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Applications of implicit constitutive theory for describing the elastic response of rocks and concrete

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An implicit constitutive relation is proposed for elastic bodies, when the gradient of the displacement is assumed to be very small, and as a result the strains are small. The resulting constitutive relation is a non-linear relationship between the linearized strain and the stress. The model is used to fit data for rock and concrete. Some boundary value problems are studied within the context of homogeneous deformations, and also a problem with inhomogeneous deformations is analyzed, namely the inflation of a circular annulus. The predictions of this new implicit constitutive relation are compared with the predictions of the constitutive equations for linearized elastic bodies.

Key words: nonlinear elasticity, small strain, isotropic solids.

1. Introduction

Many bodies exhibit the non-linear response even when they are undergoing very small strains, in that the strain is related nonlinearly with respect to the stress[1]. While this is true of several intermetallics (see Saito *et al.* [3], Li *et al.* [4], Talling *et al.* [5], Sakaguch *et al.* [6], Hao *et al.* [7], Withey *et al.* [8], Zhang *et al.* [9]), it is also true of very traditional materials like concrete (see Grasley *et al.* [10]) and rocks (see Cristescu [11]). The response of such materials cannot be described by the classical linearized elastic constitutive relation which one obtains when one linearizes any non-linear Cauchy elastic body or its sub-class Green elastic bodies (see Truesdell and Noll [12]) under the assumption that the displacement gradient is small. On the other hand, linearization of implicit constitutive relations that have been put into place to

[1]Many early scientists that include Leibniz, James Bernoulli, Ricatti, Wertheim and others did not subscribe to the elastic solids obeying Hooke's law even when the strains are exceedingly small (see discussion in Bell [1]; see also Rajagopal [2]). In fact, they considered parabola and hyperbolas to fit the experimental results.

describe the response of elastic bodies (see Rajagopal [13, 14]) do provide approximations that lead to constitutive relationship between linearized strain and stress that is non-linear. A thermodynamic framework for the description of non-dissipative (elastic bodies) has been put into place by (Rajagopal and Srinivasa [15]) within the context of which constitutive relations can be developed, and this will be the starting point for our analysis in this paper.

Using the thermodynamic framework developed by Rajagopal and Srinivasa [15] wherein the Helmholtz potential depends both on the Green–Saint Venant strain and the Cauchy stress, restricting the class of interest of the bodies to isotropic elastic bodies, and appealing to results in representation theory (Spencer [16]), one can generate the implicit constitutive relation in terms of the right Cauchy–Green tensor $\mathbf{B}$ and the Cauchy stress $\mathbf{T}$, their respective invariants and mutual invariants. Then we linearize this implicit constitutive relation under the assumption that the Frobenius norm of the displacement gradient is small in the sense that its square can be neglected in comparison to itself. This then provides us with an implicit constitutive relation between the linearized strain $\boldsymbol{\varepsilon}$ and the Cauchy stress $\mathbf{T}$ (see Eq. (3.3)) wherein the linearized strain appears linearly but the stress appears nonlinearly. We simplify this implicit relation and use it to study several boundary value problems. Two of the boundary value problems that we study make them amenable to corroboration against experiments.

We determine the material constants that appear within the context of our constitutive relation by corroborating against experimental results that are available for the uniform compression of a concrete cylinder (see the experiments results provided in Grasley *et al.* [10]) for the value of the normal axial and radial strains that are functions of the stresses. While we find very good agreement with the results for our non-linear implicit constitutive relations, the agreement is quite poor with respect to the classical linearized elastic constitutive relation (see Fig. 1). We use the same values for the material constants as used in the experimental corroboration to study several other homogeneous deformations and the inflation of a cylindrical body that leads to an inhomogeneous deformation.

In the case of rock, we consider an experiment where a cylinder of the specimen that is free of lateral loading is subject to tension/compression (see the experimental results presented in Bustamante and Ortiz [17]). Once again, we find that the implicit non-linear constitutive relation fits the experimental results much better than the classical linearized elastic constitutive relation.

In addition to studying boundary value problems within the context of the non-linear implicit constitutive relation that we develop in this paper, we also solve the boundary value problems within the context of the classical linearized elastic constitutive relation and compare the two. We find that the solutions are radically different.

The organization of the paper is as follows. After a brief review of the preliminaries in Section 2, we assume a form for the Helmholtz potential in Section 3 and derive the implicit constitutive relation that is used to describe concrete and rock. In Section 4 specific boundary value problems involving homogeneous deformations are solved using the constitutive relations developed in Section 3. In Section 5 we use the experimental data for concrete and rock that are available to corroborate against specific boundary value problems that correspond to the experiments to determine the material constants in the constitutive relation, and then use them in the solution of other boundary value problems. Section 6 is dedicated to the solution of a boundary value problem wherein a cylindrical annulus is subject to inhomogeneous deformation due to inflation. In Section 7 some final remarks are given.

2. Basic equations

The deformation gradient, the displacement field, the Green–Saint Venant strain tensor and the linearized strain tensors are defined as:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}), \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}^{\mathrm{T}}\right). \tag{2.1}$$

The Cauchy stress tensor is denoted by $\mathbf{T}$ and satisfies the balance of linear momentum

$$\rho \ddot{\mathbf{x}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{2.2}$$

where ρ is the density of the body in the current configuration, div is the divergence defined in the current configuration, and $\mathbf{b}$ corresponds to the body forces in that configuration as well. The dot denotes material time derivative. In the case of quasi-static deformations from (2.2) we have

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{0}. \tag{2.3}$$

The second Piola–Kirchhoff stress tensor is denoted by $\mathbf{S}$ and is defined as

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}, \tag{2.4}$$

where $J = \det \mathbf{F}$.

More details concerning the kinematics, concept of stress, and the basic balance equations for a continuum can be found in [18].

3. An implicit relation for elastic bodies

We use as a starting point the theory of RAJAGOPAL and SRINIVASA [15] for elastic bodies, where we assume the existence of a free energy function

$\Pi = \Pi(\mathbf{E}, \mathbf{S})$, and from the first law of thermodynamics we have (see, for example, (3.1), (3.2) in [15] and (6.19), (6.20) in [19])

$$\left[\frac{1}{2}\left(\frac{\partial^2 \Pi}{\partial \mathbf{S} \partial \mathbf{E}} + \frac{\partial^2 \Pi}{\partial \mathbf{E} \partial \mathbf{S}}\right) - \mathscr{I}\right] : \dot{\mathbf{E}} + \frac{\partial^2 \Pi}{\partial \mathbf{S} \partial \mathbf{S}} : \dot{\mathbf{S}} = \mathbf{0}, \tag{3.1}$$

where the components of the fourth order tensor $\mathscr{I}$ are given in Cartesian coordinates as

$$\mathscr{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{3.2}$$

We shall consider the special case wherein the potential Π is an isotropic function of $\mathbf{S}$ and $\mathbf{E}$, when the gradient of the displacement field is small, i.e., $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right| \sim O(\delta)$, $\delta \ll 1$, which implies that $\mathbf{S} \approx \mathbf{T}$ and $\mathbf{E} \approx \boldsymbol{\varepsilon}$, and the strains appear linearly. The particular implicit relation of interest to us is:

$$\begin{aligned} &\boldsymbol{\varepsilon} + \mathfrak{q}^{(0)}(\operatorname{tr}\mathbf{T})\boldsymbol{\varepsilon} + \mathfrak{h}^{(0)}(\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}) + \mathfrak{f}^{(1)}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + \mathfrak{f}^{(2)}(\operatorname{tr}\mathbf{T})(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} \\ &\quad + \mathfrak{q}^{(0)}\operatorname{tr}(\mathbf{T}\boldsymbol{\varepsilon})\mathbf{I} + \mathfrak{q}^{(0)}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{T} + \frac{\partial \mathfrak{p}}{\partial I_4}\mathbf{I} + \frac{\partial \mathfrak{p}}{\partial I_5}\mathbf{T} + \frac{\partial \mathfrak{p}}{\partial I_6}\mathbf{T}^2 = \mathbf{0}, \end{aligned} \tag{3.3}$$

where $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$ are material constants and $\mathfrak{p} = \mathfrak{p}(\mathbf{T}) = \mathfrak{p}(I_4, I_5, I_6)$, where $I_4 = \operatorname{tr}\mathbf{T}$, $I_5 = \frac{1}{2}\operatorname{tr}(\mathbf{T}^2)$ and $I_6 = \frac{1}{3}\operatorname{tr}(\mathbf{T}^3)$. Details concerning how the above relation (3.3) is derived, starting from (3.1) is provided in the Appendix, see in particular (7.29).

For convenience, from now onwards we shall assume that $\mathfrak{p} = \mathfrak{p}(\mathbf{T})$ depends on the principal stresses or eigenvalues of $\mathbf{T}$, i.e., if σ_i, $i = 1, 2, 3$ are such principal stresses, we assume $\mathfrak{p} = \mathfrak{p}(\mathbf{T}) = \mathfrak{p}(I_4, I_5, I_6) = \mathfrak{p}(\sigma_1, \sigma_2, \sigma_3)$, where $\mathfrak{p}(\sigma_1, \sigma_3, \sigma_2) = \mathfrak{p}(\sigma_2, \sigma_1, \sigma_3) = \mathfrak{p}(\sigma_3, \sigma_2, \sigma_1)$, and in such a situation (3.3) becomes

$$\begin{aligned} &\boldsymbol{\varepsilon} + \mathfrak{q}^{(0)}(\operatorname{tr}\mathbf{T})\boldsymbol{\varepsilon} + \mathfrak{h}^{(0)}(\mathbf{T}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\mathbf{T}) + \mathfrak{f}^{(1)}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + \mathfrak{f}^{(2)}(\operatorname{tr}\mathbf{T})(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} \\ &\quad + \mathfrak{q}^{(0)}\operatorname{tr}(\mathbf{T}\boldsymbol{\varepsilon})\mathbf{I} + \mathfrak{q}^{(0)}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{T} + \sum_{i=1}^{3}\frac{\partial \mathfrak{p}}{\partial \sigma_i}\overset{(i)}{\mathbf{t}} \otimes \overset{(i)}{\mathbf{t}} = \mathbf{0}, \end{aligned} \tag{3.4}$$

where $\overset{(i)}{\mathbf{t}}$, $i = 1, 2, 3$ are the principal directions or eigenvectors of $\mathbf{T}$.

In Section 4 we study also the simpler case, where in (3.4) we assume that $\mathfrak{q}^{(0)} = \mathfrak{h}^{(0)} = 0$, and (3.4) reduces to

$$\boldsymbol{\varepsilon} + \mathfrak{f}^{(1)}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + \mathfrak{f}^{(2)}(\operatorname{tr}\mathbf{T})(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + \sum_{i=1}^{3}\frac{\partial \mathfrak{p}}{\partial \sigma_i}\overset{(i)}{\mathbf{t}} \otimes \overset{(i)}{\mathbf{t}} = \mathbf{0}. \tag{3.5}$$

For the function $\mathfrak{p}$ we use a simplified version of the model proposed in [20], where we have

$$\begin{aligned} \mathfrak{p} = {} & \mathfrak{P}_1(\sigma_1) + \mathfrak{P}_1(\sigma_2) + \mathfrak{P}_1(\sigma_3) + \mathfrak{P}_2(\sigma_1)(\sigma_2 + \sigma_3) + \mathfrak{P}_2(\sigma_2)(\sigma_1 + \sigma_3) \\ & + \mathfrak{P}_2(\sigma_3)(\sigma_1 + \sigma_2) + \mathfrak{P}_3(\sigma_S), \end{aligned}$$

where $\sigma_S = (\sigma_1 + \sigma_2 + \sigma_3)/3$. In the present work we assume $\mathfrak{P}_2(x) = 0$, thus after using the notation $\mathfrak{F}$ and $\mathfrak{G}$ for the functions $\mathfrak{P}_1$ and $\mathfrak{P}_3$, we obtain

$$\mathfrak{p} = \mathfrak{F}(\sigma_1) + \mathfrak{F}(\sigma_2) + \mathfrak{F}(\sigma_3) + \mathfrak{G}(\sigma_S). \tag{3.6}$$

Here, since $\sigma_S = \operatorname{tr}\mathbf{T} = (\sigma_1 + \sigma_2 + \sigma_3)/3$, it follows from (3.6) that

$$\frac{\partial \mathfrak{p}}{\partial \sigma_i} = \mathfrak{F}'(\sigma_i) + \frac{1}{3}\mathfrak{G}'(\sigma_S), \tag{3.7}$$

where $\mathfrak{F}'(\sigma_i) = \frac{d\mathfrak{F}}{d\sigma_i}$ and $\mathfrak{G}'(\sigma_S) = \frac{d\mathfrak{G}}{d\sigma_S}$. From now on we assume that

$$\mathfrak{F}'(0) = 0, \quad \mathfrak{G}'(0) = 0. \tag{3.8}$$

From the above implicit relation (3.3) the classical constitutive equation for an isotropic linearly elastic body can be obtained if we assume that $\frac{\partial \mathfrak{p}}{\partial I_4} = \mathfrak{p}^{(2)}(\operatorname{tr}\mathbf{T})$, $\frac{\partial \mathfrak{p}}{\partial I_5} = \mathfrak{p}^{(3)}$, $\frac{\partial \mathfrak{p}}{\partial I_6} = 0$, where $\mathfrak{p}^{(2)}$ and $\mathfrak{p}^{(3)}$ are constants, and we assume that

$$\mathfrak{p}^{(2)} = \frac{\nu}{E}, \ \mathfrak{p}^{(3)} = -\frac{(1+\nu)}{E}, \ \mathfrak{q}^{(0)} = 0, \ \mathfrak{h}^{(0)} = 0, \ \mathfrak{f}^{(1)} = 0, \ \mathfrak{f}^{(2)} = 0, \tag{3.9}$$

where E is the Young modulus and ν is the Poisson ratio. In Sections 5 and 6 we compare some of the results of the new model with the predictions of the classical linearized isotropic elastic body

$$\boldsymbol{\varepsilon} = \frac{(1+\nu)}{E}\mathbf{T} - \frac{\nu}{E}(\operatorname{tr}\mathbf{T})\mathbf{I}. \tag{3.10}$$

In the following section we use (3.4), (3.5) to study some simple boundary value problems, wherein we have homogeneous distributions for the stresses and strains, such that (2.3) is satisfied automatically (in absence of body forces). In Section 5 we use (3.4), (3.5) to fit the experimental data for concrete from [10], and for rock from [17], considering in particular the expressions for the homogeneous uniaxial compression/tension of a cylinder without lateral load as discussed in Section 4.1. The particular expressions for the functions $\mathfrak{F}$ and $\mathfrak{G}$ are used to obtain plots for the other boundary value problems studied in Section 4.

4. Boundary value problems

In this section we study some boundary value problems, where $\mathbf{T}$ and $\boldsymbol{\varepsilon}$ do not depend on $\mathbf{x}$. In such a case if there is no body force, for the quasi-static case the equation of equilibrium (2.3) is satisfied automatically.

4.1. Uniform tension-compression of a cylinder without lateral constraints

Let us suppose that the body in cylindrical coordinates is defined through

$$0 \le r \le r_o, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le L. \tag{4.1}$$

We shall assume that the state of the stress and the strain tensors in the cylinder take the form:

$$\mathbf{T} = \sigma_z \mathbf{e}_z \otimes \mathbf{e}_z, \quad \boldsymbol{\varepsilon} = \varepsilon_r(\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \varepsilon_z \mathbf{e}_z \otimes \mathbf{e}_z, \tag{4.2}$$

where σ_z, ε_r and ε_z are constants.

We have $\operatorname{tr}\mathbf{T} = \sigma_z$, $\sigma_S = \sigma_z/3$, $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = \sigma_z$, $\overset{(i)}{\mathbf{t}} = \mathbf{e}_i$, $i = 1,2,3$ and $\operatorname{tr}\boldsymbol{\varepsilon} = 2\varepsilon_r + \varepsilon_z$.

Using (4.2) in (3.4) we obtain:

$$\varepsilon_r + \mathfrak{q}^{(0)}\sigma_z(\varepsilon_r + \varepsilon_z) + (\mathfrak{f}^{(1)} + \mathfrak{f}^{(2)}\sigma_z)(2\varepsilon_r + \varepsilon_z) + \frac{\partial \mathfrak{p}}{\partial \sigma_1} = 0, \tag{4.3}$$

$$\varepsilon_z + 2(\mathfrak{q}^{(0)} + \mathfrak{h}^{(0)})\sigma_z\varepsilon_z + [\mathfrak{f}^{(1)} + (\mathfrak{f}^{(2)} + \mathfrak{q}^{(0)})\sigma_z](2\varepsilon_r + \varepsilon_z) + \frac{\partial \mathfrak{p}}{\partial \sigma_3} = 0. \tag{4.4}$$

The above two relations can be used, for example, to obtain ε_r and ε_z in terms of σ_z.

In (3.7) using the eigenvalues of $\mathbf{T}$ given previously we have

$$\frac{\partial \mathfrak{p}}{\partial \sigma_1} = \frac{1}{3}\mathfrak{G}'(\sigma_z/3), \quad \frac{\partial \mathfrak{p}}{\partial \sigma_3} = \mathfrak{F}'(\sigma_z) + \frac{1}{3}\mathfrak{G}'(\sigma_z/3), \tag{4.5}$$

where $\mathfrak{F}'(x) = \frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}x}$ and $\mathfrak{G}'(x) = \frac{\mathrm{d}\mathfrak{G}}{\mathrm{d}x}$.

4.2. Uniform tension-compression of a cylinder with lateral load

For the same cylinder described in (4.1), we assume the presence of an axial and a radial load, which causes the same homogeneous strain $(4.2)_2$ (this test is called sometimes the triaxial test). The stress tensor is

$$\mathbf{T} = \sigma_r(\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \sigma_z \mathbf{e}_z \otimes \mathbf{e}_z, \tag{4.6}$$

where σ_r and σ_z are constants. We have $\operatorname{tr}\mathbf{T} = 2\sigma_r + \sigma_z$, $\sigma_S = (2\sigma_r + \sigma_z)/3$, $\sigma_1 = \sigma_2 = \sigma_r$, $\sigma_3 = \sigma_z$, and as before $\overset{(i)}{\mathbf{t}} = \mathbf{e}_i$, $i = 1,2,3$ and $\operatorname{tr}\boldsymbol{\varepsilon} = 2\varepsilon_r + \varepsilon_z$.

Using (4.6) and $(4.2)_2$ in (3.4) we obtain:

$$\begin{aligned}\varepsilon_r + \mathfrak{q}^{(0)}(2\sigma_r + \sigma_z)\varepsilon_r + 2\mathfrak{h}^{(0)}\sigma_r\varepsilon_r + \mathfrak{f}^{(1)}(2\varepsilon_r + \varepsilon_z) + \mathfrak{f}^{(2)}(2\sigma_r + \sigma_z)(2\varepsilon_r + \varepsilon_z)\\ + \mathfrak{q}^{(0)}(2\sigma_r\varepsilon_r + \sigma_z\varepsilon_z) + \mathfrak{q}^{(0)}(2\varepsilon_r + \varepsilon_z)\sigma_r + \frac{\partial \mathfrak{p}}{\partial \sigma_1} = 0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\varepsilon_z + \mathfrak{q}^{(0)}(2\sigma_r + \sigma_z)\varepsilon_z + 2\mathfrak{h}^{(0)}\sigma_z\varepsilon_z + \mathfrak{f}^{(1)}(2\varepsilon_r + \varepsilon_z) + \mathfrak{f}^{(2)}(2\sigma_r + \sigma_z)(2\varepsilon_r + \varepsilon_z)\\ + \mathfrak{q}^{(0)}(2\sigma_r\varepsilon_r + \sigma_z\varepsilon_z) + \mathfrak{q}^{(0)}(2\varepsilon_r + \varepsilon_z)\sigma_z + \frac{\partial \mathfrak{p}}{\partial \sigma_3} = 0.\end{aligned} \tag{4.8}$$

The above two relations can be used, for example, to obtain ε_r and ε_z in terms of σ_z and σ_r. Here, we show such explicit expressions for ε_r and ε_z:

$$\begin{aligned}(4.9)\quad \varepsilon_r = &\left\{\frac{\partial\mathfrak{p}}{\partial\sigma_1}[1+\mathfrak{f}^{(1)}+2(\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_r+(\mathfrak{f}^{(2)}+2\mathfrak{h}^{(0)}+2\mathfrak{q}^{(0)})\sigma_z]\right.\\ &\left.-\frac{\partial\mathfrak{p}}{\partial\sigma_3}[\mathfrak{f}^{(1)}+(2\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_r+(\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_z]\right\}\\ &\times\{2[\mathfrak{f}^{(1)}+\mathfrak{q}^{(0)}(\sigma_r+\sigma_z)+\mathfrak{f}^{(2)}(2\sigma_r+\sigma_z)]^2-[1+2\mathfrak{f}^{(1)}\\ &+2(\mathfrak{f}^{(2)}+\mathfrak{h}^{(0)}+3\mathfrak{q}^{(0)})\sigma_r+(2\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_z][1+\mathfrak{f}^{(1)}\\ &+2(\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_r+(\mathfrak{f}^{(2)}+2\mathfrak{h}^{(0)}+3\mathfrak{q}^{(0)})\sigma_z]\}^{-1},\end{aligned}$$

$$\begin{aligned}(4.10)\quad \varepsilon_z = &\left\{2\mathfrak{f}^{(1)}\left(\frac{\partial\mathfrak{p}}{\partial\sigma_1}-\frac{\partial\mathfrak{p}}{\partial\sigma_3}\right)+2\frac{\partial\mathfrak{p}}{\partial\sigma_1}[(2\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_r+(\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_z]\right.\\ &\left.-\frac{\partial\mathfrak{p}}{\partial\sigma_3}[1+2(2\mathfrak{f}^{(2)}+\mathfrak{h}^{(0)}+3\mathfrak{q}^{(0)})\sigma_r+(2\mathfrak{f}^{(2)}+\mathfrak{q}^{(0)})\sigma_z]\right\}\\ &\times\{1+2\sigma_r[\mathfrak{h}^{(0)}+4\mathfrak{q}^{(0)}+\mathfrak{q}^{(0)}(2\mathfrak{h}^{(0)}+5\mathfrak{q}^{(0)})\sigma_r]\\ &+2(\mathfrak{h}^{(0)}+2\mathfrak{q}^{(0)}+2{\mathfrak{h}^{(0)}}^2\sigma_r+9\mathfrak{h}^{(0)}\mathfrak{q}^{(0)}\sigma_r+8{\mathfrak{q}^{(0)}}^2\sigma_r)\sigma_z+\mathfrak{q}^{(0)}(2\mathfrak{h}^{(0)}\\ &+\mathfrak{q}^{(0)})\sigma_z^2+\mathfrak{f}^{(2)}(2\sigma_r+\sigma_z)(3+2\mathfrak{h}^{(0)}\sigma_r+6\mathfrak{q}^{(0)}\sigma_r+4\mathfrak{h}^{(0)}\sigma_z+3\mathfrak{q}^{(0)}\sigma_z)\\ &+\mathfrak{f}^{(1)}[3+6\mathfrak{q}^{(0)}\sigma_r+3\mathfrak{q}^{(0)}\sigma_z+2\mathfrak{h}^{(0)}(\sigma_r+2\sigma_z)]\}^{-1}.\end{aligned}$$

In the simpler case that $\mathfrak{q}^{(0)}=\mathfrak{h}^{(0)}=0$ (see (3.5)) (4.9), (4.10) become

$$(4.11)\qquad \varepsilon_r = \frac{1}{3}\left[\frac{\partial\mathfrak{p}}{\partial\sigma_3}-\frac{\partial\mathfrak{p}}{\partial\sigma_1}-\frac{\left(2\frac{\partial\mathfrak{p}}{\partial\sigma_1}+\frac{\partial\mathfrak{p}}{\partial\sigma_3}\right)}{(1+3\mathfrak{f}^{(1)}+6\mathfrak{f}^{(2)}\sigma_r+3\mathfrak{f}^{(2)}\sigma_z)}\right],$$

$$(4.12)\qquad \varepsilon_z = \frac{2\left(\frac{\partial\mathfrak{p}}{\partial\sigma_1}-\frac{\partial\mathfrak{p}}{\partial\sigma_3}\right)[\mathfrak{f}^{(1)}+\mathfrak{f}^{(2)}(2\sigma_r+\sigma_z)]-\frac{\partial\mathfrak{p}}{\partial\sigma_3}}{1+3\mathfrak{f}^{(1)}+3\mathfrak{f}^{(2)}(2\sigma_r+\sigma_z)}.$$

Here for both cases (4.9), (4.10) and (4.11), (4.12) from (3.7) we have:

$$(4.13)\qquad \begin{aligned}\frac{\partial\mathfrak{p}}{\partial\sigma_1} &= \mathfrak{F}'(\sigma_r)+\frac{1}{3}\mathfrak{G}'([2\sigma_r+\sigma_z]/3),\\ \frac{\partial\mathfrak{p}}{\partial\sigma_3} &= \mathfrak{F}'(\sigma_z)+\frac{1}{3}\mathfrak{G}'([2\sigma_r+\sigma_z]/3).\end{aligned}$$

4.3. Slab subject to simple shear stress

For the slab described through

$$(4.14)\qquad -\frac{L_i}{2}\le x_i\le\frac{L_i}{2},\qquad i=1,2,3,$$

we assume the presence of the stress and the strain tensors:

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \qquad \boldsymbol{\varepsilon} = \sum_{i=1}^{3} \varepsilon_i \mathbf{e}_i \otimes \mathbf{e}_i + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{4.15}$$

where τ, ε_i, $i = 1, 2, 3$ and γ are constants. In this problem $\sigma_1 = -\tau$, $\sigma_2 = 0$, $\sigma_3 = \tau$ and $\overset{(1)}{\mathbf{t}} = -\frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2$, $\overset{(2)}{\mathbf{t}} = \mathbf{e}_3$, $\overset{(3)}{\mathbf{t}} = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2$, $\sigma_S = 0$, $\operatorname{tr}\mathbf{T} = 0$, $\operatorname{tr}(\mathbf{T}\boldsymbol{\varepsilon}) = 2\gamma\tau$ and $\operatorname{tr}\boldsymbol{\varepsilon} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Using (4.15) in (3.4) we obtain:

$$\gamma + \mathfrak{h}^{(0)}(\varepsilon_1 + \varepsilon_2)\tau + \mathfrak{q}^{(0)}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\tau + \frac{1}{2}\left(\frac{\partial \mathfrak{p}}{\partial \sigma_3} - \frac{\partial \mathfrak{p}}{\partial \sigma_1}\right) = 0, \tag{4.16}$$

$$\varepsilon_1 + 2\mathfrak{h}^{(0)}\gamma\tau + \mathfrak{f}^{(1)}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mathfrak{q}^{(0)}\gamma\tau + \frac{1}{2}\left(\frac{\partial \mathfrak{p}}{\partial \sigma_1} + \frac{\partial \mathfrak{p}}{\partial \sigma_3}\right) = 0, \tag{4.17}$$

$$\varepsilon_2 + 2\mathfrak{h}^{(0)}\gamma\tau + \mathfrak{f}^{(1)}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mathfrak{q}^{(0)}\gamma\tau + \frac{1}{2}\left(\frac{\partial \mathfrak{p}}{\partial \sigma_1} + \frac{\partial \mathfrak{p}}{\partial \sigma_3}\right) = 0, \tag{4.18}$$

$$\varepsilon_3 + \mathfrak{f}^{(1)}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mathfrak{q}^{(0)}\gamma\tau + \frac{\partial \mathfrak{p}}{\partial \sigma_2} = 0. \tag{4.19}$$

Subtracting (4.18) from (4.17) we find that

$$\varepsilon_1 = \varepsilon_2 = \varepsilon. \tag{4.20}$$

Using the above in (4.16), (4.18) and (4.19) we obtain:

$$\begin{aligned} \varepsilon = \Bigg\{ & \frac{\partial \mathfrak{p}}{\partial \sigma_1}(1 + \mathfrak{f}^{(1)}) - 2\mathfrak{f}^{(1)}\frac{\partial \mathfrak{p}}{\partial \sigma_2} + \frac{\partial \mathfrak{p}}{\partial \sigma_3}(1 + \mathfrak{f}^{(1)}) \\ & + 2\left(\frac{\partial \mathfrak{p}}{\partial \sigma_1} - \frac{\partial \mathfrak{p}}{\partial \sigma_3}\right)(\mathfrak{h}^{(0)} + \mathfrak{f}^{(1)}\mathfrak{h}^{(0)} + \mathfrak{q}^{(0)})\tau \\ & + 2\mathfrak{q}^{(0)}\left[2\mathfrak{h}^{(0)}\frac{\partial \mathfrak{p}}{\partial \sigma_2} - \left(\frac{\partial \mathfrak{p}}{\partial \sigma_1} - 2\frac{\partial \mathfrak{p}}{\partial \sigma_2} + \frac{\partial \mathfrak{p}}{\partial \sigma_3}\right)\mathfrak{q}^{(0)}\right]\tau^2 \Bigg\} \\ & \times \{4[2(1 + \mathfrak{f}^{(1)}){\mathfrak{h}^{(0)}}^2 + 4\mathfrak{h}^{(0)}\mathfrak{q}^{(0)} + 3{\mathfrak{q}^{(0)}}^2]\tau^2 - 2 - 6\mathfrak{f}^{(1)}\}^{-1}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \varepsilon_3 = \Bigg\{ & \frac{\partial \mathfrak{p}}{\partial \sigma_2} - 4\frac{\partial \mathfrak{p}}{\partial \sigma_2}(\mathfrak{h}^{(0)} + \mathfrak{q}^{(0)})^2\tau^2 \\ & - \mathfrak{f}^{(1)}\left[\frac{\partial \mathfrak{p}}{\partial \sigma_1} - 2\frac{\partial \mathfrak{p}}{\partial \sigma_2} + \frac{\partial \mathfrak{p}}{\partial \sigma_3} + 2\mathfrak{h}^{(0)}\left(\frac{\partial \mathfrak{p}}{\partial \sigma_1} - \frac{\partial \mathfrak{p}}{\partial \sigma_3}\right)\tau\right] \\ & + \mathfrak{q}^{(0)}\tau\left[\frac{\partial \mathfrak{p}}{\partial \sigma_1} - \frac{\partial \mathfrak{p}}{\partial \sigma_3} + 2\left(\frac{\partial \mathfrak{p}}{\partial \sigma_1} + \frac{\partial \mathfrak{p}}{\partial \sigma_3}\right)(\mathfrak{h}^{(0)} + \mathfrak{q}^{(0)})\tau\right] \Bigg\} \\ & \times \{2[2(1 + \mathfrak{f}^{(1)}){\mathfrak{h}^{(0)}}^2 + 4\mathfrak{h}^{(0)}\mathfrak{q}^{(0)} + 3{\mathfrak{q}^{(0)}}^2]\tau^2 - 1 - 3\mathfrak{f}^{(1)}\}^{-1}, \end{aligned} \tag{4.22}$$

$$\gamma = -\Big\{(1+3\mathfrak{f}^{(1)})\Big(\frac{\partial\mathfrak{p}}{\partial\sigma_1}-\frac{\partial\mathfrak{p}}{\partial\sigma_3}\Big) + 2\Big[\mathfrak{h}^{(0)}\Big(\frac{\partial\mathfrak{p}}{\partial\sigma_1}+\mathfrak{f}^{(1)}\frac{\partial\mathfrak{p}}{\partial\sigma_1}-2\mathfrak{f}^{(1)}\frac{\partial\mathfrak{p}}{\partial\sigma_2}+\frac{\partial\mathfrak{p}}{\partial\sigma_3}+\mathfrak{f}^{(1)}\frac{\partial\mathfrak{p}}{\partial\sigma_3}\Big) + \Big(\frac{\partial\mathfrak{p}}{\partial\sigma_1}+\frac{\partial\mathfrak{p}}{\partial\sigma_2}+\frac{\partial\mathfrak{p}}{\partial\sigma_3}\Big)\mathfrak{q}^{(0)}\Big]\tau\Big\} \times\{4[2(1+\mathfrak{f}^{(1)}){\mathfrak{h}^{(0)}}^2+4\mathfrak{h}^{(0)}\mathfrak{q}^{(0)}+3{\mathfrak{q}^{(0)}}^2]\tau^2-2-6\mathfrak{f}^{(1)}\}^{-1}. \tag{4.23}$$

Since $\sigma_S = 0$ from (3.7) we have

$$\frac{\partial\mathfrak{p}}{\partial\sigma_1}=\mathfrak{F}'(-\tau),\qquad \frac{\partial\mathfrak{p}}{\partial\sigma_2}=0,\qquad \frac{\partial\mathfrak{p}}{\partial\sigma_3}=\mathfrak{F}'(\tau). \tag{4.24}$$

In the simpler case that $\mathfrak{h}^{(0)}=\mathfrak{q}^{(0)}=0$, it follows from (4.15) from (3.5), it follows from (4.25) and (3.5) that:

$$\varepsilon = -\frac{[\mathfrak{F}'(-\tau)+\mathfrak{F}'(\tau)](1+\mathfrak{f}^{(1)})}{2(1+3\mathfrak{f}^{(1)})}, \tag{4.25}$$

$$\varepsilon_3 = \frac{[\mathfrak{F}'(-\tau)+\mathfrak{F}'(\tau)]\mathfrak{f}^{(1)}}{(1+3\mathfrak{f}^{(1)})}, \tag{4.26}$$

$$\gamma = \frac{1}{2}[\mathfrak{F}'(-\tau)-\mathfrak{F}'(\tau)]. \tag{4.27}$$

4.4. Slab subject to simple shear strain

In this section we study the behaviour of the same slab described in (4.14) when it is in a state of simple shear strain.

The stress tensor and the strain tensors are assumed to take the form:

$$\mathbf{T} = \sigma_a\mathbf{e}_1\otimes\mathbf{e}_1+\sigma_b\mathbf{e}_2\otimes\mathbf{e}_2+\sigma_c\mathbf{e}_3\otimes\mathbf{e}_3+\tau(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1), \tag{4.28}$$

$$\boldsymbol{\varepsilon} = \gamma(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1), \tag{4.29}$$

where σ_a, σ_b, σ_c, τ and γ are constants.

It follows from (4.28) and (4.29) $\operatorname{tr}\boldsymbol{\varepsilon}=0$, $\operatorname{tr}\mathbf{T}=\sigma_a+\sigma_b+\sigma_c=3\sigma_S$ and $\operatorname{tr}(\mathbf{T}\boldsymbol{\varepsilon})=2\gamma\tau$. Using (4.28), (4.29) in (3.4) we obtain the relations:

$$\gamma+\mathfrak{q}^{(1)}(\sigma_a+\sigma_b+\sigma_c)+\mathfrak{h}^{(0)}(\sigma_a+\sigma_b)\gamma+\frac{\partial\mathfrak{p}}{\partial\sigma_1}\frac{1}{r_1^2}\Big(\frac{\sigma_a-\sigma_b-\ell}{2\tau}\Big) + \frac{\partial\mathfrak{p}}{\partial\sigma_2}\frac{1}{r_2^2}\Big(\frac{\sigma_a-\sigma_b+\ell}{2\tau}\Big)=0, \tag{4.30}$$

$$2\mathfrak{h}^{(0)}\gamma\tau+2\mathfrak{q}^{(0)}\gamma\tau+\frac{\partial\mathfrak{p}}{\partial\sigma_1}\frac{1}{r_1^2}\Big(\frac{\sigma_a-\sigma_b-\ell}{2\tau}\Big)^2 + \frac{\partial\mathfrak{p}}{\partial\sigma_2}\frac{1}{r_2^2}\Big(\frac{\sigma_a-\sigma_b+\ell}{2\tau}\Big)^2=0, \tag{4.31}$$

$$2\mathfrak{h}^{(0)}\gamma\tau + 2\mathfrak{q}^{(0)}\gamma\tau + \frac{\partial\mathfrak{p}}{\partial\sigma_1}\frac{1}{r_1^2} + \frac{\partial\mathfrak{p}}{\partial\sigma_2}\frac{1}{r_2^2} = 0, \tag{4.32}$$

$$2\mathfrak{q}^{(0)}\gamma\tau + \frac{\partial\mathfrak{p}}{\partial\sigma_3} = 0, \tag{4.33}$$

where

$$\sigma_1 = \frac{1}{2}(\sigma_a + \sigma_b - \ell), \qquad \sigma_2 = \frac{1}{2}(\sigma_a + \sigma_b + \ell), \qquad \sigma_3 = \sigma_c, \tag{4.34}$$

$$r_1 = \sqrt{\left(\frac{\sigma_a - \sigma_b - \ell}{2\tau}\right)^2 + 1}, \qquad r_2 = \sqrt{\left(\frac{\sigma_a - \sigma_b + \ell}{2\tau}\right)^2 + 1}, \tag{4.35}$$

and

$$\ell = \sqrt{(\sigma_a - \sigma_b)^2 + 4\tau^2}. \tag{4.36}$$

The simpler case where $\mathfrak{h}^{(0)} = \mathfrak{q}^{(0)} = 0$ can be obtained easily from (4.31)–(4.33) and is not presented here for the sake of brevity.

The four equations (4.31)–(4.33) can be used to obtain γ, σ_a, σ_b and σ_c in terms of τ. The expressions for $\frac{\partial\mathfrak{p}}{\partial\sigma_i}$ are given in (3.7).

5. Applications to the modelling of concrete and dry rock

We apply the constitutive models (3.4), (3.5) for the modelling of concrete and rock, wherein we assume rock as an isotropic, dry, homogeneous and elastic solid. The data for concrete is taken from one of the tests documented in [10], while the data for rock are taken from [17]. In both cases the data are obtained for a cylinder under uniform compression/tension and no lateral load (see Section 4.1). In the implicit model (3.4) we have four constants $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$, and the functions $\mathfrak{F}$, $\mathfrak{G}$. In this work the functions $\mathfrak{F}$, $\mathfrak{G}$ are found indirectly in the following manner. We assume that from the experimental data for a cylinder under compression/tension we can propose functions (see Section 4.1) $\hat{\varepsilon}_z(\sigma_z)$ and $\hat{\varepsilon}_r(\sigma_z)$ to fit the data for ε_z and ε_r versus σ_z for such a test, then from (4.3), (4.4) (see also (4.5)) we have (using the notation x for the arguments of the different functions):

$$\mathfrak{F}'(x) = \hat{\varepsilon}_r(x) - \mathfrak{q}^{(0)}\hat{\varepsilon}_r(x)x - \hat{\varepsilon}_z(x)[1 + 2(\mathfrak{h}^{(0)} + \mathfrak{q}^{(0)})x], \tag{5.1}$$

$$\begin{aligned}\mathfrak{G}'(x) = -3\{&\hat{\varepsilon}_r(3x)[1 + 2\mathfrak{f}^{(1)} + 3(2\mathfrak{f}^{(2)} + \mathfrak{q}^{(0)})x] \\ &+ \hat{\varepsilon}_z(3x)[\mathfrak{f}^{(1)} + 3(\mathfrak{f}^{(2)} + \mathfrak{q}^{(0)})x]\}.\end{aligned} \tag{5.2}$$

In the above expressions the constants $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$ are arbitrary. We study the effect of considering different values for such material constants for the other boundary value problems studied in Sections 4.2–4.4. If additional experimental data can be obtained, for example, for the triaxial compression of a cylinder (see Section 4.2), these constants $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$ could be used to fit that additional data.

Regarding the range of possible values for the material constants $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$, from the results shown in Sections 4.2 and 4.3 we can see that for certain values of such constants, it is possible that ε_z, ε_r (see (4.9)–(4.12)), and γ, ε and ε_3 can become infinite (see (4.21)–(4.23), (4.25), (4.26)). We choose the values for these functions so as to avoid such blow up.

5.1. Concrete

In this section we use the method described previously for the particular case of modelling the behaviour of concrete. The experimental data is taken from one of the tests presented in [10]. In that work the axial and radial components of the strain tensor are obtained for the uniform compression of a cylinder (see Section 4.1). No results are reported for the behaviour of a similar cylinder in tension. For the functions $\hat{\varepsilon}_z$ and $\hat{\varepsilon}_r$ we propose:

$$\hat{\varepsilon}_z(\sigma_z) = \xi\sigma_z + \zeta\sigma_z^5, \tag{5.3}$$

$$\hat{\varepsilon}_r(\sigma_z) = \varpi_1[\exp(\vartheta_1\sigma_z) - 1] + \varpi_2[\exp(\vartheta_2\sigma_z) - 1], \tag{5.4}$$

where ξ, ζ, ϖ_1, ϖ_2, ϑ_1 and ϑ_2 are material constants. In the case of the linearized elastic model (3.10) for this problem of a cylinder under compression without lateral load we have $\varepsilon_z = \sigma_z/E$ and $\varepsilon_r = -\nu\sigma_z/E$. The different material constants are found using least squares fitting and are shown in Table 1. In Fig. 1 we have the comparison between the experimental data called 'Exp.', the predictions of the non-linear models (5.3), (5.4) (called 'Nonlin.'), and the linearized model (3.10) (called 'Lin.'), where we compare the behaviour of the axial and radial components of the strain tensor. We show separately the results in compression and tension.

Table 1. Modeling of concrete. Values for the material constants in (5.3), (5.4) and also for the linearized model (3.10), for the case of concrete under compression (see [10]).

ξ	ζ	ϖ_1	ϑ_1
2.364×10^{-5} [MPa]$^{-1}$	1.433×10^{-13} [MPa]$^{-5}$	1.809×10^{-5}	-0.03768 [MPa]$^{-1}$
ϖ_2	ϑ_2	E	ν
1.904×10^{-16}	-0.3607 [MPa]$^{-1}$	38051.8 [MPa]	0.1452

Using the material constants presented in Table 1, we study the rest of the boundary value problems presented in Sections 4.2–4.4, considering different

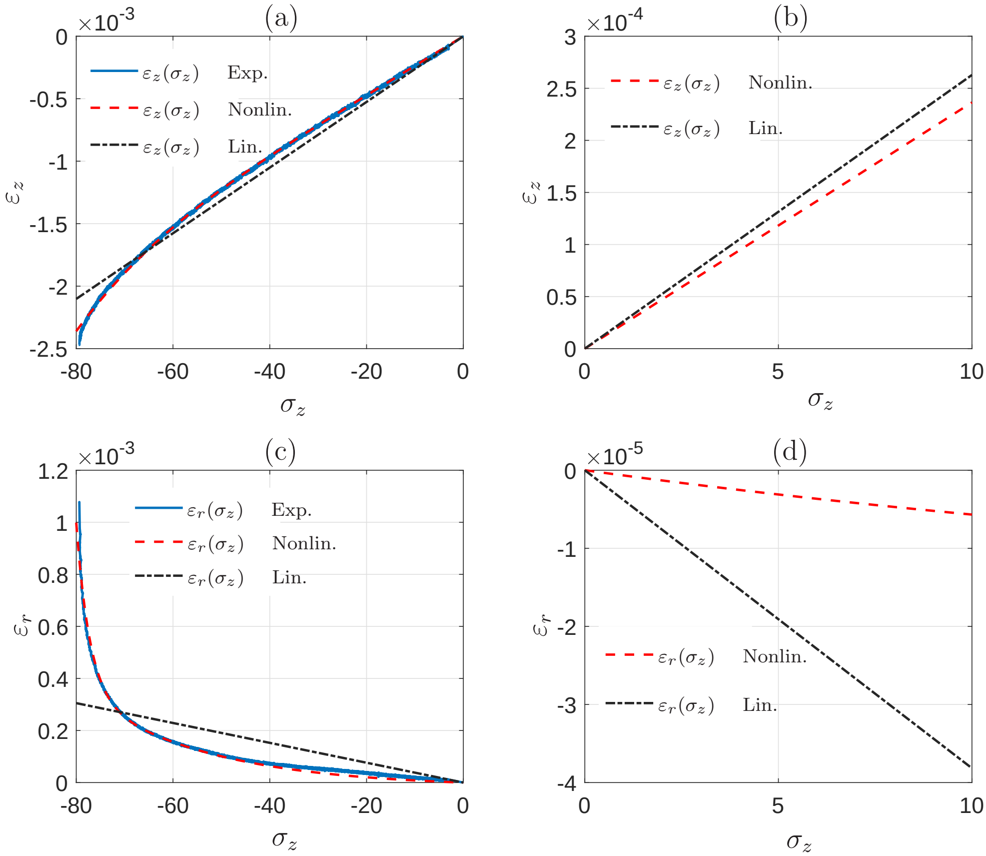

Fig. 1. Comparison of the predictions for the behaviour of concrete, considering the nonlinear model (5.3), (5.4) called 'Nonlin.', the linear model (3.10) called 'Lin.', with the experimental data from [10], called 'Exp.'. The axial stress σ_z is in [MPa]. (a,b): results for the axial component of the strain, (c,d): results for the radial component of the strain.

cases for the constants $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$. In this section from (3.4), (3.5) we can see that $\mathfrak{f}^{(1)}$ is a dimensionless constant, while $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$ are given in $1/[\text{MPa}]$. In Figs. 2–4 we show results for the triaxial compression of a cylinder, described in Section 4.2, using (3.5) and the material constants from Table 1, and assuming that $\mathfrak{h}^{(0)} = \mathfrak{q}^{(0)} = 0$. For this figure and also for Figs. 5–7 we have used the definitions

$$\bar{\sigma}_z = \frac{\sigma_z}{E}, \qquad \bar{\tau} = \frac{\tau}{E}, \qquad \bar{\sigma}_a = \frac{\sigma_a}{E}, \qquad \bar{\sigma}_c = \frac{\sigma_c}{E}. \tag{5.5}$$

In Fig. 2 results are presented for the case $\mathfrak{f}^{(1)} = 1$, $\sigma_r = -5\,[\text{MPa}]$, and where we have three cases for $\mathfrak{f}^{(2)}$, namely $\mathfrak{f}^{(2)} = 0.015\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and

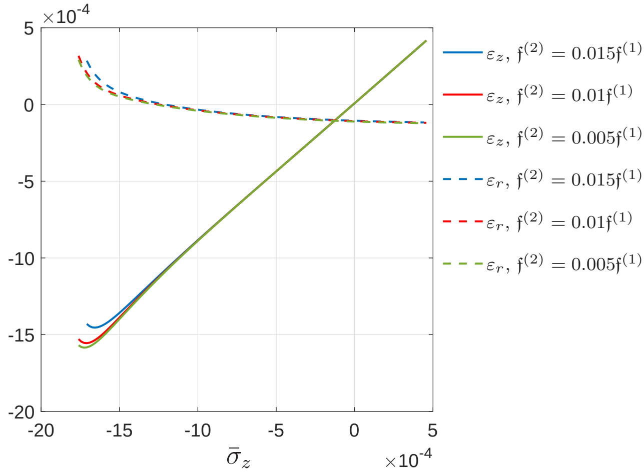

FIG. 2. Results for the triaxial compression of a cylinder considering the model for concrete (4.11), (4.12) and (5.3), (5.4). Case $\mathfrak{f}^{(1)} = 1$, $\sigma_r = -5$ [MPa] and different values for $\mathfrak{f}^{(2)}$.

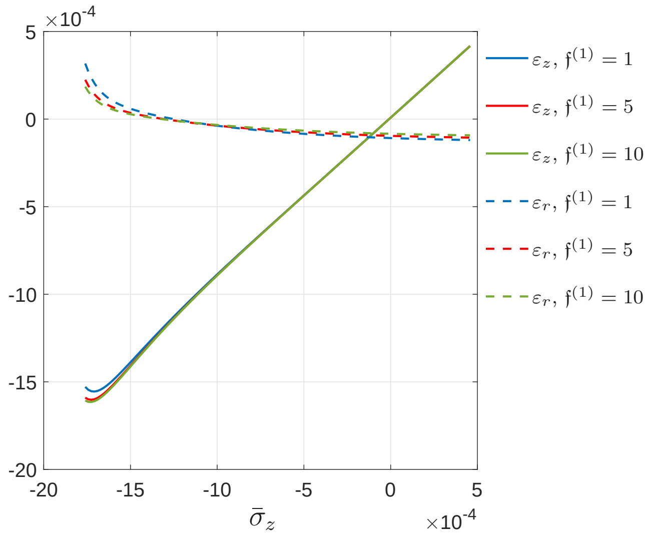

FIG. 3. Results for the triaxial compression of a cylinder considering the model for concrete (4.11), (4.12) and (5.3), (5.4). Case $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\sigma_r = -5$ [MPa] and considering three values for $\mathfrak{f}^{(1)}$.

$\mathfrak{f}^{(2)} = 0.005\mathfrak{f}^{(1)}$. In Fig. 3 we show results for the case $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\sigma_r = -5$ [MPa] and for three cases for $\mathfrak{f}^{(1)}$, namely $\mathfrak{f}^{(1)} = 1$, $\mathfrak{f}^{(1)} = 5$ and $\mathfrak{f}^{(1)} = 10$. Finally, in Fig. 4 results are presented when $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(1)} = 1$ for four values for $\sigma_r = -5, -10, -15, -20$ [MPa].

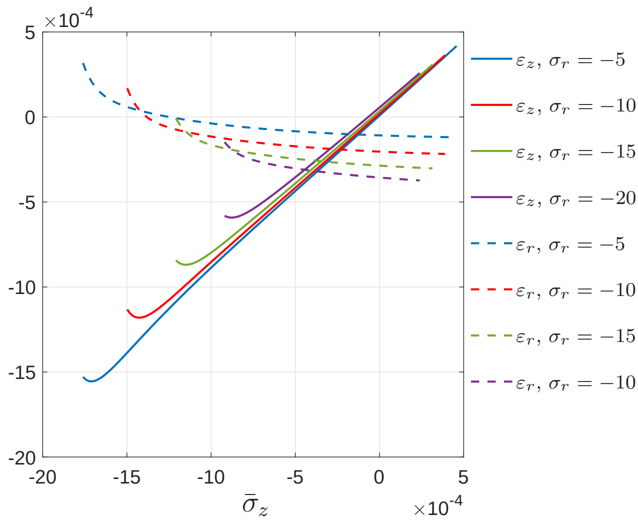

FIG. 4. Results for the triaxial compression of a cylinder considering the model for concrete (4.11), (4.12) and (5.3), (5.4). Case $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(1)} = 1$ for four values for σ_r. In the different cases σ_r is in [MPa].

In Figs. 5, 6 results are presented for the triaxial compression of a cylinder, for the case $\mathfrak{h}^{(0)} \neq 0$ or $\mathfrak{q}^{(0)} \neq 0$, assuming $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(1)} = 1$ and $\sigma_r = -5$ [MPa]. In Fig. 5 we show results for the case $\mathfrak{q}^{(0)} = 0$ and $\mathfrak{h}^{(0)} = 0$, $\mathfrak{h}^{(0)} = 0.006\mathfrak{f}^{(1)}$ and $\mathfrak{h}^{(0)} = 0.012\mathfrak{f}^{(1)}$. In Fig. 6 results are presented for the case $\mathfrak{h}^{(0)} = 0$ and $\mathfrak{q}^{(0)} = 0$, $\mathfrak{q}^{(0)} = 0.006\mathfrak{f}^{(1)}$ and $\mathfrak{q}^{(0)} = 0.012\mathfrak{f}^{(1)}$.

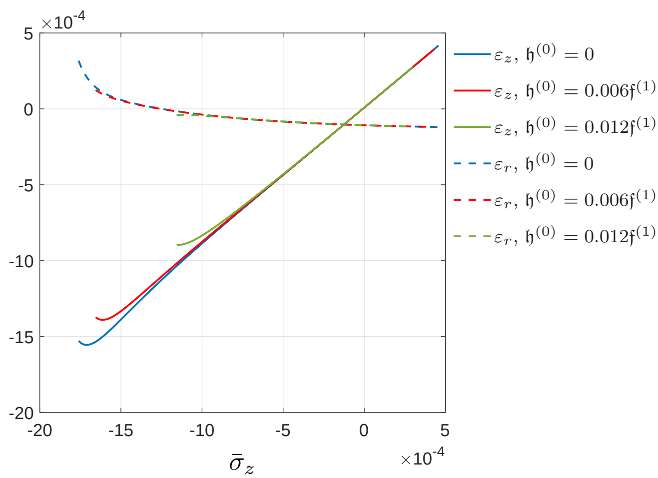

FIG. 5. Results for the triaxial compression of a cylinder considering the model for concrete (5.3), (5.4) and (3.4). Case where it is assumed that $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(1)} = 1$ and $\sigma_r = -5$ [MPa], for different values of $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)} = 0$.

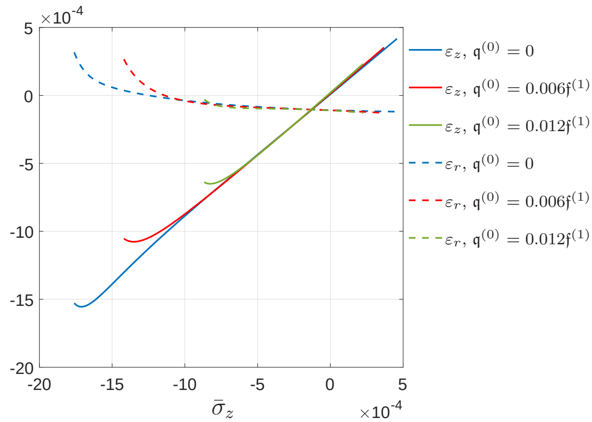

FIG. 6. Results for the triaxial compression of a cylinder considering the model for concrete (5.3), (5.4) and (3.4). Case where it is assumed that $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(1)} = 1$ and $\sigma_r = -5\,[\mathrm{MPa}]$. Case $\mathfrak{h}^{(0)} = 0$ and different values for $\mathfrak{q}^{(0)}$.

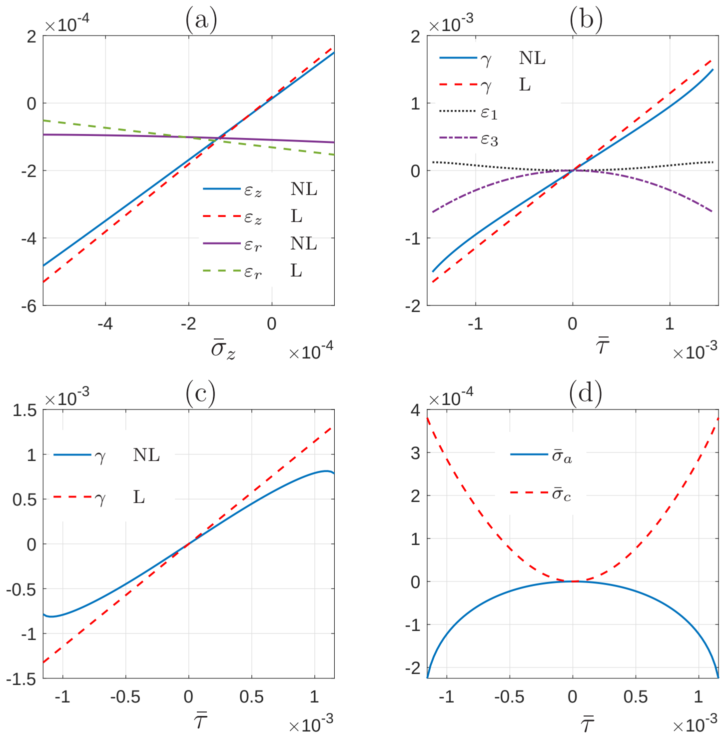

FIG. 7. Results for different homogeneous problems, for the modelling of concrete, comparing the predictions of the nonlinear model (5.3), (5.4), (3.4) denoted 'NL', and the linearized elastic model (3.10) that is denoted 'L'; (a) results for the triaxial compression/tension of a cylinder, (b) results for the simple shear stress of a slab (see Section 4.3), (c) results for the simple shear strain of a slab (see Section 4.4), results for γ, (d) results for the components of the stress $\bar{\sigma}_a$, $\bar{\sigma}_c$ for the simpler shear strain of a slab.

Finally, in Fig. 7 we present results for the problems described in Sections 4.3 and 4.4, comparing the predictions of the new constitutive relation (3.4) (see (5.1)–(5.4)), which in the figure are denoted 'NL', and the linearized model that in the figure are denoted 'L' (see (3.10)). For the three cases studied here we assume $\mathfrak{f}^{(1)} = 1$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\sigma_r = -5\,[\mathrm{MPa}]$, $\mathfrak{h}^{(0)} = 0.006\mathfrak{f}^{(1)}$ and $\mathfrak{q}^{(0)} = 0.006\mathfrak{f}^{(1)}$. In Fig. 7(a) results are shown for the triaxial compression/tension of a cylinder. In Fig. 7(b) results are shown for the simple shear stress of a slab (see 4.3), and in Figs. 7(c,d) results are presented for the simple shear strain of a slab (see 4.4). For this last problem it is necessary to observe that $\sigma_b = \sigma_a$ (see (4.31)–(4.33)).

5.2. Rock

The constitutive relations (3.4), (3.5) are also used to model the behaviour of two types of rock, namely granodiorite and diorite-monzonite. The experimental results are taken from [17]. In that paper results were obtained for a cylinder (without lateral load) under compression and also tension. In the case of the cylinder in tension, only results for the axial component of the strain versus the axial stress are reported, whereas for the cylinder in compression we have data for the axial and radial components of the strain tensor. We can observe the completely different behaviour in compression in comparison to tension.

For the functions $\hat{\varepsilon}_z$ and $\hat{\varepsilon}_z$ defined at the beginning of this Section 5, we assume that:

$$\hat{\varepsilon}_z(\sigma_z) = \hat{\varepsilon}_z^{(C)}(\sigma_z)q_C(\sigma_z) + \hat{\varepsilon}_z^{(T)}(\sigma_z)q_T(\sigma_z), \tag{5.6}$$

$$\hat{\varepsilon}_r(\sigma_z) = \hat{\varepsilon}_r^{(C)}(\sigma_z)q_C(\sigma_z) + \hat{\varepsilon}_r^{(T)}(\sigma_z)q_T(\sigma_z), \tag{5.7}$$

where:

$$q_T(\sigma_z) = \frac{1}{2}[1 + \mathrm{erf}(\mathcal{C}\sigma_z)], \qquad q_C(\sigma_z) = \frac{1}{2}[1 - \mathrm{erf}(\mathcal{C}\sigma_z)], \tag{5.8}$$

where $\mathcal{C}$ is a positive constant, and:

$$\hat{\varepsilon}_z^{(C)}(\sigma_z) = \xi_z^{(C)}\sigma_z + \zeta_z^{(C)}\sigma_z^2, \tag{5.9}$$

$$\hat{\varepsilon}_z^{(T)}(\sigma_z) = \xi_z^{(T)}\sigma_z + \zeta_z^{(T)}\sigma_z^2, \tag{5.10}$$

$$\hat{\varepsilon}_r^{(C)}(\sigma_z) = \xi_r^{(C)}\sigma_z + \zeta_r^{(C)}\sigma_z^2, \tag{5.11}$$

where $\xi_z^{(C)}$, $\zeta_z^{(C)}$, $\xi_z^{(T)}$, $\zeta_z^{(T)}$, $\xi_r^{(C)}$, and $\zeta_r^{(C)}$ are constants. In the above expressions (5.6), (5.7) we have functions $\hat{\varepsilon}_z(\sigma_z)$ and $\hat{\varepsilon}_r(\sigma_z)$ that are valid for σ_z negative or positive. Those functions depend on functions $\hat{\varepsilon}_z^{(C)}(\sigma_z)$ and $\hat{\varepsilon}_r^{(C)}(\sigma_z)$ that are obtained only using the data in compression of a cylinder, whereas the functions

$\hat{\varepsilon}_z^{(T)}(\sigma_z)$ and $\hat{\varepsilon}_r^{(T)}(\sigma_z)$ should be obtained using the data for tension of a cylinder. As mentioned before, we do not have experimental data for the radial component of the strain for the tension of a cylinder, which is the reason why in (5.9)–(5.11) we do not propose any expression for $\hat{\varepsilon}_r^{(T)}(\sigma_z)$. Instead, we simply assume that $\hat{\varepsilon}_r^{(T)}(\sigma_z) = -\nu\hat{\varepsilon}_z^{(T)}(\sigma_z)$, where ν is the Poisson ratio found during compression.

From the experimental data shown in [17] we obtain the different material constants in (5.9)–(5.11), and E, ν for the linearized model (3.10), in this last case only using the data for the compression of the cylinder. The constants are shown in Table 2. Here we assume $\mathcal{C} = 10\,[1/\text{MPa}]$.

Table 2. Modeling of rock. Values for the material constants in (5.9)–(5.11) and also for the linearized model (3.10) (see [17]).

$\xi_z^{(C)}$	$\zeta_z^{(C)}$	$\xi_r^{(C)}$	$\zeta_r^{(C)}$
1.824×10^{-6} [MPa]$^{-1}$	-3.888×10^{-7} [MPa]$^{-2}$	1.086×10^{-7} [MPa]$^{-1}$	7.631×10^{-8} [MPa]$^{-2}$
$\xi_z^{(T)}$	$\zeta_z^{(T)}$	E	ν
0.001647 [MPa]$^{-1}$	9.699×10^{-5} [MPa]$^{-2}$	46926.3 [MPa]	0.17203

In Fig. 8 we show results for the model (5.6), (5.7) and the linearized model (3.10) for the case of a cylinder under compression/tension, without lateral load, comparing with the experimental data of [17]. In the case of the linearized model

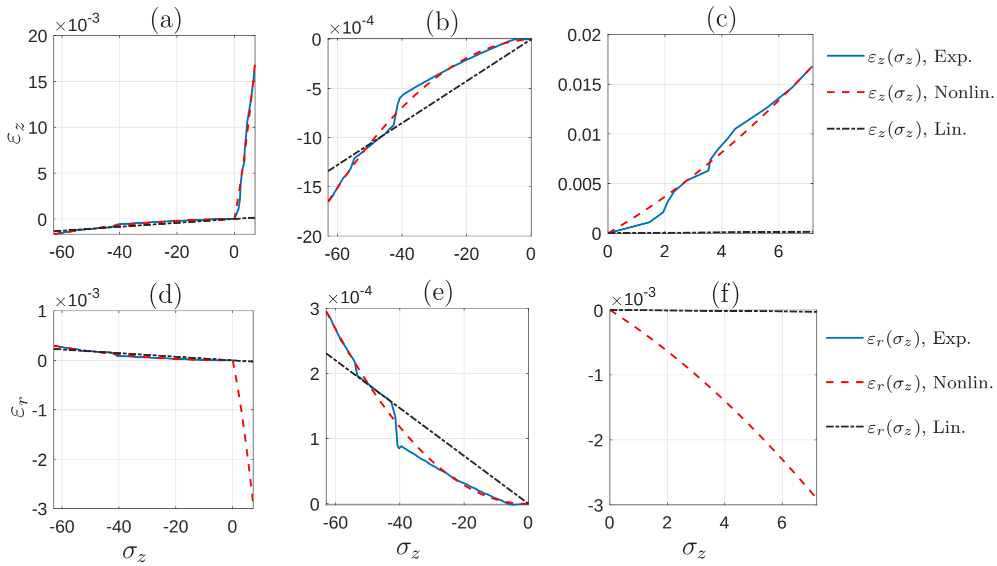

FIG. 8. Comparison of the predictions for the behaviour of rock, for the non-linear model (5.6), (5.7) called 'Nonlin.', the linear model (3.10) called 'Lin.', with the experimental data from [17], called 'Exp.'. The axial stress σ_z is in [MPa]. (a, d) Comparison for the whole range of values for σ_z. (b, e) Comparison for the case $\sigma_z \leq 0$. (c, f) Comparison for the case $\sigma_z \geq 0$.

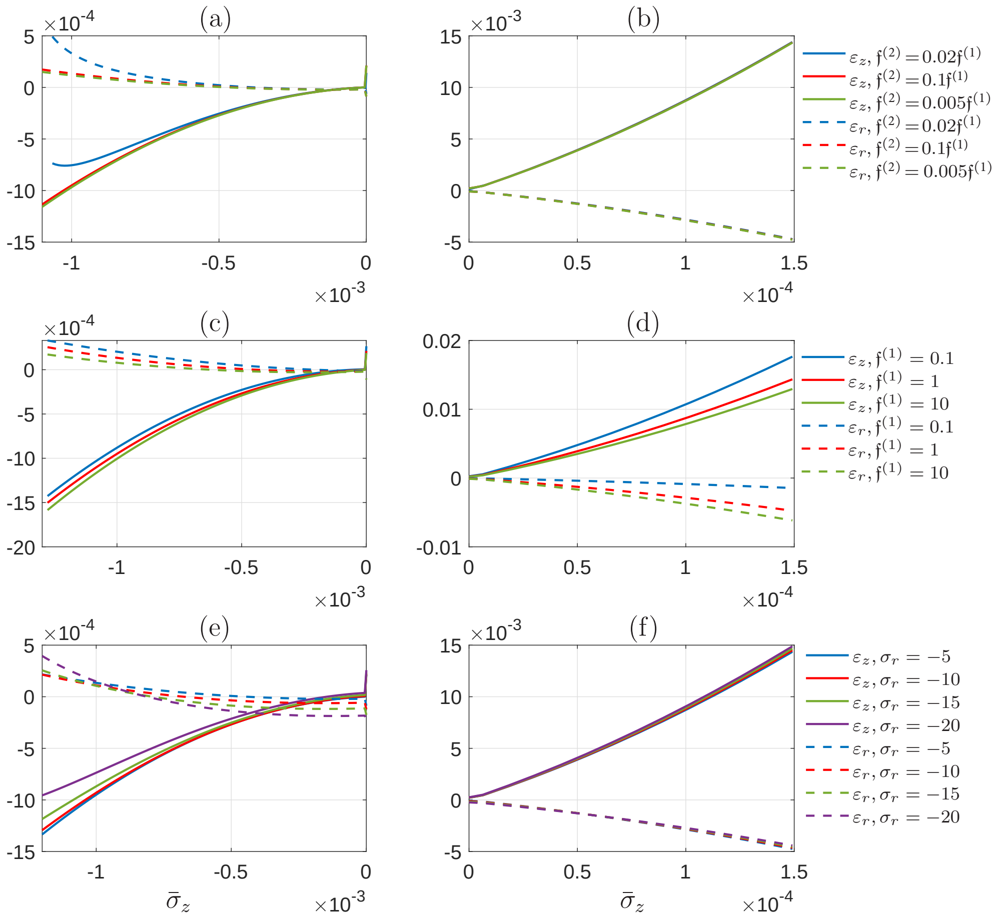

Fig. 9. Predictions of the non-linear model (3.4), (5.6), (5.7), for the triaxial load of a cylinder. Figures (a,b) results for the case $\sigma_r = -5\,[\text{MPa}]$, $\mathfrak{f}^{(1)} = 1$, and $\mathfrak{f}^{(2)} = 0.02\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and $\mathfrak{f}^{(2)} = 0.005\mathfrak{f}^{(1)}$. Figures (c,d) show results for the case $\sigma_r = -5$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and $\mathfrak{f}^{(1)} = 0.1$, $\mathfrak{f}^{(1)} = 1$ and $\mathfrak{f}^{(1)} = 10$. Figures (e,f) show results for the case $\mathfrak{f}^{(1)} = 1$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and $\sigma_r = -5$, $\sigma_r = -10$, $\sigma_r = -15$ and $\sigma_r = -20\,[\text{MPa}]$.

(3.10) for the whole range of values for σ_z we only use the constants obtained using the experimental data in compression[2]. In that Fig. 8 we show separately the behaviour of the cylinder only for compression and only for tension.

In Fig. 9 we show results for the triaxial compression of a cylinder (see Section 4.2) within the context of (3.4), assuming $\mathfrak{h}^{(0)} = \mathfrak{q}^{(0)} = 0$. Here as before $\mathfrak{f}^{(1)}$ is a dimensionless constant, while $\mathfrak{f}^{(2)}$, $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$ are given in $1/[\text{MPa}]$. In Figs. 9(a, b) results are shown for the case $\sigma_r = -5\,[\text{MPa}]$, $\mathfrak{f}^{(1)} = 1$, and

[2]See [17] for a 'bimodular' model where two sets of constants E, ν are used for the model, obtained separately using the data in compression and tension, respectively.

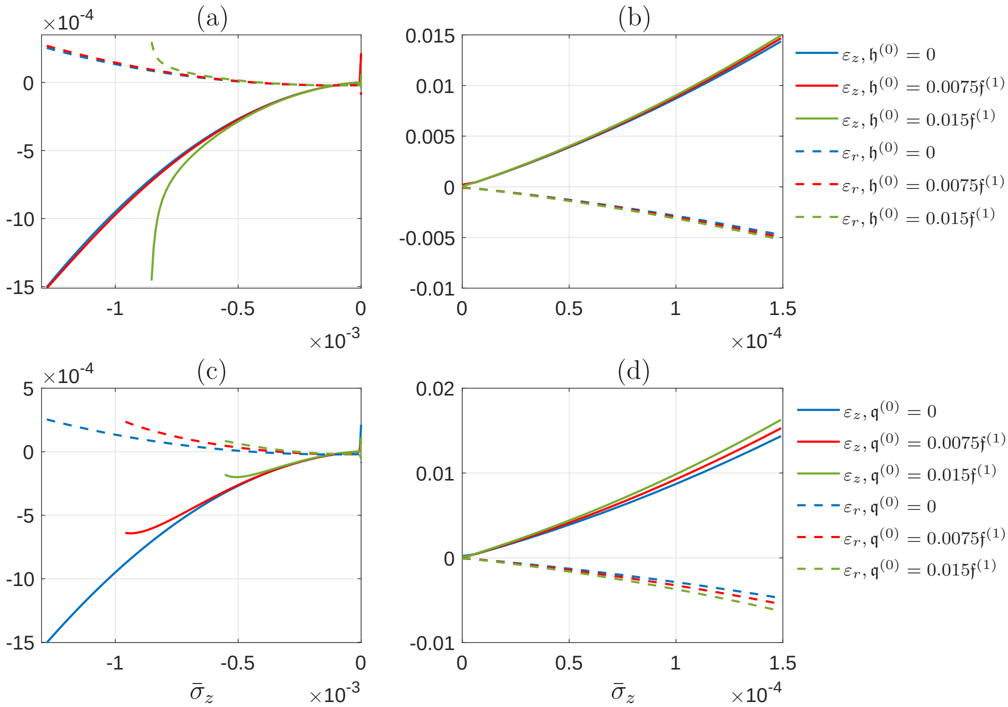

FIG. 10. Predictions of the non-linear model (5.1), (5.2), (5.6), (5.7), (3.5), for the triaxial load of a cylinder; (a,b) results for the cases $\mathfrak{q}^{(0)} = 0$, $\mathfrak{h}^{(0)} = 0$, $\mathfrak{h}^{(0)} = 0.0075\mathfrak{f}^{(1)}$ and $\mathfrak{h}^{(0)} = 0.015\mathfrak{f}^{(1)}$, (c,d) results for $\mathfrak{h}^{(0)} = 0$, $\mathfrak{q}^{(0)} = 0$, $\mathfrak{q}^{(0)} = 0.0075\mathfrak{f}^{(1)}$ and $\mathfrak{q}^{(0)} = 0.015\mathfrak{f}^{(1)}$.

$\mathfrak{f}^{(2)} = 0.02\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and $\mathfrak{f}^{(2)} = 0.005\mathfrak{f}^{(1)}$. In Figs. 9(c, d) results are presented for the case $\sigma_r = -5$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and $\mathfrak{f}^{(1)} = 0.1$, $\mathfrak{f}^{(1)} = 1$ and $\mathfrak{f}^{(1)} = 10$. Finally, in Figs. 9(e, f) results are shown for the case $\mathfrak{f}^{(1)} = 1$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ and $\sigma_r = -5$, $\sigma_r = -10$, $\sigma_r = -15$ and $\sigma_r = -20$ [MPa]. In all the cases studied above we show the behaviour of the cylinder for the whole range of values for σ_z, and also when $\sigma_z \leq 0$ and separately when $\sigma_z \geq 0$.

In Fig. 10 we portray the case $\sigma_r = -5$ [MPa], $\mathfrak{f}^{(1)} = 1$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$ for different values of $\mathfrak{h}^{(0)}$ and $\mathfrak{q}^{(0)}$. In Figs. 10(a, b) we display results for $\mathfrak{q}^{(0)} = 0$ and $\mathfrak{h}^{(0)} = 0$, $\mathfrak{h}^{(0)} = 0.0075\mathfrak{f}^{(1)}$ and $\mathfrak{h}^{(0)} = 0.015\mathfrak{f}^{(1)}$. In Figs. 10(c, d) results are presented for $\mathfrak{h}^{(0)} = 0$ and $\mathfrak{q}^{(0)} = 0$, $\mathfrak{q}^{(0)} = 0.0075\mathfrak{f}^{(1)}$ and $\mathfrak{q}^{(0)} = 0.015\mathfrak{f}^{(1)}$. In both cases we portray results for the whole range of values for σ_z, and for the particular situation $\sigma_z \leq 0$ and $\sigma_z \geq 0$.

In Figs. 11–13 we exhibit results for the different boundary value problems presented in Sections 4.2–4.4, comparing the predictions of the non-linear model (3.4), (5.1), (5.2), (5.6), (5.7), and the linearized model (3.10). Here we assume that $\sigma_r = -5$ [MPa], $\mathfrak{f}^{(1)} = 1$, $\mathfrak{f}^{(2)} = 0.01\mathfrak{f}^{(1)}$, $\mathfrak{h}^{(0)} = 0.0075\mathfrak{f}^{(1)}$ and $\mathfrak{q}^{(0)} = 0.0075\mathfrak{f}^{(1)}$. In Fig. 11 we present results for triaxial loading of a cylinder.

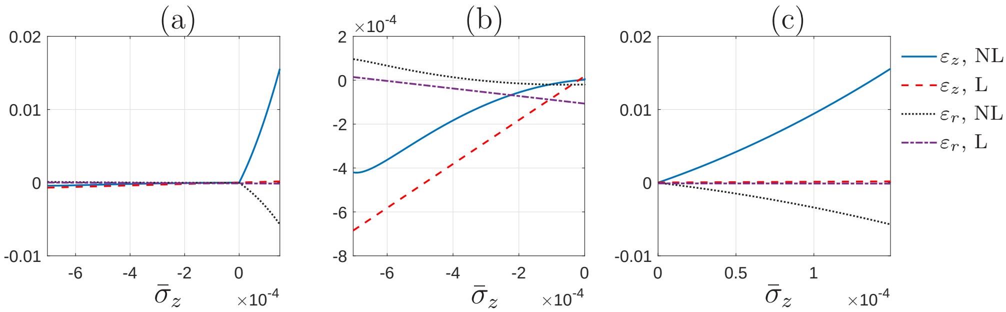

FIG. 11. Predictions for the non-linear model (5.1), (5.2), (5.6), (5.7), (3.4) denoted 'NL', and comparison with the predictions of the linearized model (3.10) that is denoted 'L', for the problem of triaxial load of a cylinder.

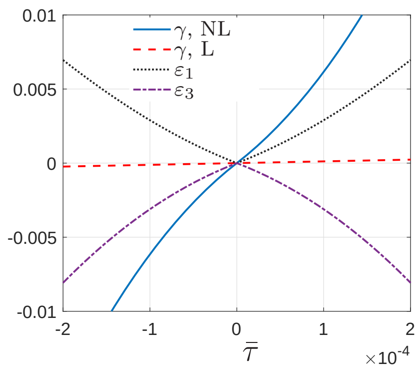

FIG. 12. Predictions for the non-linear model (5.1), (5.2), (5.6), (5.7), (3.5) denoted 'NL', and comparison with the predictions of the linearized model (3.10) that is denoted 'L', for the problem of simple shear stress of a slab.

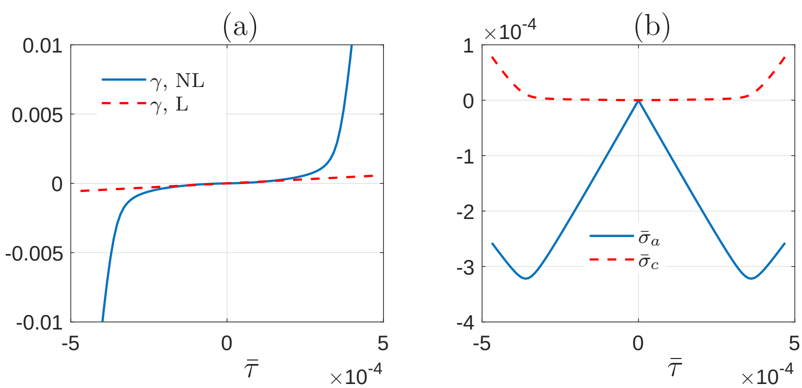

FIG. 13. Predictions for the non-linear model (5.1), (5.2), (5.6), (5.7), (3.5) denoted 'NL', and comparison with the predictions of the linearized model (3.10) that is denoted 'L', for the problem of simple shear strain of a slab.

In Fig. 11(a) we depict the predictions for the whole range of values for σ_z. In Fig. 11(b) we show details of the above plot for the case $\sigma_z \leq 0$, while in Fig. 11(c) we show details for the case $\sigma_z \geq 0$.

In Fig. 12 we document results for the case of a slab in a state of simple shear stress (see Section 4.3). In the case of the non-linear model we present results for the shear γ, and also for the longitudinal components of the strain tensor ε_1 and ε_3.

Finally, in Fig. 13 we show results corresponding to simple shear strain of a slab (see Section 4.4). In Fig. 13(a) results for γ are presented, while in Fig. 13(b) results for the normalized components of the stress $\bar{\sigma}_a$ and $\bar{\sigma}_c$ are shown for the non-linear model.

6. Inhomogeneous distributions for the stresses and strains. Inflation of a cylindrical annulus

In this section we study the behaviour of the cylindrical annulus

$$r_i \leq r \leq r_o, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq z \leq L, \tag{6.1}$$

when subject to the stresses shown below.

It is assumed that the cylinder is subject to the following distribution of stresses

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z, \tag{6.2}$$

and thus $\mathrm{tr}\,\mathbf{T} = \sigma_r(r) + \sigma_\theta(r) + \sigma_z(r)$, $\sigma_S = [\sigma_r(r) + \sigma_\theta(r) + \sigma_z(r)]/3$, $\sigma_1 = \sigma_r$, $\sigma_2 = \sigma_\theta$, $\sigma_3 = \sigma_z$ and $\overset{(1)}{\mathbf{t}} = \mathbf{e}_r$, $\overset{(2)}{\mathbf{t}} = \mathbf{e}_\theta$, $\overset{(3)}{\mathbf{t}} = \mathbf{e}_z$.

Using (6.2) in the equation of equilibrium (2.3) (assuming $\mathbf{b} = \mathbf{0}$) we obtain:

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \quad \Leftrightarrow \quad \sigma_\theta = \frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_r). \tag{6.3}$$

It is assumed that the stress distribution (6.2) produces the displacement field

$$\mathbf{u} = u_r(r)\mathbf{e}_r + (\lambda_z - 1)z\mathbf{e}_z, \tag{6.4}$$

where λ_z is a positive constant. Using (6.4) to obtain the components of the strain tensor (see $(2.1)_4$) we have

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \qquad \varepsilon_\theta = \frac{u_r}{r}, \qquad \varepsilon_z = \lambda_z - 1. \tag{6.5}$$

Using (6.2) and (6.5) in (3.4) we obtain:

$$\begin{aligned}(6.6)\quad & \varepsilon_r + \mathfrak{q}^{(0)}(\sigma_r+\sigma_\theta+\sigma_z)\varepsilon_r + 2\mathfrak{h}^{(0)}\varepsilon_r\sigma_r + \mathfrak{f}^{(1)}(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)\\ & + \mathfrak{f}^{(2)}(\sigma_r+\sigma_\theta+\sigma_z)(\varepsilon_r+\varepsilon_\theta+\varepsilon_z) + \mathfrak{q}^{(0)}(\varepsilon_r\sigma_r+\varepsilon_\theta\sigma_\theta+\varepsilon_z\sigma_z)\\ & + \mathfrak{q}^{(0)}(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)\sigma_r + \frac{\partial\mathfrak{p}}{\partial\sigma_1} = 0,\end{aligned}$$

$$\begin{aligned}(6.7)\quad & \varepsilon_\theta + \mathfrak{q}^{(0)}(\sigma_r+\sigma_\theta+\sigma_z)\varepsilon_\theta + 2\mathfrak{h}^{(0)}\varepsilon_\theta\sigma_\theta + \mathfrak{f}^{(1)}(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)\\ & + \mathfrak{f}^{(2)}(\sigma_r+\sigma_\theta+\sigma_z)(\varepsilon_r+\varepsilon_\theta+\varepsilon_z) + \mathfrak{q}^{(0)}(\varepsilon_r\sigma_r+\varepsilon_\theta\sigma_\theta+\varepsilon_z\sigma_z)\\ & + \mathfrak{q}^{(0)}(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)\sigma_\theta + \frac{\partial\mathfrak{p}}{\partial\sigma_2} = 0,\end{aligned}$$

$$\begin{aligned}(6.8)\quad & \varepsilon_z + \mathfrak{q}^{(0)}(\sigma_r+\sigma_\theta+\sigma_z)\varepsilon_z + 2\mathfrak{h}^{(0)}\varepsilon_z\sigma_z + \mathfrak{f}^{(1)}(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)\\ & + \mathfrak{f}^{(2)}(\sigma_r+\sigma_\theta+\sigma_z)(\varepsilon_r+\varepsilon_\theta+\varepsilon_z) + \mathfrak{q}^{(0)}(\varepsilon_r\sigma_r+\varepsilon_\theta\sigma_\theta+\varepsilon_z\sigma_z)\\ & + \mathfrak{q}^{(0)}(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)\sigma_z + \frac{\partial\mathfrak{p}}{\partial\sigma_3} = 0,\end{aligned}$$

where from (3.7) we have:

$$(6.9)\qquad \frac{\partial\mathfrak{p}}{\partial\sigma_1} = \mathfrak{F}'(\sigma_r(r)) + \frac{1}{3}\mathfrak{G}'([\sigma_r(r)+\sigma_\theta(r)+\sigma_z(r)]/3),$$

$$(6.10)\qquad \frac{\partial\mathfrak{p}}{\partial\sigma_2} = \mathfrak{F}'(\sigma_\theta(r)) + \frac{1}{3}\mathfrak{G}'([\sigma_r(r)+\sigma_\theta(r)+\sigma_z(r)]/3),$$

$$(6.11)\qquad \frac{\partial\mathfrak{p}}{\partial\sigma_3} = \mathfrak{F}'(\sigma_z(r)) + \frac{1}{3}\mathfrak{G}'([\sigma_r(r)+\sigma_\theta(r)+\sigma_z(r)]/3).$$

Equations (6.6)–(6.8) (recall $(6.3)_2$) are solved for $\sigma_r = \sigma_r(r)$ and σ_z using the same methodology presented in [21] and using the program Comsol[3] [22].

In subsequent analysis, for the sake of simplicity, we assume that $\lambda_z = 1$, i.e., there is no axial extension of the annulus. Regarding the boundary conditions at $r = r_i$ and at $r = r_o$ we assume that $\sigma_r(r_i) = -P$ and $\sigma_r(r_o) = 0$. In the plots shown below we use the notation:

$$(6.12)\qquad \begin{aligned} &\bar{\sigma}_r = \frac{\sigma_r}{E}, \quad \bar{\sigma}_\theta = \frac{\sigma_\theta}{E}, \quad \bar{\sigma}_z = \frac{\sigma_z}{E}, \quad \bar{r} = \frac{r}{r_i},\\ &\bar{u}_r = \frac{u_r}{r_i}, \quad \bar{u}_{r_i} = \frac{u_r(r_i)}{r_i}, \quad \bar{P} = \frac{P}{E}.\end{aligned}$$

[3]For the two constitutive relations studied in this paper, and after carrying out an analysis of the influence of the mesh, we use a finite element model with 960 elements of the same length (Lagrange quadratic) and 3842 degrees of freedom. The nonlinear equations are solved using the damped Newton method with a relative tolerance of 10^{-6}, a maximum number of iterations of 250, and the initial damping factor of 10^{-4} and the minimum damping factor of 10^{-8}.

In Figs. 14–16 results for the behaviour of the annulus are shown for the case of modelling concrete, using (5.1)–(5.4), (3.4) and comparing the results with the predictions of the linearized model (3.10). In Fig. 14 we display results for the dimensionless components of the stress versus the dimensionless radial position, for 4 values of the external traction P. In that plot the results obtained using (3.4) are denoted 'NL', and the results using the linearized model (3.10) are denoted 'L'. In Fig. 15 we depict results for the dimensionless radial displacement and the two components of the strain tensor. Finally, in Fig. 16 results are presented for the dimensionless radial displacement at $r = r_i$ versus the external dimensionless load $\bar{P}$, comparing the results obtained using the non-linear model (3.4) (which is denoted 'Nonlinear'), and the predictions of the linearized model (3.10) that is denoted 'Linear'.

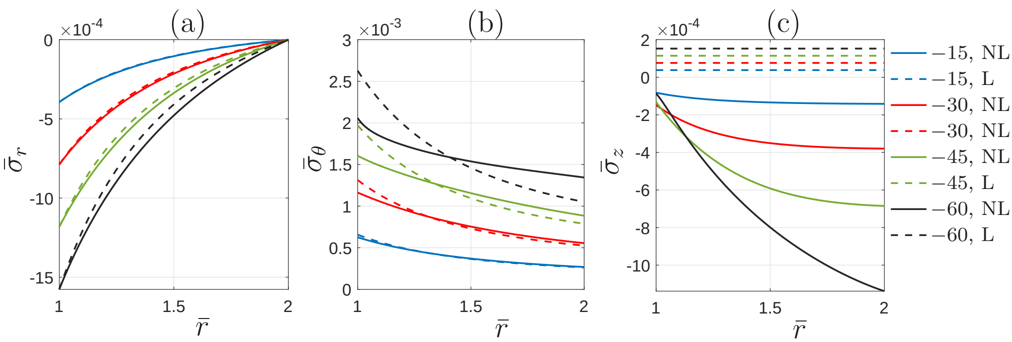

FIG. 14. Results for the dimensionless components of the stress tensor versus the dimensionless radial position for the modelling of concrete. Results are shown for different values of P [MPa]. The results obtained with (3.4) are denoted 'NL', and the results obtained using (3.10) are denoted 'L'.

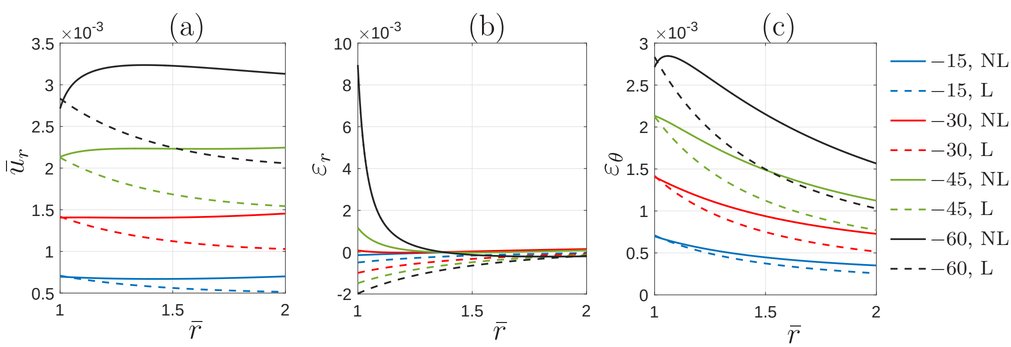

FIG. 15. Results for the dimensionless radial displacement and the components of the strain tensor versus the dimensionless radial position for the modelling of concrete. Results for different values of P [MPa]. The results obtained using (3.4) are denoted by 'NL', and the results obtained using (3.10) are denoted by 'L'.

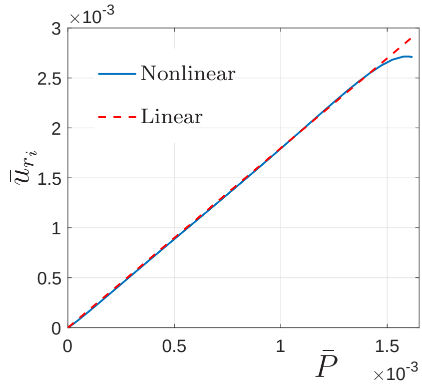

Fig. 16. Modelling of concrete. Behaviour of the dimensionless radial displacement at $r = r_i$ versus the dimensionless load $\bar{P}$.

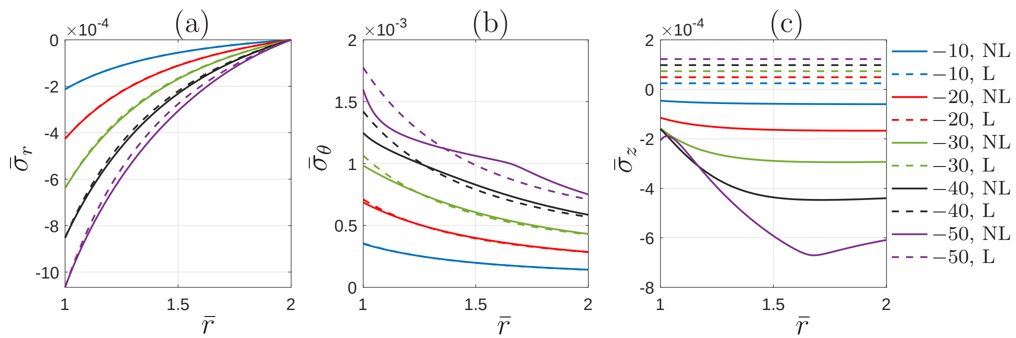

Fig. 17. Results for the dimensionless components of the stress tensor versus the dimensionless radial position in the case of rock. Results for different values of P [MPa]. The results obtained with (3.4) are denoted 'NL', and the results obtained using (3.10) are denoted 'L'.

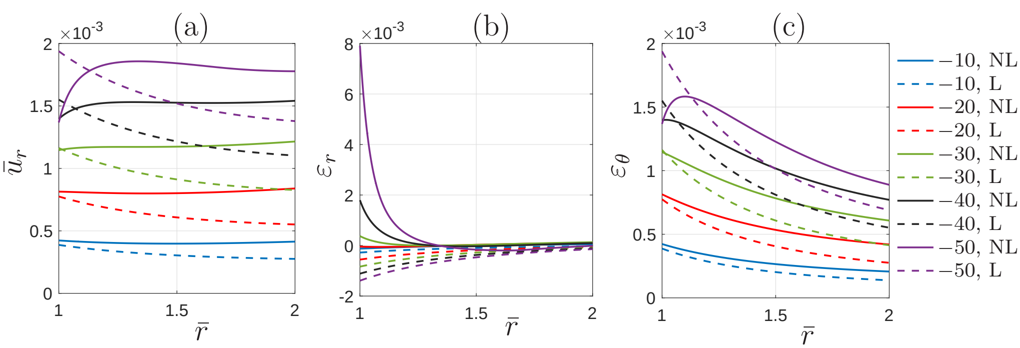

Fig. 18. Results for the dimensionless radial displacement and the components of the strain tensor versus the dimensionless radial position in the case of rock. Results for different values of P [MPa]. The results obtained with (3.4) are denoted 'NL', and the results obtained using (3.10) are denoted 'L'.

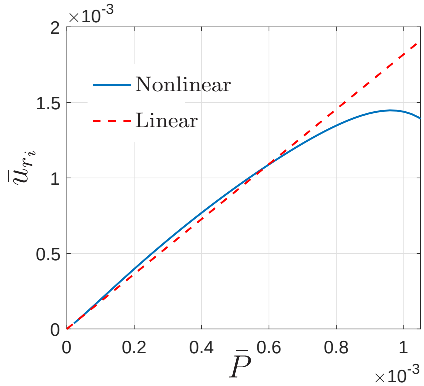

Fig. 19. Modelling of rock. Behaviour of the dimensionless radial displacement at $r = r_i$ versus the dimensionless load $\bar{P}$.

In Figs. 17–19 results for the behaviour of the annulus are shown for the case of modelling rock using (3.4), (5.1), (5.2), (5.6)–(5.11), for the non-linear model. In Fig. 17 results are presented for the dimensionless components of the stress tensor versus the dimensionless radial position, for different values of the external load P [MPa]. In Fig. 18 we show results for the dimensionless radial displacement and the two components of the strain tensor. In Fig. 19 results are presented for the dimensionless radial displacement at $r = r_i$ versus the external dimensionless load $\bar{P}$.

7. Conclusions

In the present communication we have analyzed the implicit constitutive relation (3.3), and used it to describe the response of a type of isotropic dry rock and concrete. In (3.3) we can observe the explicit dependence of the relation on $\operatorname{tr}\boldsymbol{\varepsilon}$. That term is directly related with the change in density of the body, and it can be an important factor for the modelling of porous solids, where the porosity (which is affected by the deformation) is known to have an important impact on the manner such solids behave (see, for example, [2]). We reiterate that is the main aim of this work. The constitutive relations used in [10] and [17] do not contain material moduli that depend explicitly on the density. Thus, while the constitutive relations in [10] and [17] might fit the data for one specific experiment, it might not in other experiments on porous solids whose material properties are expected to depend explicitly on density. Also, in our general constitutive relation the material moduli depend on the pressure and we expect such a dependence in porous elastic solids. Also the presence of terms like $\mathbf{T}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}$ and $\operatorname{tr}(\mathbf{T}\boldsymbol{\epsilon})$ in more general deformation would lead to terms that the other two constitutive relations cannot. The fact that in a specific experiment two different constitutive relations fit the data does not mean the philosophical underpinning of the constitutive relations are the same, and this fact cannot be overemphasized.

Regarding the modelling of the mechanical behaviour of rock, in [17] a bimodular model was proposed for rock, where in particular there are two sets of constants, a modulus resembling the Young modulus for tension and compression, and a Poisson-like modulus in tension and compression, plus another constant that is related with the transition between such states. The interested reader can compared the fitting in that work (see Fig. 8(b,c,e) in [17]) and the results from Fig. 8 here. The nonlinear model (3.3), (5.6)–(5.11) give a qualitatively better fitting for the experimental data, however the new implicit model presented here requires more material constants than that considered in [17]. On the other hand, if more experimental data would be available, for example, for the problems studied in Sections 4.2–4.4, then the new model (3.3) would be much more suitable for the fitting of that data for porous elastic solids.

Concerning the modelling of the behaviour of concrete using the experimental data from [10], it is necessary to remark that the new model (3.3), (5.3), (5.4) does not necessary give a much better fitting for the data of compression of a cylinder without lateral load (see Fig. 1). However, as explained in the introduction, the main aim of the present work is the development of an implicit constitutive model, where the effect of the density (and as a result the influence of the porosity) can be put explicitly in the constitutive relation, which we feel would be useful for describing porous elastic solids for a wide range of experiments.

Our intention is to use (3.3) or other special subclasses of (3.1) for the modelling of other porous solids, for which experimental data can be found. Another important future investigation involves the role the density plays in the stability of solutions.

Appendix A

Here, we present details concerning the derivation of (3.4) from the general implicit relation (3.1). We recall that relation (3.1)

$$\left[\frac{1}{2}\left(\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{E}}+\frac{\partial^2\Pi}{\partial\mathbf{E}\partial\mathbf{S}}\right)-\mathscr{I}\right]:\dot{\mathbf{E}}+\frac{\partial^2\Pi}{\partial\mathbf{S}\partial\mathbf{S}}:\dot{\mathbf{S}}=\mathbf{0}.$$

If we only consider the modelling of isotropic bodies, it follows from the theory of invariants (see, for example, [16]) that

$$\Pi=\Pi(\mathbf{E},\mathbf{S})=\Pi(I_1,I_2,I_3,I_4,I_5,I_6,I_7,I_8,I_9,I_{10}),$$

where

$$
\begin{aligned}
&I_1 = \operatorname{tr}\mathbf{E}, \quad I_2 = \frac{1}{2}\operatorname{tr}(\mathbf{E}^2), \quad I_3 = \frac{1}{3}\operatorname{tr}(\mathbf{E}^3), \quad I_4 = \operatorname{tr}\mathbf{S},\\
&I_5 = \frac{1}{2}\operatorname{tr}(\mathbf{S}^2), \quad I_6 = \frac{1}{3}\operatorname{tr}(\mathbf{S}^3),
\end{aligned} \tag{7.1}
$$

$$
I_7 = \operatorname{tr}(\mathbf{ES}), \quad I_8 = \operatorname{tr}(\mathbf{ES}^2), \quad I_9 = \operatorname{tr}(\mathbf{E}^2\mathbf{S}), \quad I_{10} = \operatorname{tr}(\mathbf{E}^2\mathbf{S}^2). \tag{7.2}
$$

Using the above invariants to express $\Pi = \Pi(\mathbf{E}, \mathbf{S})$

$$
\frac{\partial \Pi}{\partial \mathbf{S}} = \Pi_4\mathbf{I} + \Pi_5\mathbf{S} + \Pi_6\mathbf{S}^2 + \Pi_7\mathbf{E} + \Pi_8(\mathbf{ES}+\mathbf{SE}) + \Pi_9\mathbf{E}^2 + \Pi_{10}(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2), \tag{7.3}
$$

and

$$
\frac{\partial \Pi}{\partial \mathbf{E}} = \Pi_1\mathbf{I} + \Pi_2\mathbf{E} + \Pi_3\mathbf{E}^2 + \Pi_7\mathbf{S} + \Pi_8\mathbf{S}^2 + \Pi_9(\mathbf{ES}+\mathbf{SE}) + \Pi_{10}(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E}), \tag{7.4}
$$

where we have used the notation $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, \ldots, 10$. It follows from (7.3) that

$$
\begin{aligned}
\frac{\partial^2 \Pi}{\partial \mathbf{S}\partial \mathbf{S}} = {} & \Pi_5\mathscr{I} + \Pi_6\mathscr{S} + \Pi_8\mathscr{E}^{(1)} + \Pi_{10}\mathscr{E}^{(2)} + \Pi_{4,4}\mathbf{I}\otimes\mathbf{I}\\
& + (\Pi_{4,5}+\Pi_{7,9})(\mathbf{I}\otimes\mathbf{S}+\mathbf{S}\otimes\mathbf{I}) + \Pi_{4,6}(\mathbf{I}\otimes\mathbf{S}^2+\mathbf{S}^2\otimes\mathbf{I})\\
& + \Pi_{4,7}(\mathbf{I}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{I}) + \Pi_{4,8}[\mathbf{I}\otimes(\mathbf{ES}+\mathbf{SE})\\
& + (\mathbf{ES}+\mathbf{SE})\otimes\mathbf{I}] + \Pi_{4,9}(\mathbf{I}\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{I})\\
& + \Pi_{4,10}[\mathbf{I}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2) + (\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{I}] + \Pi_{5,5}\mathbf{S}\otimes\mathbf{S}\\
& + \Pi_{5,6}(\mathbf{S}\otimes\mathbf{S}^2+\mathbf{S}^2\otimes\mathbf{S}) + \Pi_{5,7}(\mathbf{S}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{S})\\
& + \Pi_{5,8}[\mathbf{S}\otimes(\mathbf{ES}+\mathbf{SE}) + (\mathbf{ES}+\mathbf{SE})\otimes\mathbf{S}] + \Pi_{5,9}(\mathbf{S}\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{S})\\
& + \Pi_{5,10}[\mathbf{S}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2) + (\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{S}] + \Pi_{6,6}\mathbf{S}^2\otimes\mathbf{S}^2\\
& + \Pi_{6,7}(\mathbf{S}^2\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{S}^2) + \Pi_{6,8}[\mathbf{S}^2\otimes(\mathbf{ES}+\mathbf{SE})\\
& + (\mathbf{ES}+\mathbf{SE})\otimes\mathbf{S}^2] + \Pi_{6,9}(\mathbf{S}^2\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{S}^2)\\
& + \Pi_{6,10}[\mathbf{S}^2\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2) + (\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{S}^2]\\
& + \Pi_{7,7}\mathbf{E}\otimes\mathbf{E} + \Pi_{7,8}[\mathbf{E}\otimes(\mathbf{ES}+\mathbf{SE}) + (\mathbf{ES}+\mathbf{SE})\otimes\mathbf{E}]\\
& + \Pi_{7,10}[\mathbf{E}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2) + (\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{E}]\\
& + \Pi_{8,8}(\mathbf{ES}+\mathbf{SE})\otimes(\mathbf{ES}+\mathbf{SE})\\
& + \Pi_{8,9}[\mathbf{E}^2\otimes(\mathbf{ES}+\mathbf{SE}) + (\mathbf{ES}+\mathbf{SE})\otimes\mathbf{E}^2]\\
& + \Pi_{8,10}[(\mathbf{ES}+\mathbf{SE})\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2) + (\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes(\mathbf{ES}+\mathbf{SE})]\\
& + \Pi_{9,9}\mathbf{E}^2\otimes\mathbf{E}^2 + \Pi_{9,10}[\mathbf{E}^2\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2) + (\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{E}^2]\\
& + \Pi_{10,10}(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2),
\end{aligned} \tag{7.5}
$$

where we use the notation $\Pi_{i,j} = \frac{\partial^2 \Pi}{\partial I_i \partial I_j}$, $i, j = 1, \ldots, 10$, and where the components of the fourth order tensors $\boldsymbol{\mathscr{S}}$, $\boldsymbol{\mathscr{E}}^{(1)}$ and $\boldsymbol{\mathscr{E}}^{(2)}$ are defined (in Cartesian coordinates) through

$$\mathscr{S}_{ijkl} = \frac{1}{2}(\delta_{ki} S_{lj} + S_{ik}\delta_{jl} + \delta_{il} S_{kj} + S_{il}\delta_{jk}), \tag{7.6}$$

$$\mathscr{E}^{(1)}_{ijkl} = \frac{1}{2}(E_{ik}\delta_{jl} + \delta_{ik} E_{lj} + E_{il}\delta_{jk} + \delta_{il} E_{kj}), \tag{7.7}$$

$$\mathscr{E}^{(2)}_{ijkl} = \frac{1}{2}(\overset{(2)}{E}_{ik}\delta_{jl} + \delta_{ik} \overset{(2)}{E}_{lj} + \overset{(2)}{E}_{il}\delta_{jk} + \delta_{il} \overset{(2)}{E}_{kj}), \tag{7.8}$$

where

$$\overset{(2)}{E}_{ij} = E_{im}E_{mj}. \tag{7.9}$$

On the other hand, from (7.3) and (7.4) we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial^2 \Pi}{\partial \mathbf{S} \partial \mathbf{E}} + \frac{\partial^2 \Pi}{\partial \mathbf{E} \partial \mathbf{S}}\right) \\
&= \Pi_7 \boldsymbol{\mathscr{I}} + \Pi_8 \boldsymbol{\mathscr{S}} + \Pi_9 \boldsymbol{\mathscr{E}}^{(1)} + \frac{\Pi_{10}}{2}(\boldsymbol{\mathscr{H}}^{(1)} + \boldsymbol{\mathscr{H}}^{(2)}) + \Pi_{1,4} \mathbf{I} \otimes \mathbf{I} \\
&\quad + \Pi_{2,7} \mathbf{E} \otimes \mathbf{E} + \Pi_{3,9} \mathbf{E}^2 \otimes \mathbf{E}^2 + \Pi_{5,7} \mathbf{S} \otimes \mathbf{S} + \Pi_{6,8} \mathbf{S}^2 \otimes \mathbf{S}^2 \\
&\quad + \Pi_{8,9}(\mathbf{ES} + \mathbf{SE}) \otimes (\mathbf{ES} + \mathbf{SE}) + \frac{1}{2}(\Pi_{2,4} + \Pi_{1,7})(\mathbf{I} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{I}) \\
&\quad + \frac{1}{2}(\Pi_{3,4} + \Pi_{1,9})(\mathbf{I} \otimes \mathbf{E}^2 + \mathbf{E}^2 \otimes \mathbf{I}) + \frac{1}{2}(\Pi_{4,7} + \Pi_{1,5})(\mathbf{I} \otimes \mathbf{S} + \mathbf{S} \otimes \mathbf{I}) \\
&\quad + \frac{1}{2}(\Pi_{4,8} + \Pi_{1,6})(\mathbf{I} \otimes \mathbf{S}^2 + \mathbf{S}^2 \otimes \mathbf{I}) + \frac{1}{2}(\Pi_{4,9} + \Pi_{1,8})[\mathbf{I} \otimes (\mathbf{ES} + \mathbf{SE}) \\
&\quad + (\mathbf{ES} + \mathbf{SE}) \otimes \mathbf{I}] + \frac{\Pi_{4,10}}{2}[\mathbf{I} \otimes (\mathbf{ES}^2 + \mathbf{S}^2\mathbf{E}) + (\mathbf{ES}^2 + \mathbf{S}^2\mathbf{E}) \otimes \mathbf{I}] \\
&\quad + \frac{1}{2}(\Pi_{2,5} + \Pi_{7,7})(\mathbf{E} \otimes \mathbf{S} + \mathbf{S} \otimes \mathbf{E}) + \frac{1}{2}(\Pi_{3,5} + \Pi_{7,9})(\mathbf{S} \otimes \mathbf{E}^2 + \mathbf{E}^2 \otimes \mathbf{S}) \\
&\quad + \frac{1}{2}(\Pi_{5,8} + \Pi_{6,7})(\mathbf{S} \otimes \mathbf{S}^2 + \mathbf{S}^2 \otimes \mathbf{S}) + \frac{1}{2}(\Pi_{6,9} + \Pi_{8,8})[\mathbf{S}^2 \otimes (\mathbf{ES} + \mathbf{SE}) \\
&\quad + (\mathbf{ES} + \mathbf{SE}) \otimes \mathbf{S}^2] + \frac{\Pi_{6,10}}{2}[\mathbf{S}^2 \otimes (\mathbf{ES}^2 + \mathbf{S}^2\mathbf{E}) + (\mathbf{ES}^2 + \mathbf{S}^2\mathbf{E}) \otimes \mathbf{S}^2] \\
&\quad + \frac{1}{2}(\Pi_{3,7} + \Pi_{2,9})(\mathbf{E}^2 \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{E}^2) + \frac{\Pi_{7,8}}{2}(\mathbf{E} \otimes \mathbf{S}^2 + \mathbf{S}^2 \otimes \mathbf{E}) \\
&\quad + \frac{1}{2}(\Pi_{7,9} + \Pi_{2,8})[\mathbf{E} \otimes (\mathbf{ES} + \mathbf{SE}) + (\mathbf{ES} + \mathbf{SE}) \otimes \mathbf{E}] \\
&\quad + \frac{1}{2}(\Pi_{3,8} + \Pi_{9,9})[\mathbf{E}^2 \otimes (\mathbf{ES} + \mathbf{SE}) + (\mathbf{ES} + \mathbf{SE}) \otimes \mathbf{E}^2]
\end{aligned} \tag{7.10}$$

$$
\begin{aligned}
&+\frac{\Pi_{7,8}}{2}[\mathbf{S}\otimes(\mathbf{ES}+\mathbf{SE})+(\mathbf{ES}+\mathbf{SE})\otimes\mathbf{S}]+\frac{\Pi_{8,9}}{2}(\mathbf{S}^2\otimes\mathbf{E}^2+\mathbf{E}^2\otimes\mathbf{S}^2)\\
&+\frac{\Pi_{8,10}}{2}[(\mathbf{ES}+\mathbf{SE})\otimes(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E})+(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E})\otimes(\mathbf{ES}+\mathbf{SE})]\\
&+\frac{\Pi_{9,10}}{2}[\mathbf{E}^2\otimes(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E})+(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E})\otimes\mathbf{E}^2]\\
&+\frac{\Pi_{1,10}}{2}[\mathbf{I}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{I}]\\
&+\frac{\Pi_{2,10}}{2}[\mathbf{E}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{E}]\\
&+\frac{\Pi_{3,10}}{2}[\mathbf{E}^2\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{E}^2]\\
&+\frac{\Pi_{7,10}}{2}[\mathbf{S}\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{S}]\\
&+\frac{\Pi_{8,10}}{2}[\mathbf{S}^2\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes\mathbf{S}^2]\\
&+\frac{\Pi_{9,10}}{2}[(\mathbf{ES}+\mathbf{SE})\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes(\mathbf{ES}+\mathbf{SE})]\\
&+\frac{\Pi_{10,10}}{2}[(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E})\otimes(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)+(\mathbf{E}^2\mathbf{S}+\mathbf{SE}^2)\otimes(\mathbf{ES}^2+\mathbf{S}^2\mathbf{E})],
\end{aligned}
$$

where the components of the fourth order tensors $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ are defined in Cartesian tensor components as:

$$
\begin{aligned}
\mathscr{H}^{(1)}_{ijkl}=\frac{1}{2}(&\delta_{ki}E_{lm}S_{mj}+E_{ik}S_{lj}+S_{ik}E_{lj}+S_{im}E_{mk}\delta_{lj}\\
&+\delta_{il}E_{km}S_{mj}+E_{il}S_{kj}+S_{il}E_{kj}+S_{im}E_{ml}\delta_{jk}),
\end{aligned}
\tag{7.11}
$$

$$
\begin{aligned}
\mathscr{H}^{(2)}_{ijkl}=\frac{1}{2}(&\delta_{ki}S_{lm}E_{mj}+S_{ik}E_{lj}+E_{ik}S_{lj}+E_{im}S_{mk}\delta_{lj}\\
&+\delta_{il}S_{km}E_{mj}+S_{il}E_{kj}+E_{il}S_{kj}+E_{im}S_{ml}\delta_{jk}).
\end{aligned}
\tag{7.12}
$$

Replacing (7.5) and (7.10) in (3.1) we can obtain an explicit expression for the implicit constitutive relation for isotropic bodies. For the sake of brevity we do not do that here.

Appendix B. Implicit constitutive relations for isotropic bodies. Case of small gradient of the displacement field

In this appendix we study the case when the gradient of the displacement field is small, i.e., the case $\left|\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right|\sim O(\delta)$, $\delta\ll 1$. In this case we do not distinguish between the current and the reference configurations, as well as this, we have the approximations $\mathbf{E}\approx\boldsymbol{\varepsilon}$ and $\mathbf{S}\approx\mathbf{T}$.

Neglecting terms of order δ^2 or higher for $\boldsymbol{\varepsilon}$ in (7.5) we obtain

$$
\begin{aligned}
(7.13)\qquad & \frac{\partial^2 \Pi}{\partial \mathbf{T} \partial \mathbf{T}} \\
& \approx \Pi_5 \mathscr{I} + \Pi_6 \mathscr{S} + \Pi_8 \mathscr{E}^{(1)} + \Pi_{4,4} \mathbf{I} \otimes \mathbf{I} + (\Pi_{4,5} + \Pi_{7,9})(\mathbf{I} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbf{I}) \\
& \quad + \Pi_{4,6}(\mathbf{I} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{I}) + \Pi_{4,7}(\mathbf{I} \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{I}) \\
& \quad + \Pi_{4,8}[\mathbf{I} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{I}] \\
& \quad + \Pi_{5,5} \mathbf{T} \otimes \mathbf{T} + \Pi_{5,6}(\mathbf{T} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{T}) \\
& \quad + \Pi_{5,7}(\mathbf{T} \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{T}) + \Pi_{5,8}[\mathbf{T} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{T}] \\
& \quad + \Pi_{6,6} \mathbf{T}^2 \otimes \mathbf{T}^2 + \Pi_{6,7}(\mathbf{T}^2 \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{T}^2) \\
& \quad + \Pi_{6,8}[\mathbf{T}^2 \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{T}^2],
\end{aligned}
$$

and doing the same in (7.10) we have

$$
\begin{aligned}
(7.14)\qquad & \frac{1}{2}\left(\frac{\partial^2 \Pi}{\partial \mathbf{T} \partial \boldsymbol{\varepsilon}} + \frac{\partial^2 \Pi}{\partial \boldsymbol{\varepsilon} \partial \mathbf{T}}\right) \\
\approx\ & \Pi_7 \mathscr{I} + \Pi_8 \mathscr{S} + \Pi_9 \mathscr{E}^{(1)} + \frac{\Pi_{10}}{2}(\mathscr{H}^{(1)} + \mathscr{H}^{(2)}) + \Pi_{1,4} \mathbf{I} \otimes \mathbf{I} \\
& + \Pi_{5,7} \mathbf{T} \otimes \mathbf{T} + \Pi_{6,8} \mathbf{T}^2 \otimes \mathbf{T}^2 + \frac{1}{2}(\Pi_{2,4} + \Pi_{1,7})(\mathbf{I} \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{I}) \\
& + \frac{1}{2}(\Pi_{4,7} + \Pi_{1,5})(\mathbf{I} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbf{I}) + \frac{1}{2}(\Pi_{4,8} + \Pi_{1,6})(\mathbf{I} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{I}) \\
& + \frac{1}{2}(\Pi_{4,9} + \Pi_{1,8})[\mathbf{I} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{I}] \\
& + \frac{\Pi_{4,10}}{2}[\mathbf{I} \otimes (\boldsymbol{\varepsilon}\mathbf{T}^2 + \mathbf{T}^2\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T}^2 + \mathbf{T}^2\boldsymbol{\varepsilon}) \otimes \mathbf{I}] \\
& + \frac{1}{2}(\Pi_{2,5} + \Pi_{7,7})(\boldsymbol{\varepsilon} \otimes \mathbf{T} + \mathbf{T} \otimes \boldsymbol{\varepsilon}) + \frac{1}{2}(\Pi_{5,8} + \Pi_{6,7})(\mathbf{T} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{T}) \\
& + \frac{1}{2}(\Pi_{6,9} + \Pi_{8,8})[\mathbf{T}^2 \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{T}^2] \\
& + \frac{\Pi_{6,10}}{2}[\mathbf{T}^2 \otimes (\boldsymbol{\varepsilon}\mathbf{T}^2 + \mathbf{T}^2\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T}^2 + \mathbf{T}^2\boldsymbol{\varepsilon}) \otimes \mathbf{T}^2] + \frac{\Pi_{7,8}}{2}(\boldsymbol{\varepsilon} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \boldsymbol{\varepsilon}) \\
& + \frac{\Pi_{7,8}}{2}[\mathbf{T} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \boldsymbol{\varepsilon}].
\end{aligned}
$$

In (7.13), (7.14) and thereafter the fourth order tensor $\mathscr{S}$, $\mathscr{E}^{(1)}$, $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ are defined as in (7.6), (7.7), (7.11) and (7.12), replacing $\mathbf{S}$ by $\mathbf{T}$ and $\mathbf{E}$ by $\boldsymbol{\varepsilon}$.

From (7.1), (7.2) we have the approximations:

$$
(7.15)\qquad I_1 \approx \operatorname{tr} \boldsymbol{\varepsilon},\ I_2 \approx 0,\ I_3 \approx 0,\ I_4 \approx \operatorname{tr} \mathbf{T},\ I_5 \approx \frac{1}{2} \operatorname{tr}(\mathbf{T}^2),\ I_6 \approx \frac{1}{3} \operatorname{tr}(\mathbf{T}^3),
$$

$$
(7.16)\qquad I_7 \approx \operatorname{tr}(\boldsymbol{\varepsilon}\mathbf{T}), \quad I_8 \approx \operatorname{tr}(\boldsymbol{\varepsilon}\mathbf{T}^2), \quad I_9 \approx 0, \quad I_{10} \approx 0,
$$

as a result:

$$\Pi_{7,9} = 0, \quad \Pi_9 = 0, \quad \Pi_{10} = 0, \quad \Pi_{2,4} = 0, \quad \Pi_{4,9} = 0, \quad \Pi_{4,10} = 0, \tag{7.17}$$

$$\Pi_{2,5} = 0, \quad \Pi_{6,9} = 0, \quad \Pi_{6,10} = 0, \tag{7.18}$$

thus

$$\Pi = \Pi(I_1, I_4, I_5, I_6, I_7, I_8), \tag{7.19}$$

and using this in (7.14) it becomes

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial^2 \Pi}{\partial \mathbf{T} \partial \boldsymbol{\varepsilon}} + \frac{\partial^2 \Pi}{\partial \boldsymbol{\varepsilon} \partial \mathbf{T}}\right) \\
&\quad = \Pi_7 \boldsymbol{\mathscr{I}} + \Pi_8 \boldsymbol{\mathscr{S}} + \Pi_{1,4} \mathbf{I} \otimes \mathbf{I} + \Pi_{5,7} \mathbf{T} \otimes \mathbf{T} + \Pi_{6,8} \mathbf{T}^2 \otimes \mathbf{T}^2 \\
&\qquad + \frac{\Pi_{1,7}}{2}(\mathbf{I} \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{I}) + \frac{1}{2}(\Pi_{4,7} + \Pi_{1,5})(\mathbf{I} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbf{I}) \\
&\qquad + \frac{1}{2}(\Pi_{4,8} + \Pi_{1,6})(\mathbf{I} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{I}) + \frac{\Pi_{1,8}}{2}[\mathbf{I} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \\
&\qquad + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{I}] + \frac{\Pi_{7,7}}{2}(\boldsymbol{\varepsilon} \otimes \mathbf{T} + \mathbf{T} \otimes \boldsymbol{\varepsilon}) \\
&\qquad + \frac{1}{2}(\Pi_{5,8} + \Pi_{6,7})(\mathbf{T} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{T}) + \frac{\Pi_{8,8}}{2}[\mathbf{T}^2 \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \\
&\qquad + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{T}^2] + \frac{\Pi_{7,8}}{2}(\boldsymbol{\varepsilon} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \boldsymbol{\varepsilon}) \\
&\qquad + \frac{\Pi_{7,8}}{2}[\mathbf{T} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \boldsymbol{\varepsilon}].
\end{aligned} \tag{7.20}$$

In the case $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right| \sim O(\delta)$, $\delta \ll 1$ since $\mathbf{E} \approx \boldsymbol{\varepsilon}$ and $\mathbf{S} \approx \mathbf{T}$ Eq. (3.1) can be approximated as

$$\left[\frac{1}{2}\left(\frac{\partial^2 \Pi}{\partial \mathbf{T} \partial \boldsymbol{\varepsilon}} + \frac{\partial^2 \Pi}{\partial \boldsymbol{\varepsilon} \partial \mathbf{T}}\right) - \boldsymbol{\mathscr{I}}\right] : \dot{\boldsymbol{\varepsilon}} + \frac{\partial^2 \Pi}{\partial \mathbf{T} \partial \mathbf{T}} : \dot{\mathbf{T}} = \mathbf{0}, \tag{7.21}$$

and using (7.13) and (7.20) in the above expression we have (see also the assumptions that leads to (7.23) below):

$$\begin{aligned}
&\Big[\Pi_7 \boldsymbol{\mathscr{I}} + \Pi_8 \boldsymbol{\mathscr{S}} + \Pi_{1,4} \mathbf{I} \otimes \mathbf{I} + \Pi_{5,7} \mathbf{T} \otimes \mathbf{T} + \Pi_{6,8} \mathbf{T}^2 \otimes \mathbf{T}^2 \\
&\quad + \frac{1}{2}(\Pi_{4,7} + \Pi_{1,5})(\mathbf{I} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbf{I}) + \frac{1}{2}(\Pi_{4,8} + \Pi_{1,6})(\mathbf{I} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{I}) \\
&\quad + \frac{1}{2}(\Pi_{5,8} + \Pi_{6,7})(\mathbf{T} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{T}) - \boldsymbol{\mathscr{I}}\Big] : \dot{\boldsymbol{\varepsilon}}
\end{aligned} \tag{7.22}$$

$$
\begin{aligned}
&+\{\Pi_5\boldsymbol{\mathscr{I}}+\Pi_6\boldsymbol{\mathscr{S}}+\Pi_8\boldsymbol{\mathscr{E}}^{(1)}+\Pi_{4,4}\mathbf{I}\otimes\mathbf{I}+\Pi_{4,5}(\mathbf{I}\otimes\mathbf{T}+\mathbf{T}\otimes\mathbf{I})\\
&+\Pi_{4,6}(\mathbf{I}\otimes\mathbf{T}^2+\mathbf{T}^2\otimes\mathbf{I})+\Pi_{4,7}(\mathbf{I}\otimes\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\otimes\mathbf{I})\\
&+\Pi_{4,8}[\mathbf{I}\otimes(\boldsymbol{\varepsilon}\mathbf{T}+\mathbf{T}\boldsymbol{\varepsilon})+(\boldsymbol{\varepsilon}\mathbf{T}+\mathbf{T}\boldsymbol{\varepsilon})\otimes\mathbf{I}]+\Pi_{5,5}\mathbf{T}\otimes\mathbf{T}\\
&+\Pi_{5,6}(\mathbf{T}\otimes\mathbf{T}^2+\mathbf{T}^2\otimes\mathbf{T})+\Pi_{5,7}(\mathbf{T}\otimes\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\otimes\mathbf{T})\\
&+\Pi_{5,8}[\mathbf{T}\otimes(\boldsymbol{\varepsilon}\mathbf{T}+\mathbf{T}\boldsymbol{\varepsilon})+(\boldsymbol{\varepsilon}\mathbf{T}+\mathbf{T}\boldsymbol{\varepsilon})\otimes\mathbf{T}]+\Pi_{6,6}\mathbf{T}^2\otimes\mathbf{T}^2\\
&+\Pi_{6,7}(\mathbf{T}^2\otimes\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\otimes\mathbf{T}^2)\\
&+\Pi_{6,8}[\mathbf{T}^2\otimes(\boldsymbol{\varepsilon}\mathbf{T}+\mathbf{T}\boldsymbol{\varepsilon})+(\boldsymbol{\varepsilon}\mathbf{T}+\mathbf{T}\boldsymbol{\varepsilon})\otimes\mathbf{T}^2]\}:\dot{\mathbf{T}}=\mathbf{0}.
\end{aligned}
$$

We assume that $|\dot{\boldsymbol{\varepsilon}}|\sim O(\delta^*)$, $\delta^*\ll 1$. The above makes sense if we first divide $\dot{\boldsymbol{\varepsilon}}$ by some characteristic value, which is denoted ε_t. One possible expression for that can be $\varepsilon_t=\sqrt{\frac{g}{L}}$, where L is a characteristic length for a given boundary value problem and g is the gravitational constant. In such a case the dimensionless strain rate can be defined as $\sqrt{\frac{L}{g}}\dot{\boldsymbol{\varepsilon}}$. We use the same notation for that dimensionless strain rate.

For the fourth order tensor that appears in the inner product with $\dot{\boldsymbol{\varepsilon}}$ (see the first square bracket in (7.22)) we demand it does not depend on $\boldsymbol{\varepsilon}$, whereas for the fourth order tensor that appears in a double contraction with $\dot{\mathbf{T}}$ we demand it can be at most a linear expression in $\boldsymbol{\varepsilon}$. Sufficient conditions for the above restrictions to hold are (recall the original expressions for $\frac{\partial^2\Pi}{\partial\mathbf{T}\partial\mathbf{T}}$ and $\frac{1}{2}\big(\frac{\partial^2\Pi}{\partial\mathbf{T}\partial\boldsymbol{\varepsilon}}+\frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\mathbf{T}}\big)$ given in (7.13) and (7.20)):

$$
\Pi_{1,7}=0,\qquad \Pi_{1,8}=0,\qquad \Pi_{7,7}=0,\qquad \Pi_{8,8}=0,\qquad \Pi_{7,8}=0, \tag{7.23}
$$

and

$$
\Pi_8,\quad \Pi_{4,7},\quad \Pi_{4,8},\quad \Pi_{5,7},\quad \Pi_{5,8},\quad \Pi_{6,7},\quad \Pi_{6,8}\quad \text{are constant in } \boldsymbol{\varepsilon}. \tag{7.24}
$$

A function Π that satisfies the above restrictions is

$$
\begin{aligned}
\Pi(I_1,I_4,I_5,I_6,I_7,I_8)=\mathfrak{f}(I_4,I_5,I_6)I_1+\mathfrak{g}(I_4,I_5,I_6)I_7&\\
+\mathfrak{h}(I_4,I_5,I_6)I_8+\mathfrak{p}(I_4,I_5,I_6)&.
\end{aligned} \tag{7.25}
$$

Replacing (7.25) in (7.22) we obtain

$$
\begin{aligned}
&\Big[\mathfrak{g}\boldsymbol{\mathscr{I}}+\mathfrak{h}\boldsymbol{\mathscr{S}}+\mathfrak{f}_4\mathbf{I}\otimes\mathbf{I}+\mathfrak{g}_5\mathbf{T}\otimes\mathbf{T}+\mathfrak{h}_6\mathbf{T}^2\otimes\mathbf{T}^2\\
&+\frac{1}{2}(\mathfrak{g}_4+\mathfrak{f}_5)(\mathbf{I}\otimes\mathbf{T}+\mathbf{T}\otimes\mathbf{I})+\frac{1}{2}(\mathfrak{h}_4+\mathfrak{f}_6)(\mathbf{I}\otimes\mathbf{T}^2+\mathbf{T}^2\otimes\mathbf{I})\\
&+\frac{1}{2}(\mathfrak{h}_5+\mathfrak{g}_6)(\mathbf{T}\otimes\mathbf{T}^2+\mathbf{T}^2\otimes\mathbf{T})-\boldsymbol{\mathscr{I}}\Big]:\dot{\boldsymbol{\varepsilon}}\\
&+\{(\mathfrak{f}_5I_1+\mathfrak{g}_5I_7+\mathfrak{h}_5I_8+\mathfrak{p}_5)\boldsymbol{\mathscr{I}}+(\mathfrak{f}_6I_1+\mathfrak{g}_6I_7+\mathfrak{h}_6I_8+\mathfrak{p}_6)\boldsymbol{\mathscr{S}}+\mathfrak{h}\boldsymbol{\mathscr{E}}^{(1)}
\end{aligned} \tag{7.26}
$$

$$+ (\mathfrak{f}_{4,4} I_1 + \mathfrak{g}_{4,4} I_7 + \mathfrak{h}_{4,4} I_8 + \mathfrak{p}_{4,4}) \mathbf{I} \otimes \mathbf{I}$$
$$+ (\mathfrak{f}_{4,5} I_1 + \mathfrak{g}_{4,5} I_7 + \mathfrak{h}_{4,5} I_8 + \mathfrak{p}_{4,5}) (\mathbf{I} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbf{I})$$
$$+ (\mathfrak{f}_{4,6} I_1 + \mathfrak{g}_{4,6} I_7 + \mathfrak{h}_{4,6} I_8 + \mathfrak{p}_{4,6}) (\mathbf{I} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{I})$$
$$+ \mathfrak{g}_4 (\mathbf{I} \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{I}) + \mathfrak{h}_4 [\mathbf{I} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{I}]$$
$$+ (\mathfrak{f}_{5,5} I_1 + \mathfrak{g}_{5,5} I_7 + \mathfrak{h}_{5,5} I_8 + \mathfrak{p}_{5,5}) \mathbf{T} \otimes \mathbf{T}$$
$$+ (\mathfrak{f}_{5,6} I_1 + \mathfrak{g}_{5,6} I_7 + \mathfrak{h}_{5,6} I_8 + \mathfrak{p}_{5,6}) (\mathbf{T} \otimes \mathbf{T}^2 + \mathbf{T}^2 \otimes \mathbf{T})$$
$$+ \mathfrak{g}_5 (\mathbf{T} \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{T}) + \mathfrak{h}_5 [\mathbf{T} \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{T}]$$
$$+ (\mathfrak{f}_{6,6} I_1 + \mathfrak{g}_{6,6} I_7 + \mathfrak{h}_{6,6} I_8 + \mathfrak{p}_{6,6}) \mathbf{T}^2 \otimes \mathbf{T}^2 + \mathfrak{g}_6 (\mathbf{T}^2 \otimes \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \otimes \mathbf{T}^2)$$
$$+ \mathfrak{h}_6 [\mathbf{T}^2 \otimes (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) + (\boldsymbol{\varepsilon}\mathbf{T} + \mathbf{T}\boldsymbol{\varepsilon}) \otimes \mathbf{T}^2]\} : \dot{\mathbf{T}} = \mathbf{0},$$

where we have used the notation $\mathfrak{f}_i = \frac{\partial \mathfrak{f}}{\partial I_i}$, $\mathfrak{g}_i = \frac{\partial \mathfrak{g}}{\partial I_i}$, $\mathfrak{h}_i = \frac{\partial \mathfrak{h}}{\partial I_i}$, $\mathfrak{p}_i = \frac{\partial \mathfrak{p}}{\partial I_i}$ and $\mathfrak{f}_{i,j} = \frac{\partial^2 \mathfrak{f}}{\partial I_i \partial I_j}$, $\mathfrak{g}_{i,j} = \frac{\partial^2 \mathfrak{g}}{\partial I_i \partial I_j}$, $\mathfrak{h}_{i,j} = \frac{\partial^2 \mathfrak{h}}{\partial I_i \partial I_j}$, $\mathfrak{p}_{i,j} = \frac{\partial^2 \mathfrak{p}}{\partial I_i \partial I_j}$.

Appendix C. A subclass wherein the stresses appear linearly for the expressions associated with the functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$

In this section we study (7.26) in the particular case the stresses appear linearly for the expressions associated with the functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$. That is possible if we assume that the functions $\mathfrak{f}(I_4, I_5, I_6)$, $\mathfrak{g}(I_4, I_5, I_6)$ and $\mathfrak{h}(I_4, I_5, I_6)$ defined in (7.25) are given as (recall (7.15), (7.16)):

$$\mathfrak{f}(I_4, I_5, I_6) = \mathfrak{f}^{(0)} + \mathfrak{f}^{(1)} I_4 + \mathfrak{f}^{(2)} \frac{I_4^2}{2} + \mathfrak{f}^{(3)} I_5, \ \ \mathfrak{g}(I_4, I_5, I_6) = \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)} I_4, \tag{7.27}$$

$$\mathfrak{h}(I_4, I_5, I_6) = \mathfrak{h}^{(0)}, \tag{7.28}$$

where $\mathfrak{f}^{(0)}$, $\mathfrak{f}^{(1)}$, $\mathfrak{f}^{(2)}$, $\mathfrak{f}^{(3)}$, $\mathfrak{g}^{(0)}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{h}^{(0)}$ are constants. Here we assume that $\mathfrak{p} = \mathfrak{p}(I_4, I_5, I_6)$.

Using (7.27) and (7.28) in (7.26) after some manipulations we obtain

$$\begin{aligned}(\mathfrak{g}^{(0)} - 1)\dot{\boldsymbol{\varepsilon}} &+ \mathfrak{g}^{(1)}[(\operatorname{tr}\mathbf{T})\dot{\boldsymbol{\varepsilon}} + (\operatorname{tr}\dot{\mathbf{T}})\boldsymbol{\varepsilon}] + \mathfrak{h}^{(0)}(\mathbf{T}\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}\mathbf{T} + \boldsymbol{\varepsilon}\dot{\mathbf{T}} + \dot{\mathbf{T}}\boldsymbol{\varepsilon}) \\ &+ \mathfrak{f}^{(1)}(\operatorname{tr}\dot{\boldsymbol{\varepsilon}})\mathbf{I} + \mathfrak{f}^{(2)}[(\operatorname{tr}\mathbf{T})(\operatorname{tr}\dot{\boldsymbol{\varepsilon}}) + (\operatorname{tr}\boldsymbol{\varepsilon})(\operatorname{tr}\dot{\mathbf{T}})]\mathbf{I} \\ &+ \left[\frac{(\mathfrak{g}^{(1)} + \mathfrak{f}^{(3)})}{2}\operatorname{tr}(\mathbf{T}\dot{\boldsymbol{\varepsilon}}) + \mathfrak{g}^{(1)}\operatorname{tr}(\boldsymbol{\varepsilon}\dot{\mathbf{T}})\right]\mathbf{I} \\ &+ \left[\frac{(\mathfrak{g}^{(1)} + \mathfrak{f}^{(3)})}{2}(\operatorname{tr}\dot{\boldsymbol{\varepsilon}})\mathbf{T} + \mathfrak{f}^{(3)}(\operatorname{tr}\boldsymbol{\varepsilon})\dot{\mathbf{T}}\right] + \overset{\cdot}{\overline{\mathfrak{p}_4\mathbf{I} + \mathfrak{p}_5\mathbf{T} + \mathfrak{p}_6\mathbf{T}^2}} = \mathbf{0}.\end{aligned} \tag{7.29}$$

In the above relation the number of material constants can be reduced if we divide the whole expression by $\mathfrak{g}^{(0)} - 1$ (assuming that $\mathfrak{g}^{(0)} \neq 1$). We do so and do not use a different notation for the remaining constants, and also for the derivatives of the functions $\mathfrak{p}$.

In order to obtain (3.4) we assume that

$$\mathfrak{g}^{(1)} = \mathfrak{f}^{(3)} = \mathfrak{q}^{(0)}, \tag{7.30}$$

where $\mathfrak{q}^{(0)}$ is a constant. In this case (7.29) can be integrated in time, and we obtain (3.4), where on the right side of (7.29) after integration we have a constant, which for simplicity we assume to be zero.

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