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On the existence of weakly nonlocal Rayleigh waves with impedance boundary condition

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In this paper, the existence of Rayleigh waves propagating in weakly nonlocal incompressible isotropic elastic half-spaces subject to the tangential impedance boundary condition (TIBC) (at the surface of half-spaces, the tangential stress is proportional to the horizontal displacement and the normal stress is zero) is investigated. It is shown that for the negative values of the dimensionless tangential impedance parameter and the values of the dimensionless nonlocality parameter belong to the interval $(0, 0.5)$, there exist exactly two Rayleigh waves. The first wave is the counterpart of the Rayleigh wave in local incompressible isotropic elastic half-spaces and the second is a new Rayleigh mode appearing due to the presence of nonlocality. Formulae for their velocities have been derived. Remarkably, the second wave can travel with very high velocity.

Key words: Rayleigh waves, weakly nonlocal elasticity, incompressible, impedance boundary condition, existence of Rayleigh waves.

1. Introduction

Surface Rayleigh waves in elastic half-spaces, discovered by Lord Rayleigh [1] in 1885, have been studied extensively and found a wide range of applications in various engineering fields. In the first stage, the studies of Rayleigh waves have been applied mainly in geophysics, seismology for predicting and analyzing earthquakes and Rayleigh waves were created only by earthquakes. The discovery of the Interdigital Transducer (IDT) by White and Voltmaer [2] in 1965, that can convert electric signals to surface waves and inversely, opened up new era for application of Rayleigh waves. With the IDT, Rayleigh waves can be excited by the man and it becomes a very convenient tool for nondestructively evaluating material parameters and predicting defects of structures during loading. Since then, the application of Rayleigh waves is expanded to various engineering fields where the health monitoring of structures in use is needed.

It is well-known that, nanomaterials modeled by the nonlocal continuum mechanics [3] are undergoing a rapid development due to their extraordinary thermal conductivity, mechanical and electrical properties [4], and structures made of these materials such as nano-beams, nano-plates and nano-shells are increasingly used in the modern technology [5]. Since the health of these structures is required to be monitored during their use, the study of Rayleigh waves in nonlocal elastic solids is necessary and important.

The propagation of Rayleigh waves in nonlocal elastic media was investigated by Eringen [6] for traction-free nonlocal isotropic elastic half-spaces, by Singh [7] for transversely isotropic half-spaces, by Pramanik and Biswas [8] for isotropic thermoelastic half-spaces, by Abd-Alla *et al.* [9] for isotropic magnetoelastic half-spaces with voids, by Khurana and Tomar [10], Sing and Sawhney [11] for micropolar elastic half-spaces, by Tong *et al.* [12] for porous elastic half-spaces, by Kaur *et al.* [13] for elastic half-spaces with voids, by Kaur and Singh [14] for isotropic diffusive materials and by Biswas [15] for isotropic thermoelastic half-space coated with an isotropic thermoelastic layer, by Lata and Singh [16] for isotropic magneto-thermoelastic solid with multi-dual-phase lag heat transfer. In these investigations, the authors used Eringen's fully nonlocal elasticity theory [6] to investigate the propagation of Rayleigh waves and employed the Eringen's method [6] to find the Rayleigh wave's solutions. However, it is now well-known that Eringen's fully nonlocal elasticity theory is ill-posed as shown by Romano *et al.* [17] for the beam problems, by Kaplunov *et al.* [18] for the Rayleigh wave problem, by Kaplunov *et al.* [19] for the anti-plane motion of a half-space subjected to a surface loading of a traveling harmonic wave, and recently by Vinh and Anh [20] for harmonic plane wave problems in domains with non-empty boundaries, in general. Furthermore, since the Eringen's method does not satisfy the original equations of motion, the obtained solutions are incorrect, as shown by Kaplunov *et al.* [18, 19]. This method must be replaced by a novel method proposed recently by Vinh and Anh [20]. Most of investigations mentioned above used boundary conditions in local stresses instead of nonlocal ones. However, this is unreasonable as underlined by Eringen [6] (see also Chebalov *et al.* [21]). Therefore, for studying the propagation of nonlocal Rayleigh waves we should employ a nonlocal model of elasticity which is well-posed for any problem of harmonic plane waves. Such a nonlocal elasticity model, called the weakly nonlocal model have been recommended recently by Anh and Vinh [22]. This novel model has been applied to investigate the propagation of nonlocal Stoneley waves [22], nonlocal Rayleigh waves [23, 24] and nonlocal Lamb waves [25].

It is well-known that one of the fundamental issues regarding Rayleigh waves is their existence and uniqueness. For Rayleigh waves in local elastic half-spaces, this problem has been solved even for generally anisotropic half-spaces by

Barnett and Lothe [26] and Mielke and Fu [27] with the result stating that, in generally anisotropic half-spaces there always exists a unique Rayleigh wave. However, the results of this problem for nonlocal Rayleigh waves are very poor. Khurana and Tomar [10] dealt with the necessary condition for a Rayleigh wave (propagating in nonlocal micropolar isotropic elastic half-spaces subject to traction-free boundary conditions) to satisfy the decay condition (see [10, Section 4]). The obtained result is limited to small values of micropolarity and nonlocality. Recently, the existence of Rayleigh waves in weakly nonlocal compressible isotropic elastic half-spaces subject to traction-free boundary conditions have been examined by Vinh *et al.* [28]. It is shown that one Rayleigh wave is always possible, but two or three Rayleigh waves are possible depending on the nonlocal and material parameters.

The propagation of Rayleigh waves in incompressible weakly nonlocal orthotropic elastic half-spaces subject to the full impedance boundary condition (FIBC) (at the surface of half-spaces: the tangential stress is proportional to the horizontal displacement and the normal stress is proportional to the vertical displacement) was investigated by Anh *et al.* [23]. The authors derived the secular equation of Rayleigh waves by using the incompressible limit method [24] (not the traditional method). In this paper, we consider the existence of Rayleigh waves propagating in incompressible weakly nonlocal isotropic half-spaces subject to the tangential impedance boundary condition. In order to solve this problem we use the secular equation obtained by Anh *et al.* in [23] and using the complex function method [29, 30]. It is shown that, for the negative values of the dimensionless tangential impedance parameter and the values of the dimensionless nonlocality parameter belong to the interval $(0, 1/2)$, there exist exactly two Rayleigh waves. One is the counterpart of the local Rayleigh wave and the other is a new Rayleigh mode arising due to the presence of nonlocality and the impedance boundary condition. Unlike the classical mode, the new mode can travel with very high velocity. Remarkably, explicit formulae for the velocity of two Rayleigh waves have been derived.

The propagation of Rayleigh waves in local (classical) elastic half-spaces subject to impedance boundary conditions was investigated by Gogoy *et al.* [31] for isotropic half-spaces, by Vinh and Hue [32, 33] for orthotropic and monoclinic half-spaces, by Singh and Kaur [34, 35], Kaur and Singh [36] for rotating orthotropic and monoclinic half-spaces. For isotropic and orthotropic half-spaces, the explicit secular equations were obtained using the traditional technique, while the explicit secular equations for monoclinic half-spaces were derived employing the method of the polarization vector [32, 37].

By considering directly the bijection of the impedance function derived from the secular equation, Godoy *et al.* [31] have proved that, for compressible isotropic elastic half-spaces subject to the tangential impedance boundary condition, there always exists a unique Rayleigh wave. Using the complex function method [29, 30], VINH and XUAN [38] not only established easily the existence and uniqueness but also derived formulae for the velocity of Rayleigh waves. Recently, also using the complex function method, GIANG and VINH [39] have proved that, the compressible isotropic half-spaces subject to the normal impedance boundary condition (NIBC) (at the surface of half-spaces, the normal stress is proportional to the vertical displacement and the tangential stress is zero) do not always support a Rayleigh wave, although if a Rayleigh wave exists, it is unique. It is worth noting that, the complex function method is a good tool not only for obtaining formulae for the velocity of surface waves [38, 40–45], but also for examining their existence [39, 46].

2. Rayleigh waves in weakly nonlocal incompressible isotropic elastic half-spaces with TIBC

Consider a Rayleigh wave propagating in a weakly nonlocal incompressible orthotropic elastic half-space $x_2 \geq 0$ whose principal material axes are $0x_1, 0x_2, 0x_3$ and it is subject to the full impedance boundary condition (FIBC). Suppose the Rayleigh wave propagates in the x_1-direction and decays in the x_2-direction with the velocity c (>0) and the wave number k (>0). Then, its displacements are of the form:

$$u_i = u_i(x_1, x_2, t) \quad (i = 1, 2), \qquad u_3 \equiv 0, \tag{2.1}$$

where t is the time. According to ANH and VINH [22] and ANH *et al.* [23], the motion of Rayleigh waves is governed by the following equations:

– Equations of motion (without the body force):

$$t_{11,1} + t_{12,2} = \rho \ddot{u}_1, \qquad t_{12,1} + t_{22,2} = \rho \ddot{u}_2, \tag{2.2}$$

where u_i are the displacement components, t_{ij} are the nonlocal stresses, ρ is the mass density, commas signify differentiation with respect to x_k and a dot indicates differentiation with respect to t.

– Constitutive equations:

$$(1 - \epsilon^2 \nabla^2) t_{11} = \sigma_{11}, \quad (1 - \epsilon^2 \nabla^2) t_{12} = \sigma_{12}, \quad (1 - \epsilon^2 \nabla^2) t_{22} = \sigma_{22}, \tag{2.3}$$

where σ_{ij} is the local stress corresponding to the nonlocal stress t_{ij}, $\epsilon = le_0$, l is the atomic spacing, e_0 is the material constant, $\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right]^T$ is the two dimensional gradient operator.

– Hooke relations:

$$\begin{aligned} \sigma_{11} &= -p + c_{11} u_{1,1} + c_{12} u_{2,2}, \\ \sigma_{22} &= -p + c_{12} u_{1,1} + c_{22} u_{2,2}, \\ \sigma_{12} &= c_{66}(u_{1,2} + u_{2,1}), \end{aligned} \tag{2.4}$$

where $p = p(x_1, x_2, t)$ is the hydrostatic pressure associated with the incompressibility constraint and c_{ij} are the stiffness elastic constants.

– Extra conditions:

$$\sigma_{ij} \equiv 0 \Rightarrow t_{ij} \equiv 0 \text{ (in the domain } x_2 > 0). \tag{2.5}$$

– Incompressibility condition:

$$u_{1,1} + u_{2,2} = 0 \tag{2.6}$$

along with the full impedance boundary condition:

$$t_{12} + \omega Z_1 u_1 = 0, \quad t_{22} + \omega Z_2 u_2 = 0 \quad \text{at } x_2 = 0, \tag{2.7}$$

and the decay condition:

$$u_i \to 0, \ t_{ij} \to 0 \quad \text{as } x_2 \to +\infty, \tag{2.8}$$

where ω $(= kc)$ is the wave circular frequency, $Z_1\, Z_2$ $(\in \mathbf{R})$ are the impedance parameters whose dimension is of stress/velocity.

To solve the Rayleigh wave problem that satisfies Eqs. (2.2)–(2.6), the boundary condition (2.7) and the decay condition (2.8), Anh *et al.* [23] used the incompressible limit method proposed by Vinh *et al.* [47]. That means the solution of this problem is obtained from the one of the corresponding compressible problem by taking the incompressible limit. The secular equation of Rayleigh waves propagating in a weakly nonlocal incompressible orthotropic elastic half-space whose surface is subject to the full impedance boundary condition (2.7) has been derived and it is [23, Eq. (57)], namely:

$$\begin{aligned}[x(1+2e)-e_\delta]\sqrt{P'}+x(1+2e)+\delta_1\sqrt{x}[1+e(2-e_\delta)]\sqrt{P'}\sqrt{S'+2\sqrt{P'}}\\ +\delta_2\sqrt{x}(1+2e)\sqrt{S'+2\sqrt{P'}}-\delta_1\delta_2 x[(1+e)^2-e(1+e)S'+e^2P'] = 0,\end{aligned} \tag{2.9}$$

where $\delta_k = Z_k/\sqrt{\rho c_{66}}$ (dimensionless impedance parameters), $e = k^2 e_0^2 l^2$ (dimensionless nonlocality parameter), $x = \rho c^2/c_{66}$ (squared dimensionless Rayleigh wave velocity) and:

$$\begin{aligned}&S' = \frac{e_\delta - x(1+2e) - 2}{1 - ex}, \qquad P' = 1 - \frac{x}{1-ex} \quad (>0),\\ &e_\delta = (c_{11} + c_{22} - 2c_{12})/c_{66}.\end{aligned} \tag{2.10}$$

When the half-space is isotropic, we have:

$$e_\delta = 4, \qquad \sqrt{S' + 2\sqrt{P'}} = 1 + \sqrt{P'} \tag{2.11}$$

and Eq. (2.9) is simplified to:

$$
\begin{aligned}
(2.12)\qquad & [(1+2e)x-4]b_2+(1+2e)x+\delta_1\sqrt{x}(1-2e)b_2(1+b_2) \\
& +\delta_2\sqrt{x}(1+2e)(1+b_2)-\delta_1\delta_2 x[(1+e)^2-e(1+e)(1+b_2^2)+e^2b_2^2]=0,
\end{aligned}
$$

where $b_2 := \sqrt{P'}\,(>0)$, $x=\rho c^2/\mu\,(>0)$ and $\delta_k = Z_k/\sqrt{\rho\,\mu}$, μ is the Lame constant. Taking $\delta_2=0$ we obtain from Eq. (2.12):

$$
(2.13)\qquad (1+2e)x+[(1+2e)x-4]b_2+\delta_1\sqrt{x}(1-2e)b_2(1+b_2)=0
$$

that is the secular equation of Rayleigh waves propagating in weakly nonlocal incompressible isotropic elastic half-spaces subject to the tangential impedance boundary condition.

REMARK 1. The necessary and sufficient condition for a Rayleigh wave to exist is: Eq. (2.13) has a positive real root x and with it b_2 is positive.

3. Existence of weakly nonlocal Rayleigh waves with TIBC and formulae for the wave velocity

THEOREM 1. *Let $\delta_1<0$ and $0<e<1/2$. Then, there exactly exist two Rayleigh waves and their velocities, say x_{1r} and x_{2r}, are calculated by:*

$$
(3.1)\qquad x_{1r}=\frac{(w_{1r}+1)}{2(1+e)w_{1r}},\qquad x_{2r}=\frac{(w_{2r}+1)}{2(1+e)w_{2r}},
$$

where

$$
(3.2)\qquad
\begin{aligned}
w_{1r} &= \frac{(1-P_2/P_3)+\sqrt{(1+P_2/P_3)^2-4(1+P_1/P_3)}}{2}, \\
w_{2r} &= \frac{(1-P_2/P_3)-\sqrt{(1+P_2/P_3)^2-4(1+P_1/P_3)}}{2}
\end{aligned}
$$

in which P_k $(k=1,2,3)$ are given by Eq. (A.1) *in Appendix A.*

According to Remark 1, the necessary and sufficient condition for a Rayleigh wave to exist is Eq. (2.13) has a (positive) root x and with it

$$
b_2=\sqrt{1-x/(1-ex)}>0 \Rightarrow 1-x/(1-ex)>0 \Rightarrow x\in(0,1/(1+e))\cup(1/e,+\infty).
$$

It is readily to verify that Eq. (2.13) is equivalent to the equation:

$$
\begin{aligned}
(3.3)\qquad & (1+2e)x(1-ex)+[(1+2e)x-4]\sqrt{1-ex}\sqrt{1-(1+e)x} \\
& +\delta_1(1-2e)\sqrt{x}\sqrt{1-ex}\sqrt{1-(1+e)x}+\delta_1(1-2e)\sqrt{x}[1-(1+e)x]=0
\end{aligned}
$$

for $x \in (0, 1/(1+e))$ and for $x \in (1/e, +\infty)$ it is equivalent to the equation:

$$(1+2e)x(ex-1) + [(1+2e)x - 4]\sqrt{ex-1}\,\sqrt{(1+e)x-1} + \delta_1(1-2e)\sqrt{x}\,\sqrt{ex-1}\,\sqrt{(1+e)x-1} + \delta_1(1-2e)\sqrt{x}[(1+e)x-1] = 0. \tag{3.4}$$

Now we introduce the transformation:

$$x = \frac{w+1}{2(1+e)w}, \tag{3.5}$$

that is a 1-1 mapping from $x \in (0, 1/(1+e))$ onto $w \in (-\infty, -1) \cup (1, +\infty) := S_1^*$ and from $x \in (1/e, +\infty)$ onto $w \in (0, w_2) := S_2^*$, where $w_2 = e/(2+e)$ $(0 < w_2 < 1)$. Substituting the expression (3.5) of x into Eq. (3.3) and then multiplying the resulting equation by $[2(1+e)w]^2 w$ lead to the equation for w:

$$\begin{aligned} &F_1(w) = 0, \quad w \in S_1^*, \\ &F_1(w) = h_1(w) + h_2(w)\sqrt{w-w_2}\sqrt{w-1} \\ &\qquad + h_3(w)\sqrt{w}\sqrt{w+1}\sqrt{w-w_2}\sqrt{w-1} + h_4(w)\sqrt{w}\sqrt{w+1}, \end{aligned} \tag{3.6}$$

where:

$$\begin{aligned} h_1(w) &= (1+2e)w(w+1)[(2+e)w - e], \\ h_2(w) &= \sqrt{(2+e)(1+e)}[(1+2e) - (7+6e)w]w, \\ h_3(w) &= \sqrt{2(2+e)}(1+e)\delta_1(1-2e)w, \\ h_4(w) &= \sqrt{2}(1+e)^{3/2}\delta_1(1-2e)(w-1)w. \end{aligned} \tag{3.7}$$

With the same action, Eq. (3.4) becomes:

$$\begin{aligned} &F_2(w) = 0, \quad w \in S_2^*, \\ &F_2(w) = h_1(w) - h_2(w)\sqrt{w_2-w}\sqrt{1-w} \\ &\qquad - h_3(w)\sqrt{w}\sqrt{w+1}\sqrt{w_2-w}\sqrt{1-w} + h_4(w)\sqrt{w}\sqrt{w+1}. \end{aligned} \tag{3.8}$$

Now, we apply the complex function method [29, 30] to investigate the solution existence of Eq. (3.6) in S_1^* and Eq. (3.8) in S_2^*. To this end we consider the complex equation:

$$\begin{aligned} &F(z) = 0, \quad z \in \mathbf{C}, \\ &F(z) = h_1(z) + h_2(z)\sqrt{z-w_2}\sqrt{z-1} \\ &\qquad + h_3(z)\sqrt{z}\sqrt{z+1}\sqrt{z-w_2}\sqrt{z-1} + h_4(z)\sqrt{z}\sqrt{z+1}, \end{aligned} \tag{3.9}$$

where $h_k(z)$ are defined by (3.7) in which w is replaced by z and $\sqrt{z}$, $\sqrt{z+1}$, $\sqrt{z-w_2}$, $\sqrt{z-1}$ are chosen as the principal branches of the corresponding square roots. When $z \in S_1^*$, Eq. (3.9) coincides with Eq. (3.6) and for $z \in S_2^*$ it is identical to Eq. (3.8).

PROPOSITION 1. *For $\delta_1 < 0$ and $0 < e < 1/2$, the equation $F(z) = 0$ has exactly one root in S_1^* and one root in S_2^*.*

Proof: Denote $L = L_1 \cup L_2$, $L_1 = [-1, 0]$, $L_2 = [w_2, 1]$, $S = \{z \in \mathbf{C},\, z \notin L\}$, $N(z_0) = \{z \in S : 0 < |z - z_0| < \varepsilon\}$, ε is a sufficiently small positive number, z_0 is a certain point of the complex plane $\mathbf{C}$. If a function $\phi(z)$ is holomorphic in $\Omega \subset \mathbf{C}$ we write $\phi(z) \in H(\Omega)$.

We first show that:

$$F(z) = 0 \Leftrightarrow P(z) = 0 \text{ in the domain } S, \tag{3.10}$$

where $P(z)$ is a third-order polynomial:

$$P(z) = P_3 z^3 + P_2 z^2 + P_1 z + P_0, \tag{3.11}$$

and the coefficients P_k are calculated by Eq. (A.1) in Appendix A. Then, we prove that Eq. $P(z) = 0$ has exactly one root in $S_1^* \subset S$ and one root in $S_2^* \subset S$.

From (3.7) and (3.9) it is seen that:

• *Properties of $F(z)$:*

(f_1) $F(z) \in H(S)$.

(f_2) $F(z)$ is bounded in $N(-1)$, $N(0)$, $N(w_2)$ and $N(1)$.

(f_3) $F(z) = O(z^3)$ as $|z| \to \infty$.

(f_4) $F(z)$ is continuous on L from two sides [29, 30] with the boundary values $F^+(t)$ (the above boundary value of $F(z)$), $F^-(t)$ (the below boundary value of $F(z)$) defined as follows:

$$F^{\pm}(t) = \begin{cases} F_1^{\pm}(t), & t \in L_1, \\ F_2^{\pm}(t), & t \in L_2, \end{cases} \qquad F_k^-(t) = \overline{F_k^+(t)}, \tag{3.12}$$

where $F_k^+(t) = R_k(t) + iI_k(t)$ $(k = 1, 2)$, $R_k(t)$ and $I_k(t)$ are determined as:

$$\begin{aligned} R_1(t) &= h_1(t) - h_2(t)\sqrt{w_2 - t}\sqrt{1 - t}, \\ I_1(t) &= -h_3(t)\sqrt{-t}\sqrt{t+1}\sqrt{w_2 - t}\sqrt{1 - t} + h_4(t)\sqrt{-t}\,\sqrt{t+1}, \\ R_2(t) &= h_1(t) + h_4(t)\sqrt{t}\,\sqrt{t+1}, \\ I_2(t) &= h_2(t)\sqrt{t - w_2}\sqrt{1 - t} + h_3(t)\sqrt{t}\,\sqrt{t+1}\sqrt{t - w_2}\sqrt{1 - t}. \end{aligned} \tag{3.13}$$

Now we introduce the function $g(t)$ $(t \in L)$ defined as:

$$g(t) = \begin{cases} \dfrac{F_1^+(t)}{F_1^-(t)}, & t \in L_1, \\[2ex] \dfrac{F_2^+(t)}{F_2^-(t)}, & t \in L_2. \end{cases} \tag{3.14}$$

Note that, $F_k^+(t)$ and $F_k^-(t)$ have common factor t that is needed to removed from the expression of $g(t)$ in (3.14) and $I_1(t)/t > 0\ \forall t : -1 < t < 0$, $R_2(t)/t > 0$ $\forall t : w_2 < t < 1$. From (3.12) and (3.14) it follows that:

$$F^+(t) = g(t)F^-(t), \quad t \in L. \tag{3.15}$$

Using (3.12), (3.13) and (3.14) it is not difficult to prove that:

Proposition 2. *Let $\delta_1 < 0$ and $0 < e < 1/2$, then we have:*

$$\log g(-1) = 2\pi i, \quad \log g(0) = 2\pi i, \quad \log g(w_2) = 0, \quad \log g(1) = 0. \tag{3.16}$$

Consider the function $\Gamma(z)$ defined by:

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\log g(t)}{t-z}\,dt. \tag{3.17}$$

The function $\Gamma(z)$ is called the Cauchy-type integral [29]. It is not difficult to prove the following (see [29]):

• *Properties of* $\Gamma(z)$*:*
(γ_1) $\Gamma(z) \in H(S)$.
(γ_2) $\Gamma(\infty) = 0$.
(γ_3) (i) For $z \in N(0)$: $\Gamma(z) = \log z + \Gamma_0(z)$, $\Gamma_0(z)$ is bounded in $N(0)$ and takes a defined value at $z = 0$.
(ii)

$$\Gamma(z) = \begin{cases} \log \dfrac{1}{(z+1)} + \Gamma_1(z) & \text{for } z \in N(-1), \\ \Gamma_2(z) & \text{for } z \in N(w_2), \\ \Gamma_3(z) & \text{for } z \in N(1), \end{cases} \tag{3.18}$$

where $\Gamma_1(z)$, $\Gamma_2(z)$ and $\Gamma_3(z)$ are bounded in $N(-1)$, $N(w_2)$ and $N(1)$, respectively, and takes defined values at $z = -1$, $z = w_2$ and $z = 1$.

Note that (γ_3) comes from Proposition 2 (Eq. (3.16)) and [29, formula (29.4)], namely:

$$\Gamma(z) = \pm\frac{\log g(c)}{2\pi i}\log\frac{1}{z-c} + \Gamma_0(z), \tag{3.19}$$

where the upper sign is taken for $c = -1$ and $c = w_2$ (the left end points of L_1 and L_2) and the lower for $c = 0$ and $c = 1$ (the right end points of L_1 and L_2), $\Gamma_0(z)$ is a bounded function in $N(c)$ and takes a defined value at $z = c$.

We now consider the function $Y(z)$ defined by:

$$Y(z) = F(z)/\mathrm{e}^{\Gamma(z)}. \tag{3.20}$$

From (f_1)–(f_3), (3.15), (γ_1)–(γ_3) and the Plemelj formula [29], it follows that:

• *Properties of* $Y(z)$:
(y_1) $Y(z) \in H(S)$.
(y_2) $Y(z) = O(z^3)$ as $|z| \to \infty$.
(y_3) $Y(z)$ is bounded in $N(-1)$, $N(0)$ (noting that $F(z)$ has a common factor z), $N(w_2)$, $N(1)$ and takes defined values at $z = -1, 0, w_2, 1$.
(y_4) $Y^+(t) = Y^-(t)$, $t \in L$, $Y^+(t)$ $(Y^-(t))$ is the above (below) boundary value of $Y(z)$ on L.
(y_5) $Y(-1) = 0$.

Properties (y_1) and (y_4) of the function $Y(z)$ show that $Y(z)$ is holomorphic in the entire complex plane $\mathbf{C}$ with the possible exception of points: $z = -1, 0, w_2, 1$. By (y_3) these points are removable singularity points and it may be assumed that the function $Y(z)$ is holomorphic in the entire complex $\mathbf{C}$ (see [48]). Thus, by the generalized Liouville theorem [48] and taking into account (y_2) we have: $Y(z)$ is a third-order polynomial, namely $P(z)$ defined by (3.11) and $z_1 = -1$ is its zero, according to (y_5).

The coefficients P_k $(k = \overline{0,3})$ of $P(z)$ are found by using (3.20) and the Laurent expansions at $z = \infty$ of functions $F(z)$ and $\mathrm{e}^{-\Gamma(z)}$ (see [30, 40, 42] for details). The expressions of P_1, P_2 and P_3 are given in Eq. (A.1) in Appendix A. It is easy to verify that $P_3 < 0$ for $\delta_1 < 0$ and $0 < e < 1/2$.

From (3.20) we have (noting that $P(z) := Y(z)$):

$$F(z) = \mathrm{e}^{\Gamma(z)} P(z). \tag{3.21}$$

Since $\mathrm{e}^{\Gamma(z)} \neq 0$ $\forall z \in S$, according to (γ_1), from (3.21) it implies the statement (3.10). Now we show that:

$$P(0) < 0, \qquad P(w_2) > 0, \qquad P(1) > 0. \tag{3.22}$$

Indeed, according to $(\gamma_3)(i)$: $\mathrm{e}^{\Gamma(z)} = z\,\mathrm{e}^{\Gamma_0(z)}$, where $\Gamma_0(z)$ is bounded in $N(0)$ and takes a defined value at

$$\begin{aligned} z = 0 \Rightarrow P(0) &= (F(z)/\mathrm{e}^{\Gamma(z)})|_{z=0} = (F(z)/z)|_{z=0}\mathrm{e}^{-\Gamma_0(0)} \\ &= -(1+2e)(e + \sqrt{e(1+e)})\mathrm{e}^{-\Gamma_0(0)} < 0. \end{aligned}$$

In view of (3.7) and (3.9) and taking into account the assumption: $\delta_1 < 0$, $0 < e < 1/2$ we have: $F(w_2) > 0$ and $F(1) > 0$. From these facts and (3.21) it implies: $P(w_2) > 0$ and $P(1) > 0$.

Now we consider the equation $P(z) = 0$ with $z \in \mathbf{R}$. From the first and second of (3.22) it follows that it has a root in the interval $(0, w_2)$. As $P_3 < 0$ as mentioned above, we have: $P(+\infty) < 0$. From this fact and the third of (3.22) it implies: Eq. $P(z) = 0$ has a root in the interval $(1, +\infty)$. Since $z = -1$ is a root of the cubic equation $P(z) = 0$ (according to (y_5)) that has at most three different roots, it follows that Eq. $P(z) = 0$ has exactly one root in the interval $(0, w_2)$ and one root in the domain $(1, +\infty)$ and no roots in the interval $(-\infty, -1)$. That means it has exactly one root in S_1^* and one root in S_2^*. Since $S_1^* \subset S$, $S_2^* \subset S$, from (3.10) it implies that Eq. $F(z) = 0$ has exactly one root in S_1^* and one root in S_2^*. The proof of Proposition 1 is completed.

Proof of Theorem 1.

• *Existence of two Rayleigh waves:* the existence of two Rayleigh waves is deduced from Proposition 1 and the bijective property (one-to-one correspondence) of the transformation (3.5).

• *Formulae for the wave velocity:* since $z = -1$ is a root of $P(z) = 0$, its two other roots, say w_{1r} and w_{2r}, are two roots of the quadratic equation:

$$w^2 - (1 - P_2/P_3)w + 1 + P_1/P_3 - P_2/P_3 = 0, \tag{3.23}$$

according to Vieta's formulas, where P_k $(k = 1, 2, 3)$ are given by Eq. (A.1) in the appendix A. It is clear that w_{1r} $(\in (1, +\infty))$ and w_{2r} $(\in (0, w_2))$ are calculated by (3.2). The Rayleigh wave velocities, say x_{1r} $(\in (0, 1/(1 + e)))$ and x_{2r} $(\in (1/e, +\infty))$, are therefore computed by (3.1) according to the transformation (3.5). The proof of Theorem 1 is finished.

Table 1. Some values of Rayleigh wave velocity computed by solving Eq. (3.3) in the domain: $0 < x < 1/(1 + e)$ (x_1^*) and by using formula $(3.1)_1$ (x_{1r}).

δ_1	-0.1	-0.4	-1	-5.1	-6
e	0.25	0.25	0.3	0.4	0.45
x_1^*	0.7151	0.7222	0.7003	0.6647	0.6300
x_{1r}	0.7151	0.7222	0.7003	0.6647	0.6300

Table 2. Some values of Rayleigh wave velocity computed by solving Eq. (3.4) in the domain: $1/e < x < +\infty$ (x_2^*) and by using formula $(3.1)_2$ (x_{2r}).

δ_1	-0.1	-0.4	-1	-5.1	-6
e	0.25	0.25	0.3	0.4	0.45
x_2^*	4.0330	4.3416	4.1655	4.6846	3.1906
x_{2r}	4.0330	4.3416	4.1655	4.6846	3.1906

For checking the formulae (3.1), some numerical values of the Rayleigh wave velocity are calculated by using the formulae (3.1) (denoted by x_{1r} and x_{2r}) and by solving directly the secular equation (3.3) in the domain: $0 < x < 1/(1 + e)$

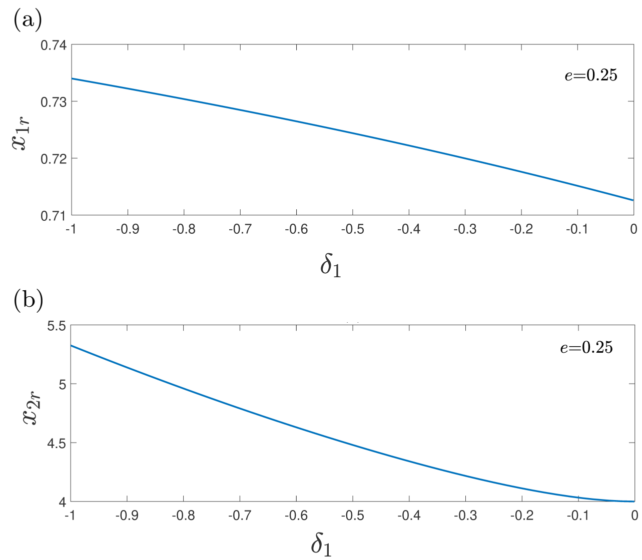

FIG. 1. The dependence of x_{1r} (a) and x_{2r} (b) on the impedance parameter δ_1.

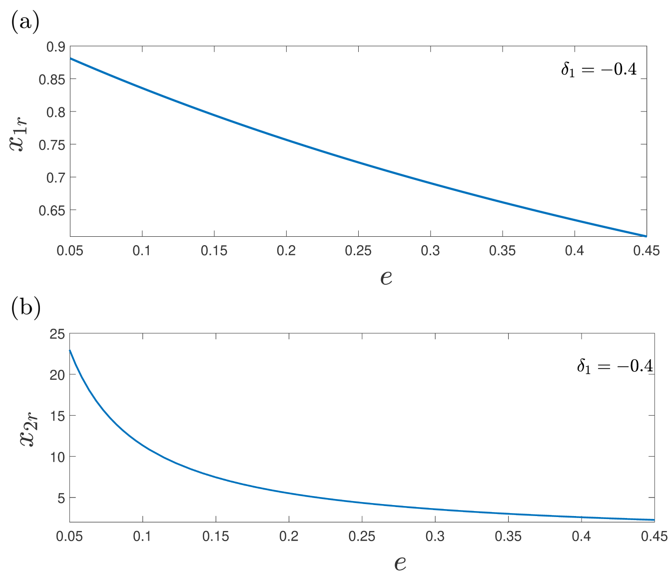

FIG. 2. The dependence of x_{1r} (a) and x_{2r} (b) on the nonlocality parameter e.

(denoted by x_1^*) and the secular equation (3.4) in the domain: $1/e < x < +\infty$ (denoted by x_2^*). It is seen from Tables 1, 2 that they are the same. We use the formulae (3.1) to draw the velocity curves depending on the impedance parameter δ_1 (Fig. 1) and the nonlocality parameter e (Fig. 2). It is seen from Figs. 1, 2 that for a given value of e (δ_1), the Rayleigh wave velocities decrease when δ_1 (e) increases. Since $x_{2r} > 1/e = 1/(k^2\epsilon^2)$, it implies that the new Rayleigh mode can travel with high velocity at small values of k (i.e. at low frequencies).

4. Conclusions

In this paper, the existence of Rayleigh waves in incompressible weakly nonlocal isotropic half-spaces subject to the tangential impedance boundary condition is considered using the complex function method. It is shown that, for the negative values of impedance parameter δ_1 and the values of the nonlocality parameter e belong to the interval (0, 1/2), there always exist two Rayleigh waves. One is the counterpart of the local Rayleigh wave and the other is a new Rayleigh mode arising due to the presence of nonlocality and the impedance boundary condition. It is remarkable that, the new Rayleigh mode can propagate with high velocity at low frequencies (at small values of wave number). Formulae for their velocity have been derived.

Appendix A. Expressions of P_k $(k = 1, 2, 3)$ of polynomial $P(z)$

$$P_3 = B_3, \quad P_2 = B_2 + B_3 J_0, \quad P_1 = B_1 + B_2 J_0 + B_3(J_1 + J_0^2/2), \tag{A.1}$$

where:

$$\begin{aligned}
B_3 &= \sqrt{2+e}\big[(1+2e)\sqrt{2+e} - (7+6e)\sqrt{1+e}\big] \\
&\quad + \sqrt{2}(1+e)\delta_1(1-2e)\big(\sqrt{1+e} + \sqrt{2+e}\big), \\
B_2 &= 2(1+2e) + (8e^2+18e+9)\sqrt{\frac{1+e}{2+e}} \\
&\quad - \frac{\sqrt{2}}{2}\,\delta_1(1-2e)(1+e)\left(\sqrt{1+e} + \frac{e}{\sqrt{2+e}}\right), \\
B_1 &= -(1+2e)e - \frac{\sqrt{1+e}}{2(2+e)^{3/2}}(4e^3+14e^2+8e-3) \\
&\quad - \frac{\sqrt{2}}{8}\delta_1(1-2e)(1+e)\left[\frac{(5e^2+16e+16)}{(2+e)^{3/2}} + 5\sqrt{1+e}\right], \\
J_0 &= \frac{1}{\pi}\left[\int_{-1}^{0}\left(\frac{\pi}{2} - \operatorname{atan}\left\{\frac{R_1(t)}{I_1(t)}\right\}\right)dt + \int_{w_2}^{1}\operatorname{atan}\left\{\frac{I_2(t)}{R_2(t)}\right\}dt\right], \\
J_1 &= \frac{1}{\pi}\left[\int_{-1}^{0} t.\left(\frac{\pi}{2} - \operatorname{atan}\left\{\frac{R_1(t)}{I_1(t)}\right\}\right)dt + \int_{w_2}^{1} t.\operatorname{atan}\left\{\frac{I_2(t)}{R_2(t)}\right\}dt\right].
\end{aligned} \tag{A.2}$$

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Conflict of interest

The authors declare that they have no conflict of interest.

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