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Uniformity of electroelastic field within a three-phase anisotropic piezoelectric elliptical inhomogeneity in anti-plane shear

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Using the Stroh quartic formalism, we prove that the internal electroelastic field is unconditionally uniform inside a three-phase anisotropic piezoelectric elliptical inhomogeneity with two confocal elliptical interfaces when the surrounding matrix is subjected to uniform remote anti-plane mechanical and in-plane electrical loading. The inhomogeneity and the matrix comprise monoclinic piezoelectric materials with symmetry plane at $x_3 = 0$ and with poling in the x_3-direction; the intermediate interphase layer is a transversely isotropic piezoelectric material with poling in the x_3-direction. Moreover, we obtain the internal uniform electroelastic field inside the elliptical inhomogeneity and the non-uniform electroelastic field in the interphase layer in real-form in terms of the fundamental piezoelectricity matrices for both the inhomogeneity and the matrix and the generalized Barnett–Lothe tensors for both the interphase layer and the matrix.

Key words: three-phase elliptical inhomogeneity, confocal ellipses; monoclinic piezoelectric material, transversely isotropic piezoelectric material, Stroh quartic formalism, real-form solution.

1. Introduction

Eshelby's uniformity property regarding stresses and strains inside a three-dimensional ellipsoidal or a two-dimensional elliptical elastic inhomogeneity when the matrix is subjected to a uniform loading at infinity is well established [1–3]. When an intermediate coating of finite thickness is inserted between the internal circular elastic inhomogeneity and the surrounding unbounded matrix, however, the field inside the inhomogeneity is intrinsically non-uniform [4]. The uniformity property inside the inhomogeneity continues to remain valid when confocal elliptical interfaces are used in the fibrous composite [5–8].

In this paper, we study the electroelastic field of a three-phase anisotropic piezoelectric composite with two confocal elliptical interfaces subjected to uniform remote anti-plane mechanical and in-plane electrical loading. The three-phase piezoelectric composite is composed of an inner piezoelectric elliptical inhomogeneity, an intermediate piezoelectric interphase layer and an outer infinite piezoelectric matrix. Both the inhomogeneity and the matrix comprise monoclinic piezoelectric materials with symmetry plane at $x_3 = 0$ and with poling in the x_3-direction, while the interphase layer is composed of a transversely isotropic piezoelectric material with poling in the x_3-direction. The complexity of the problem lies in the fact that as many as 23 electroelastic constants are involved in the analysis (10 for the inhomogeneity, 3 for the interphase layer and 10 for the matrix). Intuitively, we would expect that the ensuing analysis and resulting solution structure would be considerably complicated. On the contrary, the problem is solved quite elegantly using the Stroh quartic formalism. In this respect, we prove that the internal electroelastic field of stresses, strains, electric displacements and electric fields inside the elliptical inhomogeneity continues to remain uniform. The uniformity of the internal electroelastic field is attributed to the confocal character of the two elliptical interfaces and the transversely isotropic property of the intermediate piezoelectric interphase layer. The internal uniform electroelastic field inside the elliptical inhomogeneity is unconditional since there is no other restriction on the remote loading except the required constitutive relationship. Using the identities developed in the Stroh quartic formalism, the internal uniform electroelastic field inside the elliptical inhomogeneity and the non-uniform electroelastic field within the interphase layer are obtained in real-form in terms of the fundamental piezoelectricity matrices for the inhomogeneity and the matrix and the generalized Barnett–Lothe tensors for the interphase layer and the matrix. It is expected that the present solution can be applied in the generalized self-consistent method [4, 9] to predict the effective properties of piezoelectric composites.

2. Stroh quartic formalism

For the anti-plane shear deformations of monoclinic piezoelectric materials with symmetry plane at $x_3 = 0$ and with poling in the x_3-direction, the constitutive equations and balance laws are given by

$$\begin{gathered}
\begin{bmatrix} \sigma_{31} \\ D_1 \end{bmatrix} = \begin{bmatrix} C_{55} & e_{15} \\ e_{15} & -\in_{11} \end{bmatrix} \begin{bmatrix} 2\varepsilon_{31} \\ -E_1 \end{bmatrix} + \begin{bmatrix} C_{45} & e_{25} \\ e_{14} & -\in_{12} \end{bmatrix} \begin{bmatrix} 2\varepsilon_{32} \\ -E_2 \end{bmatrix}, \\
\begin{bmatrix} \sigma_{32} \\ D_2 \end{bmatrix} = \begin{bmatrix} C_{45} & e_{14} \\ e_{25} & -\in_{12} \end{bmatrix} \begin{bmatrix} 2\varepsilon_{31} \\ -E_1 \end{bmatrix} + \begin{bmatrix} C_{44} & e_{24} \\ e_{24} & -\in_{22} \end{bmatrix} \begin{bmatrix} 2\varepsilon_{32} \\ -E_2 \end{bmatrix}, \\
2\varepsilon_{31} = w_{,1}, \quad 2\varepsilon_{32} = w_{,2}, \quad E_1 = -\phi_{,1}, \quad E_2 = -\phi_{,2},
\end{gathered} \tag{2.1}$$

$$\begin{bmatrix} C_{55} & e_{15} \\ e_{15} & -\in_{11} \end{bmatrix} \begin{bmatrix} w_{,11} \\ \phi_{,11} \end{bmatrix} + \begin{bmatrix} 2C_{45} & e_{14}+e_{25} \\ e_{14}+e_{25} & -2\in_{12} \end{bmatrix} \begin{bmatrix} w_{,12} \\ \phi_{,12} \end{bmatrix} + \begin{bmatrix} C_{44} & e_{24} \\ e_{24} & -\in_{22} \end{bmatrix} \begin{bmatrix} w_{,22} \\ \phi_{,22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.2}$$

where σ_{31} and σ_{32} are anti-plane shear stresses, D_1 and D_2 are in-plane electric displacements, ε_{31} and ε_{32} are anti-plane strains, E_1 and E_2 are in-plane electric fields, w is the anti-plane displacement, ϕ is the electric potential, C_{44}, C_{45}, C_{55} are three elastic constants, e_{15}, e_{25}, e_{14}, e_{24} are four piezoelectric constants and $\in_{11}$, $\in_{12}$, $\in_{22}$ are three dielectric constants.

Within the framework of the Stroh formalism [10], the general solution can be expressed in the form

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} w & \phi \end{bmatrix}^T = \mathbf{A}\mathbf{f}(z) + \bar{\mathbf{A}}\overline{\mathbf{f}(z)}, \\ \boldsymbol{\varphi} &= \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}^T = \mathbf{B}\mathbf{f}(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}, \\ \mathbf{f}(z) &= \begin{bmatrix} f_1(z_1) & f_2(z_2) \end{bmatrix}^T, \\ z_k &= x_1 + p_k x_2, \qquad \mathrm{Im}\{p_k\} > 0 \quad (k = 1, 2), \end{aligned} \tag{2.4}$$

with

$$\mathbf{N} \begin{bmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{bmatrix} = p_k \begin{bmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{bmatrix} \qquad (k = 1, 2), \tag{2.5}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \tag{2.6}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}, \tag{2.7}$$

and

$$\mathbf{Q} = \begin{bmatrix} C_{55} & e_{15} \\ e_{15} & -\in_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} C_{45} & e_{25} \\ e_{14} & -\in_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} C_{44} & e_{24} \\ e_{24} & -\in_{22} \end{bmatrix}. \tag{2.8}$$

In view of the fact that both the generalized displacement vector $\mathbf{u}$ and the generalized stress function vector $\boldsymbol{\varphi}$ are two-dimensional, the foregoing general solution is referred to as the Stroh quartic formalism. Furthermore, the explicit expressions of the three 2×2 real matrices $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ comprising the 4×4

fundamental piezoelectricity matrix $\mathbf{N}$ are given by

$$
(2.9)\quad
\begin{aligned}
&\mathbf{N}_1 = -\begin{bmatrix} \dfrac{C_{45}\in_{22} + e_{24}e_{25}}{\tilde{C}_{44}\in_{22}} & \dfrac{\in_{22}e_{14} - \in_{12}e_{24}}{\tilde{C}_{44}\in_{22}} \\ \dfrac{C_{45}e_{24} - C_{44}e_{25}}{\tilde{C}_{44}\in_{22}} & \dfrac{C_{44}\in_{12} + e_{14}e_{24}}{\tilde{C}_{44}\in_{22}} \end{bmatrix},\\
&\mathbf{N}_2 = \mathbf{N}_2^T = \begin{bmatrix} \dfrac{1}{\tilde{C}_{44}} & \dfrac{e_{24}}{\tilde{C}_{44}\in_{22}} \\ \dfrac{e_{24}}{\tilde{C}_{44}\in_{22}} & -\dfrac{C_{44}}{\tilde{C}_{44}\in_{22}} \end{bmatrix},\\
&(\mathbf{N}_3)_{11} = \frac{\in_{22}(C_{45}^2 - C_{55}\tilde{C}_{44}) + 2C_{45}e_{24}e_{25} - C_{44}e_{25}^2}{\tilde{C}_{44}\in_{22}} < 0,\\
&(\mathbf{N}_3)_{12} = (\mathbf{N}_3)_{21}\\
&\qquad = \frac{e_{14}(C_{45}\in_{22} + e_{24}e_{25}) + \in_{12}(C_{44}e_{25} - C_{45}e_{24}) - \tilde{C}_{44}\in_{22}e_{15}}{\tilde{C}_{44}\in_{22}},\\
&(\mathbf{N}_3)_{22} = \frac{\in_{22}(e_{14}^2 + \tilde{C}_{44}\in_{11}) - 2\in_{12}e_{14}e_{24} - C_{44}\in_{12}^2}{\tilde{C}_{44}\in_{22}} > 0,
\end{aligned}
$$

where

$$
(2.10)\qquad \tilde{C}_{44} = C_{44} + \frac{e_{24}^2}{\in_{22}}.
$$

The generalized stress function vector $\boldsymbol{\varphi}$ is defined, in terms of the anti-plane stresses and in-plane electric displacements, as follows

$$
(2.11)\qquad
\begin{aligned}
\sigma_{31} &= -\varphi_{1,2}, & \sigma_{32} &= \varphi_{1,1},\\
D_1 &= -\varphi_{2,2}, & D_2 &= \varphi_{2,1}.
\end{aligned}
$$

The two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following orthogonality relations [10]

$$
(2.12)\qquad
\begin{aligned}
\mathbf{B}^T\mathbf{A} + \mathbf{A}^T\mathbf{B} = \mathbf{I} = \bar{\mathbf{B}}^T\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\bar{\mathbf{B}},\\
\mathbf{B}^T\bar{\mathbf{A}} + \mathbf{A}^T\bar{\mathbf{B}} = \mathbf{0} = \bar{\mathbf{B}}^T\mathbf{A} + \bar{\mathbf{A}}^T\mathbf{B},
\end{aligned}
$$

so that we can introduce the following three real generalized Barnett–Lothe tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ [10],

$$
(2.13)\qquad \mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^T.
$$

The generalized Barnett–Lothe tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ are related by

$$
(2.14)\qquad \mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{I}.
$$

In addition, the two matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric and but not positive definite. More precisely $\mathbf{H}$ and $\mathbf{L}$ are matrices of type $\mathbf{T}$ [10]. The explicit expressions of the three generalized Barnett–Lothe tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ for the anti-plane shear deformations of a monoclinic piezoelectric material have recently been obtained by Wang and Schiavone [11]. For a transversely isotropic piezoelectric material with poling in the x_3-direction, we have:

$$\mathbf{S} = \mathbf{0}, \qquad \mathbf{L} = \mathbf{H}^{-1} = \begin{bmatrix} C_{44} & e_{15} \\ e_{15} & -\in_{11} \end{bmatrix}. \tag{2.15}$$

The following identities can be proved following the idea by Ting [10]:

$$\begin{bmatrix} \mathbf{A}\left\langle z_*^n \right\rangle \mathbf{B}^T & \mathbf{A}\left\langle z_*^n \right\rangle \mathbf{A}^T \\ \mathbf{B}\left\langle z_*^n \right\rangle \mathbf{B}^T & \mathbf{B}\left\langle z_*^n \right\rangle \mathbf{A}^T \end{bmatrix} = \frac{1}{2}(x_1\mathbf{I} + x_2\mathbf{N})^n(\mathbf{I} - \mathrm{i}\tilde{\mathbf{N}}), \tag{2.16}$$

where n is a non-negative integer, $\left\langle z_*^n \right\rangle = \operatorname{diag}\left[\, z_1^n \;\; z_2^n \,\right]$ and

$$\tilde{\mathbf{N}} = \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix}, \qquad \tilde{\mathbf{N}}^2 = -\mathbf{I}. \tag{2.17}$$

3. Three-phase anisotropic piezoelectric elliptical inhomogeneity

As shown in Fig. 1, we consider a three-phase anisotropic piezoelectric elliptical inhomogeneity with two confocal elliptical interfaces. Let Ω, Θ and Ψ denote the piezoelectric elliptical inhomogeneity, the piezoelectric interphase layer and

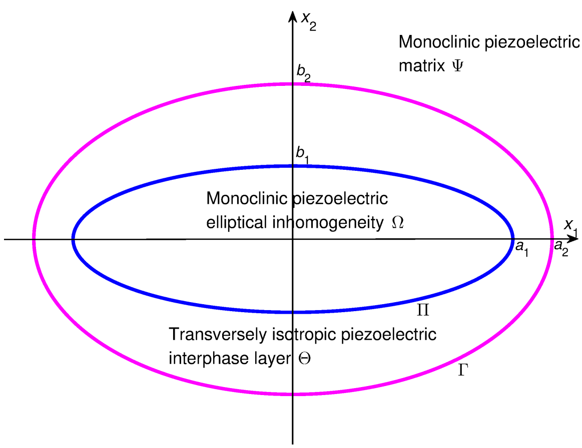

Fig. 1. A three-phase anisotropic piezoelectric elliptical inhomogeneity with two confocal elliptical interfaces under uniform remote anti-plane mechanical and in-plane electrical loading.

the infinite piezoelectric matrix, respectively, all of which are perfectly bonded through two confocal elliptical interfaces Π: $x_1^2/a_1^2 + x_2^2/b_1^2 = 1$ (with a_1 and b_1 representing, respectively, the semi-major and semi-minor axes of the ellipse Π) and Γ: $x_1^2/a_2^2 + x_2^2/b_2^2 = 1$ (a_2 and b_2 are, respectively, the semi-major and semi-minor axes of the ellipse Γ). The following condition ensures the confocal character of the two elliptical interfaces: $a_1^2 - b_1^2 = a_2^2 - b_2^2$. The inhomogeneity and the matrix are both monoclinic piezoelectric materials with symmetry plane at $x_3 = 0$ and with poling in the x_3-direction, while the interphase layer is composed of a transversely isotropic piezoelectric material with poling in the x_3-direction. The matrix is subjected to uniform remote anti-plane mechanical loading in both stresses and strains: σ_{31}^∞, σ_{32}^∞, ε_{31}^∞, ε_{32}^∞ and in-plane electrical loading in electric displacements and electric fields: D_1^∞, D_2^∞, E_1^∞, E_2^∞. It is shown in the subsequent analysis that the remote stresses, strains, electric displacements and electric fields are not independent and must satisfy a particular relationship. In what follows, the superscripts Ω and Θ refer to the elliptical inhomogeneity and the intermediate interphase layer, respectively, while quantities associated with the matrix have no attached superscript.

We first consider the following conformal mapping function:

$$\begin{aligned} z &= \omega(\xi) = R(\xi + m\xi^{-1}), \qquad 1 \le |\xi| \le \rho^{-1/2}, \\ \xi &= \omega^{-1}(z) = \frac{z + \sqrt{z^2 - 4mR^2}}{2R}, \end{aligned} \tag{3.1}$$

where $z \equiv x_1 + \mathrm{i}x_2$, and

$$R = \frac{a_1 + b_1}{2}, \; m = \frac{a_1 - b_1}{a_1 + b_1}, \; \rho = \frac{(a_1 + b_1)^2}{(a_2 + b_2)^2} = \frac{(a_2 - b_2)^2}{(a_1 - b_1)^2} \quad (0 < \rho < 1). \tag{3.2}$$

As shown in Fig. 2, using the mapping function in Eq. (3.1), the interphase layer in the z-plane is mapped onto the annulus $1 \le |\xi| \le \rho^{-1/2}$ in the ξ-plane, the inner elliptical interface Π is mapped onto the unit circle in the ξ-plane and the outer confocal elliptical interface Γ is mapped onto the circle $|\xi| = \rho^{-1/2}$ in the ξ-plane.

We then consider the following mapping functions:

$$\begin{aligned} z_k &= \omega_k(\xi_k) = \frac{a_2 - \mathrm{i}p_k b_2}{2}\xi_k + \frac{a_2 + \mathrm{i}p_k b_2}{2}\xi_k^{-1}, \qquad |\xi_k| \ge 1, \\ \xi_k &= \omega_k^{-1}(z_k) = \frac{z_k + \sqrt{z_k^2 - (a_2^2 + p_k^2 b_2^2)}}{a_2 - \mathrm{i}p_k b_2}, \qquad k = 1, 2. \end{aligned} \tag{3.3}$$

Using the mapping functions in Eq. (3.3), the outer elliptical interface Γ is mapped onto the unit circle in the ξ_k-plane, and the region outside the elliptical interface Γ is mapped onto the exterior of the unit circle in the ξ_k-plane.

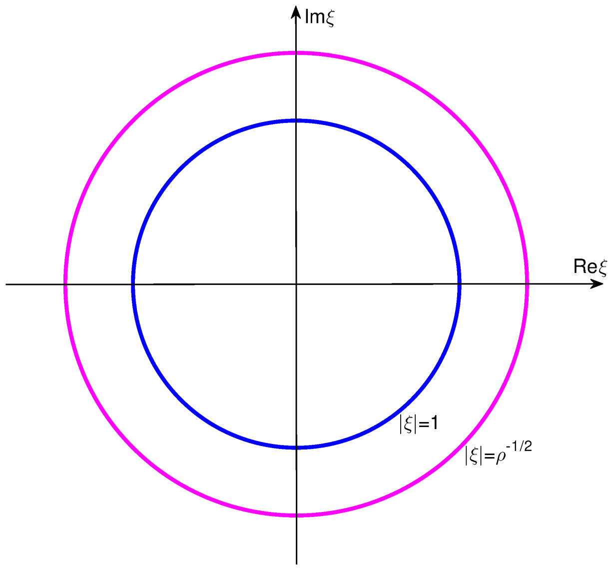

Fig. 2. The image ξ-plane.

Solutions in the three phases of the elliptical inhomogeneity, the interphase layer and the matrix can be constructed as follows:

$$\text{(3.4)}\qquad \begin{bmatrix}\mathbf{u}^{\Omega}\\ \boldsymbol{\varphi}^{\Omega}\end{bmatrix} = x_1\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\Omega}\\ \mathbf{t}_2^{\Omega}\end{bmatrix} + x_2\begin{bmatrix}\boldsymbol{\varepsilon}_2^{\Omega}\\ -\mathbf{t}_1^{\Omega}\end{bmatrix}, \qquad z\in\Omega,$$

$$\text{(3.5)}\qquad \begin{bmatrix}\mathbf{u}^{\Theta}\\ \boldsymbol{\varphi}^{\Theta}\end{bmatrix} = \mathrm{Re}\{\xi^{-1}(\mathbf{I}-\mathrm{i}\tilde{\mathbf{N}}^{\Theta})\}\begin{bmatrix}\mathbf{h}_1\\ \mathbf{g}_1\end{bmatrix} + \mathrm{Re}\{\xi(\mathbf{I}-\mathrm{i}\tilde{\mathbf{N}}^{\Theta})\}\begin{bmatrix}\mathbf{h}_2\\ \mathbf{g}_2\end{bmatrix}, \qquad z\in\Theta,$$

$$\text{(3.6)}\qquad \begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varphi}\end{bmatrix} = x_1\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix} + x_2\begin{bmatrix}\boldsymbol{\varepsilon}_2^{\infty}\\ -\mathbf{t}_1^{\infty}\end{bmatrix} + 2\mathrm{Re}\begin{bmatrix}\mathbf{A}\langle\xi_*^{-1}\rangle\mathbf{B}^T & \mathbf{A}\langle\xi_*^{-1}\rangle\mathbf{A}^T\\ \mathbf{B}\langle\xi_*^{-1}\rangle\mathbf{B}^T & \mathbf{B}\langle\xi_*^{-1}\rangle\mathbf{A}^T\end{bmatrix}\begin{bmatrix}\mathbf{h}_3\\ \mathbf{g}_3\end{bmatrix}, \qquad z\in\Psi,$$

where $\boldsymbol{\varepsilon}_1^{\Omega}$, ε_2^{Ω}, $\mathbf{t}_1^{\Omega}$, $\mathbf{t}_2^{\Omega}$, $\mathbf{h}_1$, $\mathbf{g}_1$, $\mathbf{h}_2$, $\mathbf{g}_2$, $\mathbf{h}_3$, $\mathbf{g}_3$ are ten unknown two-dimensional real constant vectors to be determined, and

$$\text{(3.7)}\quad \boldsymbol{\varepsilon}_1^{\infty} = \begin{bmatrix}2\varepsilon_{31}^{\infty}\\ -E_1^{\infty}\end{bmatrix}, \qquad \varepsilon_2^{\infty} = \begin{bmatrix}2\varepsilon_{32}^{\infty}\\ -E_2^{\infty}\end{bmatrix}, \qquad \mathbf{t}_1^{\infty} = \begin{bmatrix}\sigma_{31}^{\infty}\\ D_1^{\infty}\end{bmatrix}, \qquad \mathbf{t}_2^{\infty} = \begin{bmatrix}\sigma_{32}^{\infty}\\ D_2^{\infty}\end{bmatrix}.$$

It can be proved quite easily by using the identity in Eq. (2.16) with $n=1$ that

$$\text{(3.8)}\qquad \begin{bmatrix}\boldsymbol{\varepsilon}_2^{\Omega}\\ -\mathbf{t}_1^{\Omega}\end{bmatrix} = \mathbf{N}^{\Omega}\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\Omega}\\ \mathbf{t}_2^{\Omega}\end{bmatrix}, \qquad \begin{bmatrix}\boldsymbol{\varepsilon}_2^{\infty}\\ -\mathbf{t}_1^{\infty}\end{bmatrix} = \mathbf{N}\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix},$$

which provide a relationship among the uniform stresses, strains, electric displacements and electric fields within the piezoelectric inhomogeneity and also provide a relationship among the uniform remote stresses, strains, electric displacements and electric fields applied in the matrix. The relationships in Eq. (3.8) are equivalent to the constitutive equations in Eq. (2.1) for the inhomogeneity and the matrix.

In writing Eq. (3.5) for the interphase layer, we have utilized the identity in Eq. (2.16) with $n = 0$. In addition, the 4×4 matrix $\tilde{\mathbf{N}}^{\Theta}$ for the interphase layer composed of a transversely isotropic piezoelectric material has the following explicit expression

$$\tilde{\mathbf{N}}^{\Theta} = \begin{bmatrix} \mathbf{0} & (\mathbf{L}^{\Theta})^{-1} \\ -\mathbf{L}^{\Theta} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{L}^{\Theta} = (\mathbf{H}^{\Theta})^{-1} = \begin{bmatrix} C_{44}^{\Theta} & e_{15}^{\Theta} \\ e_{15}^{\Theta} & -\in_{11}^{\Theta} \end{bmatrix}. \tag{3.9}$$

At the elliptical boundaries Π and Γ, we have:

$$x_1 = a_1 \cos\psi, \quad x_2 = b_1 \sin\psi, \quad \xi = \mathrm{e}^{\mathrm{i}\psi}, \quad \xi^{-1} = \mathrm{e}^{-\mathrm{i}\psi}, \quad z \in \Pi; \tag{3.10}$$

$$\xi = \rho^{-1/2}\mathrm{e}^{\mathrm{i}\psi}, \quad \xi^{-1} = \rho^{1/2}\mathrm{e}^{-\mathrm{i}\psi}, \quad z \in \Gamma; \tag{3.11}$$

$$x_1 = a_2 \cos\psi, \quad x_2 = b_2 \sin\psi, \quad \xi_1^{-1} = \xi_2^{-1} = \mathrm{e}^{-\mathrm{i}\psi}, \quad z \in \Gamma. \tag{3.12}$$

Making use of the identity in Eq. (16) with n=0, Eqs. (3.4)–(3.6) on the two confocal elliptical interfaces Π and Γ have the following expressions:

$$\begin{bmatrix} \mathbf{u}_{\Pi}^{\Omega} \\ \boldsymbol{\varphi}_{\Pi}^{\Omega} \end{bmatrix} = a_1 \cos\psi \begin{bmatrix} \boldsymbol{\varepsilon}_1^{\Omega} \\ \mathbf{t}_2^{\Omega} \end{bmatrix} + b_1 \sin\psi \begin{bmatrix} \boldsymbol{\varepsilon}_2^{\Omega} \\ -\mathbf{t}_1^{\Omega} \end{bmatrix}, \tag{3.13}$$

$$\begin{bmatrix} \mathbf{u}_{\Pi}^{\Theta} \\ \boldsymbol{\varphi}_{\Pi}^{\Theta} \end{bmatrix} = \cos\psi \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} - \sin\psi \tilde{\mathbf{N}}^{\Theta} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} + \cos\psi \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} + \sin\psi \tilde{\mathbf{N}}^{\Theta} \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix}, \tag{3.14}$$

$$\begin{aligned} \begin{bmatrix} \mathbf{u}_{\Gamma}^{\Theta} \\ \boldsymbol{\varphi}_{\Gamma}^{\Theta} \end{bmatrix} &= \rho^{1/2} \cos\psi \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} - \rho^{1/2} \sin\psi \tilde{\mathbf{N}}^{\Theta} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} \\ &\quad + \rho^{-1/2} \cos\psi \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} + \rho^{-1/2} \sin\psi \tilde{\mathbf{N}}^{\Theta} \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix}, \end{aligned} \tag{3.15}$$

$$\begin{bmatrix} \mathbf{u}_{\Gamma} \\ \boldsymbol{\varphi}_{\Gamma} \end{bmatrix} = a_2 \cos\psi \begin{bmatrix} \boldsymbol{\varepsilon}_1^{\infty} \\ \mathbf{t}_2^{\infty} \end{bmatrix} + b_2 \sin\psi \begin{bmatrix} \boldsymbol{\varepsilon}_2^{\infty} \\ -\mathbf{t}_1^{\infty} \end{bmatrix} + \cos\psi \begin{bmatrix} \mathbf{h}_3 \\ \mathbf{g}_3 \end{bmatrix} - \sin\psi \tilde{\mathbf{N}} \begin{bmatrix} \mathbf{h}_3 \\ \mathbf{g}_3 \end{bmatrix}. \tag{3.16}$$

The continuity conditions of traction, displacement, normal electric displacement and electric potential across the two confocal elliptical interfaces Π and Γ require that

$$\begin{bmatrix} \mathbf{u}_{\Pi}^{\Theta} \\ \boldsymbol{\varphi}_{\Pi}^{\Theta} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_{\Pi}^{\Omega} \\ \boldsymbol{\varphi}_{\Pi}^{\Omega} \end{bmatrix}, \tag{3.17}$$

$$\begin{bmatrix} \mathbf{u}_\Gamma \\ \boldsymbol{\varphi}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{u}_\Gamma^\Theta \\ \boldsymbol{\varphi}_\Gamma^\Theta \end{bmatrix}. \tag{3.18}$$

By substituting Eqs. (3.13) and (3.14) into Eq. (3.17) and equating coefficients of $\cos\psi$ and $\sin\psi$, substituting Eqs. (3.15) and (3.16) into Eq. (3.18) and again equating coefficients of $\cos\psi$ and $\sin\psi$, we arrive at the following relationships

$$\begin{gathered} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} = a_1 \begin{bmatrix} \boldsymbol{\varepsilon}_1^\Omega \\ \mathbf{t}_2^\Omega \end{bmatrix}, \\ \tilde{\mathbf{N}}^\Theta \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} - \tilde{\mathbf{N}}^\Theta \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} = b_1 \begin{bmatrix} \boldsymbol{\varepsilon}_2^\Omega \\ -\mathbf{t}_1^\Omega \end{bmatrix}, \end{gathered} \tag{3.19}$$

and

$$\begin{gathered} \rho^{1/2} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} + \rho^{-1/2} \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} = a_2 \begin{bmatrix} \boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty \end{bmatrix} + \begin{bmatrix} \mathbf{h}_3 \\ \mathbf{g}_3 \end{bmatrix}, \\ \rho^{-1/2}\tilde{\mathbf{N}}^\Theta \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} - \rho^{1/2}\tilde{\mathbf{N}}^\Theta \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} = b_2 \begin{bmatrix} \boldsymbol{\varepsilon}_2^\infty \\ -\mathbf{t}_1^\infty \end{bmatrix} - \tilde{\mathbf{N}} \begin{bmatrix} \mathbf{h}_3 \\ \mathbf{g}_3 \end{bmatrix}. \end{gathered} \tag{3.20}$$

All the ten real constant vectors $\boldsymbol{\varepsilon}_1^\Omega$, $\boldsymbol{\varepsilon}_2^\Omega$, $\mathbf{t}_1^\Omega$, $\mathbf{t}_2^\Omega$, $\mathbf{h}_1$, $\mathbf{g}_1$, $\mathbf{h}_2$, $\mathbf{g}_2$, $\mathbf{h}_3$, $\mathbf{g}_3$ can be uniquely determined from Eqs. (3.19) and (3.20) together with the relationships in Eq. (3.8) as follows:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\varepsilon}_1^\Omega \\ \mathbf{t}_2^\Omega \end{bmatrix} &= \frac{a_2}{a_1}\rho^{1/2}\left[\mathbf{I} - \rho(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^\Theta)^{-1}(\tilde{\mathbf{N}}-\tilde{\mathbf{N}}^\Theta)\right] \\ &\times \left[\tilde{\mathbf{N}}^\Theta + \frac{b_1}{a_1}\mathbf{N}^\Omega + \rho\left(\tilde{\mathbf{N}}^\Theta - \frac{b_1}{a_1}\mathbf{N}^\Omega\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^\Theta)^{-1}(\tilde{\mathbf{N}}-\tilde{\mathbf{N}}^\Theta)\right]^{-1} \\ &\times \left(\tilde{\mathbf{N}}^\Theta - \frac{b_1}{a_1}\mathbf{N}^\Omega\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^\Theta)^{-1}\left(\tilde{\mathbf{N}} + \frac{b_2}{a_2}\mathbf{N}\right)\begin{bmatrix} \boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty \end{bmatrix} \\ &+ \frac{a_2}{a_1}\rho^{1/2}(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^\Theta)^{-1}\left(\tilde{\mathbf{N}} + \frac{b_2}{a_2}\mathbf{N}\right)\begin{bmatrix} \boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty \end{bmatrix}, \\ \begin{bmatrix} \boldsymbol{\varepsilon}_2^\Omega \\ -\mathbf{t}_1^\Omega \end{bmatrix} &= \mathbf{N}^\Omega \begin{bmatrix} \boldsymbol{\varepsilon}_1^\Omega \\ \mathbf{t}_2^\Omega \end{bmatrix}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} &= a_2\rho^{1/2}\left[\tilde{\mathbf{N}}^\Theta + \frac{b_1}{a_1}\mathbf{N}^\Omega + \rho\left(\tilde{\mathbf{N}}^\Theta - \frac{b_1}{a_1}\mathbf{N}^\Omega\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^\Theta)^{-1}(\tilde{\mathbf{N}}-\tilde{\mathbf{N}}^\Theta)\right]^{-1} \\ &\times \left(\tilde{\mathbf{N}}^\Theta - \frac{b_1}{a_1}\mathbf{N}^\Omega\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^\Theta)^{-1}\left(\tilde{\mathbf{N}} + \frac{b_2}{a_2}\mathbf{N}\right)\begin{bmatrix} \boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty \end{bmatrix}, \end{aligned} \tag{3.22}$$

$$\begin{bmatrix}\mathbf{h}_2\\ \mathbf{g}_2\end{bmatrix} = a_2\rho^{1/2}(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}\left(\tilde{\mathbf{N}}+\frac{b_2}{a_2}\mathbf{N}\right)\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix}$$
$$-a_2\rho^{\frac{3}{2}}(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}(\tilde{\mathbf{N}}-\tilde{\mathbf{N}}^{\Theta})$$
$$\times\left[\tilde{\mathbf{N}}^{\Theta}+\frac{b_1}{a_1}\mathbf{N}^{\Omega}+\rho\left(\tilde{\mathbf{N}}^{\Theta}-\frac{b_1}{a_1}\mathbf{N}^{\Omega}\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}(\tilde{\mathbf{N}}-\tilde{\mathbf{N}}^{\Theta})\right]^{-1}$$
$$\times\left(\tilde{\mathbf{N}}^{\Theta}-\frac{b_1}{a_1}\mathbf{N}^{\Omega}\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}\left(\tilde{\mathbf{N}}+\frac{b_2}{a_2}\mathbf{N}\right)\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix},$$

$$\begin{bmatrix}\mathbf{h}_3\\ \mathbf{g}_3\end{bmatrix} = 2a_2\rho(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}\tilde{\mathbf{N}}^{\Theta} \tag{3.23}$$
$$\times\left[\tilde{\mathbf{N}}^{\Theta}+\frac{b_1}{a_1}\mathbf{N}^{\Omega}+\rho\left(\tilde{\mathbf{N}}^{\Theta}-\frac{b_1}{a_1}\mathbf{N}^{\Omega}\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}(\tilde{\mathbf{N}}-\tilde{\mathbf{N}}^{\Theta})\right]^{-1}$$
$$\times\left(\tilde{\mathbf{N}}^{\Theta}-\frac{b_1}{a_1}\mathbf{N}^{\Omega}\right)(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}\left(\tilde{\mathbf{N}}+\frac{b_2}{a_2}\mathbf{N}\right)\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix}$$
$$+a_2(\tilde{\mathbf{N}}+\tilde{\mathbf{N}}^{\Theta})^{-1}\left(\frac{b_2}{a_2}\mathbf{N}-\tilde{\mathbf{N}}^{\Theta}\right)\begin{bmatrix}\boldsymbol{\varepsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix}.$$

We can see that all ten real constant vectors ε_1^{Ω}, $\boldsymbol{\varepsilon}_2^{\Omega}$, $\mathbf{t}_1^{\Omega}$, $\mathbf{t}_2^{\Omega}$, $\mathbf{h}_1$, $\mathbf{g}_1$, $\mathbf{h}_2$, $\mathbf{g}_2$, $\mathbf{h}_3$, $\mathbf{g}_3$ have been determined in real-form in terms of the four 4×4 real matrices $\mathbf{N}^{\Omega}$ for the elliptical inhomogeneity, $\tilde{\mathbf{N}}^{\Theta}$ for the interphase layer and $\mathbf{N}$, $\tilde{\mathbf{N}}$ for the matrix. The explicit expressions of the fundamental piezoelectricity matrices $\mathbf{N}^{\Omega}$ and $\mathbf{N}$ can be readily obtained from Eq. (2.9), the explicit expression of $\tilde{\mathbf{N}}^{\Theta}$ is given by Eq. (3.9), and the explicit expression of $\tilde{\mathbf{N}}$ can be found in Wang and Schiavone [11]. For example, if the piezoelectric matrix is orthotropic, we have [11]

$$\tilde{\mathbf{N}} = \begin{bmatrix}\mathbf{0} & \mathbf{H}\\ -\mathbf{H}^{-1} & \mathbf{0}\end{bmatrix}, \qquad \mathbf{H} = \mathbf{L}^{-1} = \begin{bmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{12} & -\lambda_{22}\end{bmatrix}, \tag{3.24}$$

where

$$\begin{aligned}
\lambda_{11} &= \frac{\in_{11}+\delta\in_{22}}{\sqrt{2\delta(s+1)(C_{44}\in_{22}+e_{24}^2)(C_{55}\in_{11}+e_{15}^2)}} > 0,\\
\lambda_{12} &= \frac{e_{15}+\delta e_{24}}{\sqrt{2\delta(s+1)(C_{44}\in_{22}+e_{24}^2)(C_{55}\in_{11}+e_{15}^2)}},\\
\lambda_{22} &= \frac{C_{55}+\delta C_{44}}{\sqrt{2\delta(s+1)(C_{44}\in_{22}+e_{24}^2)(C_{55}\in_{11}+e_{15}^2)}} > 0,
\end{aligned} \tag{3.25}$$

with

$$\delta = \sqrt{\frac{C_{55}\in_{11}+e_{15}^2}{C_{44}\in_{22}+e_{24}^2}}, \qquad s = \frac{C_{44}\in_{11}+C_{55}\in_{22}+2e_{15}e_{24}}{2\sqrt{(C_{55}\in_{11}+e_{15}^2)(C_{44}\in_{22}+e_{24}^2)}} > -1. \tag{3.26}$$

It is further seen from the above analysis the internal electroelastic field of stresses, strains, electric displacements and electric fields characterized by the four real vectors $\boldsymbol{\varepsilon}_1^\Omega$, $\boldsymbol{\varepsilon}_2^\Omega$, $\mathbf{t}_1^\Omega$, $\mathbf{t}_2^\Omega$ in Eq. (3.4) is uniform inside the elliptical inhomogeneity. The internal uniform electroelastic field inside the elliptical inhomogeneity is unconditional since there is no other restriction on the remote loading except the required constitutive relationship in Eq. $(3.8)_2$. The unconditional uniformity, which stems from the fact that the transversely isotropic piezoelectric material occupying the interphase layer is mathematically non-degenerate [12], is different from the conditional uniformity of the internal stresses in plane isotropic elasticity [5] and in generalized plane strain deformations for anisotropic elastic materials [8]. In addition, the internal uniform electroelastic field inside the elliptical inhomogeneity and the non-uniform electroelastic field everywhere in the interphase layer characterized by the four real vectors $\mathbf{h}_1$, $\mathbf{g}_1$, $\mathbf{h}_2$, $\mathbf{g}_2$ in Eq. (3.5) are obtained in real-form in terms of the four real matrices $\mathbf{N}^\Omega$, $\tilde{\mathbf{N}}^\Theta$, $\mathbf{N}$ and $\tilde{\mathbf{N}}$. More specifically, the internal uniform electroelastic field inside the elliptical inhomogeneity is given explicitly by

$$(3.27)\quad \begin{bmatrix} 2\varepsilon_{31} \\ -E_1 \end{bmatrix} = \boldsymbol{\varepsilon}_1^\Omega, \quad \begin{bmatrix} 2\varepsilon_{32} \\ -E_2 \end{bmatrix} = \boldsymbol{\varepsilon}_2^\Omega, \quad \begin{bmatrix} \sigma_{31} \\ D_1 \end{bmatrix} = \mathbf{t}_1^\Omega, \quad \begin{bmatrix} \sigma_{32} \\ D_2 \end{bmatrix} = \mathbf{t}_2^\Omega, \quad z \in \Omega,$$

and the non-uniform electroelastic field in the interphase layer is explicitly determined by

$$(3.28)\quad \begin{bmatrix} 2(\varepsilon_{31} - \mathrm{i}\varepsilon_{32}) \\ -E_1 + \mathrm{i}E_2 \\ \sigma_{32} + \mathrm{i}\sigma_{31} \\ D_2 + \mathrm{i}D_1 \end{bmatrix} = \frac{1}{R(\xi^2 - m)}(\mathbf{I} - \mathrm{i}\tilde{\mathbf{N}}^\Theta)\left(\xi^2 \begin{bmatrix} \mathbf{h}_2 \\ \mathbf{g}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix}\right), \quad z \in \Theta.$$

In particular, it is deduced from Eq. (3.28) with the aid of Eqs. (3.9) and (3.19) that

$$(3.29\mathrm{a})\quad \begin{aligned} \begin{bmatrix} 2(\varepsilon_{31} - \mathrm{i}\varepsilon_{32}) \\ -E_1 + \mathrm{i}E_2 \end{bmatrix} &= (\mathbf{L}^\Theta)^{-1}\mathbf{t}_1^\Omega - \mathrm{i}\boldsymbol{\varepsilon}_2^\Omega, \\ \begin{bmatrix} \sigma_{32} + \mathrm{i}\sigma_{31} \\ D_2 + \mathrm{i}D_1 \end{bmatrix} &= \mathbf{L}^\Theta\varepsilon_2^\Omega + \mathrm{i}\mathbf{t}_1^\Omega \quad \text{at } z = \pm a_1 \text{ and } z \in \Theta; \end{aligned}$$

$$(3.29\mathrm{b})\quad \begin{aligned} \begin{bmatrix} 2(\varepsilon_{31} - \mathrm{i}\varepsilon_{32}) \\ -E_1 + \mathrm{i}E_2 \end{bmatrix} &= \boldsymbol{\varepsilon}_1^\Omega - \mathrm{i}(\mathbf{L}^\Theta)^{-1}\mathbf{t}_2^\Omega, \\ \begin{bmatrix} \sigma_{32} + \mathrm{i}\sigma_{31} \\ D_2 + \mathrm{i}D_1 \end{bmatrix} &= \mathbf{t}_2^\Omega + \mathrm{i}\mathbf{L}^\Theta\boldsymbol{\varepsilon}_1^\Omega \quad \text{at } z = \pm \mathrm{i}b_1 \text{ and } z \in \Theta. \end{aligned}$$

It is clearly seen from Eqs. (3.29a) and (3.29b) that the continuity conditions of traction, displacement, normal electric displacement and electric potential have indeed been satisfied when crossing the interface Π at the four points: $z = \pm a_1, \ \pm ib_1$.

When both the inhomogeneity and the matrix are orthotropic, it is deduced from Eq. (3.21) that

$$\boldsymbol{\varepsilon}_1^{\Omega} = \mathbf{Y}_1 \varepsilon_1^{\infty}, \ \varepsilon_2^{\Omega} = \mathbf{N}_2^{\Omega} \mathbf{Y}_2 \mathbf{t}_2^{\infty}, \ \mathbf{t}_1^{\Omega} = -\mathbf{N}_3^{\Omega} \mathbf{Y}_1 \boldsymbol{\varepsilon}_1^{\infty}, \ \mathbf{t}_2^{\Omega} = \mathbf{Y}_2 \mathbf{t}_2^{\infty}, \tag{3.30}$$

where the two 2×2 real matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are defined by

$$\begin{aligned}
\mathbf{Y}_1 &= \frac{a_2}{a_1}\rho^{1/2}[\mathbf{I}+\rho(\mathbf{H}^{-1}+\mathbf{L}^{\Theta})^{-1}(\mathbf{L}^{\Theta}-\mathbf{H}^{-1})] \\
&\quad \times \left[-\mathbf{L}^{\Theta}+\frac{b_1}{a_1}\mathbf{N}_3^{\Omega}+\rho\left(\mathbf{L}^{\Theta}+\frac{b_1}{a_1}\mathbf{N}_3^{\Omega}\right)(\mathbf{H}^{-1}+\mathbf{L}^{\Theta})^{-1}(\mathbf{L}^{\Theta}-\mathbf{H}^{-1})\right]^{-1} \\
&\quad \times \left(\mathbf{L}^{\Theta}+\frac{b_1}{a_1}\mathbf{N}_3^{\Omega}\right)(\mathbf{H}^{-1}+\mathbf{L}^{\Theta})^{-1}\left(\frac{b_2}{a_2}\mathbf{N}_3-\mathbf{H}^{-1}\right) \\
&\quad + \frac{a_2}{a_1}\rho^{1/2}(\mathbf{H}^{-1}+\mathbf{L}^{\Theta})^{-1}\left(\mathbf{H}^{-1}-\frac{b_2}{a_2}\mathbf{N}_3\right),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\mathbf{Y}_2 &= \frac{a_2}{a_1}\rho^{1/2}\left[\mathbf{I}+\rho[\mathbf{H}+(\mathbf{L}^{\Theta})^{-1}]^{-1}[(\mathbf{L}^{\Theta})^{-1}-\mathbf{H}]\right] \\
&\times \left[(\mathbf{L}^{\Theta})^{-1}+\frac{b_1}{a_1}\mathbf{N}_2^{\Omega}+\rho\left[(\mathbf{L}^{\Theta})^{-1}-\frac{b_1}{a_1}\mathbf{N}_2^{\Omega}\right]\left[\mathbf{H}+(\mathbf{L}^{\Theta})^{-1}\right]^{-1}[\mathbf{H}-(\mathbf{L}^{\Theta})^{-1}]\right]^{-1} \\
&\quad \times \left[(\mathbf{L}^{\Theta})^{-1}-\frac{b_1}{a_1}\mathbf{N}_2^{\Omega}\right]\left[\mathbf{H}+(\mathbf{L}^{\Theta})^{-1}\right]^{-1}\left(\mathbf{H}+\frac{b_2}{a_2}\mathbf{N}_2\right) \\
&\quad + \frac{a_2}{a_1}\rho^{1/2}\left[\mathbf{H}+(\mathbf{L}^{\Theta})^{-1}\right]^{-1}\left(\mathbf{H}+\frac{b_2}{a_2}\mathbf{N}_2\right),
\end{aligned}$$

and

$$\mathbf{N}_2^{\Omega} = \begin{bmatrix} \dfrac{1}{\tilde{C}_{44}^{\Omega}} & \dfrac{e_{24}^{\Omega}}{\tilde{C}_{44}^{\Omega}\in_{22}^{\Omega}} \\ \dfrac{e_{24}^{\Omega}}{\tilde{C}_{44}^{\Omega}\in_{22}^{\Omega}} & -\dfrac{C_{44}^{\Omega}}{\tilde{C}_{44}^{\Omega}\in_{22}^{\Omega}} \end{bmatrix}, \ \mathbf{N}_3^{\Omega} = -\begin{bmatrix} C_{55}^{\Omega} & e_{15}^{\Omega} \\ e_{15}^{\Omega} & -\in_{11}^{\Omega} \end{bmatrix}. \tag{3.32}$$

When the matrix is transversely isotropic and has the same electroelastic constants as the interphase layer, it is deduced from Eq. (3.21) that

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\Omega} \\ \mathbf{t}_2^{\Omega} \end{bmatrix} &= \left[\tilde{\mathbf{N}}+\frac{b_1}{a_1}\mathbf{N}^{\Omega}\right]^{-1}\left(\tilde{\mathbf{N}}+\frac{b_1}{a_1}\mathbf{N}\right)\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\infty} \\ \mathbf{t}_2^{\infty} \end{bmatrix} \\
&= \left(1+\frac{b_1}{a_1}\right)\left[\mathbf{N}+\frac{b_1}{a_1}\mathbf{N}^{\Omega}\right]^{-1}\mathbf{N}\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\infty} \\ \mathbf{t}_2^{\infty} \end{bmatrix}, \\
\begin{bmatrix} \boldsymbol{\varepsilon}_2^{\Omega} \\ -\mathbf{t}_1^{\Omega} \end{bmatrix} &= \mathbf{N}^{\Omega}\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\Omega} \\ \mathbf{t}_2^{\Omega} \end{bmatrix},
\end{aligned} \tag{3.33}$$

where $\mathbf{N}^{\Omega}$ is determined by Eq. (2.9) with the electroelastic constants pertaining to the inhomogeneity, and

$$\tilde{\mathbf{N}} = \mathbf{N} = \tilde{\mathbf{N}}^{\Theta} = \mathbf{N}^{\Theta} = \begin{bmatrix} \mathbf{0} & \mathbf{L}^{-1} \\ -\mathbf{L} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{L} = \mathbf{H}^{-1} = \begin{bmatrix} C_{44} & e_{15} \\ e_{15} & -\in_{11} \end{bmatrix}. \tag{3.34}$$

Within the framework of the Stroh formalism, Eq. $(3.33)_1$ is found to be consistent with Eqs. (10.7-8) and (10.7-9) in TING [10].

4. Conclusions

We have proved the uniformity of the electroelastic field inside a monoclinic piezoelectric elliptical inhomogeneity bonded to a monoclinic piezoelectric matrix through a transversely isotropic piezoelectric interphase layer with two confocal elliptical boundaries subjected to uniform remote anti-plane and in-plane electrical loading. Using the identities in the Stroh formalism, the internal uniform electroelastic field in the elliptical inhomogeneity and the non-uniform electroelastic field in the interphase layer are obtained in real-form in terms of the four real matrices $\mathbf{N}^{\Omega}$, $\tilde{\mathbf{N}}^{\Theta}$, $\mathbf{N}$ and $\tilde{\mathbf{N}}$. The uniformity property continues to hold when an arbitrary number of transversely isotropic piezoelectric interphase layers with confocal elliptical interfaces are inserted between the elliptical inhomogeneity and the matrix, both of which remain monoclinic piezoelectric materials.

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