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Dynamics of a mass variable drum winding a heavy rope

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The paper focuses on a winch drum winding a heavy rope to present the variety of issues related to the dynamics of systems with rotating parts of time-varying mass. The dynamic motion equations for such systems are frequently either incorrect or derived under the assumption of mass constancy. When the mass variability is taken into account, simpler cases are discussed. The considerations on the angular momentum theorem enables formulating a new two time parameters form of the rotational motion equation for the system with time varying mass. Four cases are considered and different approaches of motion equations derivation are applied. The assumptions and solution methods of the analyzed systems as well as ambiguities and possible contradictions that may arise as a result of the trial of the problem formulation on the grounds of rigid body mechanics are discussed. The numerical simulations are presented.

Key words: angular momentum, two time parameters method, time-variable mass, dynamic equations, drum.

1. Introduction

The modeling of variable mass mechanical systems on the grounds of rigid body mechanics requires simplifying assumptions to be made. Careful and thorough analysis provides an understanding of how these assumptions affect the system, allows interpretation of the results obtained and specifies model limitations. Reviewing the literature on mass variable systems, one can observe that the modeling of these systems is frequently carried out improperly, the reason being that the applied laws are formulated for a rigid body with a constant mass (the distribution of which, though, may vary in time). In rigid body dynamics, the momentum and angular momentum theorems are derived as the consequences of Newton's dynamics laws, and may be applied in such an approach to the systems with constant masses. Fluid mechanics, in which the flow is analyzed and mass variability is a natural phenomenon, is presented in a slightly different way. The major law is the momentum principle, a consequence of which is the angular momentum principle. There exists, of course, a wide spectrum of papers that utilize this approach to derive the dynamic equations of variable mass systems.

One can see, however, that the elaboration of general dynamic motion equations for a body with time-varying mass, that are applicable to a wide spectrum of problems, is a difficult task, and even when they are derived, it is difficult to adapt them to a particular problem. Difficulties are also encountered when trying to make the procedures elaborated for one particular problem fit a different problem.

The variety of problems related to the dynamics of hoisting systems as well as the mistakes and simplifications made in the dynamic modeling of such a system were the direct motivation for taking on this problem. The bibliography review focuses on presenting: the variety of problems when using rope winches, simplifications made in modeling the dynamics of such systems, and approaches used when the changes in the masses of the wound and hanging parts of the rope are analyzed. For comparative purposes, the studies on modeling different mechanical systems with time-varying masses on the basis of momentum and angular momentum theorems were called up. Papers dealing with the dynamics of other variable mass mechanical systems are also discussed in order to present various approaches of derivation of the motion equations.

The dynamic analysis of hoists is important to recognize the various phenomena characterizing their work, and is therefore the objective of many analyses. Though the rope winch system is a frequent object of studies, the analysis mostly comes down to cases when either the rope weight is neglected and incorrect dynamical equations are applied, or an adequate discussion on the mathematical model utilized is not provided [1, 2]. An example of the improper application of the dynamic equation is found in [1], where the theorem of the equivalence of the mechanical work and kinetic energy $\Delta E_{\mathrm{K}} = W$ is used for the drum winding the rope, and from this theorem the relationship for powers $\mathrm{d}(J_{\mathrm{B}}\omega^2)/\mathrm{d}t = M\omega$, valid for a system with constant mass only, is derived. Analysis of transient states in drivetrain systems during acceleration, deceleration (start-up and braking) and a sudden change of operating conditions, as well as vibration arising that may cause a significant increase of the parts loads, is presented in papers [2, 3]. The Lagrangian formalism was applied to build a mathematical model of start-up processes of the drivetrain of building elevators [2]. The mass of the wound rope was taken into account in the moment of inertia and the kinetic energy of the drum, and then motion equations were derived using Lagrange's equations of the second kind. The validity of such an approach was not proved. Lagrange's equations are derived from d'Alembert's principle and mathematical transformations are valid when the mass is constant. The Lagrangian formalism in the basic formulation cannot be applied even if the mass flows between the parts of the mechanical system and the total mass of the system is constant.

The hoist is an important part of cranes [4–8], mining winding hoisters [9], winding elevators and rope winches. The objectives of studies are also – important for proper functioning of these devices – vibration in the rope-drum system. The major area of interest is the modeling of the transverse and longitudinal vibration of a hoister winding cables with varying length [10–18]. However, as can be noticed from in-depth analysis of the subject, the dynamics of the winch that determines the rope motion and normal forces is neglected. The coupled dynamic behavior between the different cables of a multi-cable winding hoister is presented in [18]. The variability of the mass moment of inertia resulting from the change of radius of the wound rope is neglected as small, and the analysis focuses on the dynamic behavior of a multi-cable double drum winding a mining hoister with flexible guides by using the Hamilton principle. In this paper it was mentioned, however, that one can take into account the change in the mass moment of inertia resulting from winding the rope. Selected phenomena closely related to the process of cable winding up onto the winch drum are considered. Rapid winch operation often causes disorderly winding of the wire rope. Paper [19] presents dynamic simulations aimed at analyzing the improper winding in the case of sudden braking of the winch. The solutions were obtained using the finite elements method. Although elements of variable length (and hence of variable mass) are applied to more accurately model the contact between the rope and drum and hydraulic system, the variability of the mass wound onto the winch drum is still neglected

With respect to the dynamic modeling the drum winding up a rope is closely related to other rotating discs or beams with time-varying mass [20–25]. The studies are extended to the vibration analysis, but the rotational motion is mostly prescribed. The objective of [20–22] is to analyze – using various techniques – the vibrations of a textile machine rotor that winds up a band of textile material into a roll. In these cases the angular velocity is assumed to be constant, then the angular momentum equation is not derived. It was stated that the mass variation significantly affects the character of the rotation stability. The paper [23] determines the vibration of the workpiece during a turning operation when the mass loss is slow. Papers [24, 25] analyze how the mass variation on the surface affects the dynamic behavior of a stretched Euler–Bernoulli beam. It can be observed that the majority of studies on time-variable mass systems utilizes the theorem of momentum only.

Machine units with a time-varying mass of processed materials perform many technological processes in the field of agricultural production, i.e. planting root-tuber fruits, grain crops, fertilizing, pressing feed materials, spraying and dusting and harvesting of various cultures. However, also in this case, the dynamics of these machines and processes is described using the theorem of momentum [26, 27]. A one-dimensional linear oscillator exchanging mass with the surroundings is described using the momentum principle [28]. On the basis of this equation, the motion equation of a pendulum exchanging mass with the surroundings was also investigated. A similar problem of linear mass motion was the objective of the paper [29]. Multi-scale dynamics of two coupled oscillators is studied in [30]. The linear one has varying mass, and a non-smooth light system is implemented for passive controlling the linear system against external impulses and/or forces. In depth balance of mass and linear momentum for variable-mass systems is presented in [26].

The derivation of the angular momentum equation for the body with time-variable mass is presented in [31, 32], and the studies on the dynamics of such bodies are summarized in [33]. A complete description of the theory of variable mass systems dynamics can be found also in [33–37]. The papers present a methodological approach to the modeling of such systems. The meaning and importance of the various terms of the motion equations of time-varying systems, and the circumstances under which each term can be neglected, are discussed in depth in [37]. It was pointed out that certain judiciously simplified versions of the equations of motion are adequate for most studies. The Lagrangian formalism was also adapted to model such systems [33]. The dynamics of multi-body mechanical systems with a continual and discontinual mass variation is presented in [35]. Up-to-date and unifying formulations comprise Lagrange and Hamiltonian approaches. Axially moving structures, like belts and chains, and pipes with an axial flow of fluid are dealt with, where the stability of motion is also considered. The interest of [38] was to determine the conditions under which the angular momentum is conserved. In [39] classical mechanics was extended to systems with variable mass by replacing the kinetic energy by a non-local-in-time kinetic energy and the velocity by a fractional velocity. This approach was applied to analyze the rotating fluid problem subject to a position-varying incompressible fluid mass which occurs in rocket dynamics. Various methods are developed for specific systems with time-varying mass.

The present paper focuses on a winch drum winding a heavy rope to present the variety of issues related to the dynamics of systems with rotating parts of time-varying mass. The angular momentum equation, which is the major point of interest, is expressed using two time parameters. At this stage of studies planar motion only is considered. A few cases are considered and different approaches of motion equations derivation are applied. The objective of the paper is, hence, to discuss the assumptions and solution methods of the analyzed system as well as the ambiguities and possible contradictions that may arise as a result of the trial of the problem formulation on the grounds of rigid body mechanics. The discussion is based on four selected examples of growing complexity. The most complex case discussed is the example of the drum with a mechanically regulated radius. The radius in this case changes as a result of mechanical action as well winding onto/out the belt on the drum. This case has not been analyzed in the literature. The works that exist present the derivation of the general form of the angular momentum equations and illustrate them on examples of the rope either reeling out of or winding up onto the drum [33, 34]. Nonetheless these equations are adopted to the simpler cases, in which a drum fails to change its radius as a result of winding a rope. More complex cases with the geometry changing not only as a result of mass variation but also as a result of mechanical regulation are not dealt with.

2. Preliminary theoretical considerations

Let us consider a mechanical system composed of two bodies that exchange mass. The general theoretical approach at the stage of the derivation of the motion equations would require additional discussion when adapting the equations to the subject of the considerations, i.e. to a winch wrapping up a heavy rope; therefore, the mechanical system directly addresses the basic physical and mechanical properties of the winch drum. Body no 1 is divided into the following elements:

– rigid body 1' of constant mass that rotates around the axis (rigid base),

– body 1" of variable mass which is transferred from body 2.

To simplify, let us assume also that mass 1" fails to change its relative position with respect to the axis of rotation. In Fig. 1 body 1' is presented as a rotating cylinder. The mass of body 1" is distributed on the cylinder circumference so that the cylinder radius may increase in time. Moreover, the distribution of the mass of 1" can be time-varying, which is shown as the deformation of the outer part of the cylinder. The mass inertia moment of body 1 with respect to O is denoted as J. The axis of rotation O is immoveable.

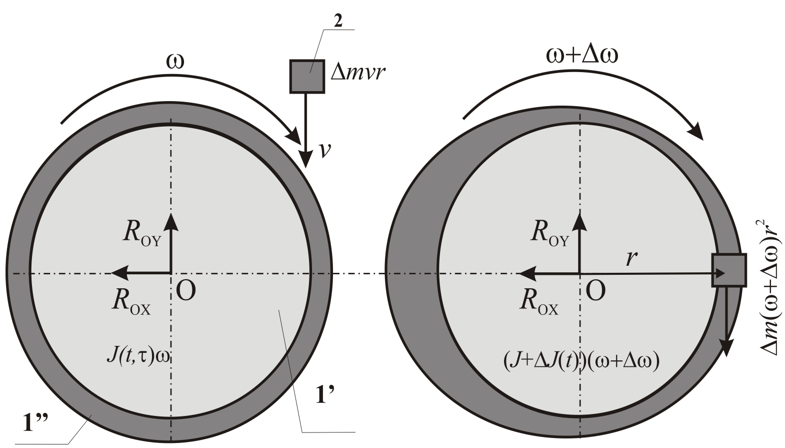

FIG. 1. Illustration to the equation expressing the change in the moment of momentum in the considered system.

Let us carry out the following analysis at this preliminary stage of consideration. We compute the change in the angular momentum at the instants $t$and $t + \Delta t$ provided that only the mass of body 1, which is therein at instant t, is taken into account. In other words, we freeze for an instant the mass of body 1 $m''(\tau)$. To reflect that fact in mathematical transformations, parameter τ is introduced.

Let us denote the increment of the angular momentum of body 1 $\Delta\mathbf{K}_{\mathrm{O}}$. The change in the angular momentum of the exchangeable mass is derived as a separate term $\Delta\mathbf{K}'_{\mathrm{O}}$. The forms of these terms depend on the character of the phenomena and are derived for all the presented cases separately. The angular momentum increment is equal to $\Delta\mathbf{K}_{\mathrm{O}} + \Delta\mathbf{K}'_{\mathrm{O}}$. The angular momentum of body 1 can be expressed in the following general form:

$$\mathbf{K}(t,\tau) = \mathbf{K}(J(\mathbf{y}(t), \mathbf{z}(\tau)), m(\mathbf{y}(t), \mathbf{z}(\tau)), \mathbf{y}(t), \mathbf{z}(\tau), \frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t}).$$

Time parameter t is the independent variable of all the geometric (vector $\mathbf{y}$) and kinematic (vector $\mathrm{d}\mathbf{y}(t)/\mathrm{d}t$) variables, the change in time of which is not related to the mass exchange process. They correspond with the mass distribution change or rigid motion of the body. Time parameter τ is the independent variable of all the geometric (vector $\mathbf{z}$) variables that change in time as the result of the mass exchange process. The parameter τ refers to all the variables in the angular momentum that causally correspond with the mass attached to or detached from body 1. It pertains to the mass of body 2 itself. *The parameter is dependent on τ when the change in that parameter causes the change of mass 1, but not a change of its distribution on body 1* (see deformation in Fig. 1).

It is used to derive the moment of momentum for the considered system. The moment of inertia of body 1 with respect to the axis of rotation is denoted as J. The moment of inertia can be expressed as the sum of two terms:

$$J(t,\tau) = J^{\mathrm{g}}(t) + J^{\mathrm{m}}(t,\tau). \tag{2.1}$$

$J^{\mathrm{g}}(t)$ – of the rigid base 1' that is immoveable with respect to the axis O; the change in this moment of inertia results from the change of the geometry, but does not result from the mass variation or geometry change resulting from the mass variation. $J^{\mathrm{m}}(t,\tau)$ – of body 1" that passes from body 2; in this term this mass is treated as instantaneously constant, then expressed as a function of parameter τ. Theoretically, this mass also can deform and change its relative position with respect to the rigid body 1', as shown in Fig. 1, then this geometric change is associated with parameter t.

Let body 1 rotate at angular velocity ω at instant t, and the mass of body 2 move straight with linear velocity v, then the mass Δm of this body moves also with velocity v. Only one component of the angular momentum, with respect to the axis of rotation of body 1, is considered, and expressed at the instant under consideration as:

$$K_{\mathrm{O1}} = J(t,\tau)\omega + \Delta m v r(t) + m_2(\tau) v r(t). \tag{2.2}$$

At instant $t+\Delta t$, mass Δm passes to the circumference of body 1, and the change of the inertia moment of rotating body 1 is a consequence of the change in the moment of inertia of body 1", whereas the change in the angular momentum of body 2 is taken into account in the angular momentum of mass Δm.

Note that the derivative of the moment of inertia of the variable mass does not appear in K_O because the way in which this mass passes to the rotating body is essential (radial or circumferential direction). Moreover, the variable mass cannot appear twice in the angular momentum equation, in terms J and Δm

$$\begin{aligned} &\Delta J(t,\tau) = \Delta J^{\mathrm{g}}(t), \\ &K_{\mathrm{O2}} = (J(t,\tau) + \Delta J(t,\tau) + \Delta m r^2(t))(\omega + \Delta\omega) + m_2(\tau) v r(t). \end{aligned} \tag{2.3}$$

The angular momentum of the remaining part of body 2 does not vary in the considered time interval. Let us note that the sum $m_2 + m_{1''}$ is constant, therefore $\Delta m = \Delta m_{1''} = -\Delta m_2$. Let the torque M act on body 1. In this torque we distinctly extract the moment of the external forces M_{EX} and the moment of the internal forces M_{IN}. The latter one expresses the internal interactions inside the body (belt) exchanging the mass with the rotating body (drum)

$$M = M_{\mathrm{EX}} + M_{\mathrm{IN}}.$$

It is assumed then the mass flowing to the body can be not only a system of separate non-interacting particles but also the continuous media with internal interactions. The internal forces, then, express the interaction between body 1" and 2 in the cross-section that separates these bodies. According to the angular momentum theorem,

$$K_{\mathrm{O2}} - K_{\mathrm{O1}} = M\Delta t.$$

By neglecting higher-order terms, we obtain:

$$\Delta J(t,\tau)\omega + J(t,\tau)\Delta\omega = M\Delta t + \Delta m_2 r(t) w, \tag{2.4}$$

where the relative change in the velocity of mass Δm is equal to $w = v - r\omega$.

Dividing Eq. (2.4) by Δt and assuming that $\Delta t \to 0$, the angular momentum equation for the system under consideration is as follows:

$$\left(\frac{\mathrm{d}J(t,\tau)}{\mathrm{d}t}\right)_{\tau=t}\omega + J(t,t)\frac{\mathrm{d}\omega}{\mathrm{d}t} = M + \mathrm{sgn}(\cdot)\frac{\mathrm{d}m_2}{\mathrm{d}t}r(t)w. \tag{2.5}$$

We further denote $J(t,t) = J(t)$. The rate of mass variation is positive when the mass passes to the body. It has been stated that body 2 moves in rectilinear motion. It follows that the moment equation is as follows [33–35]:

$$m_2 \frac{\mathrm{d}v}{\mathrm{dt}} = F + \mathrm{sgn}(\cdot)\frac{\mathrm{d}m_2}{\mathrm{dt}} w, \tag{2.6}$$

where $F$is the geometric sum of all the external and internal forces. The latter ones act in the cross-section that separates bodies 1" and 2. This equation is widely reported in the literature and does not require additional discussion.

The sign at the last term in Eqs. (2.5) and (2.6) must be specified individually for a particular problem, as it depends on how mass m_2 and relative velocity v are expressed in terms of independent variables (degrees of freedom). It follows that for each specific case it must be analyzed whether these terms are related with either flowing the angular momentum/momentum ($\mathrm{sgn}(\cdot) = -1$) out of the body or passing the angular momentum/momentum to the body ($\mathrm{sgn}(\cdot) = 1$).

3. Analysis of specific cases

3.1. Example 1

Let us consider the simple case of the cylindrical winch drum shown in Fig. 2. A heavy rope of limited length is being wound onto the drum. The rope is wound circumferentially so that the part of the rope on the winch drum does not increase the diameter of the drum.

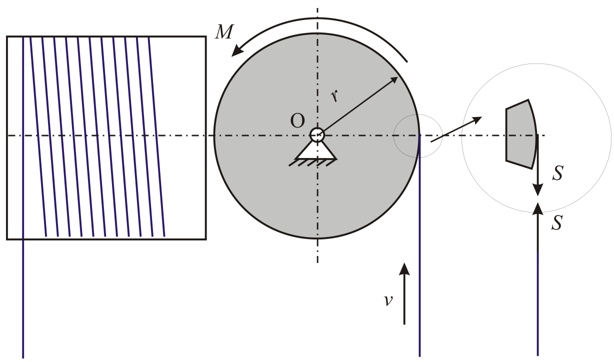

Fig. 2. The scheme of the system in case 1 with circumferential winding of the rope onto the winch drum.

Let us denote:
x – the length of the rope wound onto the winch drum;
$v = \frac{dx}{dt}$ – the rate of the rope wounding onto the winch drum;
$m_b(x)$ – the mass of the rope wound onto the winch drum;

$m_s(x)$ – the mass of the free part of the rope;
m – the total mass of the rope;
M_b – the mass of the drum
l – the total length of the rope;
M – the moment of forces subjected to the winch;
S – the internal longitudinal force in the rope at the cross-section when the rope comes onto the drum.

According to the momentum theorem, the motion equation of the free part of the rope is as follows:

$$m_{\mathrm{s}}(t)\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = S + \frac{\mathrm{d}m_{\mathrm{s}}(t)}{\mathrm{d}t}w - gm_{\mathrm{s}}(t). \tag{3.1}$$

In this case, the relative velocity of the mass passing onto the drum, i.e. of the mass flowing out of the free part of the rope, with respect to the velocity of the free part $w = 0$. Hence:

$$m_{\mathrm{s}}(t)\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = S - gm_{\mathrm{s}}(t). \tag{3.2}$$

The force in the rope is

$$S = m_{\mathrm{s}}(t)\left(\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + g\right). \tag{3.3}$$

The masses of the free and wound part of the rope are

$$m_{\mathrm{s}}(t) = m\left(1 - \frac{x(t)}{l}\right), \qquad m_{\mathrm{b}}(t) = m\left(\frac{x(t)}{l}\right).$$

The drum motion equation is derived directly from the angular momentum equation:

$$\left(\frac{\mathrm{d}J(t,\tau)}{\mathrm{d}t}\right)_{\tau=t}\omega(t) + J(t)\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = M - rS + \frac{\mathrm{d}m_{\mathrm{s}}(t)}{\mathrm{d}t}rw(t). \tag{3.4}$$

The disc is modeled as a disc with uniformly distributed mass. The inertia moment of the rotating masses is as follows:

$$J(t,\tau) = \frac{1}{2}(M_{\mathrm{b}} + 2m_{\mathrm{b}}x(\tau))r^2. \tag{3.5}$$

Let us note that the geometry of the mass distribution on the drum does not vary, then $\left(\frac{\mathrm{d}J(t,\tau)}{\mathrm{d}t}\right)_{\tau=t} = 0$. Moreover, the kinematic parameters are functions of time t only:

$$\omega(t) = \frac{1}{r}\frac{\mathrm{d}x(t)}{\mathrm{d}t}, \qquad \varepsilon(t) = \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}.$$

After simple transformation, the motion equation can be rewritten as

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\left(m+\frac{1}{2}M_\mathrm{b}\right)r = M - rm_\mathrm{s}(t)g. \tag{3.6}$$

We obtain the same equation directly from the angular momentum theorem applied to the whole system, as the angular momentum of the whole mass of the rope can be expressed as $m_2\omega r^2$. This confirms that the total moment of inertia of the drum about the axis (3.5) cannot be differentiated with respect to time when the system is divided into the drum with the wound part of the rope and the free part of the rope.

A frequently made mistake is to express the mass moment of inertia in rotational motion as follows:

$$J(t) = \frac{1}{2}(M_\mathrm{b} + 2m_\mathrm{b}x(t))r^2,$$

then $\left(\frac{\mathrm{d}J(t)}{\mathrm{d}t}\right) \neq 0$.

The solution for this case is compared to the solution obtained for the correct motion Eq. (3.6) in Fig. 3. For illustrative purposes, the following data were taken: $l = 20$ m, $M = 49.25$ Nm, $M_b = 10$ kg, $g = 9.81$ m/s^2, $r = 25$ cm, $m = 20$ kg.

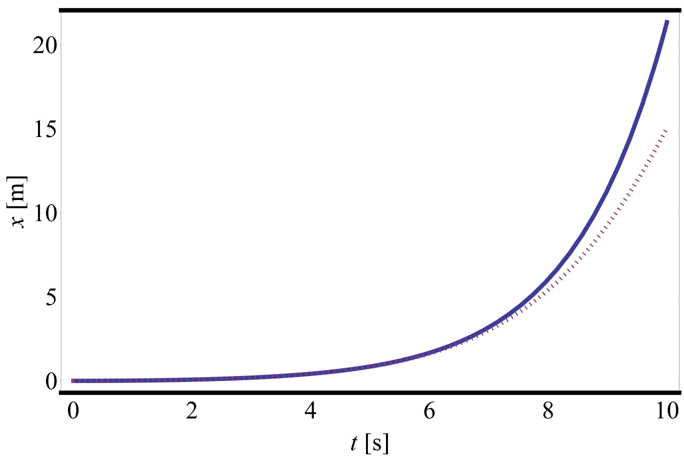

FIG. 3. The correct solution (continuous line) and the solution obtained for the incorrect form of the moment of inertia (dashed line).

The case of a rope reeling out of a drum with constant radius is presented by AWREJCEWICZ [34]. The angular dynamic equation derived for the body with variable mass has the following form:

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} = M - (\omega - \omega^*)\frac{\mathrm{d}J}{\mathrm{d}t}. \tag{3.7}$$

One can notice the fact that factor $\mathrm{d}J/\mathrm{d}t$ is the change of inertia moment of the body with variable mass assuming that the mass flowing to/out of the body fails to change the relative position with respect to the body mass centre. In the equation ω^* is the angular velocity of an elementary part of a rope separating from a drum. The drum is rotating with angular velocity ω. It is derived further that velocity of the rope on the drum $v = \omega_r$ equals to the velocity of the rope $v^* = \omega^* r$. Then $\omega^* = \omega$ and the motion equation reduces to:

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} = M.$$

It seems that more general form of the equations is derived by CVETICANIN [33]. The equation for the rotational form is as follows:

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} = M - (\omega - \omega^*)\frac{\mathrm{d}J}{\mathrm{d}t} - (\mathbf{r} \times (\mathbf{v} - \mathbf{v}^*))_3\frac{\mathrm{d}m}{\mathrm{d}t}. \tag{3.8}$$

Index 3 refers to the non-zero third component of the vector product. The vector $\mathbf{r}$ presents the location of the mass flowing to the system with respect to the mass center of the body, and $\mathbf{v} - \mathbf{v}^*$ is the increment of the mass velocity at the instant of passing to (or separating from) the body. CVETICANIN [33] also studied the similar problem. The only difference is that the constant velocity of the belt winding onto the drum is imposed, which guarantees the uniform (without wrinkles) winding up of the belt onto the drum. As a consequence all the components on the right hand side of Eq. (3.8) appear. When we adopt the equation to the case considered in Example 1 they become equivalent to (3.7) and (3.6). Nonetheless, Cveticanin takes that $\omega^* = 0$ and, opposed to Awrejcewicz, different from ω. Then, there are discrepancies in the interpretation of particular terms in the angular momentum equation and kinematic relations. That is why it is not easy task to adopt the general equations to the subsequent cases presented in this paper – especially cases 3 and 4.

3.2. Example 2

We consider a system which works in the same way as the one presented in Example 1, with the only difference that a part of the rope lies freely on the ground from which the rope is being drawn up. The drum axis is located at height h above the ground.

In this case, $m_{\mathrm{p}}(x)$ stands for the motionless part of the rope resting on the ground:

$$m_{\mathrm{p}}(t) = m\frac{(l - h - x(t))}{l}. \tag{3.9}$$

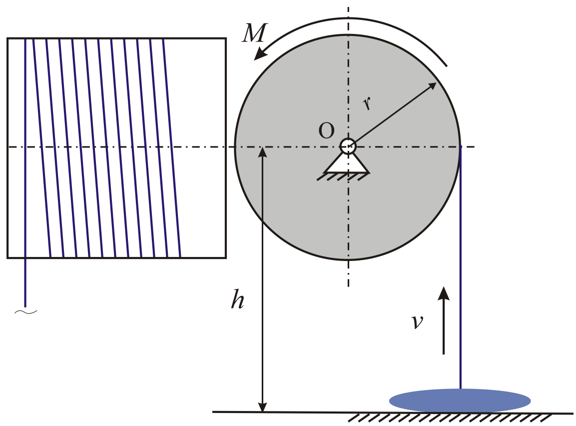

FIG. 4. Scheme of the system in Example 2 with circumferential winding of the rope onto the winch drum.

The mass of the free part of the rope is constant $m_{\mathrm{s}} = m\frac{h}{l}$. Let us note that it suffices to add an additional term resulting from the angular momentum exchange to Eq. (3.6) from Example 1. This term represents the angular momentum transported to the rope-drum system with the mass of the rope pulled in from the ground:

$$\frac{1}{r}\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\left(m_{\mathrm{s}} + m_{\mathrm{b}}(t) + \frac{1}{2}M\right)r^2 = M - rm_{\mathrm{s}}g - \operatorname{sgn}\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)r\frac{\mathrm{d}m_{\mathrm{p}}(t)}{\mathrm{d}t}w. \tag{3.10}$$

The part of the rope on the ground is motionless, so the relative velocity of the drawn up part of the rope with respect to the hanging part of the rope is equal to $w = -\frac{\mathrm{d}x(t)}{\mathrm{d}t}$. Let us note further that, according to Eq. (3.10), when $\frac{\mathrm{d}x(t)}{\mathrm{d}t} > 0$, i.e. the rope is wound onto the drum, $\frac{\mathrm{d}m_{\mathrm{p}}(t)}{\mathrm{d}t} < 0$ and $r\frac{\mathrm{d}m_{\mathrm{p}}(t)}{\mathrm{d}t}w > 0$. In this case, however, the moment of pulled rope momentum has the opposite sense to that of the torque that winds the rope onto the winch drum and, therefore, the sign of the term on the right hand side of Eq. (3.10) should be negative. When $\frac{\mathrm{d}x(t)}{\mathrm{d}t} < 0$, i.e. the rope is unwound off the drum, $\frac{\mathrm{d}m_{\mathrm{p}}(t)}{\mathrm{d}t} > 0$, and $r\frac{\mathrm{d}m_{\mathrm{p}}(t)}{\mathrm{d}t}w > 0$. Then, the rope falling on the ground decreases its own angular momentum and simultaneously increases the angular momentum of the drum, and the sign of the last term in Eq. (3.10) must be positive. That is why the term $\operatorname{sgn}\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)$ appears in Eq. (3.10).

In this case the equations presented in [33, 34] lead to the same motions equations.

For illustrative purposes, let us take the following data: $l = 80$ m, $M = 60$ Nm, $M_{\mathrm{b}} = 10$ kg, $r = 25$ cm, $m = 80$ kg, $h = 20$ m.

At the initial instant the whole rope hangs freely or rests on the ground. For this data the rope is wound up onto the drum. Let us further change the torque $M = 30$ Nm and assume that $l - h = 60$ cm of the rope is wound onto the drum at the initial instant. This case presents unwinding the rope of the drum with the resistive torque. The solutions for both cases are presented in Figs. 5 and 6.

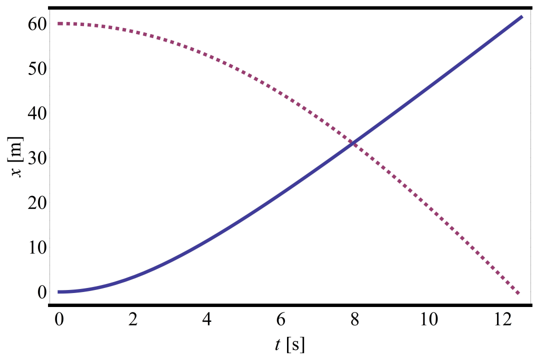

FIG. 5. The solution when the rope is wound up onto the drum (continuous line) and when the rope is unwound from the drum (dashed line).

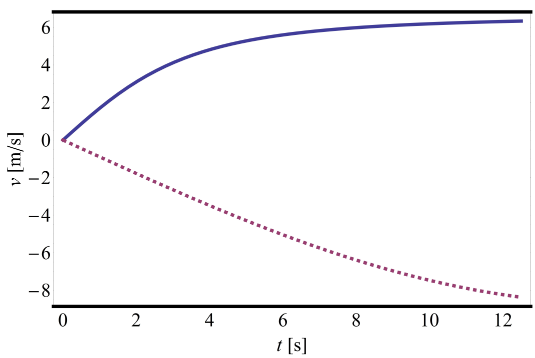

FIG. 6. The rope velocity when the rope is wound up onto the drum (continuous line) and when the rope is unwound from the drum (dashed line).

3.3. Example 3

We consider a problem like the one in Example 1, with the only difference that the rope is wound up onto the drum radially (Fig. 7).

Since the rope is wound up radially, the radius $r$of the drum with the wound part of the rope increases. r_0 is the radius of the drum itself. Let us assume

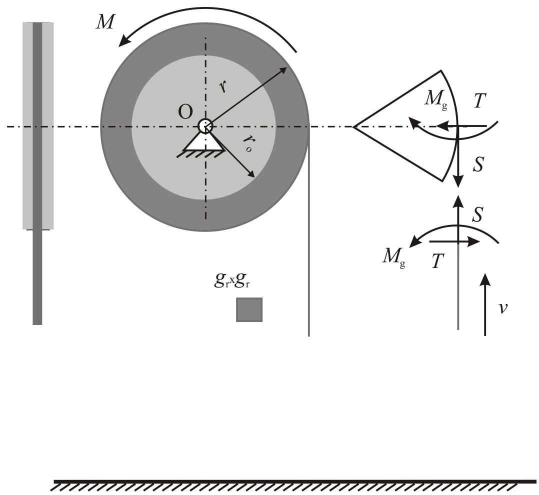

FIG. 7. Illustration for Example 3 along with the internal forces in the rope.

that the rope cross-section is square. This ensures the complete filling of the side surface of the drum by the rope, and makes the computations simpler. One can assume that the volume of the wound part of the rope is approximately equal to

$$g_{\mathrm{r}}^2 x(t) \approx \pi\big(r(x(t))^2 - r_0^2\big) g_{\mathrm{r}}, \tag{3.11}$$

where g_r is the rope thickness.

Then the time-dependent radius of the drum-rope system varies in accordance with

$$r(x(t)) = \sqrt{\frac{g_{\mathrm{r}} x(t)}{\pi} + r_0^2}. \tag{3.12}$$

Then the moment of inertia about the axis of rotation is the sum of the two terms:

$$J(t,\tau) = J^{\mathrm{m}}(\tau) + J^{\mathrm{g}}(t). \tag{3.13}$$

Let us note that, according to our previous observation, the term of the mass passing onto the drum in the drum-wound rope moment of inertia must be the function of parameter τ. The changes in length x and radius r result from the change of the total mass of the drum-wound rope system – not from the geometry or mass distribution change – and in the angular momentum they are also the functions of τ

$$J^{\mathrm{g}}(t) = \frac{1}{2} M_{\mathrm{b}} r_0^2, \tag{3.14}$$

$$J^{\mathrm{m}}(\tau) = \frac{1}{2} m_{\mathrm{b}}(\tau)(r^2(\tau) - r_0^2). \tag{3.15}$$

We have separated the two distinct terms: the mass of the drum and the mass of the rope wound on the drum. One cannot differentiate the term that varies as a result of the mass exchange. Let the drum rotate at angular velocity ω. The angular momentum of the system about the axis of rotation of the drum is being determined. We can see that the free part of the line does not move vertically only, but also horizontally and it fails to swing. Then, the angular momentum for both the motion components has to be computed. The angular momentum at instant t for the drum and the vertical component of the rope motion (the angular moment for the rope moving vertically) has been considered in Example 1, and for both Example 1 and this example the cross-vector of the distance vector and velocity is the same for the whole free length of the rope. Hence,

$$K_{\mathrm{O}1} = (J^{\mathrm{g}}(t) + J^{\mathrm{m}}(\tau))\omega + m_s(t)vr(t). \tag{3.16}$$

The angular momentum at $t + \Delta t$ is

$$K_{\mathrm{O}2} = (J^{\mathrm{g}}(t) + \Delta J^{\mathrm{g}}(t) + J^{\mathrm{m}}(\tau))(\omega + \Delta\omega) + m_s(t)(v + \Delta v)(r(t) + \Delta r). \tag{3.17}$$

The horizontal motion is related to the change in the radius, and therefore the velocity and accelerations components are

$$v_{\mathrm{r}} = \frac{\mathrm{d}r(t)}{\mathrm{d}t}, \qquad a_{\mathrm{r}} = \frac{\mathrm{d}v_{\mathrm{r}}}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}.$$

According to Fig. 8, the infinitesimal change in the angular momentum component corresponding to the horizontal motion of the rope with respect to O holds:

$$\mathrm{d}\Delta K_{\mathrm{OP}} = \mathrm{d}m_{\mathrm{s}}(v_{\mathrm{r}} + \Delta v_{\mathrm{r}})(y - \Delta x) - \mathrm{d}m_{\mathrm{s}} v_{\mathrm{r}} y. \tag{3.18}$$

By eliminating higher order terms, we have

$$\mathrm{d}\Delta K_{\mathrm{OP}} = \mathrm{d}m_{\mathrm{s}}(\Delta v_{\mathrm{r}} y - v_{\mathrm{r}}\Delta x), \tag{3.19}$$

where: $\mathrm{d}m_{\mathrm{s}} = \rho \mathrm{d}y$, $m_{\mathrm{s}} = \rho(l - x)$.

The foregoing expression is integrated along the free length of the rope:

$$\begin{aligned} \Delta K_{\mathrm{OP}} &= \rho \int_0^l (\Delta v_{\mathrm{r}} y - v_{\mathrm{r}}\Delta x)\,\mathrm{d}y = \rho\Big(\Delta v_{\mathrm{r}}\frac{y^2}{2} - v_{\mathrm{r}} y \Delta x\Big)_0^{l-x}, \\ \Delta K_{\mathrm{OP}} &= \rho\Big(\Delta v_{\mathrm{r}}\frac{(l-x)^2}{2} - v_{\mathrm{r}}\Delta x(l-x)\Big) = m_{\mathrm{s}}\Big(\Delta v_{\mathrm{r}}\frac{l-x}{2} - v_{\mathrm{r}}\Delta x\Big). \end{aligned} \tag{3.20}$$

Such a model of the rope motion imposes the existence of the binding moment in the rope cross-section, the one which straightens the free part of the rope.

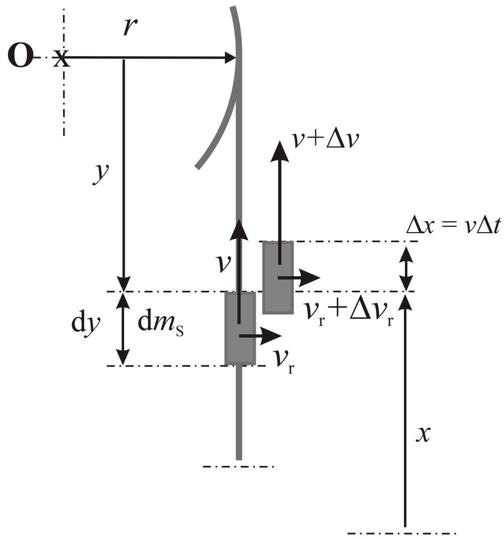

FIG. 8. The illustration for derivation of the angular moment term corresponding to the horizontal motion of the rope.

On the other hand, we have assumed that the rope is slender as it can be freely wound onto the drum without the need to perform work against the bending moment. This issue gives rise to the troubles that are encountered while trying to solve the problem using the mechanical laws of rigid body mechanics instead of continuous media, even if the variation of the mass is considered. Despite these inconsistencies, we proceed with the consideration. This is why the general equation derived for the systems with time varying mass are rarely directly adopted to particular cases.

Having eliminated the higher order terms, the finite angular momentum increment is

$$\begin{aligned}(3.21)\quad \Delta K_{\mathrm{O}} &= K_{\mathrm{O2}} - K_{\mathrm{O1}} + \Delta K_{\mathrm{OP}} \\ &= J(t,\tau)\Delta\omega(t) + \Delta J(t,\tau)\omega(t) + m_{\mathrm{s}}(t)\Delta v(t)r(t) + m_{\mathrm{s}}(t)v(t)\Delta r(t) \\ &\quad + m_{\mathrm{s}}(t)\left(-v_{\mathrm{r}}(t)\Delta x(t) + \Delta v_{\mathrm{r}}(t)\frac{(l-x(t))}{2}\right).\end{aligned}$$

Taking into account that the increment is equal to the impulse of forces moments:

$$\Delta K_{\mathrm{O}} = M\Delta t - m_{\mathrm{s}} gr\Delta t,$$

dividing by Δt, and assuming that $\Delta t \to 0$, one yields

$$(3.22)\qquad \frac{\mathrm{d}}{\mathrm{d}t}K_{\mathrm{O}} = M - m_{\mathrm{s}} gr.$$

Eventually, we have

$$J(t)\frac{\mathrm{d}\omega}{\mathrm{d}t}+\frac{\mathrm{d}J^{\mathrm{g}}(t)}{\mathrm{d}t}\omega+m_{\mathrm{s}}(t)\left(\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}r(t)+\frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}r(t)}{\mathrm{d}t}\right) + m_{\mathrm{s}}(t)\left(-\frac{\mathrm{d}r(t)}{\mathrm{d}t}\frac{\mathrm{d}x(t)}{\mathrm{d}t}+\frac{\mathrm{d}^2r(t)}{\mathrm{d}t^2}\frac{(l-x(t))}{2}\right)=M-m_{\mathrm{s}}gr(t). \tag{3.23}$$

The inertia moment of the drum and wound-rope in the angular momentum equation can change as a result of the geometry change, which fails to happen in this case, then $\Delta J(t,\tau)=0$ and $\frac{\mathrm{d}J^{\mathrm{g}}(t)}{\mathrm{d}t}=0$.

The relation coupling x with the angle of drum rotation is as follows: $x=\varphi r$. To derive the kinematic relations, let us note that the radius fails to increase along the whole circumference at a moment, and as a consequence the second term in $\Delta x = r\Delta\varphi+\varphi\Delta r$ can be neglected. In other words, the radius of the drum at instants t and $t+\Delta t$ is the same along the circumference, besides the point separating the free and wound rope parts, and the radius expressed as a function of τ computes angular velocity $\omega=\frac{1}{r}\frac{\mathrm{d}x}{\mathrm{d}t}$ and acceleration $\varepsilon=\frac{1}{r}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. Eventually, the motion equation is as follows:

$$\frac{1}{2r(t)}\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\left(M_{\mathrm{b}}r^2(t)+m_{\mathrm{b}}(t)(r^2(t)-r_0^2)\right) + m_{\mathrm{s}}(t)\left(\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}r(t)+\frac{\mathrm{d}^2r(t)}{\mathrm{d}t^2}\frac{(l-x(t))}{2}\right)=M-m_{\mathrm{s}}(t)gr(t). \tag{3.24}$$

In modeling the system, we encountered substantive problems. The assumption that the system has 1-dof imposes the translational motion of the rope. It causes the inconsistency with the real behavior of the slender line, which can swing about the point at which it passes onto the drum. The conclusion is that different mathematical models lead to different dynamical equations. The assumption made in this example cannot be completely logically justified on the grounds of classical mechanics. To justify such an approach, one can assume that the swinging motion of the rope can be neglected when analyzing the system dynamics. If we had written down the two motion equations (3.3) and (3.4) separately, for the drum with the wound part of the rope and for the free part of the rope, assuming, as in Example 1, that normal force S in the rope cross-section acts, we would have obtained the equation as follows after having eliminated force S from the two equations:

$$\frac{1}{2r(t)}\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\left(M_{\mathrm{b}}r^2(t)+m_{\mathrm{b}}(r^2(t)-r_0^2)\right) + m_{\mathrm{s}}(t)\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}r(t)=M-m_{\mathrm{s}}gr(t). \tag{3.25}$$

The motion equation (3.25) differs from Eq. (3.24) in the term

$$m_s(t)\left(\frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2}\,\frac{(l-x(t))}{2}\right).$$

This is the magnitude of bending moment M_g in the rope cross-section, when the bending moment is taken into account. The transversal force T in the cross-section fails to affect the angular momentum about point O. Hence, when the bending moment M_g is added on the right hand side of Eq. (3.4), we get the same motion equation (3.24). The problem can be solved by introducing the force equal to $m_s(t)\left(\frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2}\,\frac{(l-x(t))}{2}\right)$, and distributed along the free length of the rope, or, less accurately, subjected to the centre of the free rope part. This force would prevent the bending of the rope.

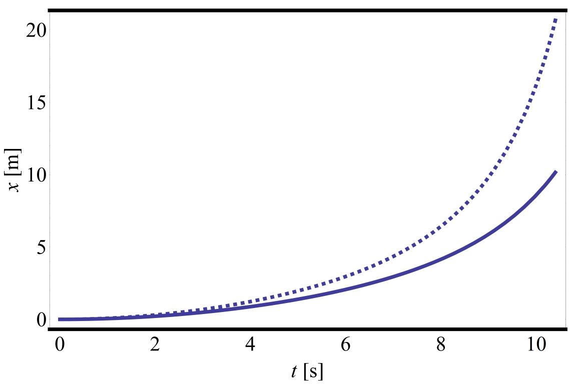

FIG. 9. The case with the radial rope winding – the variation of x in time for $g_{\mathrm{r}} = 2$ cm when M_g is taken into account (continuous line) and distegarded (dashed line).

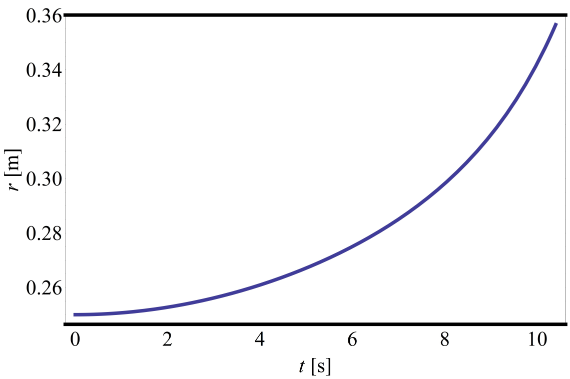

FIG. 10. The case with the radial rope winding – the variation of the drum radius in time for $g_{\mathrm{r}} = 2$ cm when M_g is taken into account.

In paper [34] term $\mathrm{d}J/\mathrm{d}t$ takes into account the rope mass variation only, whereas in this case the position of the belt passing onto the drum varies too. Then, Eq. (3.7) requires deeper analysis before adopting to this case. The similar case was not analyzed in [33] either.

The following data were taken for numerical simulations: $l = 20$ m, $M = 50$ Nm, $M_\mathrm{b} = 10$ kg, $r_0 = 25$ cm, $m = 20$ kg, $g_\mathrm{r} = 2$ cm.

In the equilibrium the internal torque M is equal to 49.05 Nm. The rope winding is impossible for the lower value. The solutions are presented in Figs. 9 and 10. The case when M_g is taken into account is compared with the case when M_g is neglected. Visible differences are observed when the rope weight must be taken into account. Faster motion appears, of course, when the stiffness of the rope is considered.

Let us increase the rope thickness $g_\mathrm{r} = 4.4$ cm, which results in increasing the rate of the radius growth, and hence, the moment of free rope gravitational weight. The mass of the rope is not changed. In this boundary case the oscillatory motion of the drum is observed, as shown in Fig. 11. Lower values of the internal torque fail to wind up the rope onto the drum.

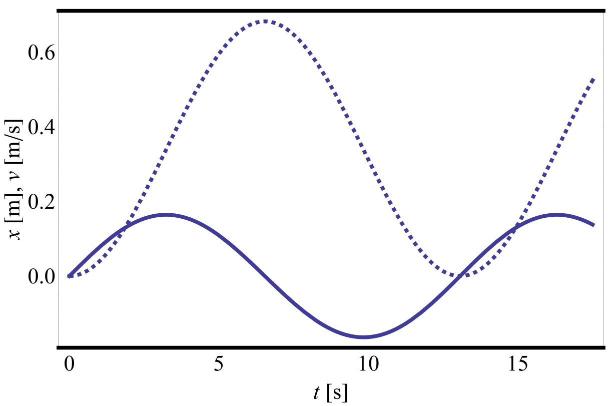

Fig. 11. The case with the radial winding up of the rope – the variation of x (dashed line) and velocity v (continuous line) in time for $g_\mathrm{r} = 4.4$ cm when M_g is taken into account.

Both cases, though, with and without rope rigidity, fail to properly describe the nature of the analyzed phenomenon. One cannot justify and fully explain the imposed dynamic behavior of the rope, which is constrained to become vertical. The conclusion is that to reliably model the rope and drum in this case, the rope has to be modeled as a continuous medium with time-variable mass. The motion equation is then expressed by means of a partial differential equation.

3.4. Example 4

The most complex case discussed is the example of the drum with a mechanically regulated radius. This drum is thought to complete the analysis, but the scheme is not inspired – to the authors' knowledge – by a real machine. The distance of the drum circumference (the drum radius) is changed by couplers 3 connected to the drum rolls 4 and sliders 2 using revolute joints. Horizontal external forces drive the sliders 2 symmetrically along the guiding link 1. The rope is wound up onto the drum radially (Fig. 12). It is assumed that the rope 5 is wound up along the circular circumference, although in fact the drum rolls separate from each other angularly as the radius increases, and the shape of the rope is polygonal.

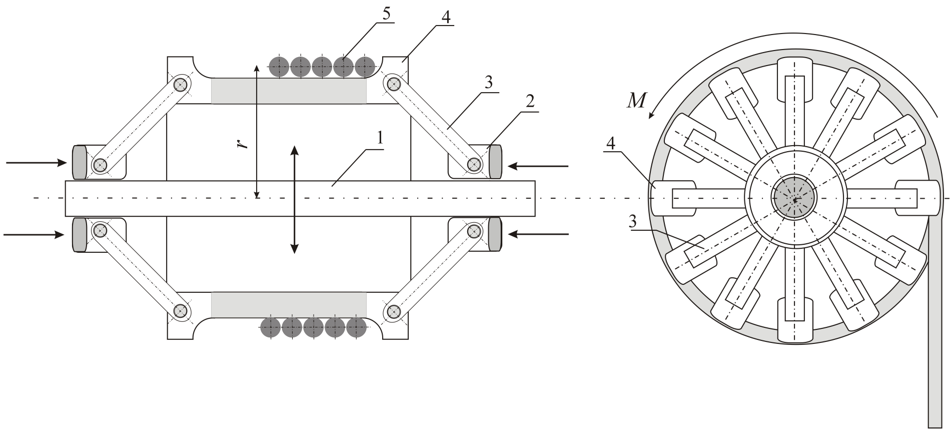

FIG. 12. Illustration of the drum with regulated radius.

The complexity causes that we find troubles with the automatic utilization of Eqs. (3.7) and (3.8) presented in [33, 34] also to this system.

Since the rope is wound up on the circumference, radius $r$of the drum with the wound part of the rope increases; r_0 is the radius of the drum in its initial position. Let us take it that the time-dependent radius of the drum-rope system varies in accordance with

$$r(t) = r_0 + \alpha t. \tag{3.26}$$

where α is constant. Then, the forces driving the sliders are not taken into account, and the system has one degree of freedom. The moment of inertia about the axis of rotation is the sum of the two terms:

$$J(t, \tau) = J^{\mathrm{m}}(\tau, t) + J^{\mathrm{g}}(t). \tag{3.27}$$

Let us note that, according to our previous observation, the term of the mass having been wound on the drum in the moment of inertia must be the function of parameters t and τ. The change in radius r results from the geometry change, and is expressed in t. On the other hand, the total mass of the wound rope is expressed in τ – the time-variable mass is instantaneously frozen. One cannot differentiate the mass that varies as a result of the mass exchange. In the angular momentum, the mass moment of inertia is the function of both time parameters

$$J^{\mathrm{g}}(t) = M_{\mathrm{b}} r(t)^2, \tag{3.28}$$

$$J^{\mathrm{m}}(\tau, t) = m_{\mathrm{b}}(\tau) r(t)^2. \tag{3.29}$$

We have separated the two distinct terms: the mass of the drum and the mass of the rope wound on the drum. Let ω stand for the drum angular velocity. Taking into account the considerations from previous cases, we have

$$\begin{aligned} J(t,\tau)\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} &+ \frac{\mathrm{d}J(t,\tau)}{\mathrm{d}t}\omega(t) \\ &+ m_{\mathrm{s}}(t)\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} r(t) + \frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2}\frac{(l - x(t))}{2}\right) = M - m_{\mathrm{s}} g r(t). \end{aligned} \tag{3.30}$$

This system differs from the previous ones in that $\frac{\mathrm{d}J(t,\tau)}{\mathrm{d}t} \neq 0$. The inertia moment of the drum with the wound part of the rope in the angular momentum equation can change as a result of the geometry change.

The absolute motion of the rope is the geometric sum of the rotational relative motion and the radial drum motion. To derive the kinematic relations, let us note that the radius increases along the whole circumference, and as a consequence increases the length of the wound rope. This length increases also as a result of the drum rotation, then

$$\Delta x = r\Delta\varphi + \varphi\Delta r. \tag{3.31}$$

Note that this way of the radius change and its influence on the x differs significantly from the one occurring in the previous section. Therefore,

$$\begin{aligned} \omega &= \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r}\left(\frac{\mathrm{d}x}{\mathrm{d}t} - \varphi\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{1}{r^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t} r - x\frac{\mathrm{d}r}{\mathrm{d}t}\right), \\ \varepsilon &= \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{r^2}\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} r - x\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - 2r\frac{\mathrm{d}r}{\mathrm{d}t}\omega\right). \end{aligned} \tag{3.32}$$

After differentiating the mass moment of inertia, $t = \tau$ is substituted into (3.30):

$$\begin{aligned} \big(M_{\mathrm{b}} r^2(t) + m_{\mathrm{b}} r(t)^2\big)\varepsilon(t) &+ 2\big(M_{\mathrm{b}} r(t) + m_{\mathrm{b}} r(t)\big)\frac{\mathrm{d}r(t)}{\mathrm{d}t}\omega(t) \\ &+ m_{\mathrm{s}}(t)\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} r(t) + \frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2}\frac{(l - x(t))}{2}\right) - M + m_{\mathrm{s}}(t) g r(t) = 0. \end{aligned} \tag{3.33}$$

The second term is the moment of the Coriolis force. In this case, the radial drum velocity is constant and equal to α. The radius second derivative is 0, which prevents consideration of the rope bending moment.

The following data were taken for numerical simulations: $l = 20$ m, $M_{\mathrm{b}} = 10$ kg, $r_0 = 10$ cm, $\alpha = 0.1$ m/s, $m = 20$ kg.

The solutions for two boundary values of the torque M are presented in Figs. 13 and 14. For $M = 40.254$ Nm, the whole rope can be wound onto the drum. The angular velocity of the drum increases at the beginning stage of motion, then it decreases to 0, and finally the rotational motion vanishes, i.e. the angular position of the drum with the wound rope is established. For the lower value of this torque, i.e. $M = 40.25$ Nm, after the angular velocity achieves 0, the drum starts rotating in the opposite direction and the rope begins to unwind.

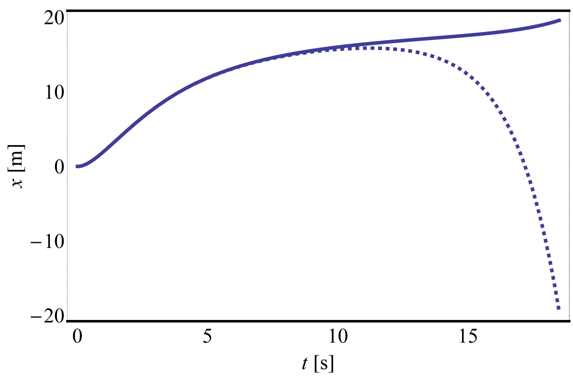

Fig. 13. The drum with the regulated radius – the variation of x in time for $M = 40.254$ Nm (continuous line) and $M = 40.25$ Nm (dashed line).

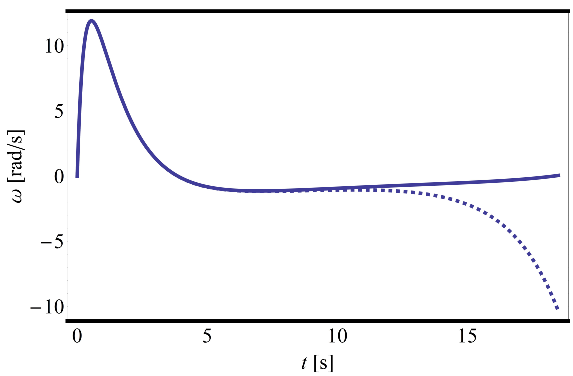

Fig. 14. The drum with the regulated radius – the variation of drum angular velocity ω in time for $M = 40.254$ Nm (continuous line) and $M = 40.25$ Nm (dashed line).

The angular velocities of the drum are shown in Fig. 14. The objective of this case is presented only to show the variety of possible cases related to variation of the mass, therefore the solution itself is not investigated in depth.

All the ordinary differential equations were solved numerically in software for symbolic and numeric computation Mathematica 8.0. The solutions were found using function *NDSolve* with default options. This function adapts the method and step size so that the estimated error of the solution is within the tolerances specified by the prescribed, 10 in the analysed cases, effective digits of accuracy. The maximum value of the absolute error expressed as the deviation of the left side of Eq. (3.33) from 0 was $5 \cdot 10^{-5}$. The errors of the solutions obtained for Examples 1–3 were much lower.

4. Conclusions

The application of theorems of rigid body dynamics leads to incorrect motion equations in the case of systems with variable masses. The general equations dedicated strictly to the system of variable masses have also to be applied carefully. The summary points of this paper can be divided into three parts:

- Comments on the introduced method.

The motion equation for a rotating body with time-variable mass has been expressed using the two time parameters. This method prevents the incorrect formulation of the angular momentum equation by separating the terms dependent on the time parameters related to the mass that is not exchanged between system elements and to the mass that is exchanged. The correctness has been proved by determining the equations for Example 1 using two approaches. The angular momentum theorem applied for the whole system of constant mass leads to the same equation as the two time parameters method applied to the two elements with mass flowing between them. The equation obtained has been compared to equations presented in literature. The method allows for dynamic analysis of much more complex problems, such as the ones dealt with in Examples 3 and 4, that have not been investigated earlier.

- Comments on modeling of the rigid bodies with time variable masses.

It can be observed from the example of the winch drum winding the heavy rope that dynamic modeling of these systems, with rotating masses in particular, frequently requires an individual approach in deriving motion equations from the angular momentum theorem, even in the case of seemingly similar systems. Such an individual approach enables analysis at each stage of motion derivation of the physical meaning of the simplifying assumption adopted. It has to be profoundly studied whether the simplifications made are acceptable and the model satisfactorily reflects the properties of the real system. That is why so many methods have appeared in the literature that have been developed for specific problems.

• Comments on the dynamics of the drum-rope system in which the variability of the drum mass is neglected.

Modeling of the winch drum is not the problem having ambiguous solutions. Nonetheless, in terms of the assumptions made, various models can be obtained. To neglect the rope weight is frequently too strong a simplification, which prevents reliable discussion of the solutions obtained.

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