\rho}^{(k)} + \tau_{33}^{(k)} - \tau_{33}^{(k)m}) + G_{33} = 0.
\end{aligned}
\tag{5.7}
$$

From (4.10), (4.13) and (5.5), the boundary conditions on L take the form:

$$
\begin{aligned}
&\sum_{k=1}^{4} a_k\tau_{\beta\alpha}^{(k)} n_\beta = \tilde{\tau}_\alpha, \qquad \sum_{k=1}^{4} a_k\tau_{\beta 3}^{(k)} n_\beta = \tilde{\tau}_3,\\
&\sum_{k=1}^{4} a_k m_{\beta ij}^{(k)} n_\beta = 0.
\end{aligned}
\tag{5.8}
$$

Equations (5.7) and (5.8) must be verified for any value of the constants a_k. First, we consider $a_1 = 1$, $a_2 = a_3 = a_4 = 0$. From (5.7) and (5.8) we obtain:

$$
\begin{aligned}
&\tau_{\beta i,\beta}^{(1)} + f_i^{(1)} = 0,\\
&m_{\rho ij,\rho}^{(1)} + \tau_{ij}^{(1)} - \tau_{ij}^{(1)m} + G_{ij}^{(1)} = 0 \quad \text{on } \Sigma,
\end{aligned}
\tag{5.9}
$$

and

$$\tau^{(1)}_{\beta i} n_\beta = \tilde{\tau_i}^{(1)}, \qquad m^{(1)}_{\beta ij} n_\beta = 0 \quad \text{on } L, \tag{5.10}$$

where

$$\begin{aligned} & f^{(1)}_1 = \lambda_e, \qquad f^{(1)}_2 = f^{(1)}_3 = 0, \\ & G^{(1)}_{\alpha\beta} = (\lambda_e - \lambda_c) x_1 \delta_{\alpha\beta}, \quad G^{(1)}_{\alpha 3} = 0, \quad G^{(1)}_{33} = (2\mu_e - 2\mu_c + \lambda_e - \lambda_c) x_1, \\ & \tilde{\tau_\alpha}^{(1)} = -\lambda_e x_1 n_\alpha, \qquad \tau^{(1)}_3 = 0. \end{aligned} \tag{5.11}$$

Equations (5.4), the constitutive equations (5.6), for $k = 1$, the equilibrium equations (5.9) and the boundary conditions (5.10) are the field equations of a plane strain problem $A^{(1)}$ in the symmetric micromorphic theory. In the same way we define the problem $A^{(2)}$ by putting $a_2 = 1$, $a_1 = a_3 = a_4 = 0$, we have:

$$\begin{aligned} & \tau^{(2)}_{\beta i,\beta} + f^{(2)}_i = 0, \\ & m^{(2)}_{\rho ij,\rho} + \tau^{(2)}_{ij} - \tau^{(2)m}_{ji} + G^{(2)}_{ij} = 0 \quad \text{on } \Sigma, \end{aligned} \tag{5.12}$$

and

$$\tau^{(2)}_{\beta i} n_\beta = \tilde{\tau_i}^{(2)}, \qquad m^{(2)}_{\beta ij} n_\beta = 0 \quad \text{on } L, \tag{5.13}$$

where

$$\begin{aligned} & f^{(2)}_2 = \lambda_e, \qquad f^{(2)}_1 = f^{(2)}_3 = 0, \\ & G^{(2)}_{\alpha\beta} = (\lambda_e - \lambda_c) x_2 \delta_{\alpha\beta}, \qquad G^{(2)}_{\alpha 3} = 0, \qquad G^{(2)}_{33} = (2\mu_e - 2\mu_c + \lambda_e - \lambda_c) x_2, \\ & \tilde{\tau_\alpha}^{(2)} = -\lambda_e x_2 n_\alpha, \qquad \tilde{\tau_3}^{(2)} = 0. \end{aligned} \tag{5.14}$$

The problem $A^{(3)}$ is given by assuming $a_3 = 1$ $a_1 = a_2 = a_4 = 0$. We get:

$$\tau^{(3)}_{\beta i,\beta} = 0, \qquad m^{(3)}_{\rho ij,\rho} + \tau^{(3)}_{ij} - \tau^{(3)m}_{ij} + G^{(3)}_{ij} = 0 \quad \text{on } \Sigma, \tag{5.15}$$

and

$$\tau^{(3)}_{\beta i} n_\beta = \tilde{\tau_i}^{(3)}, \qquad m^{(3)}_{\beta ij} n_\beta = 0 \quad \text{on } L, \tag{5.16}$$

where

$$\begin{aligned} & G^{(3)}_{\alpha\beta} = (\lambda_e - \lambda_c)\delta_{\alpha\beta}, \qquad G^{(3)}_{\alpha 3} = 0, \\ & G^{(3)}_{33} = 2\mu_e - 2\mu_c + \lambda_e - \lambda_c, \\ & \tilde{\tau_\beta}^{(3)} = -\lambda_e n_\beta, \qquad \tilde{\tau_3}^{(3)} = 0. \end{aligned} \tag{5.17}$$

The problem $A^{(4)}$ is defined by $a_1 = a_2 = a_3 = 0$, $a_4 = 1$. We obtain:

$$\tau^{(4)}_{\beta i,\beta} = 0, \qquad m^{(4)}_{\rho ij,\rho} + \tau^{(4)}_{ij} - \tau^{(4)m}_{ij} + G^{(4)}_{ij} = 0 \quad \text{on } \Sigma, \tag{5.18}$$

and

$$\tau^{(4)}_{\beta i} n_\beta = \tilde{\tau_i}^{(4)}, \qquad m^{(4)}_{\beta ij} n_\beta = 0 \quad \text{on } L, \tag{5.19}$$

where

$$\begin{gathered} G^{(4)}_{\alpha\beta} = 0, \qquad G^{(4)}_{\alpha 3} = \epsilon_{\beta\alpha 3}(\mu_e - \mu_c)x_3, \qquad G^{(4)}_{33} = 0, \\ \tilde{\tau_1}^{(4)} = \tilde{\tau_2}^{(4)} = 0, \qquad \tilde{\tau_3}^{(4)} = \mu_e(x_2 n_1 - x_1 n_2). \end{gathered} \tag{5.20}$$

The problem $A^{(k)}$ $(k = 1, 2, 3, 4)$ is independent of the constants a_k and depend only on the cross-section of the cylinder. The solution of the problem P_1 has been reduced to the solutions of four plane strain problems $A^{(k)}$. Now, we consider the boundary conditions (3.6) on Σ_1. Taking into account Eqs. (3.5), (3.7) and (5.25), we rewrite the conditions (3.6) in the form:

$$\begin{aligned} &\sum_{k=1}^{4} a_k \int_{\Sigma_1} \tau^{(k)}_{3\alpha}\, da = 0, \\ &\sum_{k=1}^{4} a_k \int_{\Sigma_1} \tau^{(k)}_{33}\, da + a_3(2\mu_e + \lambda_e)A = -F_3, \\ &\sum_{k=1}^{4} a_k \int_{\Sigma_1} x_2 \tau^{(k)}_{33}\, da + a_2(2\mu_e + \lambda_e)I_1 = -M_1, \\ &\sum_{k=1}^{4} a_k \int_{\Sigma_1} x_1 \tau^{(k)}_{33}\, da + a_1(2\mu_e + \lambda_e)I_2 = -M_2, \\ &\sum_{k=1}^{4} a_k \int_{\Sigma_1} (x_1 \tau^{(k)}_{32} - x_2 \tau^{(k)}_{31})\, da + \mu_e a_4 I_0 = -M_3, \end{aligned} \tag{5.21}$$

where we have used the notations:

$$\begin{gathered} \int_{\Sigma_1} da = A, \qquad \int_{\Sigma_1} x_2^2\, da = I_1, \qquad \int_{\Sigma_1} x_1^2\, da = I_2, \\ \int_{\Sigma_1} (x_1^2 + x_2^2)\, da = I_0. \end{gathered} \tag{5.22}$$

The first two equations of (5.21) are identically satisfied. For simplicity we give the proof for $\alpha = 1$. Since $v_{,3}$ is a rigid displacement, this implies that $\mathcal{M}(v_{,3}) = 0$. From (3.5) we have

$$\begin{aligned}
0 &= \int_{\Sigma_1} [x_2\tau_{33}(v_{,3}) + m_{332}(v_{,3}) - m_{323}(v_{,3})]\, da \\
&= \int_{\Sigma_1} [x_2\tau_{33,3}(v) + m_{332,3}(v) - m_{323,3}(v)]\, da \\
&= -\int_{\Sigma_1} x_2(\tau_{13,1}(v) + \tau_{23,2}(v))\, da \\
&= -\int_{\Sigma_1} (x_2\tau_{13}(v))_{,1} + (x_2\tau_{23}(v))_{,2} + \int_{\Sigma_1} \tau_{31}\, da \\
&= -\int_{L} x_2(\tau_{13}(v)n_1 + \tau_{23}(v_{,m}))\, ds + \int_{\Sigma_1} \tau_{31} da = \int_{\Sigma_1} \tau_{31}\, da.
\end{aligned}$$

The remaining four equations of (5.21) constitute a non homogeneous system for the constants a_k $(k = 1, 2, 3, 4)$. If we use the notations:

$$\begin{aligned}
&A_{1\alpha} = \int_{\Sigma_1} \overset{(\alpha)}{\tau}_{33}\, da, \qquad A_{13} = \int_{\Sigma_1} \overset{(3)}{\tau}_{33}\, da + (2\mu_e + \lambda_e)A, \\
&A_{14} = \int_{\Sigma_1} \overset{(4)}{\tau}_{33}\, da, \qquad A_{21} = \int_{\Sigma_1} x_2\overset{(1)}{\tau}_{33}\, da, \\
&A_{22} = \int_{\Sigma_1} x_2\overset{(2)}{\tau}_{33}\, da + (2\mu_e + \lambda_e)I_1, \\
&A_{2s} = \int_{\Sigma_1} x_2\overset{(s)}{\tau}_{33}\, da \quad (s = 3, 4), \\
&A_{31} = \int_{\Sigma_1} x_1\overset{(1)}{\tau}_{33}\, da + (2\mu_e + \lambda_e)I_2, \\
&A_{3r} = \int_{\Sigma_1} x_1\overset{(r)}{\tau}_{33}\, da \quad (r = 2, 3, 4), \\
&A_{4j} = \int_{\Sigma_1} (x_1\overset{(j)}{\tau}_{32} - x_2\overset{(j)}{\tau}_{31})\, da \quad (j = 1, 2, 3), \\
&A_{44} = \int_{\Sigma_1} (x_1\overset{(4)}{\tau}_{32} - x_2\overset{(4)}{\tau}_{31})\, da + \mu_e I_0.
\end{aligned} \tag{5.23}$$

The system (5.21) takes the following form:

$$
\begin{aligned}
&\sum_{k=1}^{k} A_{1k}a_k = -F_3, \qquad \sum_{k=1}^{k} A_{2k}a_k = -M_1,\\
&\sum_{k=1}^{4} A_{3k}a_k = -M_2, \qquad \sum_{k=1}^{4} A_{4k}a_k = -M_3.
\end{aligned} \tag{5.24}
$$

The positive definiteness of the elastic potential and the reciprocal theorem imply

$$
\det A_{ij} > 0, \qquad A_{ij} = A_{ji}. \tag{5.25}
$$

We conclude that the system (5.24) admits a unique solution and the constant a_k are uniquely determined. Thus the problem of extension, bending and torsion of an elastic isotropic cylinder has been solved. The results established in the previous sections, can be used to derive the solutions for special cases of simplified micromorphic theories. For instance we consider the case in which $\lambda_c = 0$, $\mu_c = 0$. The constitutive equations (2.4) become:

$$
\begin{aligned}
\tau_{ij}(v) &= 2\mu_e e_{ij}(v) + \lambda_e e_{rr}(v)\delta_{ij},\\
\tau^m_{ij}(v) &= 2\mu_m \varepsilon_{ij}(v) + \lambda_m \varepsilon_{kk}(v)\delta_{ij},\\
m_{kij}(v) &= l^2(2\mu_m \kappa_{kij}(v) + \lambda_m \kappa_{krr}(v)\delta_{ij});
\end{aligned} \tag{5.26}
$$

In this case the problems $A^{(k)}$ are characterized by the following values of $f^{(k)}$, $G^{(k)}_{ij}$ and $\tilde{\tau}^{(k)}$:

Problem $A^{(1)}$:

$$
\begin{aligned}
&f^{(1)}_1 = \lambda_e, \quad f^{(1)}_2 = f^{(1)}_3 = 0, \quad G^{(1)}_{\alpha\beta} = \lambda_e x_1 \delta_{\alpha\beta},\\
&G^{(1)}_{\alpha 3} = 0, \quad G^{(1)}_{33} = (2\mu_e + \lambda_e)x_1, \quad \tilde{\tau}^{(1)}_1 = -\lambda_e x_1 n_1,\\
&\tilde{\tau}^{(1)}_2 = -\lambda_e x_1 n_2, \quad \tilde{\tau}^{(1)}_3 = 0;
\end{aligned} \tag{5.27}
$$

Problem $A^{(2)}$:

$$
\begin{aligned}
&f^{(2)}_2 = \lambda_e, \quad f^{(1)}_1 = f^{(3)}_3 = 0, \quad G^{(2)}_{\alpha\beta} = \lambda_e x_2 \delta_{\alpha\beta},\\
&G^{(2)}_{\alpha 3} = 0, \quad G^{(2)}_{33} = (2\mu_e + \lambda_e)x_2, \quad \tilde{\tau}^{(2)}_1 = -\lambda_e x_2 n_1,\\
&\tilde{\tau}^{(2)}_2 = -\lambda_e x_2 n_2, \quad \tilde{\tau}^{(2)}_3 = 0.
\end{aligned} \tag{5.28}
$$

Problem $A^{(3)}$:

$$
\begin{aligned}
&f^{(3)}_1 = f^{(3)}_2 = f^{(3)}_3 = 0, \quad G^{(3)}_{\alpha\beta} = \lambda_e \delta_{\alpha\beta}, \quad G^{(3)}_{\alpha 3} = 0,\\
&G^{(3)}_{33} = 2\mu_e + \lambda_e, \quad \tilde{\tau}^{(3)}_1 = -\lambda_e n_1, \quad \tilde{\tau}^{(3)}_2 = -\lambda_e n_2, \quad \tilde{\tau}^{(3)}_3 = 0;
\end{aligned} \tag{5.29}
$$

Problem $A^{(4)}$:

$$f_1^{(4)} = f_2^{(4)} = f_3^{(4)} = 0, \quad G_{\alpha\beta}^{(4)} = 0, \quad G_{\alpha 3}^{(4)} = \mu_e x_3,$$
$$G_{33}^{(4)} = 0, \quad \tilde{\tau}_1^{(4)} = \tilde{\tau}_2^{(4)} = 0, \quad \tilde{\tau}_3^{(4)} = -\mu_e(x_2 n_1 - x_1 n_2). \tag{5.30}$$

The constants a_k are determined by solving the system (5.24). In this model, as in the previous model, the constants A_{ij} $(i, j = 1, 2, 3, 4)$, are independent of the material constants μ_c and λ_c.

6. The problem P_2: flexure

In this Section we generalize a method established by Iesan [28, 29] to construct the solution of the flexure. The method is based on the following propositions:

1. *If* $\boldsymbol{u}$ *is a solution of the problem of flexure* $P_2 = (F_1, F_2)$ *then* $u_{,3}$ *is a solution of the bending problem* $P_1 = (0, F_2, -F_1, 0)$.

2. *Let* v *be a solution of the bending problem* $P_1 = (0, F_2, -F_1, 0)$. *By integrating* v *with respect to the axial coordinate* x_3, *we obtain a solution of the problem* $P_2 = (F_1, F_2)$.

Thus, we seek the solution of the problem of flexure in the form:

$$\begin{aligned}
u_\alpha &= -\tfrac{1}{6} c_\alpha x_3^3 - \tfrac{1}{2} b_\alpha x_3^2 + \epsilon_{3\beta\alpha}\big(b_4 x_3 + \tfrac{1}{2} c_4 x_3^2\big) \\
&\quad + \sum_{k=1}^{4} (b_k + c_k x_3)\omega_\alpha^{(k)}(x_1, x_2) + \psi_\alpha(x_1, x_2), \\
u_3 &= (b_1 x_1 + b_2 x_2 + b_3) x_3 + \tfrac{1}{2}(c_1 x_1 + c_2 x_2 + c_3) x_3^2 \\
&\quad + \sum_{k=1}^{4} (b_k + c_k x_3)\omega_3^{(k)}(x_1, x_2) + \psi_3(x_1, x_2),
\end{aligned}$$

$$P_{\alpha\beta} = \epsilon_{3\beta\alpha}(b_4 x_3 + \frac{1}{2} c_4 x_3^2) + \sum_{k=1}^{4} (b_k + c_k x_3) Q_{\alpha\beta}^{(k)}(x_1, x_2) + \chi_{\alpha\beta}(x_1, x_2), \tag{6.1}$$

$$P_{\alpha 3} = -\big(b_\alpha x_3 + \tfrac{1}{2} c_\alpha x_3^2\big) + \sum_{k=1}^{4} (b_k + c_k x_3) Q_{\alpha 3}^{(k)}(x_1, x_2) + \chi_{\alpha 3}(x_1, x_2),$$

$$P_{3\alpha} = b_\alpha x_3 + \tfrac{1}{2} c_\alpha x_3^2 + \sum_{k=1}^{4} (b_k + c_k x_3) Q_{3\alpha}^{(k)}(x_1, x_2) + \chi_{3\alpha}(x_1, x_2),$$

$$P_{33} = \sum_{k=1}^{4} (b_k + c_k x_3) Q_{33}^{(k)}(x_1, x_2) + \chi_{33}(x_1, x_2),$$

where c_k and b_k are unknown constants and, ψ_i and χ_{ij} are unknown functions independent of x_3. In view of (2.3), (5.4) and (6.1) we have:

$$
\begin{aligned}
E_{\alpha\beta} &= \sum_{k=1}^{4}(b_k + c_k x_3)e_{\alpha\beta}^{(k)} + \gamma_{\alpha\beta},\\
E_{3\alpha} &= \sum_{k=1}^{4}\left[(b_k + c_k x_3)e_{3\alpha}^{(k)} + \tfrac{1}{2}c_k\omega_\alpha^{(k)}\right] + \tfrac{1}{2}\epsilon_{3\beta\alpha}(b_4 + c_4 x_3)x_\beta + \gamma_{3\alpha},\\
E_{33} &= \sum_{k=1}^{4}[(b_k + c_k x_3)e_{33}^{(k)} + c_k\omega_3^{(k)}]\\
&\quad + b_1x_1 + b_2x_2 + b_3 + (c_1x_1 + c_2x_2 + c_3)x_3 + \gamma_{33},\\
F_{ij} &= \sum_{k=1}^{4}(b_k + c_k x_3)\varepsilon_{ij}^{(k)} + \nu_{ij},\\
K_{\rho ij} &= \sum_{k=1}^{4}(b_k + c_k x_3)\kappa_{\rho ij}^{(k)} + \nu_{ij,\rho}, \qquad K_{3ij} = \sum_{k=1}^{4} c_k\varepsilon_{ij}^{(k)},
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{aligned}
\gamma_{\alpha\beta} &= \tfrac{1}{2}(\psi_{\alpha,\beta} + \psi_{\beta,\alpha} - \chi_{\alpha\beta} - \chi_{\beta\alpha}), \qquad \gamma_{33} = -\chi_{33},\\
\gamma_{3\alpha} &= \tfrac{1}{2}(\psi_{3,\alpha} - \chi_{\alpha 3} - \chi_{3\alpha}), \qquad \nu_{ij} = \tfrac{1}{2}(\chi_{ij} + \chi_{ji}).
\end{aligned}
\tag{6.3}
$$

From (2.4), (5.6) and (6.2) we obtain the stress tensors:

$$
\begin{aligned}
T_{\alpha\beta} &= \sum_{k=1}^{4}(b_k + c_k x_3)\tau_{\alpha\beta}^{(k)} + \sigma_{\alpha\beta} + \lambda_e h\delta_{\alpha\beta},\\
T_{3\alpha} &= \sum_{k=1}^{4}(b_k + c_k x_3)\tau_{3\alpha}^{(k)} + \sigma_{3\alpha} + \mu_e g_\alpha,\\
T_{33} &= \sum_{k=1}^{4}(b_k + c_k x_3)\tau_{33}^{(k)} + \sigma_{33} + (2\mu_e + \lambda_e)h,\\
T_{\alpha\beta}^{m} &= \sum_{k=1}^{4}(b_k + c_k x_3)\tau_{\alpha\beta}^{(k)m} + \sigma_{\alpha\beta}^{m} + \lambda_c h\delta_{\alpha\beta},\\
T_{3\alpha}^{m} &= \sum_{k=1}^{4}(b_k + c_k x_3)\tau_{3\alpha}^{(k)m} + \sigma_{3\alpha}^{m} + \mu_c g_\alpha,\\
T_{33}^{m} &= \sum_{k=1}^{4}(b_k + c_k x_3)\tau_{33}^{(k)m} + \sigma_{33}^{m} + (2\mu_c + \lambda_c)h,\\
M_{\rho ij} &= \sum_{k=1}^{4}(b_k + c_k x_3)m_{\rho ij}^{(k)} + \mu_{\rho ij}, \qquad M_{3ij} = \sum_{k=1}^{4} c_4\rho_{ij},
\end{aligned}
\tag{6.4}
$$

where

$$
\begin{aligned}
\sigma_{ij} &= 2\mu_e\gamma_{\alpha\beta} + \lambda_e\gamma_{rr}\delta_{ij} + 2\mu_c\nu_{ij} + \lambda_c\nu_{rr}\delta_{ij},\\
\sigma^m_{ij} &= 2\mu_c\gamma_{\alpha\beta} + \lambda_c\gamma_{rr}\delta_{ij} + 2\mu_m\nu_{ij} + \lambda_m\nu_{rr}\delta_{ij},\\
h &= b_1x_1 + b_2x_2 + b_3 + (c_1x_1 + c_2x_2 + c_3)x_3 + \sum_{k=1}^{4} c_k\omega_3^{(k)},\\
g_\alpha &= \epsilon_{3\beta\alpha}(b_4 + c_4x_3)x_\beta + \sum_{k=1}^{4} c_k\omega_\alpha^{(k)},\\
\mu_{\rho ij} &= l^2(2\mu_m\nu_{ij,\rho} + \lambda_m\nu_{rr,\rho}\delta_{ij}),\\
\rho_{ij} &= l^2(2\mu_m\varepsilon_{ij}^{(k)} + \lambda_m\varepsilon^k_{rr}\delta_{ij}).
\end{aligned}
\tag{6.5}
$$

The equilibrium equations (2.7), taking into account Eqs. (6.4), became:

$$
\sigma_{\beta i,\beta} + N_i = 0, \qquad \mu_{\rho ij,\rho} + \sigma_{ij} - \sigma^m_{ij} + R_{ij} = 0,
\tag{6.6}
$$

where

$$
\begin{aligned}
N_\alpha &= \sum_{k=1}^{4} c_k(\tau_{3\alpha}^{(k)} + \lambda_e\omega_{3,\alpha}^{(k)}) + \varepsilon_{3\beta\alpha}\mu_e c_4 x_\beta,\\
N_3 &= \sum_{k=1}^{4} c_k(\tau_{33}^{(k)} + \mu_e\omega_{\beta,\beta}^{(k)}) + (2\mu_e + \lambda_e)(c_1x_1 + c_2x_2 + c_3),\\
R_{\alpha\beta} &= (\lambda_e - \lambda_c)\sum_{k=1}^{4} c_k\omega_3^{(k)}\delta_{\alpha\beta},\\
R_{3\alpha} &= (\mu_e - \mu_c)\sum_{k=1}^{4} c_k\omega_\alpha^{(k)},\\
R_{33} &= (2\mu_e + \lambda_e - 2\mu_c - \lambda_c)\sum_{k=1}^{4} c_k\omega_3^{(k)}.
\end{aligned}
\tag{6.7}
$$

In view of (5.8) and (6.4), the boundary conditions (2.8) reduce to:

$$
\sigma_{\alpha\beta}n_\alpha = \tilde{\sigma_\beta}, \qquad \sigma_{3\alpha}n_\alpha = \tilde{\sigma_3}, \qquad \mu_{\rho ij}n_\rho = 0,
\tag{6.8}
$$

where

$$
\tilde{\sigma_\beta} = -\lambda_e\sum_{k=1}^{4} c_k\omega_3^{(k)}n_\beta, \qquad \tilde{\sigma_3} = -\mu_e\sum_{k=1}^{4} c_k\omega_\beta^{(k)}n_\beta.
\tag{6.9}
$$

The plane strain problem (6.6), (6.8) admits a solution if and only if:

$$\begin{aligned}
&\int_{\Sigma_1} N_i da + \int_L \tilde{\sigma}_i\, ds = 0,\\
&\int_{\Sigma_1} \epsilon_{3\beta\alpha}(x_\beta N_\alpha + R_{\alpha\beta})da + \int_L \epsilon_{3\beta\alpha} x_\beta \tilde{\sigma_\alpha}\, ds = 0.
\end{aligned} \tag{6.10}$$

Taking into account (5.5), (5.22), (6.7) and the divergence theorem, Eqs. (6.10) became:

$$\begin{aligned}
&\sum_{k=1}^{4} c_k \int_{\Sigma_1} \tau_{3\alpha}^{(k)}\, da = 0,\\
&\sum_{k=1}^{4} c_k \int_{\Sigma_1} \tau_{33}^{(k)}\, da + c_3(2\mu_e + \lambda_e)A = 0,\\
&\sum_{k=1}^{4} c_k \int_{\Sigma_1} \epsilon_{3\beta\alpha} x_\beta \tau_{3\alpha}^{(k)}\, da + \mu_e c_4 I_0 = 0.
\end{aligned} \tag{6.11}$$

Equations $(6.11)_1$ are identically satisfied. The remaining two equations of (6.11) take the form:

$$\sum_{k=1}^{4} c_k A_{1k} = 0, \qquad \sum_{k=1}^{4} c_k A_{4k} = 0, \tag{6.12}$$

where A_{1k} $(k = 1, 2, 3, 4)$ are given by (5.23). The boundary conditions on Σ_1 are:

$$\begin{aligned}
&\int_{\Sigma_1} T_{3\alpha}\, da = -F_\alpha, && \int_{\Sigma_1} T_{33}\, da = 0,\\
&\int_{\Sigma_1} \epsilon_{\alpha\beta 3} x_\beta T_{33}\, da = 0, && \int_{\Sigma_1} \epsilon_{3\alpha\beta}(x_\alpha T_{3\beta} + M_{3\beta\alpha})\, da = 0.
\end{aligned} \tag{6.13}$$

Now we consider the first equation of (6.13). We have:

$$\begin{aligned}
\int_{\Sigma_1} T_{31}\, da &= \int_{\Sigma_1} [(x_1 T_{\rho 3})_{,\rho} + x_1 T_{33,3}]\, da = \int_{\Sigma_1} (x_1 T_{33,3})\, da\\
&= \sum_{k=1}^{4} c_k \int_{\Sigma_1} x_1 \tau_{33}^{(k)}\, da + c_1(2\mu_e + \lambda_e) I_2.
\end{aligned} \tag{6.14}$$

Similarly,

$$\int_{\Sigma_1} T_{32}\, da = \sum_{k=1}^{4} c_k \int x_2 \tau_{33}^{(k)}\, da + c_2(2\mu_e + \lambda_e) I_1. \tag{6.15}$$

It follows from (6.14), (6.15) and (5.23) that the first two equations of (6.13) take the form:

$$\sum_{k=1}^{4} c_k A_{3k} = -F_1, \qquad \sum_{k=1}^{4} c_k A_{2k} = -F_2. \tag{6.16}$$

The system given by Eqs. (6.12) and (6.16) determines the constants c_k ($k = 1, 2, 3, 4$). In the following, we suppose that the functions σ_{ij}, $\omega_i^{(k)}$ and the constants c_k are known. From the remaining equations (6.13) we have:

$$\begin{aligned} \sum_{k=1}^{4} b_k A_{1k} &= -F_3^*, \qquad & \sum_{k=1}^{4} b_k A_{2k} &= -M_1^*, \\ \sum_{k=1}^{4} b_k A_{3k} &= -M_2^*, \qquad & \sum_{k=1}^{4} b_k A_{4k} &= -M_3^*, \end{aligned} \tag{6.17}$$

where

$$\begin{aligned} F_3^* &= \int_{\Sigma_1} \sigma_{33}\, da - (2\mu_e + \lambda_e) \sum_{k=1}^{4} c_k \int_{\Sigma_1} \omega_3^{(k)}\, da, \\ M_1^* &= \int_{\Sigma_1} x_2 \sigma_{33}\, da - (2\mu_e + \lambda_e) \sum_{k=1}^{4} c_k \int_{\Sigma_1} x_2 \omega_3^{(k)}\, da, \\ M_2^* &= \int_{\Sigma_1} x_1 \sigma_{33}\, da + (2\mu_e + \lambda_e) \sum_{k=1}^{4} c_4 \int_{\Sigma_1} x_1 \omega_3^{(k)}\, da, \\ M_3^* &= -\int_{\Sigma_1} (x_1 \sigma_{32} - x_2 \sigma_{31})\, da - \mu_e \sum_{k=1}^{4} c_k \int_{\Sigma_1} (x_1 \omega_2^{(k)} - x_2 \omega_1^{(k)})\, da. \end{aligned} \tag{6.18}$$

The system (6.17) determines the constants b_k and the problem of flexure of an elastic and isotropic rod is solved.

7. Conclusions

In this paper, we study the deformation of a beam made of a micromorphic elastic material. We have considered a simplified symmetric micromorphic theory with a reduced number of material constants. As in classical elasticity, the problem is decomposed in the four basic problems of extension, bending, torsion and flexure. The solution is constructed with a method that avoids a priori assumptions and with the help of some plane strain problems. The results can be used to obtain explicit formulas of the deformation of the beam for each basic problem when the cross section is assigned. It might be interesting to compare these analytical explicit formulas with those existing in literature deduced from technical arguments. Moreover, the results presented in this paper are also useful in investigating the problem of a cylinder loaded on the lateral surface.

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References

1. S. Forest, R. Sievert, *Nonlinear microstrain theories*, International Journal of Solids and Structres, **43**, 24, 7224–7245, 2006.
2. A.C. Eringen, E.S. Suhubi, *Nonlinear theory of simple micro-elastic solids I*, International Journal of Engineering Sciences, **2**, 189–203, 1964.
3. A.C. Eringen, *Microcontinuum Field Theories I: Foundations and Solids*, Springer, New York, 1999.
4. Y. Chen, J.D. Lee, *Determining material constants in micromorphic theory through phonon dispersion relations*, International Journal of Engineering Sciences, **41**, 871–886, 2003.
5. P. Neff, *On material constants for micromorphic continua*, [in:] Y. Wang, K. Hutter [eds.], *Trends in Applications of Mathematics to Mechanics, STAMM 2004 Proceedings*, pp. 337–348, 2005.
6. R. Teisseyre, *Earthquake process in a micromorphic continuum*, Pure and Applied Geophysics, **102**, 1, 15–28, 1973.
7. R. Teisseyre, *Symmetric Micromorphic Continuum: Wave Propagation, Point Source Solutions and some Applications to Earthquake Processes*, Continuum Mechanics Aspects of Geodynamics and Rock Fracture Mechanics, NATO Advanced Study Institute Series, **12**, Springer, 1974.
8. P. Neff, I.D. Ghiba, A. Madeo, L. Placidi, G. Rosi, *A unifying perspective: the relaxed linear micromorphic continuum*, Continuum Mechanics and Thermodynamics, **26**, 639–681, 2014.
9. P. Neff, A. Madeo, G. Barbagallo, M.V. d'Agostino, R. Abreu, I.D. Ghiba, *Real wave propagation in the isotropic-relaxed micromorphic model*, The Royal Society Publishing, Proceedings A, **473**, 20160790, 2017.
10. S. De Cicco, *On the deformation of porous spherical bodies under radial surface traction*, Journal of Theoretical and Applied Mechanics, **61**, 2, 305–316, 2023.

11. D. Iesan, L. Nappa, *Saint-Venant's problem for microstretch elastic solids*, International Journal of Engineering Sciences, **32**, 2, 229–236, 1994.

12. D. Iesan, L. Nappa, *Extension and bending of microstretch elastic circular cylinders*, International Journal of Engineering Sciences, **33**, 8, 1139–1151, 1995.

13. S. De Cicco, L. Nappa, *Torsion and flexure of microstretch elastic circular cylinders*, International Journal of Engineering Sciences, **35**, 6, 573–583, 1997.

14. L. Nappa, S. Pesce, *Thermoelastic deformation of microstretch elastic beams*, Journal of Thermal Stresses, **19**, 8, 763–777, 1996.

15. R. Lakes, W.J. Drugan, *Bending of Cosserat elastic bar of square cross section: theory and experiment*, Journal of Applied Mechanics, **82**, 091002, 2015.

16. A. Taliercio, D. Veber, *Torsion of elastic anisotropic micropolar cylindrical bars*, European Journal of Mechanics – A/Solids, **55**, 45–56, 2016.

17. A. Taliercio, *Torsion of micropolar hollow circular cylinders*, Mechanics Research Communications, **37**, 4, 406–411, 2010.

18. S. De Cicco, *Non-simple elastic materials with double porosity structure*, Archives of Mechanics, **74**, 2-3, 127–142, 2022.

19. S. De Cicco, *Explicit formulas for the deformation of chiral porous circular beams in gradient thermoelasticity*, Symmetry, **16**, 1, 2024, doi: 10.3390/sym16010129.

20. S. De Cicco, D. Iesan, *On the thermal stresses in chiral porous elastic beams*, Continuum Mechanics and Thermodynamics, **35**, 5, 2095–2115, 2023.

21. D. Iesan, *On the deformation of micromorphic elastic beams*, Mathematics and Mechanics of Solids, **26**, 12, 1779–1797, 2021.

22. A. Norouzzadeh, M.F. Oskonie, R. Ausari, H. Rouhi, *Integral and differential nonlocal micromorphic theory: finite element bending analysis of Timoshenko micro-nano-beams*, Engineering Compututations, **37**, 566–590, 2019.

23. M. Shaat, E. Ghavauloo, S. Eman, *A micromorphic beam theory for beams with elongated microstructures*, Scientific Reports, **10**, 1–18, 2020.

24. R.S. Lakes, *Experimental evaluation of micromorphic elastic constants in foams and lattices*, Zeitschrift für angewandte Mathematik und Physik, **74**, 1, 31, 2023.

25. G. Rizzi, G. Hutter, A. Madeo, P. Neff, *Analytical solutions of the cylindrical bending problem for the relaxed micromorphic continuum and other generalized continua*, Continuum Mechanics and Thermodynamics, **33**, 1505–1539, 2021.

26. G. Rizzi, G. Hutter, H. Khan, I.D. Ghiba, A. Madeo, P. Neff, *Analytical solution of the cylindrical torsion problem for the relaxed micromorphic continuum and other generalized continua (including full derivations)*, Mathathematics and Mechanics of Solids, **27**, 3, 507–553, 2022.

27. G.Y. Zhang, X.L. Gao, C.Y. Zheng, C.W. Mi, *A non-classical Bernoulli–Euler beam model based on a simplified micromorphic elasticity theory*, Mechanics of Materials, **161**, 103967, 2021.

28. D. Iesan, *On Saint-Venant's problem*, Archive for Rational Mechanics and Analysis, **91**, 363–373, 1986.

29. D. Iesan, *On Saint-Venat's Problem*, Springer, Berlin, Heidelberg, 1987.

30. D. Iesan, L. Nappa, *Extremum principles and existence results in micromorphic elasticity*, International Journal of Engineering Sciences, **39**, 18, 2051–2070, 2001.

31. I. Hlavacek, M. Hlavacek, *On the existence and uniqueness of solution and some variational principles in linear theories of elasticity with couple stresses*, Applications of Mathematics, **14**, 411–427, 1969.

32. D. Iesan, R. Quintanilla, *Existence and continuous dependence results in the theory of microstretch elastic bodies*, International Journal of Engineering Sciences, **32**, 6, 991–1001, 1994.

33. G. Fichera, *Existence Theorems in Elasticity*, [in:] C. Truesdell [ed.], *Linear Theories of Elasticity and Thermoelasticity*, Springer, Berlin, pp. 347–389, 1975.

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