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Anisotropic turbulent viscosity and large-scale motive force in thermally driven turbulence at low Prandtl number

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The fully developed turbulent Boussinesq convection is known to form large-scale rolls, often termed the 'large-scale circulation' (LSC). It is an interesting question how such a large-scale flow is created, in particular in systems when the energy input occurs at small scales, when inverse cascade is required in order to transfer energy into the large-scale modes. Here, the small-scale driving is introduced through stochastic, randomly distributed heat source (say radiational). The mean flow equations are derived by means of simplified renormalization group technique, which can be termed a 'weakly nonlinear renormalization procedure' based on consideration of only the leading order terms at each step of the recursion procedure, as full renormalization in the studied anisotropic case turns out unattainable. The effective, anisotropic viscosity is obtained and it is shown that the inverse energy cascade occurs via an effective 'motive force' which takes the form of transient negative, vertical diffusion.

Key words: thermal convection, heat transport, Rayleigh number, Prandtl number, random heat source.

1. Introduction

The investigation of turbulent flows involves description of a very complicated nonlinear dynamics of small scale fluctuations, hence it is extremely difficult and requires sophisticated mathematical tools. In particular the emergence of large-scale coherent structures is a topic of interest, i.e. the transfer of energy from small scales to the large scales termed the inverse energy cascade. To simplify the problem various assumptions have been put forward and in particular a common simplification in the theoretical approaches is the assumption of the so-called weak turbulence, which corresponds to weak amplitude of turbulent pulsations and linearization of their evolution, cf. [1–3] where this idea has been thoroughly explained and related to the general context of fully developed turbulence; moreover, the latter references consider also the nonlinear wave turbulence with resonant wave interactions in the limit of a small interaction parameter. As argued in [4] or [5] in some cases the regime of weak turbulence can be sustained for long times, nevertheless, it is much more common for natural systems to develop into the strong turbulence regime, where the evolution of turbulent fluctuations becomes non-linear. An interested reader is directed to [6], for a review of various approaches and a physical description of turbulent flows, in particular chapter 10 of that book, concerned with buoyancy driven turbulence.

In order to treat the fully nonlinear regime and reliably estimate turbulent diffusion the renormalization technique has been developed and applied to strongly turbulent flows. This is a statistical closure approximation which is based on systematic, subsequent (iterative) elimination of thin wave-number bands from the Fourier spectrum of rapidly evolving variables (cf. [7–9]). Notable contributions come from [10–12] who have published comprehensive works on renormalization of hydrodynamic equations. This approach has also been applied recently to rotating and thus anisotropic turbulence by [13].

Another powerful method which allows to relate the turbulent diffusion to the turbulent energy tensor is the so-called two-scale direct-interaction approximation (TSDIA) dating back to Kraichnan [14, 15]. It involves introduction of a tensorial response function to an infinitesimal impulse-force and application of a two-scale approach in space and time related by the same parameter of expansion. Despite its limitations it allows to describe the turbulent viscosity in strong turbulence once the statistical properties of the underlying small-scale chaotic flow are known, see [16, 17] for a review.

Recent investigations of [18, 19] have involved applications of the renormalization group method to study the effect of non-stationarity and anisotropy on the magnetohydrodynamic turbulence in what could be called an 'intermediate regime of turbulence' or 'weakly nonlinear turbulence', in contrast to a simple, linear, weak turbulence regime. Due to high complexity of the mathematical approach in the case of non-stationary and non-isotropic turbulence the effect of nonlinear evolution of the fluctuations has been included only at leading order at each step of the renormalization procedure. As a result, although reliable estimates of the turbulent electromotive force could be made the wave number dependence of all the turbulent coefficients likewise of the energy and helicity spectra was not fully resolved. Because the problem of turbulent convection is also anisotropic due to action of vertical buoyancy, full renormalization of the Boussinesq equations turns out unattainable thus the latter approach corresponding to 'intermediate turbulence' is adopted here in order to study the physical mechanism of formation of the large-scale convection cells from a small-scale energy input. The nonlinear evolution of turbulent pulsations is thus included through calculation of leading order expressions for the effective turbulent viscosity at each step of the iterative renormalization procedure, which corresponds to expansion of the full, renormalized coefficients up to the first order in the Reynolds number. The turbulence is assumed to be driven by a small-scale, statistically random (Gaussian) heat source and the Prandtl number, that is the ratio of the viscosity to thermal diffusivity is assumed small, so that the temperature dynamics is dominated by rapid diffusion and the heat source. Renormalized dynamical equations for the mean flow are obtained which contain turbulent coefficients describing the net nonlinear effect of short-wavelength fluctuations, such as the turbulent viscosity and the turbulent coefficient describing the effective 'motive force' at large scales, which takes the form of negative vertical diffusion in the studied regime. The results correspond to a somewhat initial stage of formation of the large-scale convective flow, as the 'intermediate turbulence' regime is necessarily eventually destroyed and strong turbulence must emerge. It is worth mentioning that random heat sources are sometimes considered e.g. in the dynamics of dusty media as stochastically heated dust grains play an important role in transport of radiation (cf. [20]) and in the problem of stochastic heat engines (cf. [21]); also [22, 23] have designed and experimental set-up and studied the development of radiatively driven convection to the 'ultimate' turbulent regime defined by the scaling of the heat flux with square root of the Rayleigh number.

The dynamics of the turbulent Boussinesq convection involves formation of large-scale circulation (LSC) or the so-called 'wind of turbulence' (cf. [24–28]). In cylindrical geometry with comparable vertical and horizontal size, the LSC is believed to result from a quasi-two-dimensional, coupled horizontal sloshing and torsional (ST) oscillatory mode. Roche [29] studied the physics of transition to the 'ultimate state' of convection at a very high Rayleigh number (a measure of relative strength of the buoyancy force with respect to diffusion) and developed a model based on boundary layer stability. Zwirner *et al.* [30] suggested that such transitions could occur through the development of elliptical instabilities and showed that states with smaller amount of large-scale rolls built on top of each other transport heat more efficiently than states with more complex roll-structure. Vasiliev *et al.* [31] discovered for the first time spontaneous formation of large-scale azimuthal flow. Here, we analyse the system driven thermally by a random heat source and in that way we avoid the problem of boundary conditions and thus the effect of boundary layers; we do not study the LSC structure, but simply study the physical mechanism of LSC formation, i.e. derive a formula for the effective 'motive force' driving LSC, which is shown to take the form of negative vertical diffusion. This exact form is valid only as long as the regime of weak and 'intermediate' turbulence persists, that is at the initial and intermediate stages of evolution and turbulence development, but once the turbulence becomes strong the structure of the 'motive' force is expected to change. This is a fundamental study and the result sheds light on the physics of the process of energy transfer to large scales in thermal convection. The introduced simplification can be viewed as an advantage in the sense, that the problem of inverse turbulent energy cascade in convection is extracted and studied in isolation from the influence of velocity boundary conditions.

2. Dynamical equations and mathematical formulation

To study the thermally driven turbulence in an incompressible fluid we consider a fluid layer between two flat, parallel boundaries distant L apart, with gravity $\mathbf{g} = -g\hat{\mathbf{e}}_z$ pointing downwards and volume heat sources delivering heat to the system at a rate $\mathcal{Q}(\mathbf{x}, t)$. Such a system is governed by the following dynamical equations describing the evolution of the velocity field of the fluid flow $\mathbf{u}(\mathbf{x}, t)$ and the temperature field $T(\mathbf{x}, t)$ under the Boussinesq approximation (cf. [32, 33]):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \Pi + g\bar{\alpha}T\hat{\mathbf{e}}_z + \nu\nabla^2\mathbf{u}, \tag{2.1a}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa\nabla^2 T + Q, \tag{2.1b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1c}$$

where $\Pi = p/\bar{\rho}$ is the pressure divided by the mean density $\bar{\rho}$, $\kappa = k/\bar{\rho}\bar{c}_p = \text{const}$, is the thermal diffusivity of the fluid, uniform by the assumption (k is the fluid's thermal conductivity) and $\bar{T}$ is the mean temperature of the system which within the Boussinesq approximation is much greater than any temperature variations; $\bar{c}_p$, $\bar{\alpha}$ are the mean values of the specific heat at constant pressure and the coefficient of thermal expansion, respectively. The kinematic viscosity of the fluid is denoted by ν and $Q = \mathcal{Q}/\bar{\rho}\bar{c}_p$ is the heat source in K/s. The solenoidal constraint for the velocity field (2.1c) simply expresses the law of mass conservation. As typically done in the case of Boussinesq convection we have also assumed, that the adiabatic temperature gradient $g\bar{\alpha}\bar{T}/\bar{c}_p \ll \|\nabla T\|$ is much smaller than the typical temperature gradients in the fluid flow (typically about a thousand times smaller in laboratory flows, cf. e.g. [33]).

We concentrate on fluids with low Prandtl numbers:

$$\mathrm{Pr} = \frac{\nu}{\kappa} \ll 1, \tag{2.2}$$

such as e.g. liquid gallium and write down the equations in the following non-dimensional form:

$$\frac{\partial \mathbf{u}}{\partial t} + \epsilon(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \Pi + T\hat{\mathbf{e}}_z + \nabla^2\mathbf{u}, \tag{2.3a}$$

$$\mathrm{Pr}\,\frac{\partial T}{\partial t} + \mathrm{Pe}\,\mathbf{u} \cdot \nabla T = \nabla^2 T + Q, \tag{2.3b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.3c}$$

where the Reynolds number (Re =)ϵ and the Péclet number Pe are defined in a standard form:

$$\epsilon = \frac{UL}{\nu}, \qquad \mathrm{Pe} = \epsilon\,\mathrm{Pr} = \frac{UL}{\kappa}, \tag{2.4}$$

and we have chosen L^2/ν for the time scale, L for the spatial scale, $\nu U/g\bar{\alpha}L^2$ for the temperature scale, $\kappa\nu U/g\bar{\alpha}L^4$ for the heat source scale and finally pressure was scaled with $\bar{\rho}\nu U/L$. We seek for the form of the large-scale flow equations in the limit of 'intermediate turbulence' (described in general terms in the introduction and in detail below and in [18, 19]), through expansions in the Reynolds number ϵ; since $\mathrm{Pr} \ll 1$ it follows that the Péclet number must also be small, $\mathrm{Pe} \ll 1$. Hence the final set of equations describing convection at the low Prandtl number takes the form:

$$\frac{\partial \mathbf{u}}{\partial t} + \epsilon(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\Pi + T\hat{\mathbf{e}}_z + \nabla^2\mathbf{u}, \tag{2.5a}$$

$$\nabla^2 T = -Q, \qquad \nabla\cdot\mathbf{u} = 0. \tag{2.5b}$$

Furthermore, we introduce the Fourier transforms defined in the following way:

$$u_i(\mathbf{x},t) = \int_0^\Lambda \mathrm{d}^3k \int_{-\infty}^{\infty} \mathrm{d}\omega\, \hat{u}_i(\mathbf{k},\omega)\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{2.6a}$$

$$T(\mathbf{x},t) = \int_0^\Lambda \mathrm{d}^3k \int_{-\infty}^{\infty} \mathrm{d}\omega\, \hat{T}(\mathbf{k},\omega)\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{2.6b}$$

$$\Pi(\mathbf{x},t) = \int_0^\Lambda \mathrm{d}^3k \int_{-\infty}^{\infty} \mathrm{d}\omega \hat{\Pi}(\mathbf{k},\omega)\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{2.6c}$$

$$Q(\mathbf{x},t) = \int_0^\Lambda \mathrm{d}^3k \int_{-\infty}^{\infty} \mathrm{d}\omega \hat{Q}(\mathbf{k},\omega)\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{2.6d}$$

where according to the standard renormalization approach we have introduced the upper cut-off for the Fourier spectra Λ, which in natural systems appears due to enhanced energy dissipation at very small scales of turbulence.

The aim of this analysis is to study the large-scale flows in turbulent convection at low Pr. In order to model developed turbulence we assume a stochastic heat source, Gaussian with zero mean

$$\langle Q\rangle = 0, \tag{2.7}$$

statistically homogeneous and stationary, fully defined by the following correlation function

$$\langle \hat{Q}(\mathbf{k},\omega)\hat{Q}(\mathbf{k}',\omega')\rangle = \Xi(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'), \tag{2.8}$$

where the function $\Xi(k)$ is specified later and angular brackets $\langle\cdot\rangle$ denote the ensemble mean. Note that so-induced turbulence will be *anisotropic* because of action of vertical gravity (buoyancy force). We can calculate a positive definite quantity

$$\int k^2 \mathrm{d}\mathring{\Omega}_{\mathbf{k}} \int \mathrm{d}^4 q' \langle \hat{Q}(\mathbf{k},\omega)\hat{Q}(\mathbf{k}',\omega')\rangle = 4\pi k^2 \Xi(k) > 0, \tag{2.9}$$

where $\mathring{\Omega}_{\mathbf{k}}$ denotes a solid angle associated with the vector $\mathbf{k}$, which implies $\Xi(k) > 0$.

The approach is based on the renormalization group technique, which is an iterative procedure of successive elimination of thin wave-number bands from the Fourier spectrum of fluctuating fields. In this way the effect of thin bands of modes with shortest wavelengths on the remaining modes is calculated at each step of the procedure. The final aim of this approach is to obtain recursion equations for coefficients describing the effective mean Reynolds stress $\langle\mathbf{uu}\rangle$ as a function of the wave number at each step of the procedure. The Reynolds stresses are responsible for creation of the turbulent viscosity and what can be called a 'motive force' for the large-scale flows. The recursion equations (provided in (A.43)) are then solved for $k \to 0$ in order to obtain the final forms of the large-scale viscosity and the motive force which appears in the mean-field equations and includes the effect of nonlinear evolution of turbulent fluctuations.

3. The iterative weakly nonlinear renormalization procedure

Introducing new shorter four-component vector notation

$$\mathbf{q} = (\mathbf{k},\omega), \qquad \int_0^\Lambda \mathrm{d}^3 k \int_{-\infty}^{\infty} \mathrm{d}\omega(\cdot) = \int \mathrm{d}^4 q(\cdot), \tag{3.1}$$

so that e.g.

$$u_i(\mathbf{x},t) = \int \mathrm{d}^4 q \hat{u}_i(\mathbf{q}) \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{3.2}$$

the equations take the form:

$$(-\mathrm{i}\omega + k^2)\hat{u}_i(\mathbf{q}) - \hat{T}(\mathbf{q})\delta_{i3} + \mathrm{i}k_i\hat{\Pi} = -\mathrm{i}\epsilon k_j \mathbb{I}^{(u)}_{ij}(\mathbf{q}), \tag{3.3a}$$

$$k^2\hat{T}(\mathbf{q}) = \hat{Q}(\mathbf{q}), \qquad \mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{q}) = 0. \tag{3.3b}$$

In the above we have also defined the following convolution integral

$$\mathbb{I}_{ij}^{(u)} = \int \mathrm{d}^4 q' \hat{u}_i(\mathbf{q}-\mathbf{q}')\hat{u}_j(\mathbf{q}'), \tag{3.4}$$

which possesses the symmetry property

$$\mathbb{I}_{ij}^{(u)} = \mathbb{I}_{ji}^{(u)}. \tag{3.5}$$

It should be noted, at this stage, that the convolution integrals, which represent the non-linear interactions between fluctuating turbulent fields, are *not* neglected in the evolutional equations for fluctuations (3.3a, 3.3b), and their effect is included within the scope of the 'intermediate turbulence regime', based on the iterative renormalization procedure introduced in [10]. Thus we go beyond the weak turbulence regime and quantitatively express the effect of this nonlinearity on the dynamics of the large-scale flow. The main aim here is to study the development of large scale structures in turbulent convection, i.e. the effect of nonlinear evolution of the fluctuations is crucial. Of course full nonlinear solutions are not achievable. We need the assumption of $\epsilon \ll 1$ in order to make analytic progress, however, the situation is somewhat rectified by the fact, that under the specified assumptions, solving the final differential recursion relations obtained via the renormalization method has the meaning of contraction of the perturbational series in ϵ, thus it can be expected that the solutions may be valid also beyond the asymptotic limit of $\epsilon \ll 1$. However, contrary to the standard renormalization approach at each step of the iterative procedure based on a step-by-step elimination of thin wave number bands from the Fourier spectrum, only the terms of the leading order in ϵ will be retained; in such a way the Taylor series in ϵ will not be finally contracted as in the renormalization approaches, but instead we will obtain a weakly nonlinear expressions on effective coefficients appearing in the final 'renormalized' equation for the large-scale flow, i.e. up to the first order in ϵ.

In order to eliminate pressure we apply the projection operator

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{3.6}$$

to both sides of the Navier–Stokes equation (3.3a):

$$\gamma \hat{u}_i(\mathbf{q}) = \frac{P_{i3}}{k^2}\hat{Q}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}\mathbb{I}_{mn}^{(u)}(\mathbf{q}), \tag{3.7a}$$

$$\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{q}) = 0, \tag{3.7b}$$

where we have already introduced $\hat{T}(\mathbf{q}) = \hat{Q}(\mathbf{q})/k^2$ from (3.3b) and:

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k}), \tag{3.8}$$

$$\gamma = -\mathrm{i}\omega + k^2. \tag{3.9}$$

The smallness of the amplitude of turbulence $\epsilon \ll 1$ allows for a proper mathematical formulation of the problem, since the velocity field is expressed at the leading order by the driving $\hat{Q}$ and the nonlinearity, which is of the order $\mathcal{O}(\epsilon)$, can be treated in the perturbational sense. The iterational procedure of renormalization is then applicable. The final recursion differential relations for the coefficients describing the Reynolds stresses can be solved analytically, thus in particular the turbulent viscosity and the motive force for the large-scale flow can be determined.

We now start the iterative procedure of taking successive little bites off the Fourier spectrum from the short-wavelength side in order to obtain the final nonlinear effect of the fluctuations on the means. At the first step of the procedure we introduce the parameter λ_1, which satisfies

$$\delta\lambda = \Lambda - \lambda_1 \ll 1, \tag{3.10}$$

and divides the Fourier spectrum into two parts:

$$\text{(3.11a)}\quad \hat{u}_i^{>}(\mathbf{k},\omega) = \theta(k-\lambda_1)\hat{u}_i(\mathbf{k},\omega) \ \text{ or } \ \hat{u}_i^{>}(\mathbf{k},\omega) = \hat{u}_i(\mathbf{k}^{>},\omega),\ \lambda_1 < |\mathbf{k}^{>}| < \Lambda,$$
$$\text{(3.11b)}\quad \hat{u}_i^{<}(\mathbf{k},\omega) = \theta(\lambda_1-k)\hat{u}_i(\mathbf{k},\omega) \ \text{ or } \ \hat{u}_i^{<}(\mathbf{k},\omega) = \hat{u}_i(\mathbf{k}^{<},\omega),\ |\mathbf{k}^{<}| < \lambda_1,$$

and the same way for $\hat{Q}$. The equation for the field $\hat{u}_i^{<}(\mathbf{k},\omega)$ is obtained by averaging (3.7a) over the first shell ($\lambda_1 < k < \Lambda$)

$$(-\mathrm{i}\omega + k^2)\hat{u}_i^{<}(\mathbf{q}) = \frac{P_{i3}}{k^2}\hat{Q}^{<}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}\mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}(\mathbf{k})\int \mathrm{d}^4q'\langle \hat{u}_m^{>}(\mathbf{q}')\hat{u}_n^{>}(\mathbf{q}-\mathbf{q}')\rangle_c. \tag{3.12}$$

To get the equation for $\hat{u}_i^{>}(\mathbf{k},\omega)$ we utilize (3.7a) again

$$\hat{u}_i^{>}(\mathbf{q}) = \frac{P_{i3}}{k^2\gamma}\hat{Q}^{>}(\mathbf{q}) - \mathrm{i}\epsilon\frac{P_{imn}}{\gamma}\mathbb{J}_{mn}^{(u)}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon\frac{P_{imn}}{\gamma}\mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) + R_i, \tag{3.13}$$

where $\langle\cdot\rangle_c$ denotes conditional average over the first shell ($\lambda_1 < k < \Lambda$) statistical subensemble, described at the beginning of the Appendix (cf. [34–36]); furthermore, we have defined

$$\mathbb{J}_{mn}^{(u)}(\mathbf{q}) = \int \mathrm{d}^4q'\hat{u}_m^{<}(\mathbf{q}')\hat{u}_n^{>}(\mathbf{q}-\mathbf{q}'), \tag{3.14}$$

and the rest in (3.13) is given by

$$R_i^{(u)} = -\frac{1}{2}\mathrm{i}\epsilon\frac{P_{imn}}{\gamma}\mathbb{I}_{mn}^{(u^>)}. \tag{3.15}$$

The rests is neglected on the basis of generating triple order statistical correlations since they involve only terms second order in $\mathbf{u}^{>}$ or because of the kept order of accuracy in the asymptotic limit $\epsilon \ll 1$, which allows to neglect terms of the order $\mathcal{O}(\epsilon^3)$; for details the reader is referred to the Appendix.

In what follows we provide a short description of the asymptotic iterative procedure, described in detail in the Appendix. Firstly, we introduce (3.13) into (3.12) and calculate the dynamical effect of short-wavelength components $\hat{u}_i^>(\mathbf{k},\omega)$ on the evolution of $\hat{u}_i^<(\mathbf{k},\omega)$ (long wavelength modes). This results in corrections to some of the terms in Eq. (3.12), but also generates terms with a new structure. Therefore, a next step is necessary, involving calculation of the effect of the next shell $\lambda_2 = \lambda_1 - \delta\lambda < k < \lambda_1$ (new short-wavelength modes) on the modes with $k < \lambda_1 - \delta\lambda$ (new long wavelength modes); this is continued until invariance of the equations for long-wavelength modes is achieved, i.e. the equations do not change from one iterational step to the next one. We can then take the limit of infinitesimally narrow wave number bands $\delta\lambda \to 0$, which leads to differential recursion relations for all the coupling constants introduced into the equations for long wavelength modes by couplings of the short wavelength ones. In the isothermal, fully isotropic case YAKHOT and ORSZAG [10] calculated the correction from short wavelength modes in the Navier–Stokes equation which was proportional to $k^2\hat{u}_i^<\delta\lambda$ thus creating viscosity correction; the turbulent viscosity was then obtained from an equation of the form $\mathrm{d}\nu_{turb}/\mathrm{d}\lambda = f(\lambda)$ with an 'initial' condition $\nu_{\mathit{eff}}(\lambda = \Lambda) = \nu$. The case at hand is *anisotropic* because of the vertical gravity (buoyancy force); explicit calculation of two initial steps of the renormalization procedure is enough to derive the final differential recursion relations with satisfactory accuracy. The details of the procedure are provided in the Appendix, however, we note here that at each step of the procedure the contributions from the velocity gradients are included up to the order $(k/k')^2$, where k' belongs to the thin wave number band considered at this step. In other words, although the heat source (2.8) is homogeneous, the non-homogeneity resulting from mean velocity gradients and the anisotropy due to the effect of buoyancy are included in the renormalization procedure. In turn, the resulting motive force is anisotropic and proportional to the velocity gradients.

4. Dynamics of the large-scale flow

The mean-field equations are derived in the Appendix A and take the following form:

$$\begin{aligned} \text{(4.1a)} \qquad & \frac{\partial\langle\mathbf{U}\rangle}{\partial t} + (\langle\mathbf{U}\rangle\cdot\nabla)\langle\mathbf{U}\rangle \\ & \quad = -\nabla\langle\Pi\rangle - 9\varsigma\nabla^2\langle U\rangle_z\hat{\mathbf{e}}_z + (\nu+6\varsigma)\nabla_h^2\langle\mathbf{U}\rangle + (\nu+4\varsigma)\frac{\partial^2\langle\mathbf{U}\rangle}{\partial z^2}, \\ \text{(4.1b)} \qquad & \nabla\cdot\langle\mathbf{U}\rangle = 0, \end{aligned}$$

where the coefficient

$$\varsigma = \frac{2\pi^2}{735}\frac{g^2\bar{\alpha}^2L^3\mathbb{Q}^2}{\nu^3\kappa^2k_\ell^7}, \tag{4.2}$$

includes the effect of the turbulent fluctuations on the means; in the above $\mathbb{Q}$ is the magnitude of heat delivery rate (in K/s) and k_ℓ is the wave number based on the length scale of most energetic turbulent eddies. The general differential recursion relations for the turbulent coefficients are solved in the Appendix A, see (A.43a–c). It is evident that the term $-9\varsigma\nabla^2\langle U\rangle_z\hat{\mathbf{e}}_z$ is the large-scale motive force responsible for energy transfer from small scales to large scales, i.e. the inverse energy cascade; it takes the form of negative diffusion in the vertical direction, which drives the large-scale flow.

For the sake of a rough estimate we may take the Kolmogorov cut-off value $\Lambda \sim (\sqrt{gL}/\nu)^{3/4}L^{-1/4}$, where the free-fall velocity $\sqrt{gL}$ was used as the convective velocity scale, which yields $\varsigma \sim \nu\mathcal{G}^{-13/4}\mathcal{H}^2(k_\ell/\Lambda)^{-7}$, with $\mathcal{H} = g^{1/2}\bar{\alpha}L^{7/2}\mathbb{Q}/\kappa\nu$ and $\mathcal{G} = \sqrt{gL^3/\nu^2}$. Since $k_\ell/\Lambda \ll 1$, this coefficient is expected to be much larger than the molecular viscosity, in particular when $\mathcal{H}$ exceeds unity. It is, in fact a typical situation when the turbulent coefficients greatly exceed the molecular ones.

The turbulent coefficient (4.2) can also be expressed in terms of the Rayleigh number defined as $\mathrm{Ra}_\ell = g\bar{\alpha}\mathbb{T}\ell^3/\kappa\nu$ with $\mathbb{T} = \mathbb{Q}\ell^2/\kappa$, which is a non-dimensional measure of the driving force (the magnitude of the heat source) and is based on the length scale of the most energetic eddies ℓ. This yields

$$\varsigma \sim \nu\,\mathrm{Ra}_\ell^2\,\mathrm{Pr}^{-2}\left(\frac{L}{\ell}\right)^3, \tag{4.3}$$

hence the turbulent coefficient ς scales with the ratio $\mathrm{Ra}_\ell\,/\,\mathrm{Pr}$ to the power of two. This is a rather strong dependence, especially in the considered limit of a low Prandtl number.

4.1. Linear regime

Linearization of (4.1a) and substitution of normal modes in the form:

$$\langle U\rangle_z = \mathrm{e}^{\sigma t}\cos(\mathbf{K}_h\cdot\mathbf{x})\sin(K_z z)\hat{U}_z, \tag{4.4a}$$

$$\langle\mathbf{U}\rangle_h = \mathrm{e}^{\sigma t}\sin(\mathbf{K}_h\cdot\mathbf{x})\cos(K_z z)\hat{\mathbf{U}}, \tag{4.4b}$$

$$\mathbf{K}\cdot\hat{\mathbf{U}} = 0, \tag{4.4c}$$

$$\langle\Pi\rangle = \mathrm{e}^{\sigma t}\cos(\mathbf{K}_h\cdot\mathbf{x})\cos(K_z z)\hat{\mathbf{\Pi}}, \tag{4.4d}$$

leads to

$$\sigma\langle\mathbf{U}\rangle = -\mathrm{i}\mathbf{K}\hat{\Pi} - \nabla\langle\Pi\rangle + 9\varsigma K^2\langle U\rangle_z\hat{\mathbf{e}}_z - (\nu+6\varsigma)K_h^2\langle\mathbf{U}\rangle - (\nu+4\varsigma)K_z^2\langle\mathbf{U}\rangle, \tag{4.5}$$

which under action of the projection operator $\mathbf{1} - \mathbf{KK}/K^2$ where $\mathbf{1}$ is the unity matrix transforms into

$$(4.6)\quad \sigma\langle U\rangle_i = 9\varsigma K^2\left(\delta_{i3} - \frac{K_i K_z}{K^2}\right)\langle U\rangle_z - (\nu + 6\varsigma)K_h^2\langle U\rangle_i - (\nu + 4\varsigma)K_z^2\langle U\rangle_i,$$

so that the pressure is eliminated. This yields for the growth rate of the mean flow

$$(4.7)\qquad \sigma = (3\varsigma - \nu)K_h^2 - (\nu + 4\varsigma)K_z^2.$$

It follows that in turbulent convection driven by strong stochastic heat sources $\mathcal{H} \gg 1$ the growth rate takes the approximate form

$$(4.8)\qquad \sigma \approx 3\varsigma K_h^2 - 4\varsigma K_z^2,$$

and thus turbulence excites large-scale modes with horizontal wavelengths shorter than vertical ones,

$$(4.9)\qquad \sigma > 0 \;\Leftrightarrow\; K_z^2 < \frac{3}{4}K_h^2 \;\Leftrightarrow\; L_h < \frac{\sqrt{3}}{2}L_z \approx 0.87L_z.$$

In other words in the studied problem the large scale flow is expected to form vertically elongated rolls. The growth rate increases unboundedly with K_h^2, but since the Eq. (4.1a) describes the large-scale flow only, there is a natural upper bound on the horizontal wave number of the large-scale modes and thus on the growth rate. As argued in the Appendix B there also exists an additional term on the r.h.s. of the mean flow Eq. (4.1a) of the form $\varrho\partial_z^2\langle U\rangle_z\hat{\mathbf{e}}_z$, which is of smaller (asymptotically negligible) magnitude than the other turbulent terms proportional to the coefficient ς, i.e. $\varrho < \varsigma$. Thus the growth rate is modified to $\sigma = 3\varsigma K_h^2 - 4\varsigma K_z^2 - \varrho\frac{K_z^2 K_h^2}{K^2}$, but the sign of the turbulent coefficient ϱ remains undetermined, hence it is not clear whether it acts as additional diffusion (if $\varrho > 0$) or additional motive force (if $\varrho < 0$).

The normal modes in the form (4.4a) are individually also solutions of the nonlinear Eq. (4.1a). Of course in the discussed problem, the energy is transferred from the small scale fluctuations, where the flow is thermally driven, to the large-scale modes and thus in the limit of small K the dynamics naturally involves wave packets, rather than individual modes, which evolve nonlinearly. Still it is possible for the most unstable modes to dominate the dynamics, in which case the amplitude of convection grows unboundedly in time until the initial assumption of a small Reynolds number ceases to be valid and saturation may occur. In other words, the analysis of weakly nonlinear turbulence does not lead to saturation of large-scale modes, which is possible only beyond the scope of this approach, that is in fully developed, strong turbulence.

5. Conclusions

The presented analysis was focused on derivation of the effective equation describing the dynamics of the large-scale flow (circulation) in turbulent convection driven by a random heat source at low Pr. The applied technique was based on the renormalization approach of [10] and [34] (see also [11] for a review of the method), which allowed to incorporate the effect of the nonlinear terms in the dynamical equations for small-scale turbulent fluctuations, and calculate the anisotropic turbulent viscosity and 'motive force' induced by the fluctuations and experienced by the large-scale flows. The renormalized mean-flow equation was derived and it was shown that the 'motive force' acts in the form of negative vertical diffusion, $\partial_t\langle\mathbf{U}\rangle = -9\varsigma\nabla^2\langle U\rangle_z\hat{\mathbf{e}}_z + \cdots$, where the turbulent coefficient ς is given in (4.2), leading to enhancement of the mean flow energy. The general recursion differential equations for all the turbulent coefficients are provided in (A.40) for any form of the random heat-source function $\Xi(k)$. This motive force transfers energy from the small-scale fluctuations into the large-scale flow, i.e. drives formation of the large-scale cells. In other words the physics of the energy transfer from small scales to large ones during the process of the development of turbulent convection can be understood via formation of an effective, transient negative diffusion. However, such a structure of the motive force, in the form of negative, anisotropic diffusion is eventually destroyed when the amplitude of turbulence grows and the regime of fully nonlinear strong turbulence is reached.

Data availability statement: all data generated or analysed during this study are included in this published article.

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Appendix A. Details of the iterative, weakly nonlinear renormalization procedure

The details of the renormalization procedure applied in order to obtain the mean field equations are given in here. First of all we clarify how the ensemble averaging should be understood and explain the concept of a conditional average over a statistical subensemble for short-wavelength modes. We adopt the approach of McComb *et al.* [34] (cf. also [35, 36]). The essential idea of this approach is the introduction of a subensemble of flow realizations including near-chaotic statistical properties for the short-wavelength shell $\lambda_1 < k \leq \Lambda$, but remaining quasi-deterministic for $k \leq \lambda_1$. The subensemble average can be precisely defined and then, utilizing the assumption, that in the turbulent cascade the energy transfer in the Fourier space is local (i.e. the assumption of ergodicity of the system), the following crucial properties can be proved:

$$\langle \hat{\mathbf{u}}^{<}(\mathbf{q})\rangle_c = \hat{\mathbf{u}}^{<}(\mathbf{q}), \quad \langle \hat{\mathbf{u}}^{<}(\mathbf{q})\hat{\mathbf{u}}^{<}(\mathbf{q}')\rangle_c \approx \hat{\mathbf{u}}^{<}(\mathbf{q})\hat{\mathbf{u}}^{<}(\mathbf{q}'), \tag{A.1a}$$

$$\langle \hat{\mathbf{u}}^{>}(\mathbf{q}')\rangle_c \approx \langle \hat{\mathbf{u}}^{>}(\mathbf{q}')\rangle = 0, \quad \langle \hat{\mathbf{u}}^{<}(\mathbf{q})\hat{\mathbf{u}}^{>}(\mathbf{q}')\rangle_c \approx \hat{\mathbf{u}}^{<}(\mathbf{q})\langle \hat{\mathbf{u}}^{>}(\mathbf{q}')\rangle_c \approx 0, \tag{A.1b}$$

$$\langle \hat{\mathbf{u}}^{>}(\mathbf{q})\hat{\mathbf{u}}^{>}(\mathbf{q}')\rangle_c \approx \langle \hat{\mathbf{u}}^{>}(\mathbf{q})\hat{\mathbf{u}}^{>}(\mathbf{q}')\rangle. \tag{A.1c}$$

For details see particularly section IV and the beginning of section V in [34].

We now substitute the expressions for short wavelength modes from (3.13) into the conditional averages in the equations for long wavelength modes in (3.12). Neglecting higher order correlations of the type $\langle \hat{u}_i^{>}\hat{u}_j^{>}\hat{Q}^{>}\rangle_c$ etc. (which eliminates the rests in (3.13)) and using $\langle \hat{Q}^{>}\rangle_c = 0$ and $\langle \hat{u}_i^{<}\rangle_c = \hat{u}_i^{<}$ one obtains

$$\begin{aligned}\langle \hat{u}_m^{>}(\mathbf{q}')\hat{u}_n^{>}(\mathbf{q}-\mathbf{q}')\rangle_c &\\ = \frac{P_{m3}(\mathbf{k}')P_{n3}(\mathbf{k}-\mathbf{k}')}{k'^2|\mathbf{k}-\mathbf{k}'|^2\gamma(\mathbf{q}')\gamma(\mathbf{q}-\mathbf{q}')}&\langle \hat{Q}^{>}(\mathbf{q}')\hat{Q}^{>}(\mathbf{q}-\mathbf{q}')\rangle_c\\ - \frac{\mathrm{i}\epsilon P_{m3}(\mathbf{k}')}{k'^2\gamma(\mathbf{q}')}\frac{P_{npq}(\mathbf{k}-\mathbf{k}')}{\gamma(\mathbf{q}-\mathbf{q}')}&\langle \hat{Q}^{>}(\mathbf{q}')\mathbb{J}_{pq}^{(u)}(\mathbf{q}-\mathbf{q}')\rangle_c\\ - \frac{\mathrm{i}\epsilon P_{n3}(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2\gamma(\mathbf{q}-\mathbf{q}')}\frac{P_{mpq}(\mathbf{k}')}{\gamma(\mathbf{q}')}&\langle \hat{Q}^{>}(\mathbf{q}-\mathbf{q}')\mathbb{J}_{pq}^{(u)}(\mathbf{q}')\rangle_c + \mathcal{O}(\epsilon^2).\end{aligned} \tag{A.2}$$

The first term in (A.2) is proportional to $\langle \hat{Q}^{>}(\mathbf{q}')\hat{Q}^{>}(\mathbf{q}-\mathbf{q}')\rangle_c \sim \delta(\mathbf{k})\delta(\omega)$, hence on taking the inverse Fourier transform of $\frac{1}{2}\mathrm{i}\epsilon P_{imn}(\mathbf{k})\int \mathrm{d}^4q'\langle \hat{u}_m^{>}(\mathbf{q}')\hat{u}_n^{>}(\mathbf{q}-\mathbf{q}')\rangle_c$ in the Navier–Stokes equation to return to the real space, this term vanishes and thus does not contribute to the dynamics of large-scale fields; it follows that this term is disregarded. Substituting once again for $\hat{\mathbf{u}}^{>}$ from (3.13) into the $\mathbb{J}^{(u)}$-terms in (A.2) and making use of the symmetry $\mathbf{q}' \mapsto \mathbf{q}-\mathbf{q}'$ under the integral $\int \mathrm{d}^4q'$ one obtains

$$\begin{aligned}&\int \mathrm{d}^4q'\langle \hat{u}_m^{>}(\mathbf{q}')\hat{u}_n^{>}(\mathbf{q}-\mathbf{q}')\rangle_c\\ &= -\mathrm{i}\epsilon\int \mathrm{d}^4q'\\ &\times\int \mathrm{d}^4q''\frac{\hat{u}_p^{<}(\mathbf{q}'')P_{m3}(\mathbf{k}')P_{npq}(\mathbf{k}-\mathbf{k}')P_{q3}(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')}{k'^2|\mathbf{k}-\mathbf{k}'-\mathbf{k}''|^2\gamma(\mathbf{q}')\gamma(\mathbf{q}-\mathbf{q}')\gamma(\mathbf{q}-\mathbf{q}'-\mathbf{q}'')}\langle \hat{Q}^{>}(\mathbf{q}')\hat{Q}^{>}(\mathbf{q}-\mathbf{q}'-\mathbf{q}'')\rangle_c\\ &+(m\leftrightarrow n)+\mathcal{O}(\epsilon^2),\end{aligned} \tag{A.3}$$

where $(m \leftrightarrow n)$ in (A.3) denotes a term of the same structure as the previous one but with exchanged indices m and n. We can now substitute for the heat-source correlations

$$\langle \hat{Q}(\mathbf{k},\omega)\hat{Q}(\mathbf{k}',\omega')\rangle = \Xi(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'), \tag{A.4}$$

cf. (2.8), into (A.3) and perform the $\mathbf{q}''$ integral which yields

$$\begin{aligned}\int \mathrm{d}^4q' \langle \hat{u}^{>}_m(\mathbf{q}')\hat{u}^{>}_n(\mathbf{q}-\mathbf{q}')\rangle_c \\ = -\mathrm{i}\epsilon\hat{u}^{<}_p(\mathbf{q})\int \mathrm{d}^4q' \frac{\Xi(k')P_{m3}(\mathbf{k}')P_{npq}(\mathbf{k}-\mathbf{k}')P_{q3}(\mathbf{k}')}{k'^4|\gamma(\mathbf{q}')|^2\gamma(\mathbf{q}-\mathbf{q}')} + (m\leftrightarrow n) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{A.5}$$

The q'-integrals are taken over an intersection of the domains $\lambda_1 < k' < \Lambda$ and $\lambda_1 < |\mathbf{k}-\mathbf{k}'| < \Lambda$, i.e.

$$\{\mathbf{k}' :\ \lambda_1 < k' < \Lambda,\ \lambda_1 < |\mathbf{k}-\mathbf{k}'| < \Lambda\}. \tag{A.6}$$

Following the approach of YAKHOT and ORSZAG [10] and SMITH and WOODRUFF [11] we calculate the q'-integrals to lowest nontrivial order in the distant-interaction limit

$$\frac{k}{k'} \to 0, \qquad \frac{\omega}{\omega'} \to 0, \tag{A.7}$$

which stems from the assumption of local energy transfer in the Fourier spectrum of a turbulent cascade. The integrals are then calculated by setting $\omega = 0$ and substitution $\mathbf{k}' \mapsto \mathbf{k}'+\mathbf{k}/2$ hence by symmetrization of the integration domain; in the case at hand, when the zeroth order term $\sim (k/k')^0$ vanishes no corrections of the order k (and higher) from the integration domain are then necessary, and it simplifies to

$$\{\mathbf{k}' :\ \lambda_1 < k' < \Lambda\}. \tag{A.8}$$

This way the total renormalized corrections from short-wavelength modes are proportional to k^2, which implies that the lowest non-trivial order in distant interactions produces corrections to diffusivities.

Therefore the corrections from short-wavelength modes to the equations for long wavelength fluctuations in (A.5) can be expressed as follows. Making the aforementioned substitution $\mathbf{k}' \to \mathbf{k}' + \frac{1}{2}\mathbf{k}$ to symmetrize the domain of integration one obtains

$$\begin{aligned}&\int \mathrm{d}^4q' \langle \hat{u}^{>}_m(\mathbf{q}')\hat{u}^{>}_n(\mathbf{q}-\mathbf{q}')\rangle_c \\ &= -\mathrm{i}\epsilon\hat{u}^{<}_p(\mathbf{q})\int_{\lambda_1}^{\Lambda}\mathrm{d}k'\int \mathrm{d}\mathring{\Omega}\int_{-\infty}^{\infty}\mathrm{d}\omega' \frac{\Xi(|\mathbf{k}'+\frac{1}{2}\mathbf{k}|)P_{m3}(\mathbf{k}'+\frac{1}{2}\mathbf{k})P_{npq}(\frac{1}{2}\mathbf{k}-\mathbf{k}')P_{q3}(\mathbf{k}'+\frac{1}{2}\mathbf{k})}{(k'^2+k'_r k_r)|\gamma(\mathbf{k}'+\frac{1}{2}\mathbf{k},\omega')|^2\gamma(\mathbf{k}'-\frac{1}{2}\mathbf{k},-\omega')} \\ &\quad + (m\leftrightarrow n) + \mathcal{O}(\epsilon^2),\end{aligned} \tag{A.9}$$

where $\mathring{\Omega}$ denotes a solid angle.

Next we use the symmetry property such that $\int_{-\infty}^{\infty}\omega' f_s(\omega')\,\mathrm{d}\omega' = 0$ for any function $f_s(\omega')$ symmetric about $\omega' = 0$ and the following expansions in k/k' up to the first order:

$$\text{(A.10a)}\qquad P_{mp}\left(\mathbf{k}'+\frac{1}{2}\mathbf{k}\right) = P_{mp}(\mathbf{k}') + \frac{k'_m k'_p k'_r}{k'^4}k_r - \frac{k'_m k_p + k_m k'_p}{2k'^2} + \mathcal{O}(k^2),$$

$$\text{(A.10b)}\qquad P_{nqp}\left(\frac{1}{2}\mathbf{k}-\mathbf{k}'\right) = -P_{nqp}(\mathbf{k}') + 2\frac{k'_n k'_p k'_q k'_r}{k'^4}k_r - \frac{k'_n k'_q k_p + k'_n k'_p k_q + 2k_n k'_p k'_q}{2k'^2} + \frac{1}{2}k_q P_{np}(\mathbf{k}') + \frac{1}{2}k_p P_{nq}(\mathbf{k}') + \mathcal{O}(k^2),$$

$$\text{(A.10c)}\qquad \frac{1}{\left|\gamma\left(\mathbf{k}'+\frac{1}{2}\mathbf{k},\omega'\right)\right|^2} = \frac{1}{\omega'^2 + k'^4 + 2k'^2 k'_t k_t} = \frac{1}{\omega'^2+k'^4} - \frac{2k'^2 k'_t k_t}{(\omega'^2+k'^4)^2} + \mathcal{O}(k^2),$$

$$\text{(A.10d)}\qquad \Xi\left(\left|\mathbf{k}'+\frac{1}{2}\mathbf{k}\right|\right) = \Xi\left(k' + \frac{1}{2k'}\mathbf{k}'\cdot\mathbf{k} + \mathcal{O}(k^2)\right) = \Xi(k') + \frac{1}{2k'}k'_t k_t\frac{\partial\Xi}{\partial k'} + \mathcal{O}(k^2),$$

which yields

$$\text{(A.11)}\qquad P_{imn}\int \mathrm{d}^4 q'\langle \hat{u}^{>}_m(\mathbf{q}')\hat{u}^{>}_n(\mathbf{q}-\mathbf{q}')\rangle_c = -2\mathrm{i}\epsilon P_{imn}\hat{u}^{<}_p(\mathbf{q})\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^2}\int \mathrm{d}\mathring{\Omega} \times\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{[k'^2\Xi(k') - 2k'_t k_t(\Xi(k') - \frac{1}{4}k'\frac{\partial\Xi}{\partial k'})]D_{npm}(\mathbf{k}',\mathbf{k})}{(\omega'^2+k'^4)^2} + \mathcal{O}(\epsilon^2),$$

with

$$\text{(A.12)}\qquad D_{npm}(\mathbf{k}',\mathbf{k}) = \left[-k'_p P_{n3}(\mathbf{k}') + \delta_{np}k_z - \frac{k'_p k'_n}{k'^2}k_z - \delta_{np}\frac{k'_z k'_q}{k'^2}k_q + \frac{k'_p k'_z}{k'^2}k_n\right]P_{m3}(\mathbf{k}').$$

Furthermore, by the use of:

$$\text{(A.13a)}\qquad P_{imn}(\mathbf{k})\delta_{mn} = 0,\quad k_p\hat{u}^{<}_p(\mathbf{q}) = 0,$$

$$\text{(A.13b)}\qquad \int \mathrm{d}\mathring{\Omega}\underbrace{k_m\ldots k_n k_k}_{N} = 0,\quad \text{for any odd } N \text{ and all } m,\ldots,n,\,k,$$

(A.13c) $$\int \mathrm{d}\mathring{\Omega}\frac{k_m k_n}{k^2} = \frac{4\pi}{3}\delta_{mn},$$

(A.13d) $$\int \frac{k_j k_n}{k^2}\cos^2\theta \mathrm{d}\mathring{\Omega} = \frac{4\pi}{15}(\delta_{jn} + 2\delta_{j3}\delta_{n3}),$$

(A.13e) $$\int \mathrm{d}\mathring{\Omega}\frac{k_m k_n k_p k_q}{k^4} = \frac{4\pi}{15}(\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}),$$

(A.13f) $$\int \frac{k_t k_p k_m k_n}{k^4}\cos^2\theta \mathrm{d}\mathring{\Omega} = \frac{4\pi}{105}(\delta_{tp}\delta_{mn} + \delta_{tm}\delta_{pn} + \delta_{tn}\delta_{pm}) + \frac{8\pi}{105}(\delta_{tp}\delta_{m3}\delta_{n3} + \delta_{tm}\delta_{p3}\delta_{n3} + \delta_{tn}\delta_{p3}\delta_{m3} + \delta_{pm}\delta_{t3}\delta_{n3} + \delta_{pn}\delta_{t3}\delta_{m3} + \delta_{mn}\delta_{t3}\delta_{p3}),$$

(A.13g) $$\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{(\omega'^2 + k'^4)^2} = \frac{\pi}{2k'^6},$$

where θ is the polar angle in spherical coordinates (k, θ, ϕ) one obtains

(A.14)
$$\begin{aligned}
&-\frac{1}{2}\mathrm{i}\epsilon P_{imn}\int \mathrm{d}^4 q' \langle \hat{u}^{>}_m(\mathbf{q}')\hat{u}^{>}_n(\mathbf{q}-\mathbf{q}')\rangle_c \\
&= -\epsilon^2\frac{2\pi^2}{105}P_{imn}\hat{u}^{<}_p(\mathbf{q}) \\
&\quad\times \int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^6}\left\{\left[4\Xi(k') - k'\frac{\partial\Xi}{\partial k'}\right]\delta_{pn}k_m + \left[8\Xi(k') + 5k'\frac{\partial\Xi}{\partial k'}\right]\delta_{pn}\delta_{m3}k_z\right\} \\
&\quad - \epsilon^2\frac{2\pi^2}{105}P_{imn}\hat{u}^{<}_p(\mathbf{q})\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^6}\left[\Xi(k') + 5k'\frac{\partial\Xi}{\partial k'}\right]\delta_{p3}\delta_{m3}k_n + \mathcal{O}(\epsilon^3) \\
&= -\epsilon^2\frac{2\pi^2}{105}\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^6}\left[4\Xi(k') - k'\frac{\partial\Xi}{\partial k'}\right]k^2\hat{u}^{<}_i(\mathbf{q}) \\
&\quad - \epsilon^2\frac{2\pi^2}{105}\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^6}\left[8\Xi(k') + 5k'\frac{\partial\Xi}{\partial k'}\right]k_z^2\hat{u}^{<}_i(\mathbf{q}) \\
&\quad - \epsilon^2\frac{2\pi^2}{105}\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^6}\left[\Xi(k') + 5k'\frac{\partial\Xi}{\partial k'}\right]P_{i3}k^2\hat{u}^{<}_z(\mathbf{q}) + \mathcal{O}(\epsilon^3).
\end{aligned}$$

We now utilize the assumption of narrowness of the first spectral bite $\Lambda - \lambda_1 = \delta\lambda \ll 1$ and define the following coefficients which describe the average effect of the short-wavelength fluctuations with wave numbers from the narrow band

$\lambda_1 \le k \le \Lambda$ on the long-wavelength fluctuations corresponding to the band $0 < k < \lambda_1$ (cf. (A.14)):

$$\breve{\xi}(\lambda_1) = 1 + \epsilon^2 \frac{2\pi^2}{105} \frac{\delta\lambda}{\lambda_1^6} \left[4\Xi(\lambda_1) - \lambda_1 \frac{\partial \Xi}{\partial \lambda}(\lambda_1) \right], \tag{A.15a}$$

$$\breve{\zeta}(\lambda_1) = \epsilon^2 \frac{2\pi^2}{105} \frac{\delta\lambda}{\lambda_1^6} \left[8\Xi(\lambda_1) + 5\lambda_1 \frac{\partial \Xi}{\partial \lambda}(\lambda_1) \right], \tag{A.15b}$$

$$\breve{\chi}(\lambda_1) = \epsilon^2 \frac{2\pi^2}{105} \frac{\delta\lambda}{\lambda_1^6} \left[\Xi(\lambda_1) + 5\lambda_1 \frac{\partial \Xi}{\partial \lambda}(\lambda_1) \right]. \tag{A.15c}$$

With the use of those definitions we can write down the dynamical equation (3.12) in the new form, with the effect of the short-wavelength modes $\mathbf{u}^{>}$ expressed through the effective Reynolds stresses (anisotropic turbulent viscosity)

$$\begin{aligned} \left[-\mathrm{i}\omega + \breve{\xi}(\lambda_1)k^2 + \breve{\zeta}(\lambda_1)k_z^2\right] \hat{u}_i^{<}(\mathbf{q}) + \breve{\chi}(\lambda_1) P_{i3} k^2 \hat{u}_z^{<}(\mathbf{q}) \\ = \frac{P_{i3}}{k^2} \hat{Q}^{<}(\mathbf{q}) - \frac{1}{2} \mathrm{i}\epsilon P_{imn} \mathbb{I}_{mn}^{(u^<)}(\mathbf{q}). \end{aligned} \tag{A.16}$$

In order to proceed to the second step of the procedure we introduce a short notation

$$\breve{\gamma} = -\mathrm{i}\omega + \breve{\xi}(\lambda_1)k^2 + \breve{\zeta}(\lambda_1)k_z^2, \tag{A.17}$$

which yields

$$\hat{u}_i^{<}(\mathbf{q}) + \frac{\breve{\chi}(\lambda_1)k^2}{\breve{\gamma}} P_{i3} \hat{u}_z^{<}(\mathbf{q}) = \frac{P_{i3}}{k^2 \breve{\gamma}} \hat{Q}^{<}(\mathbf{q}) - \frac{1}{2} \mathrm{i}\epsilon \frac{P_{imn}}{\breve{\gamma}} \mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) \stackrel{\text{def}}{=} r.h.s._i. \tag{A.18}$$

Since

$$\hat{u}_z^{<}(\mathbf{q}) = \frac{\mathbf{k}}{k_z} \cdot [\hat{\mathbf{e}}_z \times (\hat{\mathbf{e}}_z \times \hat{\mathbf{u}}^{<}(\mathbf{q}))], \tag{A.19}$$

we may take the cross-product of (A.18) with $\hat{\mathbf{e}}_z$ twice and then the dot-product with $\mathbf{k}$ to obtain

$$\hat{u}_z^{<}(\mathbf{q}) = \frac{\breve{\gamma}}{(\breve{\gamma} + \breve{\chi}(\lambda_1)k_h^2)} \frac{\mathbf{k}}{k_z} \cdot [\hat{\mathbf{e}}_z \times (\hat{\mathbf{e}}_z \times \mathbf{r.h.s})] = \frac{\breve{\gamma}}{\breve{\gamma} + \breve{\chi}(\lambda_1)k_h^2} r.h.s._z, \tag{A.20}$$

where $k_h^2 = k^2 - k_z^2$. This leads to

$$\begin{aligned} \hat{u}_i^{<}(\mathbf{q}) = \frac{P_{i3}}{k^2 \breve{\gamma}} \hat{Q}^{<}(\mathbf{q}) - \frac{1}{2} \mathrm{i}\epsilon \frac{P_{imn}}{\breve{\gamma}} \mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) \\ - \frac{\breve{\chi}(\lambda_1)k^2 P_{i3}}{\breve{\gamma} + \breve{\chi}(\lambda_1)k_h^2} \left[\frac{1 - \frac{k_z^2}{k^2}}{k^2 \breve{\gamma}} \hat{Q}^{<}(\mathbf{q}) - \frac{1}{2} \mathrm{i}\epsilon \frac{P_{3mn}}{\breve{\gamma}} \mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) \right], \end{aligned} \tag{A.21}$$

and hence

$$\text{(A.22)}\quad \hat{u}_i^<(\mathbf{q}) = \frac{P_{i3}}{k^2\breve{\Gamma}}\hat{Q}^<(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon\frac{P_{imn}}{\breve{\gamma}}\mathbb{I}_{mn}^{(u<)}(\mathbf{q}) + \frac{1}{2}\mathrm{i}\epsilon\frac{\breve{\chi}(\lambda_1)k^2}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\mathbb{I}_{mn}^{(u<)}(\mathbf{q}),$$

where we have defined

$$\text{(A.23)}\quad \begin{aligned}\breve{\Gamma} \overset{\text{def}}{=} \breve{\gamma} + \breve{\chi}(\lambda_1)k_h^2 &= -\mathrm{i}\omega + [\breve{\xi}(\lambda_1) + \breve{\chi}(\lambda_1)]k_h^2 + [\breve{\xi}(\lambda_1) + \breve{\zeta}(\lambda_1)]k_z^2\\ &= -\mathrm{i}\omega + [\breve{\xi}(\lambda_1) + \breve{\chi}(\lambda_1)]k^2 + [\breve{\zeta}(\lambda_1) - \breve{\chi}(\lambda_1)]k_z^2.\end{aligned}$$

We now proceed to the next step of the iterative procedure which consists of a step-by-step elimination of infinitesimally small wave-number bands from the Fourier spectrum from the short-wavelength side. We introduce λ_2, which satisfies

$$\text{(A.24)}\quad \delta\lambda = \lambda_1 - \lambda_2 \ll 1,$$

and, again, split the remaining fluctuational Fourier spectrum $0 \leq k \leq \lambda_1$ into two parts by defining new variables (but keeping the same notation):

$$\text{(A.25)}\quad \theta(k-\lambda_2)\hat{u}_i^<(\mathbf{k},\omega) \mapsto \hat{u}_i^>(\mathbf{k},\omega), \qquad \text{for } \lambda_2 < k < \lambda_1,$$

$$\text{(A.26)}\quad \theta(\lambda_2-k)\hat{u}_i^<(\mathbf{k},\omega) \mapsto \hat{u}_i^<(\mathbf{k},\omega), \qquad \text{for } k < \lambda_2,$$

and thesame way for $\hat{Q}$. The equations are also split, as in the first step (cf. (3.12) and (3.13)), i.e. we have

$$\text{(A.27)}\quad \begin{aligned}&\left[-\mathrm{i}\omega + \breve{\xi}(\lambda_1)k^2 + \breve{\zeta}(\lambda_1)k_z^2\right]\hat{u}_i^<(\mathbf{q}) + \breve{\chi}(\lambda_1)P_{i3}k^2\hat{u}_z^<(\mathbf{q})\\ &= \frac{P_{i3}}{k^2}\hat{Q}^<(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}\mathbb{I}_{mn}^{(u<)}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}\int \mathrm{d}^4q'\langle\hat{u}_m^>(\mathbf{q}')\hat{u}_n^>(\mathbf{q}-\mathbf{q}')\rangle_c,\end{aligned}$$

for the new long-wavelength modes and for the new short-wavelength ones we get

$$\text{(A.28)}\quad \begin{aligned}\hat{u}_i^>(\mathbf{q}) = \frac{P_{i3}}{k^2\breve{\Gamma}}\hat{Q}^>(\mathbf{q}) &- \mathrm{i}\epsilon\frac{P_{imn}}{\breve{\gamma}}\mathbb{J}_{mn}^{(u)}(\mathbf{q}) + \mathrm{i}\epsilon\frac{\breve{\chi}(\lambda_1)k^2}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\mathbb{J}_{mn}^{(u)}(\mathbf{q})\\ &- \frac{1}{2}\mathrm{i}\epsilon\frac{P_{imn}}{\breve{\gamma}}\mathbb{I}_{mn}^{(u<)}(\mathbf{q}) + \frac{1}{2}\mathrm{i}\epsilon\frac{\breve{\chi}(\lambda_1)k^2}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\mathbb{I}_{mn}^{(u<)}(\mathbf{q}) + R_i,\end{aligned}$$

where:

$$\text{(A.29)}\quad \mathbb{J}_{mn}^{(u)}(\mathbf{q}) = \int \mathrm{d}^4q''\hat{u}_m^<(\mathbf{q}'')\hat{u}_n^>(\mathbf{q}-\mathbf{q}''),$$

$$\text{(A.30)}\quad R_i = -\frac{1}{2}\mathrm{i}\epsilon\left[\frac{P_{imn}}{\breve{\gamma}} - \frac{\breve{\chi}(\lambda_1)k^2}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\right]\mathbb{I}_{mn}^{(u>)}(\mathbf{q}).$$

Of course now $\langle\cdot\rangle_c$ denotes conditional average over the second shell ($\lambda_2 \le k \le \lambda_1$) statistical subensemble.

Repetition of the sub-steps undertaken in the first step of the iterative procedure, but with the modified expression for the short-wavelength modes $\hat{u}_i^>(\mathbf{q})$, in general leads to a new expression for the mean Reynolds stress. However, it becomes clear that at the leading order the Reynolds stress remains uninfluenced by the corrections $\breve{\xi}-1$, $\breve{\zeta}$ and $\breve{\chi}$, which are all of the order ϵ^2, thus unaltered with respect to the previous step of the procedure (recall that we neglect the terms of the order $\mathcal{O}(\epsilon^2)$ in the fluctuational corrections $\langle\hat{u}_m^>\hat{u}_n^>\rangle_c$). To demonstrate this explicitly we calculate

$$\begin{aligned}
\text{(A.31)}\qquad &\langle\hat{u}_m^>(\mathbf{q}')\hat{u}_n^>(\mathbf{q}-\mathbf{q}')\rangle_c \\
&= \frac{P_{m3}(\mathbf{k}')P_{n3}(\mathbf{k}-\mathbf{k}')}{k'^2|\mathbf{k}-\mathbf{k}'|^2\breve{\Gamma}(\mathbf{q}')\breve{\Gamma}(\mathbf{q}-\mathbf{q}')}\langle\hat{Q}^>(\mathbf{q}')\hat{Q}^>(\mathbf{q}-\mathbf{q}')\rangle_c \\
&\quad - \frac{\mathrm{i}\epsilon P_{m3}(\mathbf{k}')}{k'^2\breve{\Gamma}(\mathbf{q}')}\frac{P_{npq}(\mathbf{k}-\mathbf{k}')}{\breve{\gamma}(\mathbf{q}-\mathbf{q}')}\langle\hat{Q}^>(\mathbf{q}')\mathbb{J}_{pq}^{(u)}(\mathbf{q}-\mathbf{q}')\rangle_c \\
&\quad - \frac{\mathrm{i}\epsilon P_{m3}(\mathbf{k}')}{k'^2\breve{\Gamma}(\mathbf{q}')}\frac{\breve{\chi}|\mathbf{k}-\mathbf{k}'|^2}{\breve{\Gamma}(\mathbf{q}-\mathbf{q}')}\frac{P_{n3}(\mathbf{k}-\mathbf{k}')P_{3pq}(\mathbf{k}-\mathbf{k}')}{\breve{\gamma}(\mathbf{q}-\mathbf{q}')}\langle\hat{Q}^>(\mathbf{q}')\mathbb{J}_{pq}^{(u)}(\mathbf{q}-\mathbf{q}')\rangle_c \\
&\quad - \frac{\mathrm{i}\epsilon P_{n3}(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2\breve{\Gamma}(\mathbf{q}-\mathbf{q}')}\frac{P_{mpq}(\mathbf{k}')}{\breve{\gamma}(\mathbf{q}')}\langle\hat{Q}^>(\mathbf{q}-\mathbf{q}')\mathbb{J}_{pq}^{(u)}(\mathbf{q}')\rangle_c \\
&\quad - \frac{\mathrm{i}\epsilon P_{n3}(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2\breve{\Gamma}(\mathbf{q}-\mathbf{q}')}\frac{\breve{\chi}k'^2}{\breve{\Gamma}(\mathbf{q}')}\frac{P_{m3}(\mathbf{k}')P_{3pq}(\mathbf{k}')}{\breve{\gamma}(\mathbf{q}')}\langle\hat{Q}^>(\mathbf{q}-\mathbf{q}')\mathbb{J}_{pq}^{(u)}(\mathbf{q}')\rangle_c + \mathcal{O}(\epsilon^2).
\end{aligned}$$

The first term in (A.31) is proportional to $\langle\hat{Q}^>(\mathbf{q}')\hat{Q}^>(\mathbf{q}-\mathbf{q}')\rangle_c \sim \delta(\mathbf{k})\delta(\omega)$, hence it does not contribute to the large-scale dynamics and it is disregarded, as in the first step. Substituting once again for $\hat{\mathbf{u}}^>$ from (A.28) into the $\mathbb{J}^{(u)}$-terms in (A.31) and making use of the symmetry $\mathbf{q}'\mapsto\mathbf{q}-\mathbf{q}'$ under the integral $\int\mathrm{d}^4q'$ one obtains

$$\begin{aligned}
\text{(A.32)}\qquad &\int\mathrm{d}^4q'\langle\hat{u}_m^>(\mathbf{q}')\hat{u}_n^>(\mathbf{q}-\mathbf{q}')\rangle_c \\
&= -\mathrm{i}\epsilon\int\mathrm{d}^4q'\int\mathrm{d}^4q''\frac{\hat{u}_p^<(\mathbf{q}'')P_{m3}(\mathbf{k}')P_{npq}(\mathbf{k}-\mathbf{k}')P_{q3}(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')}{k'^2|\mathbf{k}-\mathbf{k}'-\mathbf{k}''|^2\breve{\Gamma}(\mathbf{q}')\breve{\Gamma}(\mathbf{q}-\mathbf{q}'-\mathbf{q}'')\breve{\gamma}(\mathbf{q}-\mathbf{q}')}\langle\hat{Q}^{>2}\rangle_c \\
&\quad -\mathrm{i}\epsilon\breve{\chi}\int\mathrm{d}^4q' \\
&\quad \times\int\mathrm{d}^4q''\frac{\hat{u}_p^<(\mathbf{q}'')|\mathbf{k}-\mathbf{k}'|^2P_{m3}(\mathbf{k}')P_{n3}(\mathbf{k}-\mathbf{k}')P_{3pq}(\mathbf{k}-\mathbf{k}')P_{q3}(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')}{k'^2|\mathbf{k}-\mathbf{k}'-\mathbf{k}''|^2\breve{\Gamma}(\mathbf{q}')\breve{\Gamma}(\mathbf{q}-\mathbf{q}')\breve{\Gamma}(\mathbf{q}-\mathbf{q}'-\mathbf{q}'')\breve{\gamma}(\mathbf{q}-\mathbf{q}')}\langle\hat{Q}^{>2}\rangle_c \\
&\quad +(m\leftrightarrow n)+\mathcal{O}(\epsilon^2),
\end{aligned}$$

where $(m \leftrightarrow n)$ in (A.32) denotes terms of the same structure as the two previous ones but with exchanged indices m and n and

$$\langle \hat{Q}^{>2} \rangle_c = \langle \hat{Q}^{>}(\mathbf{q}') \hat{Q}^{>}(\mathbf{q} - \mathbf{q}' - \mathbf{q}'') \rangle_c. \tag{A.33}$$

We can now substitute for the heat-source correlations

$$\langle \hat{Q}(\mathbf{k}, \omega) \hat{Q}(\mathbf{k}', \omega') \rangle = \Xi(k) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'), \tag{A.34}$$

cf. (2.8), into (A.32) and perform the $\mathbf{q}''$ integral which yields

$$\begin{aligned} &-\frac{1}{2} \mathrm{i} \epsilon P_{imn} \int \mathrm{d}^4 q' \langle \hat{u}_m^{>}(\mathbf{q}') \hat{u}_n^{>}(\mathbf{q} - \mathbf{q}') \rangle_c \\ = &-\epsilon^2 P_{imn} \hat{u}_p^{<}(\mathbf{q}) \int \mathrm{d}^4 q' \frac{\Xi(k') P_{m3}(\mathbf{k}') P_{npq}(\mathbf{k} - \mathbf{k}') P_{q3}(\mathbf{k}')}{k'^4 |\breve{\Gamma}(\mathbf{q}')|^2 \breve{\gamma}(\mathbf{q} - \mathbf{q}')} \\ &- \epsilon^2 \breve{\chi} P_{imn} \hat{u}_p^{<}(\mathbf{q}) \int \mathrm{d}^4 q' \frac{\Xi(k') |\mathbf{k} - \mathbf{k}'|^2 P_{m3}(\mathbf{k}') P_{n3}(\mathbf{k} - \mathbf{k}') P_{3pq}(\mathbf{k} - \mathbf{k}') P_{q3}(\mathbf{k}')}{k'^4 |\breve{\Gamma}(\mathbf{q}')|^2 \breve{\Gamma}(\mathbf{q} - \mathbf{q}') \breve{\gamma}(\mathbf{q} - \mathbf{q}')} \\ &+ \mathcal{O}(\epsilon^3), \end{aligned} \tag{A.35}$$

where:

$$\breve{\gamma} = -\mathrm{i}\omega + \breve{\xi} k^2 + \breve{\zeta} k_z^2 = \gamma + \mathcal{O}(\epsilon^2), \tag{A.36a}$$

$$\breve{\Gamma} = -\mathrm{i}\omega + (\breve{\xi} + \breve{\chi}) k^2 + (\breve{\zeta} - \breve{\chi}) k_z^2 = \gamma + \mathcal{O}(\epsilon^2), \tag{A.36b}$$

$$\breve{\chi} = \mathcal{O}(\epsilon^2). \tag{A.36c}$$

Hence we can write at leading order

$$\begin{aligned} &-\frac{1}{2} \mathrm{i} \epsilon P_{imn} \int \mathrm{d}^4 q' \langle \hat{u}_m^{>}(\mathbf{q}') \hat{u}_n^{>}(\mathbf{q} - \mathbf{q}') \rangle_c \\ &\qquad = -\epsilon^2 P_{imn} \hat{u}_p^{<}(\mathbf{q}) \int \mathrm{d}^4 q' \frac{\Xi(k') P_{m3}(\mathbf{k}') P_{npq}(\mathbf{k} - \mathbf{k}') P_{q3}(\mathbf{k}')}{k'^4 |\gamma(\mathbf{q}')|^2 \gamma(\mathbf{q} - \mathbf{q}')} + \mathcal{O}(\epsilon^3), \end{aligned} \tag{A.37}$$

which is exactly the same as in the first step (cf. (A.5)) and therefore the resulting corrections must have the same form as in (A.16), (A.15). It follows that:

$$\breve{\xi}(\lambda_2) = \breve{\xi}(\lambda_1) + \epsilon^2 \frac{2\pi^2}{105} \frac{\delta\lambda}{\lambda_2^6} \left[4\Xi(\lambda_2) - \lambda_2 \frac{\partial \Xi}{\partial \lambda}(\lambda_2) \right], \tag{A.38a}$$

$$\breve{\zeta}(\lambda_2) = \breve{\zeta}(\lambda_1) + \epsilon^2 \frac{2\pi^2}{105} \frac{\delta\lambda}{\lambda_2^6} \left[8\Xi(\lambda_2) + 5\lambda_2 \frac{\partial \Xi}{\partial \lambda}(\lambda_2) \right], \tag{A.38b}$$

$$\breve{\chi}(\lambda_2) = \breve{\chi}(\lambda_1) + \epsilon^2 \frac{2\pi^2}{105} \frac{\delta\lambda}{\lambda_2^6} \left[\Xi(\lambda_2) + 5\lambda_2 \frac{\partial \Xi}{\partial \lambda}(\lambda_2) \right]. \tag{A.38c}$$

If we now return to the dimensional units (recall, that we had chosen L^2/ν for the time scale, L for the spatial scale and $\kappa\nu U/g\bar{\alpha}L^4$ for the heat source scale; the latter implies $\kappa^2\nu U^2/g^2\bar{\alpha}^2L^3$ for the scale of $\Xi(k)$)

$$\text{(A.39)}\qquad \left[-\mathrm{i}\omega + \nu\breve{\xi}(\lambda_1)k^2 + \nu\breve{\zeta}(\lambda_1)k_z^2\right]\hat{u}_i^<(\mathbf{q}) + \nu\breve{\chi}(\lambda_1)P_{i3}k^2\hat{u}_z^<(\mathbf{q}) = \frac{g\bar{\alpha}}{\kappa}\frac{P_{i3}}{k^2}\hat{Q}^<(\mathbf{q}) - \frac{1}{2}\mathrm{i}P_{imn}\mathbb{I}_{mn}^{(u^<)}(\mathbf{q}),$$

we obtain:

$$\text{(A.40a)}\qquad \frac{\breve{\xi}(\lambda_1)-\breve{\xi}(\lambda_2)}{\delta\lambda} = -\frac{2\pi^2}{105}\frac{g^2\bar{\alpha}^2}{\nu^3\kappa^2}\frac{1}{\lambda_2^6}\left[4\Xi(\lambda_2) - \lambda_2\frac{\partial\Xi}{\partial\lambda}(\lambda_2)\right],$$

$$\text{(A.40b)}\qquad \frac{\breve{\zeta}(\lambda_1)-\breve{\zeta}(\lambda_2)}{\delta\lambda} = -\frac{2\pi^2}{105}\frac{g^2\bar{\alpha}^2}{\nu^3\kappa^2}\frac{1}{\lambda_2^6}\left[8\Xi(\lambda_2) + 5\lambda_2\frac{\partial\Xi}{\partial\lambda}(\lambda_2)\right],$$

$$\text{(A.40c)}\qquad \frac{\breve{\chi}(\lambda_1)-\breve{\chi}(\lambda_2)}{\delta\lambda} = -\frac{2\pi^2}{105}\frac{g^2\bar{\alpha}^2}{\nu^3\kappa^2}\frac{1}{\lambda_2^6}\left[\Xi(\lambda_2) + 5\lambda_2\frac{\partial\Xi}{\partial\lambda}(\lambda_2)\right].$$

It is now clear, that in all the following steps of the iterative, asymptotic procedure no terms with new structure can appear in the velocity equation and thus we can now take the continuous limit $\delta\lambda \to 0$ of the obtained recursions. Let us introduce the following simple form of the heat source correlation function (note that in an isotropic case driven by a random forcing $\mathbf{f}$, a scaling of the form $\langle f_i f_j\rangle \sim k^2$ was shown by LIFSHITZ and PITAEVSKII [37] to describe systems in thermal equilibrium thus to study non-equilibrium flows we consider forcing with significantly smaller scaling exponent)

$$\text{(A.41)}\qquad \Xi(k) = \frac{\mathbb{Q}^2L^3}{\nu k^2},$$

which ensures, that the spectral density of the heat source Q^2

$$\text{(A.42)}\qquad \int_0^\Lambda k^2\mathrm{d}k\int \mathrm{d}\mathring{\Omega}_{\mathbf{k}}\int \mathrm{d}^4q'\langle\hat{Q}(\mathbf{k},\omega)\hat{Q}(\mathbf{k}',\omega')\rangle = \int_0^\Lambda \frac{4\pi\mathbb{Q}^2L^3}{\nu}\mathrm{d}k,$$

is uniform and where $\mathbb{Q}$ is the magnitude of the heat delivery rate (in K/s); this yields:

$$\text{(A.43a)}\qquad \frac{\mathrm{d}\breve{\xi}}{\mathrm{d}\lambda} = -\frac{4\pi^2}{35}\frac{g^2\bar{\alpha}^2L^3}{\nu^4\kappa^2}\frac{\mathbb{Q}^2}{\lambda^8} \;\Rightarrow\; \breve{\xi}(\Lambda)-\breve{\xi}(\lambda) = -\frac{4\pi^2}{245}\frac{g^2\bar{\alpha}^2L^3\mathbb{Q}^2}{\nu^4\kappa^2}\left(\frac{1}{\lambda^7}-\frac{1}{\Lambda^7}\right),$$

$$\text{(A.43b)}\qquad \frac{\mathrm{d}\breve{\zeta}}{\mathrm{d}\lambda} = \frac{4\pi^2}{105}\frac{g^2\bar{\alpha}^2L^3}{\nu^4\kappa^2}\frac{\mathbb{Q}^2}{\lambda^8} \;\Rightarrow\; \breve{\zeta}(\Lambda)-\breve{\zeta}(\lambda) = \frac{4\pi^2}{735}\frac{g^2\bar{\alpha}^2L^3\mathbb{Q}^2}{\nu^4\kappa^2}\left(\frac{1}{\lambda^7}-\frac{1}{\Lambda^7}\right),$$

$$\text{(A.43c)}\qquad \frac{\mathrm{d}\breve{\chi}}{\mathrm{d}\lambda} = \frac{6\pi^2}{35}\frac{g^2\bar{\alpha}^2L^3}{\nu^4\kappa^2}\frac{\mathbb{Q}^2}{\lambda^8} \;\Rightarrow\; \breve{\chi}(\Lambda)-\breve{\chi}(\lambda) = \frac{6\pi^2}{245}\frac{g^2\bar{\alpha}^2L^3\mathbb{Q}^2}{\nu^4\kappa^2}\left(\frac{1}{\lambda^7}-\frac{1}{\Lambda^7}\right).$$

The application of the “initial” conditions $\breve{\xi}(\Lambda) = 1$, $\breve{\zeta}(\Lambda) = 0$, $\breve{\chi}(\Lambda) = 0$ and the limit $\lambda = k_\ell = 2\pi/\ell \ll \Lambda$ leads to:

(A.44a) $$\breve{\xi}(k_\ell) = 1 + \frac{4\pi^2}{245}\frac{g^2\bar{\alpha}^2 L^3 \mathbb{Q}^2}{\nu^4\kappa^2 k_\ell^7},$$

(A.44b) $$\breve{\zeta}(k_\ell) = -\frac{4\pi^2}{735}\frac{g^2\bar{\alpha}^2 L^3 \mathbb{Q}^2}{\nu^4\kappa^2 k_\ell^7},$$

(A.44c) $$\breve{\chi}(k_\ell) = -\frac{6\pi^2}{245}\frac{g^2\bar{\alpha}^2 L^3 \mathbb{Q}^2}{\nu^4\kappa^2 k_\ell^7},$$

where ℓ can be thought of as the length-scale of most energetic eddies in the turbulent flow. Defining:

(A.45a) $$\varsigma = \frac{2\pi^2}{735}\frac{g^2\bar{\alpha}^2 L^3 \mathbb{Q}^2}{\nu^3\kappa^2 k_\ell^7},$$

(A.45b) $$\xi = \nu\breve{\xi}(k_\ell) = \nu + 6\varsigma,$$

(A.45c) $$\zeta = -\nu\breve{\zeta}(k_\ell) = 2\varsigma,$$

(A.45d) $$\chi = -\nu\breve{\chi}(k_\ell) = 9\varsigma,$$

the large-scale flow is governed by:

(A.46) $$\frac{\partial\langle\mathbf{U}\rangle}{\partial t} + (\langle\mathbf{U}\rangle\cdot\nabla)\langle\mathbf{U}\rangle = -\nabla\langle\Pi\rangle - 9\varsigma\nabla^2\langle U\rangle_z\hat{\mathbf{e}}_z + (\nu + 6\varsigma)\nabla^2\langle\mathbf{U}\rangle - 2\varsigma\frac{\partial^2\langle\mathbf{U}\rangle}{\partial z^2},$$

or

(A.47) $$\frac{\partial\langle\mathbf{U}\rangle}{\partial t} + (\langle\mathbf{U}\rangle\cdot\nabla)\langle\mathbf{U}\rangle = -\nabla\langle\Pi\rangle - 9\varsigma\nabla^2\langle U\rangle_z\hat{\mathbf{e}}_z + (\nu + 6\varsigma)\nabla_h^2\langle\mathbf{U}\rangle + (\nu + 4\varsigma)\frac{\partial^2\langle\mathbf{U}\rangle}{\partial z^2}.$$

Appendix B. Comments on full renormalization

Full renormalization of the anisotropic problem at hand is not possible, because the integrals in (A.35) without the simplification (A.36) coming from neglection of higher order terms in ϵ cannot be evaluated analytically. Moreover, as it is demonstrated below, the full integrals (A.35) (not approximated to the leading order in ϵ) lead to introduction of yet another term in the equation of the form $\sim P_{i3}k_z^2\hat{u}_z^<$, thus making the further steps even more complicated; it can be demonstrated however, that further steps of the renormalization procedure do not introduce terms of a new structure. Therefore, the most general form of the mean flow equation is the following:

(B.1) $$\frac{\partial\langle\mathbf{U}\rangle}{\partial t} + (\langle\mathbf{U}\rangle\cdot\nabla)\langle\mathbf{U}\rangle = -\nabla\langle\Pi\rangle - \chi\nabla^2\langle U\rangle_z\hat{\mathbf{e}}_z + \varrho\frac{\partial^2\langle U\rangle_z}{\partial z^2}\hat{\mathbf{e}}_z + \xi\nabla_h^2\langle\mathbf{U}\rangle + \zeta\frac{\partial^2\langle\mathbf{U}\rangle}{\partial z^2},$$

where based on our weakly nonlinear results we may suppose that $\xi > 0$, $\zeta > 0$ and $\chi > 0$, but the sign of the coefficient ϱ remains undetermined. Within the weakly nonlinear approach this coefficient is negligibly small $\varrho = \mathcal{O}(\epsilon^4)$, however in fully developed strong turbulence it might be of a comparable magnitude with all the remaining coefficients.

We now demonstrate that indeed, the renormalization leads to the above general form of the mean flow equation. To that end we need to return to the second step of the procedure and consider the full integrals in (A.35). Making the aforementioned substitution $\mathbf{k}' \to \mathbf{k}' + \frac{1}{2}\mathbf{k}$ to symmetrize the domain of integration one obtains:

$$\begin{aligned}
\text{(B.2)}\qquad & -\frac{1}{2}\mathrm{i}\epsilon P_{imn}\int \mathrm{d}^4q' \langle \hat{u}^>_m(\mathbf{q}')\hat{u}^>_n(\mathbf{q}-\mathbf{q}')\rangle_c \\
=& -\epsilon^2\frac{\mathbb{Q}^2L^3}{\nu}P_{imn}\hat{u}^<_p(\mathbf{q})\int \mathrm{d}k'\int \mathrm{d}\mathring{\Omega} \\
&\times\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\mathbb{P}_{mq}(\mathbf{k}'+\frac{1}{2}\mathbf{k})P_{npq}(\frac{1}{2}\mathbf{k}-\mathbf{k}')}{(k'^2+k'_rk_r)^2\left|\breve{\Gamma}(\mathbf{k}'+\frac{1}{2}\mathbf{k},\omega')\right|^2\breve{\gamma}(\mathbf{k}'-\frac{1}{2}\mathbf{k},-\omega')} \\
&-\epsilon^2\frac{\mathbb{Q}^2L^3}{\nu}\breve{\chi}P_{imn}\hat{u}^<_p(\mathbf{q})\int \mathrm{d}k'\int \mathrm{d}\mathring{\Omega} \\
&\times\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\left|\mathbf{k}'-\frac{1}{2}\mathbf{k}\right|^2\mathbb{P}_{mq}(\mathbf{k}'+\frac{1}{2}\mathbf{k})P_{n3}(\mathbf{k}'-\frac{1}{2}\mathbf{k})P_{3pq}(\frac{1}{2}\mathbf{k}-\mathbf{k}')}{k'^4\left|\breve{\Gamma}(\mathbf{k}'+\frac{1}{2}\mathbf{k},\omega')\right|^2\breve{\Gamma}(\mathbf{k}'-\frac{1}{2}\mathbf{k},-\omega')\breve{\gamma}(\mathbf{k}'-\frac{1}{2}\mathbf{k},-\omega')}+\mathcal{O}(\epsilon^3),
\end{aligned}$$

where:

$$\text{(B.3)}\qquad \mathbb{P}_{mq}\big(\mathbf{k}'+\tfrac{1}{2}\mathbf{k}\big) = P_{m3}\big(\mathbf{k}'+\tfrac{1}{2}\mathbf{k}\big)P_{q3}\big(\mathbf{k}'+\tfrac{1}{2}\mathbf{k}\big).$$

Expansion in k/k' up to the first order leads to:

$$\begin{aligned}
\text{(B.4)}\qquad & -\frac{1}{2}\mathrm{i}\epsilon P_{imn}\int \mathrm{d}^4q' \langle \hat{u}^>_m(\mathbf{q}')\hat{u}^>_n(\mathbf{q}-\mathbf{q}')\rangle_c \\
=& -\epsilon^2\frac{\mathbb{Q}^2L^3}{\nu}P_{imn}\hat{u}^<_p(\mathbf{q})\int \frac{\mathrm{d}k'}{k'^8}\int \mathrm{d}\mathring{\Omega} \\
&\times\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\mathcal{F}(\mathbf{k}',\omega')\mathbb{P}_{mq}(\mathbf{k}'+\frac{1}{2}\mathbf{k})P_{npq}(\frac{1}{2}\mathbf{k}-\mathbf{k}')}{\left|\breve{\Gamma}(\mathbf{k}'+\frac{1}{2}\mathbf{k},\omega')\right|^2\left|\breve{\gamma}(\mathbf{k}'-\frac{1}{2}\mathbf{k},\omega')\right|^2} \\
&-\epsilon^2\frac{\mathbb{Q}^2L^3}{\nu}\breve{\chi}P_{imn}\hat{u}^<_p(\mathbf{q})\int \frac{\mathrm{d}k'}{k'^8}\int \mathrm{d}\mathring{\Omega} \\
&\times\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\mathcal{G}(\mathbf{k}',\omega')\mathbb{P}_{mq}(\mathbf{k}'+\frac{1}{2}\mathbf{k})P_{n3}(\mathbf{k}'-\frac{1}{2}\mathbf{k})P_{3pq}(\frac{1}{2}\mathbf{k}-\mathbf{k}')}{\left|\breve{\Gamma}(\mathbf{k}',\omega')\right|^4\left|\breve{\gamma}(\mathbf{k}'-\frac{1}{2}\mathbf{k},\omega')\right|^2}+\mathcal{O}(\epsilon^3),
\end{aligned}$$

with:

$$\begin{aligned}\text{(B.5)}\qquad \mathcal{F}(\mathbf{k}',\omega') &\approx [\breve{\xi}k'^2-\breve{\zeta}k_z'^2-\breve{\xi}k_r'k_r+\breve{\zeta}k_z'k_z](k'^2-k_r'k_r)^2\\ &\approx k'^4[k'^2(\breve{\xi}-\breve{\zeta}X^2)+\breve{\zeta}k_z'k_z+(2\breve{\zeta}X^2-3\breve{\xi})k_r'k_r],\end{aligned}$$

$$\begin{aligned}\text{(B.6)}\qquad \mathcal{G}(\mathbf{k}',\omega') \approx{}& \{-\omega'^2+k'^4(\breve{\xi}-\breve{\zeta}X^2)[(\breve{\xi}-\breve{\chi})-(\breve{\zeta}-\breve{\chi})X^2]\\ &-[2\breve{\xi}(\breve{\xi}-\breve{\chi})-(2\breve{\xi}\breve{\zeta}-\breve{\chi}(\breve{\xi}+\breve{\zeta}))X^2]k'^2k_r'k_r\\ &+[2\breve{\xi}\breve{\zeta}-\breve{\chi}(\breve{\xi}+\breve{\zeta})-2\breve{\zeta}(\breve{\zeta}-\breve{\chi})X^2]k'^2k_z'k_z\}(k'^2-k_r'k_r)^3\\ \approx{}& -\omega'^2k'^6+k'^{10}(\breve{\xi}-\breve{\zeta}X^2)[(\breve{\xi}-\breve{\chi})-(\breve{\zeta}-\breve{\chi})X^2]\\ &+\{3\omega'^2-[5\breve{\xi}(\breve{\xi}-\breve{\chi})-4(2\breve{\xi}\breve{\zeta}-\breve{\chi}(\breve{\xi}+\breve{\zeta}))X^2\\ &+3\breve{\zeta}(\breve{\zeta}-\breve{\chi})X^4]k'^4\}k'^4k_r'k_r\\ &+[2\breve{\xi}\breve{\zeta}-\breve{\chi}(\breve{\xi}+\breve{\zeta})-2\breve{\zeta}(\breve{\zeta}-\breve{\chi})X^2]k'^8k_z'k_z,\end{aligned}$$

$$\begin{aligned}\text{(B.7)}\qquad &\frac{1}{|\breve{\gamma}(\mathbf{k}'-\frac{1}{2}\mathbf{k},\omega')|^2}\frac{1}{|\breve{\Gamma}(\mathbf{k}'+\frac{1}{2}\mathbf{k},\omega')|^2}\\ &\approx\frac{1}{[\omega'^2+k'^4(\breve{\xi}-\breve{\zeta}X^2)^2][\omega'^2+k'^4((\breve{\xi}-\breve{\chi})-(\breve{\zeta}-\breve{\chi})X^2)^2]}\\ &+2\breve{\chi}k'^2\frac{\omega'^2\{[\breve{\xi}(2-X^2)-\breve{\zeta}X^2-\breve{\chi}(1-X^2)]k_r'k_r+[\breve{\zeta}(2X^2-1)-\breve{\xi}+\breve{\chi}(1-X^2)]k_z'k_z\}}{[\omega'^2+k'^4((\breve{\xi}-\breve{\chi})-(\breve{\zeta}-\breve{\chi})X^2)^2]^2[\omega'^2+k'^4(\breve{\xi}-\breve{\zeta}X^2)^2]^2}\\ &+2\breve{\chi}k'^2\frac{k'^4(\breve{\xi}-\breve{\zeta})(\breve{\xi}-\breve{\zeta}X^2)((\breve{\xi}-\breve{\chi})-(\breve{\zeta}-\breve{\chi})X^2)(X^2k_r'k_r-k_z'k_z)}{[\omega'^2+k'^4((\breve{\xi}-\breve{\chi})-(\breve{\zeta}-\breve{\chi})X^2)^2]^2[\omega'^2+k'^4(\breve{\xi}-\breve{\zeta}X^2)^2]^2},\end{aligned}$$

and under integration over the azimuthal angle $\int_0^{2\pi}\mathrm{d}\varphi$ different terms in the integrands transform into:

$$\begin{aligned}\text{(B.8)}\qquad &P_{imn}\hat{u}_p^<(\mathbf{q})f(X^2)P_{m3}\left(\mathbf{k}'+\frac{1}{2}\mathbf{k}\right)P_{npq}\left(\frac{1}{2}\mathbf{k}-\mathbf{k}'\right)P_{q3}\left(\mathbf{k}'+\frac{1}{2}\mathbf{k}\right)\\ &\to P_{imn}\hat{u}_p^<(\mathbf{q})f(X^2)\left[-\delta_{np}\frac{k_z'k_q'}{k'^2}k_q+\delta_{np}k_z-\frac{k_p'k_n'}{k'^2}k_z+\frac{k_p'k_z'}{k'^2}k_n\right]P_{m3}(\mathbf{k}')\\ &\to \pi f(X^2)(1-X^2)^2k_z^2\hat{u}_i^<(\mathbf{q})+3\pi f(X^2)X^2(1-X^2)k^2P_{i3}\hat{u}_z^<(\mathbf{q})\\ &\quad+\pi f(X^2)\left[\frac{35}{2}X^4-21X^2+2\right]k_z^2P_{i3}\hat{u}_z^<(\mathbf{q}),\end{aligned}$$

(B.9)

$$\begin{aligned}&P_{imn}\hat{u}_p^<(\mathbf{q})f(X^2)P_{m3}\left(\mathbf{k}'+\frac{1}{2}\mathbf{k}\right)P_{n3}\left(\mathbf{k}'-\frac{1}{2}\mathbf{k}\right)P_{3pq}\left(\frac{1}{2}\mathbf{k}-\mathbf{k}'\right)P_{q3}\left(\mathbf{k}'+\frac{1}{2}\mathbf{k}\right)\\ &\to P_{imn}\hat{u}_p^<(\mathbf{q})f(X^2)\left[P_{m3}(\mathbf{k}')P_{n3}(\mathbf{k}')P_{p3}(\mathbf{k}')k_z-P_{m3}(\mathbf{k}')P_{n3}(\mathbf{k}')P_{p3}(\mathbf{k}')\frac{k_z'k_q'}{k'^2}k_q\right]\end{aligned}$$

$$
\begin{aligned}
&\to \frac{\pi}{2} f(X^2) X^4 (1-X^2)^2 k^2 \hat{u}_i^<(\mathbf{q}) \\
&+ \frac{\pi}{2} f(X^2) X^2 (4 - 17X^2 + 18X^4 - 5X^6) k_z^2 \hat{u}_i^<(\mathbf{q}) \\
&+ \frac{\pi}{2} f(X^2) X^2 (4 - 9X^2 + 10X^4 - 5X^6) P_{i3} k^2 \hat{u}_z^<(\mathbf{q}) \\
&+ \frac{\pi}{2} f(X^2) (8 - 56X^2 + 123X^4 - 110X^6 + 35X^8) P_{i3} k_z^2 \hat{u}_z^<(\mathbf{q}),
\end{aligned}
$$

(B.10) $$P_{imn} \hat{u}_p^<(\mathbf{q}) f(X^2) k'_p P_{n3}(\mathbf{k}') P_{m3}(\mathbf{k}') k'_z k_z$$

$$
\begin{aligned}
&\to P_{imn} k_z \hat{u}_p^<(\mathbf{q}) k'^2 f(X^2) (\delta_{m3}\delta_{n3} \frac{k'_z k'_p}{k'^2} - 2X^2 \delta_{m3} \frac{k'_n k'_p}{k'^2} + X^2 \frac{k'_m k'_n k'_p k'_z}{k'^4}) \\
&\to 2\pi k'^2 f(X^2) X^2 (5X^4 - 9X^2 + 4) P_{i3} k_z^2 \hat{u}_z^<(\mathbf{q}) - 2\pi k'^2 f(X^2) X^2 (1-X^2)^2 k_z^2 \hat{u}_i^<(\mathbf{q}),
\end{aligned}
$$

(B.11) $$P_{imn} \hat{u}_p^<(\mathbf{q}) f(X^2) k'_p P_{n3}(\mathbf{k}') P_{m3}(\mathbf{k}') k'_r k_r$$

$$
\begin{aligned}
&\to k_r P_{imn} \hat{u}_p^<(\mathbf{q}) k'^2 f(X^2) (\delta_{n3}\delta_{m3} \frac{k'_p k'_r}{k'^2} - 2\delta_{m3} \frac{k'_p k'_r k'_n k'_z}{k'^4} + X^2 \frac{k'_p k'_r k'_m k'_n}{k'^4}) \\
&\to \frac{\pi}{2} k'^2 f(X^2) X^2 (1-X^2)^2 k^2 \hat{u}_i^<(\mathbf{q}) - \frac{5}{2}\pi k'^2 f(X^2) X^2 (1-X^2)^2 P_{i3} k^2 \hat{u}_z^<(\mathbf{q}) \\
&- \frac{5}{2}\pi k'^2 f(X^2) X^2 (1-X^2)^2 k_z^2 \hat{u}_i^<(\mathbf{q}) \\
&- \frac{\pi}{2} k'^2 f(X^2) (1-X^2)(35X^4 - 35X^2 + 4) P_{i3} k_z^2 \hat{u}_z^<(\mathbf{q}).
\end{aligned}
$$

It is clear from the latter formulae that a new term of the form $P_{i3}k_z^2\hat{u}_z^<(\mathbf{q})$ appears in the equation for the long-wavelength modes in the second step (A.27) and hence we need to perform one more step, until invariance of the equations for long-wavelength modes is achieved at each step and the procedure can be closed and reduced to the form of recursion differential equations. Hence we introduce λ_3, which satisfies

(B.12) $$\delta\lambda = \lambda_2 - \lambda_3 \ll 1,$$

and once again split the remaining fluctuational Fourier spectrum $0 \le k \le \lambda_2$ into two parts by defining new variables (but keeping the same notation):

(B.13) $$\theta(k-\lambda_3)\hat{u}_i^<(\mathbf{k},\omega) \mapsto \hat{u}_i^>(\mathbf{k},\omega), \quad \text{for } \lambda_3 < k < \lambda_2,$$

(B.14) $$\theta(\lambda_3 - k)\hat{u}_i^<(\mathbf{k},\omega) \mapsto \hat{u}_i^<(\mathbf{k},\omega), \quad \text{for } k < \lambda_3,$$

and the same way for $\hat{Q}$. The equations are also split, as in the first and second steps (cf. (3.12) and (3.13)), i.e. for the new long-wavelength modes we have

(B.15) $$
\begin{aligned}
&[-\mathrm{i}\omega + \breve{\xi}(\lambda_2)k^2 + \breve{\zeta}(\lambda_2)k_z^2]\hat{u}_i^<(\mathbf{q}) + \breve{\chi}(\lambda_2)P_{i3}k^2\hat{u}_z^<(\mathbf{q}) + \breve{\varrho}(\lambda_2)P_{i3}k_z^2\hat{u}_z^<(\mathbf{q}) \\
&= \frac{P_{i3}}{k^2}\hat{Q}^<(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}\mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon P_{imn}\int \mathrm{d}^4q' \langle \hat{u}_m^>(\mathbf{q}')\hat{u}_n^>(\mathbf{q}-\mathbf{q}')\rangle_c.
\end{aligned}
$$

Of course now $\langle\cdot\rangle_c$ denotes conditional average over the second shell ($\lambda_3 \leq k \leq \lambda_2$) statistical subensemble but the coefficients $\breve{\xi}(\lambda_2)$, $\breve{\zeta}(\lambda_2)$, $\breve{\chi}(\lambda_2)$ and $\breve{\varrho}(\lambda_2)$ are now strongly nonlinear functions of $\breve{\xi}(\lambda_1)$, $\breve{\zeta}(\lambda_1)$, $\breve{\chi}(\lambda_1)$. A third step is now necessary, in order to verify that in the following steps of the recursion procedure the new term $\breve{\varrho}(\lambda_2)P_{i3}k_z^2\hat{u}_z^{<}(\mathbf{q})$ does not lead to appearance of yet other terms with a distinct structure. This however, is obvious since the new term can be treated as a correction to the $\breve{\chi}(\lambda_2)$-term, thus for the new short-wavelength modes we get (cf. (A.28))

$$\begin{aligned}\text{(B.16)}\quad \hat{u}_i^{>}(\mathbf{q}) = {} & \frac{P_{i3}}{k^2\breve{\Gamma}}\hat{Q}^{>}(\mathbf{q}) - \mathrm{i}\epsilon\frac{P_{imn}}{\breve{\gamma}}\mathbb{J}_{mn}^{(u)}(\mathbf{q}) \\ & + \mathrm{i}\epsilon\frac{(\breve{\chi}(\lambda_1)k^2 + \breve{\varrho}(\lambda_2)k_z^2)}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\mathbb{J}_{mn}^{(u)}(\mathbf{q}) - \frac{1}{2}\mathrm{i}\epsilon\frac{P_{imn}}{\breve{\gamma}}\mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) \\ & + \frac{1}{2}\mathrm{i}\epsilon\frac{(\breve{\chi}(\lambda_1)k^2 + \breve{\varrho}(\lambda_2)k_z^2)}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\mathbb{I}_{mn}^{(u^<)}(\mathbf{q}) + R_i,\end{aligned}$$

where

$$\text{(B.17)}\qquad \mathbb{J}_{mn}^{(u)}(\mathbf{q}) = \int \mathrm{d}^4q''\hat{u}_m^{<}(\mathbf{q}'')\hat{u}_n^{>}(\mathbf{q}-\mathbf{q}''),$$

$$\text{(B.18)}\qquad R_i = -\frac{1}{2}\mathrm{i}\epsilon\left[\frac{P_{imn}}{\breve{\gamma}} - \frac{(\breve{\chi}(\lambda_1)k^2 + \breve{\varrho}(\lambda_2)k_z^2)}{\breve{\Gamma}}\frac{P_{i3}P_{3mn}}{\breve{\gamma}}\right]\mathbb{I}_{mn}^{(u^>)}(\mathbf{q}).$$

It follows that although the new coefficient $\breve{\varrho}(\lambda_2)$ influences the dependencies of $\breve{\xi}(\lambda_3)$, $\breve{\zeta}(\lambda_3)$, $\breve{\chi}(\lambda_3)$, $\breve{\varrho}(\lambda_3)$ on $\breve{\xi}(\lambda_2)$, $\breve{\zeta}(\lambda_2)$, $\breve{\chi}(\lambda_2)$ and $\breve{\varrho}(\lambda_2)$ and makes them even more complex, no terms with new structure appear in the following steps of the procedure. This allows to close the recursion problem which results in some strongly nonlinear equations for the four coefficients $\breve{\xi}(\lambda)$, $\breve{\zeta}(\lambda)$, $\breve{\chi}(\lambda)$ and $\breve{\varrho}(\lambda)$ and hence the large-scale equations take the general form (B.1).

As a final note, it is to be stressed once again that the entire technique fundamentally relies on two important assumptions, regarding the properties of the flow. Firstly, we recall that the statistical correlations between short-wavelength fluctuations of the order higher than second, i.e. terms of the type $\langle\hat{u}_i^{>}\hat{u}_j^{>}\hat{Q}^{>}\rangle_c$ have been neglected. Secondly, the limit of distant interactions (A.7) corresponding to an assumption of ergodicity of the system has greatly simplified the calculations.

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