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Optimum design of elastic moduli for the multiple load problems

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The paper deals with minimization of the weighted average of compliances of structures, made of an elastic material of spatially varying elasticity moduli, subjected to n load variants acting non-simultaneously. The trace of the Hooke tensor is assumed as the unit cost of the design. Three versions of the *free material design* are discussed: designing the moduli of arbitrary anisotropy (AMD), designing the moduli of an isotropic material (IMD), designing of Young's modulus for a fixed Poisson ratio (YMD). The problem is in all cases reduced to the Linear Constrained Problem (LCP) of Bouchitté and Fragalà consisting of two mutually dual problems: stress based and strain based, the former one being characterized by the integrand of linear growth depending on the trial statically admissible stresses. The paper shows equivalence of the stress fields solving the (LCP) problem and those appearing in the optimal body subjected to subsequent load cases.

Key words: structural optimization, free material design, elasticity.

1. Introduction

1.1. Overview of the free material design problems for n load conditions

The optimal anisotropy emerging in the least compliant designs of non-homogeneous d-dimensional structures subjected to a single load variant turns out to be highly singular. Of the six eigenvalues of the optimal Hooke tensor only one is nonzero, as noted in the paper by Bendsøe *et al.* [1], where the unit cost had been chosen as the trace of the Hooke tensor. One of the method to make the optimal Hooke tensor non-singular is to consider additional loads, to be applied non-simultaneously, and assume the *weighted average of compliances* as the merit function, see Bendsøe *et al.* [2] and Weldeyesus and Stolpe [3]. If the loads are appropriately chosen such that the whole design domain is used to carry the loads and if the number n of the loads is bigger or equal $m = d(d+1)/2$, then there appears a chance to make the Hooke tensor nonsingular (i.e. positive definite), cf. Czarnecki and Lewiński [4]. The second method is to impose new constraints on the Hooke tensor, e.g. *cubic symmetry*. If a single load is applied, the optimal Hooke tensor of cubic symmetry becomes singular, but the number of positive eigenvalues of the Hooke tensor in 3D is three and remaining three are zero, cf. Czubacki and Lewiński [5]. The stronger condition is *isotropy*; the optimal isotropic designs corresponding to a single load variant are characterized by two, in general non-zero, bulk and shear moduli, cf. Czarnecki [6] and Czarnecki and Wawruch [7]. But still there subdomains can appear where the bulk modulus is zero and other places where the shear modulus vanishes. In the former case the Poisson ratio attains its lower bound equal -1 (if $d = 2$ or $d = 3$) and in the latter case the Poisson ratio attains its upper bound: 1/2 for $d = 3$ and 1 for $d = 2$.

The methods aimed at designing the elastic moduli pointwise, keeping the cost condition expressed directly in terms of the components of the Hooke tensor, are called the methods of Free Material Design (FMD), as proposed in the original papers by Ringertz [8] and Bendsøe *et al.* [2]. Later, since 1990's, the same family of methods has been called Free Material Optimization (FMO), see the comprehensive study by Haslinger *et al.* [9] and the bibliography therein. A detailed history of the development of the FMD methods (concerning mainly the case of a single load condition) has been recently published in Bołbotowski and Lewiński [10]. The aim of the present paper is to discuss the FMD methods in their selected versions for constructing the designs for the arbitrary finite number of the load conditions.

Let us stress that the present paper concerns the FMD approaches only, while the majority of papers on the material distribution problems aim at constructing an optimal layout of a single isotropic material (of given bulk and shear moduli) within a given design domain, the volume to be occupied by the given material being the cost (or the isoperimetric) condition. The genuine papers on this topic, dated back to the late 1970's, have dealt with the optimal distribution of two isotropic materials of given moduli. The properties of the material to be distributed are there treated as given, while only the necessary *relaxation by the homogenization* process introduces the composite zones where the material properties become spatially graded, yet complying with the *G-closure* constraints. The theoretical background of this approach has been given in the fundamental papers by Luc Tartar on the scalar problem, then extended to the vectorial elasticity setting by François Murat, Andrej Cherkaev, Konstantin Lur'e, Robert Kohn, Graeme Milton, Robert Lipton, etc., see the monographs by Andrej Cherkaev [11] and Gregoire Allaire [12]. More recent works originating from these ideas concern the three-material design (e.g. two isotropic materials and a void) and just this approach opens new perspectives in the optimum design, see Cherkaev and Dzierżanowski [13] and the literature cited therein.

In the FMD approach the unknown anisotropy (or its special classes) is not designed with using given materials to be optimally placed but it is itself treated as a compound design variable, subject only to the necessary symmetry conditions and to the condition of positive semi-definiteness of the Hooke tensor field. Consequently, applying the FMD approaches we are not concerned for the final results being related or not to the mechanics of composites' setting within which any material point should be given a *representative volume element* (RVE), where the stress field (called also a micro-stress field) should be constructable as statically admissible. Indeed, a weak point of the FMD methods is that this connection with an underlying microstructure is broken. However, by this simplification the FMD approaches make the design variables free of the underlying local equilibrium constraints.

As noted above, the Hooke tensor field will be the design variable of the optimum design problem; it will only be subjected to the pointwise indispensable symmetry conditions, and to the condition of positive semi-definiteness. The latter condition means that in some sub-regions of the design domain all the components of the Hooke tensor may vanish; just there voids emerge. Usually, these empty (or non-material) domains appear close to the boundaries, far from the places where the loads are applied. Thus, the assumption of positive semi-definiteness means endowing this optimization method with the tool of cutting off the unnecessary parts of the design domain; the method becomes the topology optimization method, since no *a priori* assumptions on connectivity of the final design are imposed: the voids appear where all the design variables vanish. Moreover, the optimal design is determined by the minimal set of data: the design domain, the loads and their weights, and the boundary where the support is possible. This idealized formulation reduces the problem of minimization of the weighted average of compliances to the pair of two mutually dual problems, the theory of which has been put forward in Bouchitté and Fragalà [14] and named there the *Linear Constrained Problem* (LCP). However, the paper [14], along with the former paper by Bouchitté and Buttazzo [15] did not deal with the free material design, but with the optimal distribution of the measure being the carrier of the structure, which, in particular, may be alternatively interpreted as the problem of optimal distribution of Young's modulus of an isotropic body of a fixed Poisson ratio, see Czarnecki and Lewiński [16]. It is surprising indeed that the free material design problems in various settings reduce to such pairs of mutually dual problems. A rigorous passage from the free material design to the LCP problem has been recently shown in [10] in the context of a single load variant.

The present paper discusses the three versions of the FMD problem of designing of an elastic non-homogeneous material locally being:

a) anisotropic, b) isotropic, c) isotropic with the predefined Poisson ratio

in a structure subjected to n load variants. The merit function is chosen as the weighted compliance, while the unit cost is assumed as equal to the trace of the

Hooke tensor. The weighted average of compliances is a convex combination of the compliances corresponding to the subsequent (non-simultaneously applied) load variants; the weights are denoted by $\eta_1, \ldots, \eta_n$, $0 \le \eta_\alpha \le 1$, $\eta_1 + \cdots + \eta_n = 1$. On the unknown Hooke tensor we impose pointwise the well-known symmetry conditions as well as the conditions of positive semi-definiteness, as mentioned earlier.

In the context of n variants of loads the problem (a), named here the *anisotropic material design* (AMD) has been for the first time formulated in [2]. In [4] the problem AMD has been reduced to the one part of the LCP problem, yet the dual (or strain-based) problem has not been specified with all necessary details. The present paper puts forward both the problems constituting the linear constrained problem (LCP) implied by AMD method.

In the context of n load variants the problem (b) has been for the first time proposed in Czarnecki and Lewiński [17] and there named the *isotropic material design* (IMD). In the same paper the problem (c) has been discussed in which the Young modulus is the only design variable. Both the IMD and YMD (or the *Young modulus design*) problems have been there reduced to one part of the LCP problem in which the minimization process concerns the trial stress fields. The present paper complements the hitherto published results by delivering the explicit formulations of both the mutually dual problems constituting the LCP problem for the IMD and YMD methods. Moreover, the paper delivers arguments confirming that the stress fields in the optimal structure induced by the subsequent loads coincide with the relevant components of the minimizer of the LCP problems.

The present paper sets forth the vectorial optimization as minimization of the convex combination of compliances for the subsequent load variants, the loads being viewed as acting non-simultaneously. As the general scheme of such problems discussed in Marler and Arora [18] applies here, the complete solution to this problem paves the way for constructing the Edgeworth–Pareto front for the least compliant designs of elastic structures. This justifies the chosen method of scalarization. Alternatively, one can consider minimization of the *principal compliance*, as proposed in Cherkaev and Cherkaev [19]. This concept introduces a priori a certain integral constraint on the possible loads. Hence, the final design becomes insensitive to the selected load variants. This approach has much in common with Segev's [20] concept of the load capacity of structures. The capacity of a structure coincides with the Lagrange multiplier corresponding to the integral constraint in Cherkaevs' theory. Introduction of these concepts into the free material design will be the subject of the forthcoming papers.

The FMD methods deliver the tools of predicting the optimal distribution of elastic moduli, necessary for the further treatment: constructing the underlying spatially varying microstructure of a given symmetry class and then programing the 3D printing process. The recent papers by Goda *et al.* [21], Czubacki *et al.* [22], Lewiński *et al.* [23], Ganghoffer *et al.* [24] and Czarnecki and Łukasiak [25] deliver an ample variety of examples of making use of the free material design tools in the process of designing materials of prescribed effective properties.

The present paper is aimed at constructing:

(i) the strain-based components of the LCP problems for the IMD and YMD settings. In the hitherto available literature only the stress-based components of the LCP problems have been constructed and only basing on them the numerical methods have been developed,

(ii) the strain-based component of the LCP problem for the AMD method. The hitherto known formulation was only a general scheme; the present work shows that the locking locus of the problem is the unit ball with respect to Schatten's ∞-norm for collection of virtual strains corresponding to the subsequent load cases,

(iii) the auxeticity regime predicted by the IMD method,

(iv) the algorithm of computing the stress fields in the optimal body corresponding to the subsequent load cases.

The present paper does not deal with the regularity issues concerning the variational problems considered. In particular, the problems of existence of solutions will not be dealt with, hence, for instance, we shall not use the operators *sup* or *inf*, understanding that the operations *max* and *min* suffice to convey the idea. The correct setting of the FMD problems requires subtle tools of the modern variational calculus along with measure-theoretic methods, see the recent paper by Bołbotowski and Lewiński [10] on the free material design for the case of a single load condition.

1.2. Selected mathematical tools and adopted conventions

The aim of the present paper is reducing the AMD, IMD and YMD problems concerning the design for n load variants to the pairs of mutually dual problems forming the LCP problem of Bouchitté and Fragalà [14], see problems (9) and (11) therein. It turns out that in the LCP problems corresponding to the AMD problem the specific norms of matrices appear, so called Schatten p-norms. The main feature of these norms is their property of being unitarily invariant on the space of square matrices, see Ch. IV.2 of the book by Bhatia [26]. Let us recall the definition of Schatten's norms. Let $\mu_i(\mathbf{B})$ denote the i-th eigenvalue of a square matrix $\mathbf{B}$. Let the $t \times l$ matrix $\mathbf{A}$ be of the rank: $r = \operatorname{rank}(\mathbf{A})$. Let us define

$$s_k(\mathbf{A}) = \sqrt{\mu_k(\mathbf{A}\mathbf{A}^T)}. \tag{1.1}$$

These quantities are either positive (if $1 \le k \le r$) or zero, if $r < k \le \min(t, l)$. We introduce the ordering: $s_1 \ge s_2 \ge \cdots \ge s_r$; the quantities s_j are called singular values of $\mathbf{A}$. The Schatten p-norm of the matrix $\mathbf{A}$ is defined by

$$\|\mathbf{A}\|_p = \Big(\sum_{j=1}^{r} (s_j(\mathbf{A}))^p\Big)^{1/p} \quad \text{if } 1 \le p < \infty \quad \text{and}$$
$$\|\mathbf{A}\|_\infty = \max_{1\le j\le r} s_j(\mathbf{A}) = s_1(\mathbf{A}). \tag{1.2}$$

The Schatten 1-norm of a matrix $\mathbf{A}$ is (for future convenience) denoted by

$$\rho(\mathbf{A}) = \|\mathbf{A}\|_1 = \sum_{j=1}^{\min(t,l)} s_j(\mathbf{A}) \quad \text{or} \quad \rho(\mathbf{A}) = \sum_{j=1}^{\min(t,l)} \sqrt{\mu_j(\mathbf{A}\mathbf{A}^T)}. \tag{1.3}$$

Throughout the paper a conventional notation is applied: the design domain in $\mathbb{R}^d$ is denoted by Ω; in case of $d = 3$ the domain is parameterized by the Cartesian system (x_1, x_2, x_3) with the orthogonal basis $\mathbf{e}_i$, $i = 1, 2, 3$; $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where $\cdot$ is the scalar product in $\mathbb{R}^d$. The Euclidean norm of $\mathbf{p} \in \mathbb{R}^d$ is defined by $\|\mathbf{p}\| = \sqrt{\mathbf{p} \cdot \mathbf{p}}$.

The set of second rank symmetric tensors is denoted by E_s^2. The identity tensor in E_s^2 is $\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$; repetition of indices implies summation. The set of fourth rank tensors $\mathbf{C} = C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ satisfying the symmetry conditions $C_{ijkl} = C_{klij}$, $C_{ijkl} = C_{jikl}$ is denoted by E_s^4. The identity tensor in E_s^4 is represented by: $\mathbf{II} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$. The scalar product of $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in E_s^2$ is defined by: $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$. The Euclidean norm of $\boldsymbol{\sigma} \in E_s^2$ is defined by $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}$. Any tensor from the set E_s^2 may be decomposed as follows: $\boldsymbol{\sigma} = \frac{1}{d}(\operatorname{tr} \boldsymbol{\sigma})\mathbf{I} + \operatorname{dev} \boldsymbol{\sigma}$; where $\operatorname{tr} \boldsymbol{\sigma} = \sigma_{ij}\delta_{ij}$. Note that for $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in E_s^2$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \left(\frac{1}{\sqrt{d}} \operatorname{tr} \boldsymbol{\sigma}\right)\left(\frac{1}{\sqrt{d}} \operatorname{tr} \boldsymbol{\varepsilon}\right) + \operatorname{dev} \boldsymbol{\sigma} \cdot \operatorname{dev} \boldsymbol{\varepsilon}, \tag{1.4}$$

which suggests introducing a new operator: $\operatorname{Tr} \boldsymbol{\sigma} = (\operatorname{tr} \boldsymbol{\sigma})/\sqrt{d}$ to simplify further formulae. Let $\sigma_I = \mu_1(\boldsymbol{\sigma}) \ge \sigma_{II} = \mu_2(\boldsymbol{\sigma}) \ge \sigma_{III} = \mu_3(\boldsymbol{\sigma})$ be principal stresses. Then

$$\operatorname{Tr} \boldsymbol{\sigma} = \begin{cases} \dfrac{1}{\sqrt{2}}(\sigma_I + \sigma_{II}) & \text{if } d = 2, \\ \dfrac{1}{\sqrt{3}}(\sigma_I + \sigma_{II} + \sigma_{III}) & \text{if } d = 3, \end{cases} \tag{1.5}$$

$$\|\operatorname{dev} \boldsymbol{\sigma}\| = \begin{cases} \dfrac{1}{\sqrt{2}}|\sigma_I - \sigma_{II}| & \text{if } d = 2, \\ \dfrac{1}{\sqrt{3}}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_I - \sigma_{III})^2 + (\sigma_{II} - \sigma_{III})^2} & \text{if } d = 3. \end{cases} \tag{1.6}$$

A comma before an index implies partial differentiation, e.g.: $\partial(\cdot)/\partial x_i = (\cdot)_{,i}$, $i = 1, 2, 3$. The symmetric part of the gradient of the vector field $\mathbf{v}$ is denoted by $\boldsymbol{\varepsilon}_{ij}(\mathbf{v}) = (\nu_{i,j} + \nu_{j,i})/2$. By averaging over Ω we understand the operation $\langle \ldots \rangle$ defined by $\langle f \rangle = (\int_\Omega f\, dx)/|\Omega|$; $dx = dx_1\, dx_2\, dx_3$ if $d = 3$ and $dx = dx_1\, dx_2$ if $d = 2$. The components of $\boldsymbol{\sigma} \in E_s^2$ may be viewed as a column vector:

$$\boldsymbol{\sigma} \sim [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{13}, \sqrt{2}\sigma_{12}]^T \quad \text{in } \mathbb{R}^6 \text{ if } d = 3, \tag{1.7}$$

$$\boldsymbol{\sigma} \sim [\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12}]^T \quad \text{in } \mathbb{R}^3 \text{ if } d = 2. \tag{1.8}$$

These column vectors are still denoted by $\boldsymbol{\sigma}$, which shall not lead to misunderstandings. The vectorization technique above has been proposed by P. Bechterew in 1920's, see Annin and Ostrosablin [27], and has been rediscovered in Mehrabadi and Cowin [28], Blinowski *et al.* [29], Moakher [30].

Let us recall that for any norm $\phi(\cdot)$ in $\mathbb{R}^m$ one may introduce the dual norm by

$$\phi^o(\mathbf{q}) = \max_{\substack{\mathbf{p}\in\mathbb{R}^m,\\ \mathbf{p}\neq\mathbf{0}}} \frac{\mathbf{p}\cdot\mathbf{q}}{\phi(\mathbf{p})} \tag{1.9}$$

or

$$\phi^o(\mathbf{q}) = \max\{\mathbf{p}\cdot\mathbf{q} \mid \phi(\mathbf{p}) \le 1,\ \mathbf{p} \in \mathbb{R}^m\}. \tag{1.10}$$

Similarly, for any norm $\phi(\cdot)$ in E_s^2 one may introduce the dual norm by

$$\phi^o(\boldsymbol{\varepsilon}) = \max_{\substack{\sigma\in E_s^2\\ \boldsymbol{\sigma}\neq\mathbf{0}}} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}}{\phi(\boldsymbol{\sigma})} \tag{1.11}$$

or

$$\phi^o(\boldsymbol{\varepsilon}) = \max\{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} \mid \boldsymbol{\sigma} \in E_s^2,\ \phi(\boldsymbol{\sigma}) \le 1\}. \tag{1.12}$$

The functions $\mathbf{q} \to \phi^o(\mathbf{q})$, $\boldsymbol{\varepsilon} \to \phi^o(\boldsymbol{\varepsilon})$ are called functions being polar to $\mathbf{q} \to \phi(\mathbf{q})$, $\boldsymbol{\varepsilon} \to \phi(\boldsymbol{\varepsilon})$.

If $\phi(\mathbf{p}) = \|\mathbf{p}\|$, i.e. if $\phi(\cdot)$ is the Euclidean norm, then $\phi^o(\mathbf{p}) = \phi(\mathbf{p}) = \|\mathbf{p}\|$.

The following estimates hold

$$\mathbf{p}\cdot\mathbf{q} \le \phi(\mathbf{p})\phi^o(\mathbf{q}), \qquad \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} \le \phi(\boldsymbol{\sigma})\phi^o(\boldsymbol{\varepsilon}) \tag{1.13}$$

and the equalities are attainable. Let us recall that the Schatten q-norm is dual to the Schatten p-norm if $1/p + 1/q = 1$. In particular, see Bhatia [26]

$$(\|\mathbf{A}\|_1)^o = \|\mathbf{A}\|_\infty. \tag{1.14}$$

The following convention of expressing the minimization problems is adopted: $\min_{x\in X} f(x) = \min\{f(x) \mid \text{over } x \in X\}$, which is very convenient if the expression $x \in X$ is complicated. The set X is not strictly defined, since the regularity conditions are not specified.

1.3. A useful minimization result

In the optimum design problems to be analyzed the following minimization problem appears:

$$J = \min\Big\{ \int_\Omega \Big(\sum_{i=1}^{n} \frac{a_i(x)}{u_i(x)} \Big)\, dx \;\Big|\; \text{over } u_i \text{ such that: } u_i \ge 0,\ \int_\Omega \Big(\sum_{i=1}^{n} u_i(x) \Big)\, dx \le \Lambda \Big\} \tag{1.15}$$

where $a_i(x) > 0$ are given functions in the domain Ω, Λ is a given positive constant while the functions $u_i(x)$, $i = 1, \ldots, n$ are unknown. Let us define $E_0 = \Lambda/|\Omega|$. The solution to the problem (1.15) reads

$$\hat{u}_i(x) = E_0 \frac{\sqrt{a_i(x)}}{\langle \sum_{j=1}^{n} \sqrt{a_j} \rangle}, \tag{1.16}$$

which delivers the explicit formula for J

$$J = \frac{1}{\Lambda} \Big(\int_\Omega \Big(\sum_{j=1}^{n} \sqrt{a_j} \Big)\, dx \Big)^2 . \tag{1.17}$$

Since the result (1.16, 1.17) is crucial in the sequel, it is thought appropriate to deliver its proof. Let us introduce the vector fields $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ in Ω, with the components:

$$f_i = \sqrt{\frac{a_i}{u_i}}, \qquad g_i = \sqrt{u_i}. \tag{1.18}$$

Let us apply now the Cauchy–Schwarz inequality:

$$(\mathbf{f} \mid \mathbf{g}) \le \sqrt{(\mathbf{f}|\mathbf{f})}\sqrt{(\mathbf{g}|\mathbf{g})} \tag{1.19}$$

where the scalar product is defined by

$$(\mathbf{f}|\mathbf{g}) = \int_\Omega \mathbf{f} \cdot \mathbf{g}\, dx. \tag{1.20}$$

Upon computing the scalar products

$$(\mathbf{f}|\mathbf{g}) = \int_\Omega \Big(\sum_{i=1}^{n} \sqrt{a_i} \Big)\, dx, \quad (\mathbf{f}|\mathbf{f}) = \int_\Omega \Big(\sum_{i=1}^{n} \frac{a_i}{u_i} \Big)\, dx, \quad (\mathbf{g}|\mathbf{g}) = \int_\Omega \Big(\sum_{i=1}^{n} u_i \Big)\, dx \tag{1.21}$$

and inserting the above results into the inequality (1.19) one gets

$$\int_\Omega \Big(\sum_{j=1}^{n} \frac{a_i}{u_i}\Big)\,dx \geq \frac{1}{\int_\Omega \left(\sum_{j=1}^{n} u_j\right)dx}\bigg(\int_\Omega \Big(\sum_{j=1}^{n}\sqrt{a_j}\Big)\,dx\bigg)^2 \geq \frac{1}{\Lambda}\bigg(\int_\Omega \Big(\sum_{j=1}^{n}\sqrt{a_j}\Big)\,dx\bigg)^2. \tag{1.22}$$

The inequalities above become equalities for $u_i = \hat{u}_i$ given by (1.16), which ends the proof of the formulae (1.16) and (1.17).

2. Anisotropic Material Design (AMD)

The aim of the present section is rearranging the AMD method by reducing it to the pair of mutually dual problems forming the relevant Linear Constrained Problems (LCP), thus constituting the theory for both single and multiple load cases. As stressed in the Introduction, the AMD method discussed imposes the weakest possible constraints on the Hooke tensor components: known symmetry conditions and positive semi-definiteness, thus constructing the AMD as a topology optimization tool. The complete solution provides the Edgeworth–Pareto front of the problem considered, see [18].

The stress-based formulation is recalled after Czarnecki and Lewiński [4]; the explicitly written strain-based formulation is the novelty of this part of the present paper.

2.1. Elasticity problem for subsequent n load variants

Consider a structure made of a linearly elastic material, subjected, non-simultaneously, to $n \geq 1$ traction loads $\mathbf{T}^{(\alpha)}$ acted on the given part Γ_1 of the boundary Γ of the given spatial ($d = 3$), or planar ($d = 2$) design domain Ω. The body is fixed on the boundary Γ_2, a part of Γ. The traction load $\mathbf{T}^{(\alpha)}$, $\alpha = 1,\ldots,n$, deforms the body and induces the displacement field $\mathbf{u}^{(\alpha)} = (u_1^{(\alpha)}, u_2^{(\alpha)}, u_3^{(\alpha)})$ (case of $d = 3$) or $\mathbf{u}^{(\alpha)} = (u_1^{(\alpha)}, u_2^{(\alpha)})$ (case of $d = 2$) and the stress field $\boldsymbol{\tau}^{(\alpha)} = (\tau_{ij}^{(\alpha)})$. The stress field $\boldsymbol{\tau}^{(\alpha)}$ is linked with the load $\mathbf{T}^{(\alpha)}$ by the variational equation of equilibrium

$$\forall \mathbf{v} \in V(\Omega) \qquad \int_\Omega \boldsymbol{\tau}^{(\alpha)} \cdot \boldsymbol{\varepsilon}(\mathbf{v})dx = f^{(\alpha)}(\mathbf{v}) \tag{2.1}$$

where the virtual work of the traction load is represented by the linear form

$$f^{(\alpha)}(\mathbf{v}) = \int_{\Gamma_1} \mathbf{T}^{(\alpha)} \cdot \mathbf{v}\,ds \tag{2.2}$$

and $V(\Omega)$ is the set of kinematically admissible displacement fields $\mathbf{v} = (v_1, \ldots, v_d)$ satisfying the condition: $\mathbf{v} = \mathbf{0}$ on the boundary Γ_2. The stress fields satisfying (2.1) form the set of statically admissible stresses $\Sigma_\alpha(\Omega)$. These statical problems are the starting point for the anisotropic material design, see Fig. 1 illustrating the case of $d = 2$, $n = 2$.

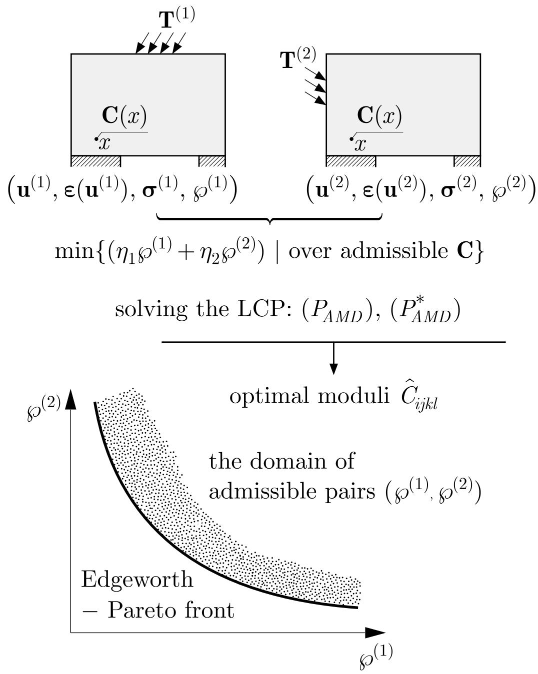

Fig. 1. A scheme of the anisotropic material design (AMD) problem for $d = 2$ and two load variants ($n = 2$).

The elastic moduli C_{ijkl} of the Hooke tensor $\mathbf{C}$, $\mathbf{C} \in E_s^4$, are the design variables. Let us recall its spectral decomposition

$$\mathbf{C}(x) = \sum_{K=1}^{m} \lambda_K(x)\boldsymbol{\omega}_K(x) \otimes \boldsymbol{\omega}_K(x) \tag{2.3}$$

where $m = d(d+1)/2$, λ_K are eigenvalues of the Hooke tensor $\mathbf{C}$, while $\boldsymbol{\omega}_K \in E_s^2$ are so-called eigenstates satisfying the orthogonality conditions: $\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}$, $K, L = 1, \ldots, m$, see Rychlewski [31]. We assume that λ_K are non-negative, i.e. the case of $\mathbf{C} = \mathbf{0}$ in some subdomains of the design domain is admitted. The trace of the Hooke tensor is defined by

$$\operatorname{tr} \mathbf{C} = \lambda_1 + \cdots + \lambda_m. \tag{2.4}$$

The stress field $\boldsymbol{\sigma}^{(\alpha)} = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}^{(\alpha)})$ associated with the displacement field $\mathbf{u}^{(\alpha)}$ satisfies the variational equation of equilibrium (2.1). The quantity

$$\wp^{(\alpha)} = f^{(\alpha)}(\mathbf{u}^{(\alpha)}) \tag{2.5}$$

represents the compliance of the structure corresponding to the load variant no α. The compliance will be viewed as a functional of the Hooke tensor field: $\wp^{(\alpha)} = \wp^{(\alpha)}(\mathbf{C})$.

2.2. Setting the AMD problem for n load variants

The aim of the approach is rational designing a structure to make it as stiff as possible if subjected to the traction loads $\mathbf{T}^{(1)} \ldots, \mathbf{T}^{(n)}$ acting non-simultaneously. Let the weight coefficient η_α ,$0 \leq \eta_\alpha \leq 1$, reflect the significance of the αth load and let $\eta_1 + \cdots + \eta_n = 1$, $0 \leq \eta_\alpha \leq 1$. One of the method of scalarization of the vectorial optimization problems is choosing a single merit function as the convex combination of the objective functions, here – compliances. The problem is reduced to minimization of the functional

$$F_\eta(\mathbf{C}) = \sum_{\alpha=1}^{n} \eta_\alpha \wp^{(\alpha)}(\mathbf{C}) \tag{2.6}$$

over the Hooke tensor fields $\mathbf{C}$ satisfying the cost condition in the form:

$$\int_\Omega \operatorname{tr}\mathbf{C}\,dx \leq \Lambda \tag{2.7}$$

where $\Lambda = E_0|\Omega|$, E_0 being a referential elasticity modulus. The isoperimetric condition (2.7) may be expressed as $\langle \operatorname{tr}\mathbf{C} \rangle \leq E_0$. We admit a degeneration of the Hooke tensor fields: not only some of their eigenvalues may vanish, but it is allowed that all of them vanish in some subdomain of the design domain. Thus, the set of admissible Hooke tensor fields in Ω consists of the tensors satisfying pointwise the symmetry conditions expressed as $\mathbf{C}(x) \in E_s^4$ (see Section 1.2), the eigenvalues of $\mathbf{C}$ being nonnegative, or $\lambda_K(x) \geq 0$, $K = 1, \ldots, m$. Such tensors form a set $H(\Omega)$. The tensor fields from this set satisfying (2.7) form the set $H_\Lambda(\Omega)$.

Now we are ready to formulate the *anisotropic material design problem* (AMD):

$$J_{\Lambda,\eta} = \min\{F_\eta(\mathbf{C}) \mid \text{over } \mathbf{C} \text{ such that: } \mathbf{C} \in H_\Lambda(\Omega)\} \tag{2.8}$$

If $\hat{\mathbf{C}}_\eta$ is the solution to this problem, the compliances $\wp^{(\alpha)}(\hat{\mathbf{C}}_\eta)$ characterize the optimum structure. The mapping

$$\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n) \to [\wp^{(1)}(\hat{\mathbf{C}}_\eta), \ldots, \wp^{(n)}(\hat{\mathbf{C}}_\eta)] \tag{2.9}$$

forms a hypersurface in $\mathbb{R}^n$. If $n = 2$, taking $\eta_1 = \eta$, $\eta_2 = 1-\eta$ we obtain a plane curve

$$\eta \to [\wp^{(1)}(\hat{\mathbf{C}}_\eta), \wp^{(2)}(\hat{\mathbf{C}}_\eta)].$$

The hypersurface (2.9) is the boundary of the set of points $[\wp^{(1)}(\mathbf{C}), \ldots, \wp^{(n)}(\mathbf{C})]$ corresponding to the all tensors $\mathbf{C}$ satisfying (2.7). This boundary is the Edgeworth–Pareto front of the problem (2.8), as noted in [4, 17]. Thus, by solving the problem (2.8) for subsequent weights $(\eta_1, \ldots, \eta_n)$ we build the Edgeworth–Pareto front, which delivers the complete information on the best designs for the given load variants. The designs lying on the Edgeworth–Pareto front cannot be corrected without increasing at least one of the compliances, see Fig. 1 explaining the construction of the Edgeworth–Pareto front in case of $n = 2$.

2.3. The stress-based reformulation of the AMD problem

The terms in the decomposition (2.3) of tensor $\mathbf{C}$ are mutually orthogonal; hence the density of the complementary energy $W^{(\alpha)}$ of the body subjected to the load variant α can be written as below

$$W^{(\alpha)} = \frac{1}{2}\boldsymbol{\tau}^{(\alpha)} \cdot \left(\sum_{K\in I} \frac{1}{\lambda_K} \boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_K\right)\boldsymbol{\tau}^{(\alpha)} = \frac{1}{2}\sum_{K\in I} \frac{1}{\lambda_K}(\boldsymbol{\omega}_K \cdot \boldsymbol{\tau}^{(\alpha)})^2 \tag{2.10}$$

where I is the collection of indices such that $\lambda_K > 0$. The α-th compliance is expressed by

$$\wp^{(\alpha)} = \min\left\{\sum_{K\in I} \int_\Omega \frac{1}{\lambda_K}(\boldsymbol{\omega}_K \cdot \boldsymbol{\tau})^2\, dx \mid \text{over } \boldsymbol{\tau} \text{ such that: } \boldsymbol{\tau} \in \Sigma_\alpha(\Omega)\right\}. \tag{2.11}$$

Let

$$G_\eta(\lambda_1, \ldots, \lambda_m; \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m; \boldsymbol{\tau}^{(1)}, \ldots, \boldsymbol{\tau}^{(n)}) = \sum_{\alpha=1,\ldots,n} \eta_\alpha \sum_{K\in I} \frac{1}{\lambda_K}(\boldsymbol{\omega}_K \cdot \boldsymbol{\tau}^{(\alpha)})^2\, dx. \tag{2.12}$$

The problem (2.8) may be written as below

$$J_{\Lambda,\eta} = \min_{\boldsymbol{\tau}^{(\alpha)}\in\Sigma_\alpha(\Omega)} \min_{\substack{\lambda_K\geq 0\\ \int_\Omega(\lambda_1+\cdots+\lambda_m)\,dx\leq\Lambda}} \min_{\substack{\boldsymbol{\omega}_K\in E_s^2\\ \boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L=\delta_{KL}}} \int_\Omega G_\eta(\lambda_1, \ldots, \lambda_m; \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m; \boldsymbol{\tau}^{(1)}, \ldots, \boldsymbol{\tau}^{(n)})\, dx. \tag{2.13}$$

The minimization operations over λ_K and over $\boldsymbol{\omega}_K$ can be performed analytically, cf. [4]. This process requires Bechterew's vectorization formalism of replacing the tensors by the column vectors (see (1.7)) and forming a stress matrix from the column vectors of stresses corresponding to the subsequent load variants. Namely, in the 3D case, if $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \ldots, \boldsymbol{\sigma}^{(n)}$ are the states of stress corresponding to the subsequent loads, we form a 6 by n matrix

$$[\boldsymbol{\sigma}^{(1)} \boldsymbol{\sigma}^{(2)} \ldots \boldsymbol{\sigma}^{(n)}] = \begin{bmatrix} \sigma_{11}^{(1)} & \sigma_{11}^{(2)} & \ldots & \sigma_{11}^{(n)} \\ \sigma_{22}^{(1)} & \sigma_{22}^{(2)} & \ldots & \sigma_{22}^{(n)} \\ \sigma_{33}^{(1)} & \sigma_{33}^{(2)} & \ldots & \sigma_{33}^{(n)} \\ \sqrt{2}\sigma_{23}^{(1)} & \sqrt{2}\sigma_{23}^{(2)} & \ldots & \sqrt{2}\sigma_{23}^{(n)} \\ \sqrt{2}\sigma_{13}^{(1)} & \sqrt{2}\sigma_{13}^{(2)} & \ldots & \sqrt{2}\sigma_{13}^{(n)} \\ \sqrt{2}\sigma_{12}^{(1)} & \sqrt{2}\sigma_{12}^{(2)} & \ldots & \sqrt{2}\sigma_{12}^{(n)} \end{bmatrix}. \tag{2.14}$$

In case of the 2D setting the matrix above has dimensions 3 by n and is constructed by using (1.8). By $\rho([\boldsymbol{\sigma}^{(1)}\ \boldsymbol{\sigma}^{(2)} \ldots \boldsymbol{\sigma}^{(n)}])$ we understand the Schatten 1-norm (1.3) of the matrix (2.14).

By performing in (2.13) minimization over λ_K, $\boldsymbol{\omega}_K$ one finds, see [4]

$$J_{\Lambda,\eta} = \frac{1}{\Lambda}(\hat{Z}_\eta)^2 \tag{2.15}$$

where $\hat{Z}_\eta$ is expressed by the linear constrained problem (LCP): find statically admissible stress fields $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ minimizing the functional below

$$\hat{Z}_\eta = \min_{\substack{\boldsymbol{\tau}^{(\alpha)} \in \Sigma_\alpha(\Omega) \\ \alpha=1,\ldots,n}} \int_\Omega \rho([\sqrt{\eta_1}\boldsymbol{\tau}^{(1)} \ldots \sqrt{\eta_n}\boldsymbol{\tau}^{(n)}])\, dx \qquad (P_{AMD}). \tag{2.16}$$

The optimum design problem (2.8, 2.13) has been thus reduced to solving the problem (2.16) which is a purely static problem of finding optimal stress fields, the problem being unaffected by elastic moduli. The optimal moduli $\hat{\lambda}_K$ (or the eigenvalues of $\hat{\mathbf{C}}$) are expressed by the minimizer $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ of the problem (2.16):

$$\hat{\lambda}_K(x) = E_0 \frac{s_K([\sqrt{\eta_1}\hat{\boldsymbol{\tau}}^{(1)}(x) \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}(x)])}{\langle \rho([\sqrt{\eta_1}\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}]) \rangle} \tag{2.17}$$

where $K = 1, \ldots, m$; some of the moduli may vanish.

The optimal eigenstates $\hat{\boldsymbol{\omega}}_K$ are eigenvectors of the eigenvalue problem: find $(\mu, \mathbf{y})$, $\mu \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^m$ such that

$$(\hat{\mathbf{S}}_\eta \hat{\mathbf{S}}_\eta^T)\mathbf{y} = \mu \mathbf{y} \tag{2.18}$$

where

$$\hat{\mathbf{S}}_\eta = [\sqrt{\eta_1}\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}]. \tag{2.19}$$

In case of 3D the vector $\mathbf{y} \in \mathbb{R}^6$ determines the components of the tensor $\hat{\boldsymbol{\omega}}$ according to (1.7), or $\hat{\omega}_{11} = y_1$, $\hat{\omega}_{22} = y_2$, $\hat{\omega}_{33} = y_3$, $\hat{\omega}_{23} = y_4/\sqrt{2}$, $\hat{\omega}_{13} = y_5/\sqrt{2}$, $\hat{\omega}_{12} = y_6/\sqrt{2}$.

The fields $\hat{\lambda}_K$, $\hat{\boldsymbol{\omega}}_K$ determine the optimal Hooke tensor

$$\hat{\mathbf{C}}(x) = \sum_{K=1}^{m} \hat{\lambda}_K(x)\hat{\boldsymbol{\omega}}_K(x) \otimes \hat{\boldsymbol{\omega}}_K(x). \tag{2.20}$$

If $1 \le n < m$ tensor $\hat{\mathbf{C}}$ is singular, independently of the form of the loads. If $n \ge m$ there exist loads such that all eigenvalues $\hat{\lambda}_K$, $K = 1, \ldots, m$, are positive at each point of the design domain.

2.4. The strain-based reformulation of the AMD problem

The general theory put forward in [14] teaches us that the minimization problem (2.16) should be analyzed jointly with its dual formulation, thus forming the LCP-type problem. The dual version of the problem (2.16) requires construction of the function polar to the function $\rho([\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)} \ldots \boldsymbol{\sigma}^{(n)}])$, or

$$\rho^o([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max\{(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\varepsilon}^{(1)} + \cdots + \boldsymbol{\tau}^{(n)} \cdot \boldsymbol{\varepsilon}^{(n)}) \mid \text{over } \boldsymbol{\tau}^{(\alpha)} \in E_s^2,\ \alpha = 1, \ldots, n;\ \rho([\boldsymbol{\tau}^{(1)} \ldots \boldsymbol{\tau}^{(n)}]) \le 1\}. \tag{2.21}$$

The maximization operation can be performed analytically, resulting in, see (1.14)

$$\rho^o([\boldsymbol{\varepsilon}^{(1)}\boldsymbol{\varepsilon}^{(2)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max_{1 \le j \le \min(m,n)} \{s_j[\boldsymbol{\varepsilon}^{(1)}\boldsymbol{\varepsilon}^{(2)} \ldots \boldsymbol{\varepsilon}^{(n)}]\} = \|[\boldsymbol{\varepsilon}^{(1)}\boldsymbol{\varepsilon}^{(2)} \ldots \boldsymbol{\varepsilon}^{(n)}]\|_\infty. \tag{2.22}$$

Thus, the problem dual to (2.16) assumes the form of the LCP-type problem: find the maximizer $(\hat{\mathbf{v}}^{(1)}, \hat{\mathbf{v}}^{(2)}, \ldots, \hat{\mathbf{v}}^{(n)})$ of:

$$\hat{Z}_\eta = \max\Big\{\sum_{\alpha=1}^{n} \sqrt{\eta_\alpha} f^{(\alpha)}(\mathbf{v}^{(\alpha)}) \ \Big|\ \text{over } \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)} \in V(\Omega) \text{ such that } \|[\boldsymbol{\varepsilon}(\mathbf{v}^{(1)}(x))\boldsymbol{\varepsilon}(\mathbf{v}^{(2)}(x)) \ldots \boldsymbol{\varepsilon}(\mathbf{v}^{(n)}(x))]\|_\infty \le 1 \text{ for a.e. } x \in \Omega\Big\} \quad (P^*_{AMD}). \tag{2.23}$$

The problems (2.16), (2.23) should be viewed as one problem and solved simultaneously.

Remark 2.1. Let us note that the problem (2.23) introduces the locking conditions on virtual strains, like in the theory of materials with locking, see [32]. These locking conditions impose pointwise constraints on the whole collection of virtual strains with using the Schatten ∞-norm. On the other hand, similar locking conditions on the virtual strain (for a single load variant) was discovered already by Michell in 1904 in the context of minimization of weight of the fully stressed frameworks. Just the locking conditions on the virtual strain are the typical starting point for constructing the optimal Michell structures in plane, see [33] and [34]. The first mathematical and very clear explanation of appearance of the locking phenomenon has been delivered by Strang and Kohn [35]. Nowadays, more general treatment of Bouchitté and Fragalà [14] confirms that such conditions appear inevitably within certain classes of optimum design problems.

2.5. Specification for the case of $n = 2$ load variants

2.5.1. Discussion of the problem (P_{AMD}). In the case of two load variants ($n = 2$) some steps of the procedure leading to (2.16) can be done by geometrical methods treating stress tensors as vectors in $\mathbb{R}^3$.This delivers a deeper understanding of the result (2.16). First we shall specify the problem (2.16) for $n = 2$, given $\eta_1 = \eta$, $\eta_2 = 1 - \eta$, $0 \leq \eta \leq 1$; Ω is a plane or a spatial domain. Let us start with computing the Schatten 1-norm of the matrix $\mathbf{A} = [\boldsymbol{\sigma}\ \boldsymbol{\tau}]$, $\boldsymbol{\sigma}, \boldsymbol{\tau} \in E_s^2$. We make use of the vectorial representation of tensors from E_s^2, cf. (1.7, 1.8), hence

$$\text{(2.24)}\quad \mathbf{A}^T = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sqrt{2}\sigma_{23} & \sqrt{2}\sigma_{13} & \sqrt{2}\sigma_{12} \\ \tau_{11} & \tau_{22} & \tau_{33} & \sqrt{2}\tau_{23} & \sqrt{2}\tau_{13} & \sqrt{2}\tau_{12} \end{bmatrix}, \quad \mathbf{A}^T = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sqrt{2}\sigma_{12} \\ \tau_{11} & \tau_{22} & \sqrt{2}\tau_{12} \end{bmatrix}$$

for $d = 3$ and $d = 2$, respectively. We shall find an explicit representation of the Schatten 1-norm of this matrix. The matrix $\mathbf{A}^T\mathbf{A}$ reads

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} & \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \\ \boldsymbol{\tau}\cdot\boldsymbol{\sigma} & \boldsymbol{\tau}\cdot\boldsymbol{\tau} \end{bmatrix}.$$

The eigenvalues $\mu_\alpha = \mu_\alpha(\mathbf{A}^T\mathbf{A})$ of this matrix are the roots of the equation

$$\text{(2.25)}\qquad \mu^2 - (\|\boldsymbol{\sigma}\|^2 + \|\boldsymbol{\tau}\|^2)\mu + [\|\boldsymbol{\sigma}\|^2\|\boldsymbol{\tau}\|^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2] = 0.$$

The singular values are given by: $s_1(\mathbf{A}) = \sqrt{\mu_1}$, $s_2(\mathbf{A}) = \sqrt{\mu_2}$, where

$$\text{(2.26)}\qquad \mu_{1,2} = \frac{1}{2}(\|\boldsymbol{\sigma}\|^2 + \|\boldsymbol{\tau}\|^2) \pm \frac{1}{2}\sqrt{(\|\boldsymbol{\sigma}\|^2 - \|\boldsymbol{\tau}\|^2)^2 + 4(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2}.$$

The Schatten 1-norm of $\mathbf{A}$ equals $\rho(\mathbf{A}) = s_1(\mathbf{A}) + s_2(\mathbf{A})$, or

$$\rho([\boldsymbol{\sigma}\ \boldsymbol{\tau}]) = \sqrt{\mu_1} + \sqrt{\mu_2}. \tag{2.27}$$

One can write $\rho([\boldsymbol{\sigma}\ \boldsymbol{\tau}]) = U(\boldsymbol{\sigma}, \boldsymbol{\tau})$ and the function $U(\boldsymbol{\sigma}, \boldsymbol{\tau})$ may be reduced to the form

$$U(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \sqrt{\|\boldsymbol{\sigma}\|^2 + \|\boldsymbol{\tau}\|^2 + 2\sqrt{\|\boldsymbol{\sigma}\|^2\|\boldsymbol{\tau}\|^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2}}. \tag{2.28}$$

In case of $d = 2$ one may write:

$$U(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \sqrt{\|\boldsymbol{\sigma}\|^2 + 2\|\boldsymbol{\sigma}\times\boldsymbol{\tau}\| + \|\boldsymbol{\tau}\|^2}$$

or

$$U(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \max\{\|\boldsymbol{\sigma} + \mathbf{R}_{-\pi/2}\boldsymbol{\tau}\|, \|\boldsymbol{\sigma} + \mathbf{R}_{\pi/2}\boldsymbol{\tau}\|\},$$

where $\mathbf{R}_{\pm\pi/2}$ represents the matrix of rotation by $\pm\pi/2$ in the plane determined by the vectors $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$.

Thus, for n=2, the problem (2.16) can be put in the form: find $\hat{\boldsymbol{\tau}}^{(1)} \in \Sigma_1(\Omega)$, $\hat{\boldsymbol{\tau}}^{(2)} \in \Sigma_2(\Omega)$ attaining minimum in:

$$\hat{Z}_\eta = \min\Big\{\int_\Omega U(\sqrt{\eta}\,\boldsymbol{\tau}^{(1)}, \sqrt{1-\eta}\,\boldsymbol{\tau}^{(2)})\,dx \ \Big|\ \text{over } \boldsymbol{\tau}^{(1)} \in \Sigma_1(\Omega), \boldsymbol{\tau}^{(2)} \in \Sigma_2(\Omega)\Big\}. \tag{2.29}$$

Let us recall now the geometric approach by CZARNECKI and LEWIŃSKI [36] and DZIERŻANOWSKI and LEWIŃSKI [37, 38] for the planar case: $d = 2$, $m = 3$. We shall show that the problem (2.29) can be deduced from (2.13) by a direct minimization over the eigenstates $\boldsymbol{\omega}_K$ and over the eigenvalues λ_K. We minimize G_η (see 2.12) over the tensors $\boldsymbol{\omega}_K$ of eigenstates, treated as vectors in $\mathbb{R}^3$, see (1.8); the result reads

$$\begin{aligned}\min\{G_\eta(\lambda_1,\lambda_2,\lambda_3;\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3;\boldsymbol{\tau}^{(1)},\boldsymbol{\tau}^{(2)}) \mid \\ \boldsymbol{\omega}_K \in E^2_s,\ \boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L = \delta_{KL},\ K = 1,2,3\} = W_\lambda(\sqrt{\eta}\,\boldsymbol{\tau}^{(1)}, \sqrt{1-\eta}\,\boldsymbol{\tau}^{(2)})\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}W_\lambda(\boldsymbol{\sigma},\boldsymbol{\tau}) = {}&\frac{1}{2}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big)(\|\boldsymbol{\sigma}\|^2 + \|\boldsymbol{\tau}\|^2) \\ &- \frac{1}{2}\Big(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\Big)\sqrt{(\|\boldsymbol{\sigma}\|^2 - \|\boldsymbol{\tau}\|^2)^2 + 4(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2},\end{aligned} \tag{2.31a}$$

see Eq. (4.48) in [36]. Let us note that the smallest eigenvalue λ_3 does not enter the r.h.s. of (2.30). Moreover, the potential (2.31a) can be expressed by the eigenvalues of the matrix

$$[\sqrt{\eta}\,\boldsymbol{\tau}^{(1)}\ \sqrt{1-\eta}\boldsymbol{\tau}^{(2)}]^T[\sqrt{\eta}\,\boldsymbol{\tau}^{(1)}\ \sqrt{1-\eta}\,\boldsymbol{\tau}^{(2)}]$$

according to the formula

$$W_\lambda(\boldsymbol{\sigma},\boldsymbol{\tau}) = \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}, \tag{2.31b}$$

where μ_1, μ_2 are given by the formulae (2.26) for $\boldsymbol{\sigma} = \sqrt{\eta}\boldsymbol{\tau}^{(1)}$, $\boldsymbol{\tau} = \sqrt{1-\eta}\boldsymbol{\tau}^{(2)}$.

Now we are ready to perform minimization in (2.13) over $\boldsymbol{\omega}_K$. This operation can be shifted under the integral. By using (2.30) we find

$$J_{\Lambda,\eta} = \min_{\substack{\boldsymbol{\tau}^{(\alpha)}\in\Sigma_\alpha(\Omega)\\ \alpha=1,2}} \min_{\substack{\lambda_K\geq 0\\ \int_\Omega(\lambda_1+\lambda_2+\lambda_3)\,dx\leq\Lambda}} \int_\Omega W_\lambda(\sqrt{\eta}\,\boldsymbol{\tau}^{(1)}, \sqrt{1-\eta}\,\boldsymbol{\tau}^{(2)})\,dx. \tag{2.32}$$

In the next step we perform minimization over $\lambda_1, \lambda_2, \lambda_3$. By using (2.31b) we write the nested problem of (2.32) in the form

$$\min\Big\{\int_\Omega \Big(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}\Big)\,dx \;\Big|\; \text{over } \lambda_1,\lambda_2,\lambda_3 \text{ such that:} \quad \lambda_1\geq\lambda_2\geq\lambda_3\geq 0,\ \int_\Omega(\lambda_1+\lambda_2+\lambda_3)\,dx \leq \Lambda\Big\}. \tag{2.33}$$

First, note that the optimal λ_3 vanishes (to extend maximally the range of variation of the moduli λ_1, λ_2). In the next step we make use of the result (1.15)–(1.17) and find

$$\min\Big\{\int_\Omega \Big(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}\Big)\,dx \;\Big|\; \text{over } \lambda_1,\lambda_2 \text{ such that:} \quad \lambda_1\geq\lambda_2\geq 0,\ \int_\Omega(\lambda_1+\lambda_2)\,dx \leq \Lambda\Big\} = \frac{1}{\Lambda}\Big(\int_\Omega(\sqrt{\mu_1}+\sqrt{\mu_2})\,dx\Big)^2. \tag{2.34}$$

The above result reduces (2.32) to the form (2.15), (2.29). The minimum in (2.34) is attained for, see (1.16)

$$\lambda_K = E_0\frac{\sqrt{\mu_K}}{\langle\sqrt{\mu_1}+\sqrt{\mu_2}\rangle}, \qquad K=1,2. \tag{2.35}$$

Thus, the eigenvalues of the optimal Hooke tensor are expressed by

$$\hat{\lambda}_K(x) = E_0 \frac{s_K([\sqrt{\eta}\,\hat{\boldsymbol{\tau}}^{(1)}(x)\ \sqrt{1-\eta}\,\hat{\boldsymbol{\tau}}^{(2)}(x)])}{\langle \rho([\sqrt{\eta}\,\hat{\boldsymbol{\tau}}^{(1)}\ \sqrt{1-\eta}\,\hat{\boldsymbol{\tau}}^{(2)}])\rangle}, \qquad K = 1,2,\ \hat{\lambda}_3(x) = 0, \tag{2.36}$$

where $(\hat{\boldsymbol{\tau}}^{(1)}, \hat{\boldsymbol{\tau}}^{(2)})$ is the minimizer of the problem (2.29). We see that the method developed in [36, 37, 38] – to solve the problem (2.30) geometrically – turns out to be effective. The result (2.16) (for $d = 2$, $n = 2$) has been derived in two different ways.

2.5.2. Discussion of the problem (P^*_{AMD})**.** Let us specify the problem (2.23) for the considered case of the two load variant ($n = 2$). The pointwise condition nested in this problem assumes the form

$$(\boldsymbol{\varepsilon}(\mathbf{v}^{(1)}(x)), \boldsymbol{\varepsilon}(\mathbf{v}^{(2)}(x))) \in B, \tag{2.37}$$

where B is the unit ball in $E^2_s \times E^2_s$ with respect to the Schatten ∞-norm:

$$\|[\boldsymbol{\xi}, \boldsymbol{\zeta}]\|_\infty = \frac{1}{\sqrt{2}}\sqrt{(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2) + \sqrt{(\|\boldsymbol{\xi}\|^2 - \|\boldsymbol{\zeta}\|^2)^2 + 4(\boldsymbol{\xi}\cdot\boldsymbol{\zeta})^2}}. \tag{2.38}$$

Problem (2.23) assumes the form

$$\hat{Z}_\eta = \max\Big\{\sqrt{\eta}\,f^{(1)}(\mathbf{v}^{(1)}) + \sqrt{1-\eta}\,f^{(2)}(\mathbf{v}^{(2)}) \mid \text{over } \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in V(\Omega)$$
$$\text{such that } (\boldsymbol{\varepsilon}(\mathbf{v}^{(1)}(x)), \boldsymbol{\varepsilon}(\mathbf{v}^{(2)}(x))) \in B \text{ for a.e. } x \in \Omega\Big\}. \tag{2.39}$$

The ball B is equivalently expressed by

$$B = \Big\{(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in E^2_s \times E^2_s \mid \|\boldsymbol{\xi}\| \le 1,\ \|\boldsymbol{\zeta}\| \le 1,$$
$$\underline{(\boldsymbol{\xi}\cdot\boldsymbol{\zeta})^2 \le (1 - \|\boldsymbol{\xi}\|^2)(1 - \|\boldsymbol{\zeta}\|^2)}\Big\} \tag{2.40}$$

or, more specifically, $B = B_1 \cup B_2$ and

$$B_1 = \{(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in E^2_s \times E^2_s \mid \|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2 \le 1\},$$
$$B_2 = \{(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in E^2_s \times E^2_s \mid \|\boldsymbol{\xi}\| \le 1,\ \|\boldsymbol{\zeta}\| \le 1,\ \|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2 \ge 1,$$
$$(\boldsymbol{\xi}\cdot\boldsymbol{\zeta})^2 \le (1 - \|\boldsymbol{\xi}\|^2)(1 - \|\boldsymbol{\zeta}\|^2)\}. \tag{2.41}$$

The underlined term in (2.40) is inactive (or redundant) if $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in B_1$ because then $(1 - \|\boldsymbol{\xi}\|^2)(1 - \|\boldsymbol{\zeta}\|^2) > \|\boldsymbol{\xi}\|^2\|\boldsymbol{\zeta}\|^2$. In the planar case ($d = 2$) the tensors $\boldsymbol{\xi}, \boldsymbol{\zeta}$

can be viewed as vectors in $\mathbb{R}^3$ (see the rule (1.8)); the set B_2 can be redefined as below

$$B_2 = \Big\{(\boldsymbol{\xi},\boldsymbol{\zeta}) \in E_s^2 \times E_s^2 \mid \|\boldsymbol{\xi}\| \le 1,\ \|\boldsymbol{\zeta}\| \le 1,\ \|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2 \ge 1, \\ \|\boldsymbol{\xi}\times\boldsymbol{\zeta}\| \ge \sqrt{\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2 - 1}\Big\}. \tag{2.42}$$

Let $\alpha = \min(\angle\boldsymbol{\xi},\boldsymbol{\zeta}), \angle(-\boldsymbol{\xi},\boldsymbol{\zeta}))$; $0 \le \alpha \le \pi/2$. The last condition in (2.42) imposes the lower bound on the angle α, i.e.

$$\alpha \ge \arcsin\left(\sqrt{\frac{1}{\|\boldsymbol{\xi}\|^2} + \frac{1}{\|\boldsymbol{\zeta}\|^2} - \frac{1}{\|\boldsymbol{\xi}\|^2\|\boldsymbol{\zeta}\|^2}}\right). \tag{2.43}$$

Moreover, note that if $\|\boldsymbol{\xi}\| = 1$ or $\|\boldsymbol{\zeta}\| = 1$ then $\alpha = \pi/2$ and $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ are orthogonal. If $(\boldsymbol{\xi},\boldsymbol{\zeta}) \in B_1$ the angle α is arbitrary.

2.6. Case of a single variant of the load

In the case considered we specify: $\eta_1 = 1$, $A = [\boldsymbol{\sigma}]$, $\mu_1 = \mu_1(\mathbf{A}^T\mathbf{A}) = \mu_1(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = \|\boldsymbol{\sigma}\|^2$. Thus, $s_1(\mathbf{A}) = \|\boldsymbol{\sigma}\|$, $\rho([\boldsymbol{\sigma}]) = \|\boldsymbol{\sigma}\|$.

The problem (2.16) assumes the form: find the statically admissible stress field $\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}$ solving the minimization problem:

$$\hat{Z} = \min_{\boldsymbol{\tau}\in\Sigma(\Omega)} \int_\Omega \|\boldsymbol{\tau}\|\, dx. \tag{2.44}$$

Having found the minimizer $\hat{\boldsymbol{\tau}}$ one can compute the optimal moduli by:

$$\hat{\lambda}_1(x) = E_0 \frac{\|\hat{\boldsymbol{\tau}}(x)\|}{\langle\|\hat{\boldsymbol{\tau}}\|\rangle}, \quad \hat{\lambda}_K(x) = 0, \qquad K = 2,\dots,m;\ m = d(d+1)/2. \tag{2.45}$$

The eigenstate corresponding to the eigenvalue $\hat{\lambda}_1$ is $\hat{\boldsymbol{\omega}}_1 = \hat{\boldsymbol{\tau}}/\|\hat{\boldsymbol{\tau}}\|$, while other eigenstates are constructed such that $\hat{\boldsymbol{\omega}}_K \cdot \hat{\boldsymbol{\omega}}_L = \delta_{KL}$. Hence

$$\hat{\mathbf{C}}(x) = \hat{\lambda}_1(x)\hat{\boldsymbol{\omega}}_1(x) \otimes \hat{\boldsymbol{\omega}}_1(x). \tag{2.46}$$

The problem dual to (2.37) has the form

$$\hat{Z} = \max\{f(\mathbf{v}) \mid \mathbf{v} \text{ kinematically admissible such that} \\ \|\boldsymbol{\varepsilon}(\mathbf{v}(x))\| \le 1 \text{ a.e. in } \Omega\} \tag{2.47}$$

where $f(\cdot)$ represents the virtual work of a given load.

The result (2.47) follows from the property of the Euclidean norm: its dual is also the Euclidean norm, cf. Section 1.2. The problem (2.44) appeared for the first time in Czarnecki and Lewiński [39], while the problem (2.47) has been for the first time reported in [4] inspired by the papers of Strang and Kohn [35] on Michell structures, and Golay and Seppecher [40] on optimal thickness distribution of in-plane loaded plates. The mathematical theory of the problems (2.44, 2.47) has been recently put forward in [10].

Remark 2.2. The final result (2.46) discloses that, in case of a single load variant, assuming all the 21 components of the Hooke tensor as independent design variables, leads to the highly unstable optimum material design. Two possible remedies are possible: consider more load conditions (if its number is bigger than 6 (in 3D) and 3 (in 2D) the material becomes stable) or impose some material symmetry conditions a priori.

Remark 2.3. Problem (2.37) is the tensorial counterpart of the celebrated *M. Beckmann's problem*, involving a vector unknown:

$$\min\left\{\int_\Omega \|\mathbf{p}\|\,dx \mid \text{over the vector field } \mathbf{p}=(p_1,\dots,p_d) \text{ satisfying:} \right.$$
$$\left. \operatorname{div}\mathbf{p}=0 \text{ in } \Omega \text{ and } \mathbf{p}\cdot\mathbf{n}=g \text{ on } \Gamma\right\} \quad (P_s)$$

where g is subject to the condition of its integral over Γ being zero; the problem dual to (P_s) involves one scalar unknown

$$\max\left\{\int_\Gamma gv\,d\Gamma \;\middle|\; \text{over the scalar fields } v \text{ satisfying: } \|\nabla v\|\le 1 \text{ a.e. in } \Omega\right\} \quad (P_s^*).$$

The history as well as the theory of the Beckmann problem can be found in [41, p. 115]. The Beckmann problem is closely linked with the optimum transportation problem. Its theory is now well developed, see [42], while the numerical methods are still in progress. It is also worth noting that the pair of problems $((P_s),(P_s^*))$ constitutes the LCP problem of the free material design problem within the conductivity setting, i.e. for the scalar version of FMD, cf. [10, Section 7.2].

2.7. Stress fields in the optimal structure

The aim of the present section is to prove that the stress fields in the optimal structure induced by the subsequent loads coincide with the relevant components of the collection of the stress fields minimizing the functional in the auxiliary problem (2.16). This is not obvious – note that the displacement field in the optimal structure subjected to a given load variant does not coincide with the relevant component of the collection of the displacement fields maximizing the functional of the auxiliary problem (2.23).

Let us consider a structure made of a material of optimal moduli $\hat{\lambda}_K$ and optimal eigenstates $\hat{\boldsymbol{\omega}}_K$ given by (2.17, 2.18). Le I be the set of indices such that $\hat{\lambda}_K > 0$. Assume that the load $\mathbf{T}^{(\alpha)}$ is applied to the optimal structure. The emerging stress field is the minimizer of the functional (2.11) where $\boldsymbol{\omega}_K = \hat{\boldsymbol{\omega}}_K$ are optimal eigenstates and $\lambda_K = \hat{\lambda}_K$are optimal elastic moduli. For this load variant the compliance $\tilde{\wp}^{(\alpha)} = \wp^{(\alpha)}(\hat{\mathbf{C}})$ equals

$$(2.48)\qquad \tilde{\wp}^{(\alpha)} = \min\left\{ \sum_{K\in I} \int\limits_{\Omega} \frac{1}{\hat{\lambda}_K} (\hat{\boldsymbol{\omega}}_K \cdot \boldsymbol{\tau})^2 \, dx \mid \text{over } \boldsymbol{\tau} \text{ such that: } \boldsymbol{\tau} \in \Sigma_\alpha(\Omega) \right\}.$$

Let $\boldsymbol{\tau}^{(\alpha)} = \tilde{\boldsymbol{\tau}}^{(\alpha)}$ be the minimizer of this functional; by using (2.17) we find

$$(2.49)\qquad \tilde{\wp}^{(\alpha)} = \frac{1}{E_0} \left\langle \rho([\sqrt{\eta_1}\, \hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\, \hat{\boldsymbol{\tau}}^{(n)}]) \right\rangle \times \int\limits_{\Omega} \sum_{K\in I} \frac{(\hat{\boldsymbol{\omega}}_K \cdot \tilde{\boldsymbol{\tau}}^{(\alpha)})^2}{s_K([\sqrt{\eta_1}\hat{\boldsymbol{\tau}}^{(1)}(x) \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}(x)])} \, dx,$$

hence the expression for the optimal weighted compliance given by

$$\tilde{F}_\eta = \sum_{\alpha=1}^{n} \eta_\alpha \tilde{\wp}^{(\alpha)}$$

assumes the form

$$(2.50)\qquad \tilde{F}_\eta = \frac{1}{E_0} \left\langle \rho([\sqrt{\eta_1}\, \hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}]) \right\rangle \times \int\limits_{\Omega} \sum_{K\in I} \frac{\sum_{\alpha=1}^{n} (\hat{\boldsymbol{\omega}}_K \cdot \sqrt{\eta_\alpha}\, \tilde{\boldsymbol{\tau}}^{(\alpha)})^2}{s_K([\sqrt{\eta_1}\, \hat{\boldsymbol{\tau}}^{(1)}(x) \ldots \sqrt{\eta_n}\, \hat{\boldsymbol{\tau}}^{(n)}(x)])} \, dx.$$

We shall prove that $\tilde{\boldsymbol{\tau}}^{(\alpha)} = \hat{\boldsymbol{\tau}}^{(\alpha)}$. At first we shall show that

$$(2.51)\qquad \tilde{F}_\eta \geq \frac{1}{\Lambda} (\hat{Z}_\eta)^2$$

where $\hat{Z}_\eta$ is given by (2.16).

Proof. We make use of the inequality (1.22), where now $i = K$ and

$$(2.52)\qquad a_K = \sum_{\alpha=1}^{n} (\hat{\boldsymbol{\omega}}_K \cdot \sqrt{\eta_\alpha}\, \tilde{\boldsymbol{\tau}}^{(\alpha)})^2, \quad u_K = s_K([\sqrt{\eta_1}\, \hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}]).$$

The inequality (1.22) implies

$$(2.53)\qquad W \overset{df}{=} \int_\Omega \sum_{K\in I} \frac{\sum_{\alpha=1}^n (\hat{\boldsymbol{\omega}}_K \cdot \sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)})^2}{s_K([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}(x)\dots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}(x)])}\,dx \geq \frac{\left(\sum_{K\in I}\int_\Omega \sqrt{\sum_{\alpha=1}^n (\hat{\boldsymbol{\omega}}_K\cdot\sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)})^2\,dx}\right)^2}{\int_\Omega\left(\sum_{K\in I} s_K([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}(x)\dots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}(x)])\right)}\,dx.$$

Let $\tilde{\mathbf{S}}_\eta = [\sqrt{\eta_1}\,\tilde{\boldsymbol{\tau}}^{(1)}\dots\sqrt{\eta_n}\tilde{\boldsymbol{\tau}}^{(n)}]$, then

$$(2.54)\qquad \sum_{\alpha=1}^n (\hat{\boldsymbol{\omega}}_K\cdot\sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)})^2 = \hat{\boldsymbol{\omega}}_K\cdot(\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\hat{\boldsymbol{\omega}}_K).$$

It is sufficient to prove:

$$(2.55)\qquad \int_\Omega\Big(\sum_{K\in I}\sqrt{\hat{\boldsymbol{\omega}}_K\cdot(\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\hat{\boldsymbol{\omega}}_K)}\Big)\,dx \geq \int_\Omega\Big(\sum_{K\in I} s_K([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}(x)\dots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}(x)])\Big)\,dx.$$

Note that there exists an orthogonal matrix $\mathbf{Q}_{6\times 6}$ such that $\hat{\boldsymbol{\omega}}_K = \mathbf{Q}^T\tilde{\boldsymbol{\omega}}_K$. We compute

$$(2.56)\qquad \hat{\boldsymbol{\omega}}_K\cdot(\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\hat{\boldsymbol{\omega}}_K) = \tilde{\boldsymbol{\omega}}_K\cdot(\mathbf{Q}\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\mathbf{Q}^T\tilde{\boldsymbol{\omega}}_K) = \mu_K(\mathbf{Q}\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\mathbf{Q}^T) = \mu_K(\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T) = (s_K(\tilde{\mathbf{S}}_\eta))^2$$

or

$$(2.57)\qquad \sqrt{\hat{\boldsymbol{\omega}}_K\cdot(\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\hat{\boldsymbol{\omega}}_K)} = s_K(\tilde{\mathbf{S}}_\eta),$$

hence

$$(2.58)\qquad \sum_{K\in I}\sqrt{\hat{\boldsymbol{\omega}}_K\cdot(\tilde{\mathbf{S}}_\eta\tilde{\mathbf{S}}_\eta^T\hat{\boldsymbol{\omega}}_K)} = \sum_{K\in I} s_K(\tilde{\mathbf{S}}_\eta) = \|\tilde{\mathbf{S}}_\eta\|_1 = \sum_{K\in I} s_K([\sqrt{\eta_1}\,\tilde{\boldsymbol{\tau}}^{(1)}(x)\dots\sqrt{\eta_n}\tilde{\boldsymbol{\tau}}^{(n)}(x)]) = \rho([\sqrt{\eta_1}\,\tilde{\boldsymbol{\tau}}^{(1)}(x)\dots\sqrt{\eta_n}\,\tilde{\boldsymbol{\tau}}^{(n)}(x)]).$$

On the other hand,

$$(2.59)\qquad \int_\Omega \rho([\sqrt{\eta_1}\,\tilde{\boldsymbol{\tau}}^{(1)}\dots\sqrt{\eta_n}\tilde{\boldsymbol{\tau}}^{(n)}])\,dx \geq \int_\Omega \rho([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}\dots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\,dx,$$

which confirms the estimate (2.55) and, consequently, implies

$$W \geq \int_{\Omega} \rho([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])dx = \hat{Z}_\eta. \tag{2.60}$$

Now we come back to (2.50) and conclude that

$$\tilde{F}_\eta \geq \frac{1}{E_0|\Omega|}(\hat{Z}_\eta)^2 = \frac{1}{\Lambda}(\hat{Z}_\eta)^2. \tag{2.61}$$

Thus, indeed, the αth load induces the stress field $\hat{\boldsymbol{\tau}}^{(\alpha)}$, an element of the collection $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ being the minimizer of (2.16).

3. Isotropic Material Design (IMD)

The present section discusses the *Isotropic Material Design* method, proposed by Czarnecki [6] and Czarnecki and Wawruch [7] for the single load variant. For the multiple load variants, for the case of loads being applied non-simultaneously, the stress-based setting of the IMD has been put forward in [17]. The aim of the present section is to make the theory of IMD complete by delivering the pairs of the primal and dual problems forming this theory and discuss the elastic properties of the optimal structure. In this method the *bulk modulus* and the *shear modulus* are *design variables*. The final optimal design is composed of three subdomains: (i) where both the moduli are positive, (ii) where the bulk modulus is positive and the shear modulus vanishes, and (iii) where the bulk modulus vanishes and the shear modulus is positive. The domains where both the moduli vanish are cut out from the final design as non-material. Due to this cutting-out property the IMD method can be viewed as a topology optimization method, solving simultaneously the problem of optimal shape and of optimal material distribution.

3.1. The 3D stress-based formulation for n load variants

We consider the problem (2.8), where now tensor $\mathbf{C}$ reflects the isotropic properties of the material. The design variables are the bulk and shear moduli: k, μ. The Hooke tensor enjoys the celebrated Hill representation

$$\mathbf{C} = 3k(x)\boldsymbol{\Lambda}_1 + 2\mu(x)\boldsymbol{\Lambda}_2, \tag{3.1}$$

where

$$\boldsymbol{\Lambda}_1 = \frac{1}{3}\delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \qquad \boldsymbol{\Lambda}_2 = \mathbf{II} - \boldsymbol{\Lambda}_1. \tag{3.2}$$

The cost condition is assumed in the form (2.7) and now $\operatorname{tr}\mathbf{C} = 3k + 10\mu$, since the eigenvalues of $\mathbf{C}$ are: $\{3k, 2\mu, 2\mu, 2\mu, 2\mu, 2\mu\}$. Let us compute first the contributions to the elastic energy:

$$\boldsymbol{\tau}\cdot(\boldsymbol{\Lambda}_1\boldsymbol{\tau}) = (\operatorname{Tr}\boldsymbol{\tau})^2, \qquad \boldsymbol{\tau}\cdot(\boldsymbol{\Lambda}_2\boldsymbol{\tau}) = \|\operatorname{dev}\boldsymbol{\tau}\|^2. \tag{3.3}$$

Thus

$$\boldsymbol{\tau}\cdot(\mathbf{C}^{-1}\boldsymbol{\tau}) = \frac{1}{3k}(\operatorname{Tr}\boldsymbol{\tau})^2 + \frac{1}{2\mu}\|\operatorname{dev}\boldsymbol{\tau}\|^2 \tag{3.4}$$

and, consequently

$$\sum_{\alpha=1}^{n}\eta_\alpha\boldsymbol{\tau}^{(\alpha)}\cdot(\mathbf{C}^{-1}\boldsymbol{\tau}^{(\alpha)}) = \frac{a_1}{u_1} + \frac{a_2}{u_2} \tag{3.5}$$

where

$$\begin{aligned} &u_1 = 3k, \qquad u_2 = 10\mu,\\ &a_1 = \sum_{\alpha=1}^{n}\big(\operatorname{Tr}(\sqrt{\eta_\alpha}\boldsymbol{\tau}^{(\alpha)})\big)^2, \qquad a_2 = \sum_{\alpha=1}^{n}\|\beta\operatorname{dev}(\sqrt{\eta_\alpha}\boldsymbol{\tau}^{(\alpha)})\|^2 \end{aligned} \tag{3.6}$$

with $\beta = \sqrt{5}$.

The problem of minimization of the weighted compliance (with weights η_α, $0 \le \eta_\alpha \le 1$, $\eta_1 + \cdots + \eta_n = 1$) has the form

$$\begin{aligned} J_{\Lambda,\eta} = \min\Big\{\int_\Omega\Big(\sum_{\alpha=1}^{n}\eta_\alpha\boldsymbol{\tau}^{(\alpha)}\cdot(\mathbf{C}^{-1}\boldsymbol{\tau}^{(\alpha)})\Big)\,dx \;\Big|\; &\text{over } \boldsymbol{\tau}^{(1)},\dots,\boldsymbol{\tau}^{(n)},\ k,\mu\\ &\text{such that: } \boldsymbol{\tau}^{(\alpha)} \in \Sigma_\alpha(\Omega);\ k \ge 0,\ \mu \ge 0,\ \int_\Omega (3k+10\mu)\,dx \le \Lambda\Big\}. \end{aligned} \tag{3.7}$$

By using the results (3.5), (3.6) we rewrite (3.7) as follows

$$\begin{aligned} J_{\Lambda,\eta} = \min\Big\{\int_\Omega\Big(\frac{a_1}{u_1} + \frac{a_2}{u_2}\Big)\,dx \;\Big|\; &\text{over } \boldsymbol{\tau}^{(\alpha)} \in \Sigma_\alpha(\Omega),\ \alpha = 1,\dots,n;\\ &u_1 \ge 0,\ u_2 \ge 0,\ \int_\Omega (u_1+u_2)\,dx \le \Lambda\Big\}. \end{aligned} \tag{3.8}$$

The nested problem of minimization over u_1, u_2 can be solved by using (1.15–1.17), hence

$$J_{\Lambda,\eta} = \frac{1}{\Lambda}\bigg(\min\Big\{\int_\Omega(\sqrt{a_1}+\sqrt{a_2})\,dx \;\Big|\; \text{over } \boldsymbol{\tau}^{(\alpha)} \in \Sigma_\alpha(\Omega),\ \alpha = 1,\dots,n\Big\}\bigg)^2 \tag{3.9}$$

where the dependence of a_1, a_2 on $\boldsymbol{\tau}^{(1)},\ldots,\boldsymbol{\tau}^{(n)}$ has been suppressed. The integrand above can be expressed in terms of the virtual stress fields by

$$\sqrt{a_1}+\sqrt{a_2}=\rho_\beta\big([\sqrt{\eta_1}\,\boldsymbol{\tau}^{(1)}\ldots\sqrt{\eta_n}\,\boldsymbol{\tau}^{(n)}]\big) \tag{3.10}$$

with using the norm

$$\rho_\beta([\boldsymbol{\sigma}^{(1)}\ \boldsymbol{\sigma}^{(2)}\ \ldots\ \boldsymbol{\sigma}^{(n)}])=\sqrt{\sum_{\alpha=1}^{n}(\operatorname{Tr}\boldsymbol{\sigma}^{(\alpha)})^2}+\beta\sqrt{\sum_{\alpha=1}^{n}\|\operatorname{dev}\boldsymbol{\sigma}^{(\alpha)}\|^2} \tag{3.11}$$

and $\beta=\sqrt{5}$. The optimal u_1, u_2 read, see (1.16) and (2.35)

$$u_1=E_0\frac{\sqrt{a_1}}{\langle\sqrt{a_1}+\sqrt{a_2}\rangle},\qquad u_2=E_0\frac{\sqrt{a_2}}{\langle\sqrt{a_1}+\sqrt{a_2}\rangle}. \tag{3.12}$$

We come back to (3.9) and write $J_{\Lambda,\eta}=\frac{1}{\Lambda}(\hat{Z}_\eta)^2$, where $\hat{Z}_\eta$ is given by the problem: find $(\hat{\boldsymbol{\tau}}^{(1)},\ldots,\hat{\boldsymbol{\tau}}^{(n)})$, the minimizer of the problem

$$\hat{Z}_\eta=\min_{\substack{\boldsymbol{\tau}^{(\alpha)}\in\Sigma_\alpha(\Omega)\\ \alpha=1,\ldots,n}}\int_\Omega\rho_\beta\big([\sqrt{\eta_1}\,\boldsymbol{\tau}^{(1)}\ldots\sqrt{\eta_n}\,\boldsymbol{\tau}^{(n)}]\big)\,dx\qquad (P_{IMD}) \tag{3.13}$$

where $\beta=\sqrt{5}$.

The minimizer $(\hat{\boldsymbol{\tau}}^{(1)},\ldots,\hat{\boldsymbol{\tau}}^{(n)})$ determines the optimal moduli of isotropy

$$\begin{aligned}3\hat{k}(x)&=E_0\frac{\sqrt{\sum_{\alpha=1}^{n}(\operatorname{Tr}(\sqrt{\eta_\alpha}\,\hat{\boldsymbol{\tau}}^{(\alpha)}))^2}}{\langle\rho_{\sqrt{5}}([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}\ldots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\rangle},\\ 10\hat{\mu}(x)&=E_0\frac{\sqrt{5}\sqrt{\sum_{\alpha=1}^{n}\|\operatorname{dev}(\sqrt{\eta_\alpha}\,\hat{\boldsymbol{\tau}}^{(\alpha)})\|^2}}{\langle\rho_{\sqrt{5}}([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}\ldots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\rangle}.\end{aligned} \tag{3.14}$$

It is seen that $\langle 3\hat{k}+10\hat{\mu}\rangle=E_0$; the cost condition is satisfied sharply.

3.2. The 3D strain-based formulation for n load variants

The problem (3.13) can be rearranged to its dual form; it reads: find $(\hat{\mathbf{v}}^{(1)},\hat{\mathbf{v}}^{(2)},\ldots,\hat{\mathbf{v}}^{(n)})$, the maximizer of the problem

$$\hat{Z}_\eta=\max\Big\{\sum_{\alpha=1}^{n}\sqrt{\eta_\alpha}\,f^{(\alpha)}(\mathbf{v}^{(\alpha)})\ \Big|\ \text{over } \mathbf{v}^{(1)},\ldots,\mathbf{v}^{(n)}\in V(\Omega) \text{ such that}$$
$$\rho^o_\beta([\boldsymbol{\varepsilon}(\mathbf{v}^{(1)}(x))\ldots\boldsymbol{\varepsilon}(\mathbf{v}^{(n)}(x))])\le 1 \text{ a.e. in } \Omega\Big\}\qquad (P^*_{IMD}) \tag{3.15}$$

where $\beta = \sqrt{5}$ and $\rho^o_\beta(\cdot)$ is a function polar to the norm $\rho_\beta(\cdot)$, see (1.11), or

$$\rho^o_\beta([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max_{\substack{\boldsymbol{\sigma}^{(\alpha)} \in E^2_s \\ \boldsymbol{\sigma}^{(\alpha)} \neq \mathbf{0} \\ \alpha = 1,\ldots,n}} \frac{\sum_{\alpha=1}^n \boldsymbol{\varepsilon}^{(\alpha)} \cdot \boldsymbol{\sigma}^{(\alpha)}}{\rho_\beta([\boldsymbol{\sigma}^{(1)} \ldots \boldsymbol{\sigma}^{(n)}])}. \tag{3.16}$$

We shall find the explicit form of this function. According to (1.4):

$$\boldsymbol{\varepsilon}^{(\alpha)} \cdot \boldsymbol{\sigma}^{(\alpha)} = (\operatorname{Tr} \boldsymbol{\sigma}^{(\alpha)})(\operatorname{Tr} \boldsymbol{\varepsilon}^{(\alpha)}) + \operatorname{dev} \boldsymbol{\sigma}^{(\alpha)} \cdot \operatorname{dev} \boldsymbol{\varepsilon}^{(\alpha)}. \tag{3.17}$$

Let us introduce: $\mathbf{p} = (p^{(1)}, \ldots, p^{(n)})$, $\mathbf{e} = (e^{(1)}, \ldots, e^{(n)})$ with $p^{(\alpha)} = \operatorname{Tr} \boldsymbol{\sigma}^{(\alpha)}$, $e^{(\alpha)} = \operatorname{Tr} \boldsymbol{\varepsilon}^{(\alpha)}$ and $\mathbf{q} = (\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(n)})$, $\mathbf{b} = (\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(n)})$ with $\mathbf{q}^{(\alpha)} = \operatorname{dev} \boldsymbol{\sigma}^{(\alpha)}$, $\mathbf{b}^{(\alpha)} = \operatorname{dev} \boldsymbol{\varepsilon}^{(\alpha)}$. We rewrite (3.17)

$$\sum_{\alpha=1}^n \boldsymbol{\varepsilon}^{(\alpha)} \cdot \boldsymbol{\sigma}^{(\alpha)} = \mathbf{p} \cdot \mathbf{e} + \mathbf{q} \cdot \mathbf{b} \tag{3.18}$$

and express the norm (3.11) as follows

$$\rho_\beta([\boldsymbol{\sigma}^{(1)} \ldots \boldsymbol{\sigma}^{(n)}]) = \|\mathbf{p}\| + \beta \|\mathbf{q}\|. \tag{3.19}$$

The dual norm is given by

$$\rho^o_\beta([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max_{\substack{\mathbf{p} \in R^n,\, \mathbf{p} \neq \mathbf{0} \\ \mathbf{q} \in R^{6n},\, \mathbf{q} \neq \mathbf{0}}} \frac{\mathbf{p} \cdot \mathbf{e} + \mathbf{q} \cdot \mathbf{b}}{\|\mathbf{p}\| + \beta \|\mathbf{q}\|}. \tag{3.20}$$

To maximize the numerator we put:

$$\mathbf{p} = \|\mathbf{p}\| \frac{\mathbf{e}}{\|\mathbf{e}\|}, \qquad \mathbf{q} = \|\mathbf{q}\| \frac{\mathbf{b}}{\|\mathbf{b}\|}. \tag{3.21}$$

Hence

$$\rho^o_\beta([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max_{x>0,\, y>0} \frac{\|\mathbf{e}\| x + \|\mathbf{b}\| y}{x + \beta y}, \tag{3.22}$$

where $x = \|\mathbf{p}\|$, $y = \|\mathbf{q}\|$; by performing the maximization operation we get

$$\rho^o_\beta([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max\left\{ \|\mathbf{e}\|, \frac{1}{\beta} \|\mathbf{b}\| \right\}. \tag{3.23}$$

Thus, the norm dual to the norm (3.11) has the form

$$\rho^o_\beta([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max\left\{ \sqrt{\sum_{\alpha=1}^n (\operatorname{Tr} \boldsymbol{\varepsilon}^{(\alpha)})^2}, \frac{1}{\beta} \sqrt{\sum_{\alpha=1}^n \|\operatorname{dev} \boldsymbol{\varepsilon}^{(\alpha)}\|^2} \right\}. \tag{3.24}$$

Just this norm specifies the point-wise condition involved in the problem (3.15).

3.3. The 2D formulation for n load variants

In the 2D setting the representation of the isotropic Hooke tensor reads

$$\mathbf{C} = 2k(x)\mathbf{\Lambda}_1 + 2\mu(x)\mathbf{\Lambda}_2 \tag{3.25}$$

where the projectors are given by

$$\mathbf{\Lambda}_1 = \tfrac{1}{2}\delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \qquad \mathbf{\Lambda}_2 = \mathbf{II} - \mathbf{\Lambda}_1. \tag{3.26}$$

The eigenvalues of tensor $\mathbf{C}$ are: $\{2k, 2\mu, 2\mu\}$, hence $\operatorname{tr}\mathbf{C} = 2k + 4\mu$. The cost condition (2.7) has the form $\langle 2k + 4\mu\rangle \le E_0$. In the 2D setting we write: $\operatorname{Tr}\boldsymbol{\sigma} = (\operatorname{tr}\boldsymbol{\sigma})/\sqrt{2}$ and then the formulae (3.3) hold true. The density of complementary energy equals

$$\boldsymbol{\tau} \cdot \mathbf{C}^{-1}\boldsymbol{\tau} = \frac{1}{2k}(\operatorname{Tr}\boldsymbol{\tau})^2 + \frac{1}{2\mu}\|\operatorname{dev}\boldsymbol{\tau}\|^2. \tag{3.27}$$

Thus, the weighted energy density is expressed by (3.5), (3.6) with $\beta = \sqrt{2}$. Proceeding as in Section 3.1 we get $J_\eta = \frac{1}{\Lambda}(\hat{Z}_\eta)^2$, where $\hat{Z}_\eta$ is given as in (3.13), only now $\beta = \sqrt{2}$. Having solved the latter problem, one can compute the optimal moduli of isotropy by

$$\begin{aligned} 2\hat{k}(x) &= E_0 \frac{\sqrt{\sum_{\alpha=1}^{n}\left(\operatorname{Tr}(\sqrt{\eta_\alpha}\,\hat{\boldsymbol{\tau}}^{(\alpha)})\right)^2}}{\left\langle \rho_{\sqrt{2}}([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\right\rangle}, \\ 4\hat{\mu}(x) &= E_0 \frac{\sqrt{2}\sqrt{\sum_{\alpha=1}^{n}\|\operatorname{dev}(\sqrt{\eta_\alpha}\,\hat{\boldsymbol{\tau}}^{(\alpha)})\|^2}}{\left\langle \rho_{\sqrt{2}}([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\right\rangle}. \end{aligned} \tag{3.28}$$

We note that $\langle 2\hat{k} + 4\hat{\mu}\rangle = E_0$; the cost condition is satisfied sharply. The problem dual to (3.13) has the form (3.15), where $\beta = \sqrt{2}$.

3.4. On emerging the auxetic properties of the optimal material

The bulk and shear moduli are two independent design variables of the IMD method, subjected only to the conditions of being non-negative. Consequently, the optimal Poisson ratio $\hat{\nu}$ is not restricted to be positive. Thus, the optimum design method IMD extends the possible values of the Poisson ratio beyond the range observed in nature (the nature teaches us that the Poisson ratio is positive) toward the broader ranges implied by the condition of non-negative definiteness of elastic energy: $-1 \le \hat{\nu} \le 1/2$, $-1 \le \hat{\nu} \le 1$, for $d = 3$, $d = 2$, respectively. The optimal bulk and shear moduli are determined by the stress fields solving the problem (3.13), where $\beta = \sqrt{5}$, $\beta = \sqrt{2}$ for $d = 3$, $d = 2$, respectively. Since these stress fields are independent of any data concerning elastic moduli, the shape of the stress regime (using Cherkaev's nomenclature [11]) corresponding to the condition of auxeticity $\hat{\nu}(x) \leq 0$ will be free of any parameters. These regimes for 3D and 2D settings are constructed below.

A. The 3D setting

In the spatial problem the Poisson ratio is expressed by the bulk and shear moduli according to the known rule:

$$\nu = \frac{3k - 2\mu}{2(3k + \mu)}, \tag{3.29}$$

which, by the way, becomes recently the most fundamental formula of the mechanics of composites.

The following subdomains of the design domain are of special interest:

(a) $\hat{k}(x) = 0$, $\hat{\mu}(x) > 0$. Then, due to the numerator in $(3.28)_1$ being a norm, the minimizer $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ is such that $\operatorname{tr} \hat{\boldsymbol{\tau}}^{(\alpha)}(x) = 0$, $\alpha = 1, \ldots, n$ and there $\hat{\nu}(x) = -1$.

(b) $\hat{k}(x) > 0$, $\hat{\mu}(x) = 0$. Then, due to the numerator in $(3.28)_2$ being a norm, the minimizer $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ is such that $\operatorname{dev} \hat{\boldsymbol{\tau}}^{(\alpha)}(x) = \mathbf{0}$, $\alpha = 1, \ldots, n$ and there $\hat{\nu}(x) = 1/2$.

(c) $3\hat{k}(x) - 2\hat{\mu}(x) < 0$. Then $\hat{\nu}(x) < 0$. Let us substitute (3.14) into the inequality above. We note that the subdomain (c) enjoys the auxetic property at a point x, if the solution $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ of the problem (3.13) satisfies the property:

$$\frac{\sum_{\alpha=1}^{n} \eta_\alpha (\operatorname{tr}(\hat{\boldsymbol{\tau}}^{(\alpha)}(x)))^2}{\sum_{\alpha=1}^{n} \eta_\alpha \|\operatorname{dev} \hat{\boldsymbol{\tau}}^{(\alpha)}(x))\|^2} < \frac{3}{5}. \tag{3.30}$$

If $n = 1$, then the above condition reduces to the inequality

$$\frac{|\operatorname{tr} \hat{\boldsymbol{\tau}}(x)|}{\|\operatorname{dev} \hat{\boldsymbol{\tau}}(x)\|} < \sqrt{\frac{3}{5}} \tag{3.31}$$

which can be expressed in terms of principal stresses as follows (see (1.5, 1.6))

$$\frac{|\hat{\tau}_I(x) + \hat{\tau}_{II}(x) + \hat{\tau}_{III}(x)|}{\sqrt{(\hat{\tau}_I(x) - \hat{\tau}_{II}(x))^2 + (\hat{\tau}_I(x) - \hat{\tau}_{III}(x))^2 + (\hat{\tau}_{II}(x) - \hat{\tau}_{III}(x))^2}} < \frac{1}{\sqrt{5}} \tag{3.32}$$

or, equivalently

$$(\hat{\tau}_I(x))^2 + (\hat{\tau}_{II}(x))^2 + (\hat{\tau}_{III}(x))^2 + 4(\hat{\tau}_I(x)\hat{\tau}_{II}(x) + \hat{\tau}_{III}(x)\hat{\tau}_I(x) + \hat{\tau}_{II}(x)\hat{\tau}_{III}(x)) < 0. \tag{3.33}$$

The regime (3.33) lies between the two surfaces shown in Fig. 2.

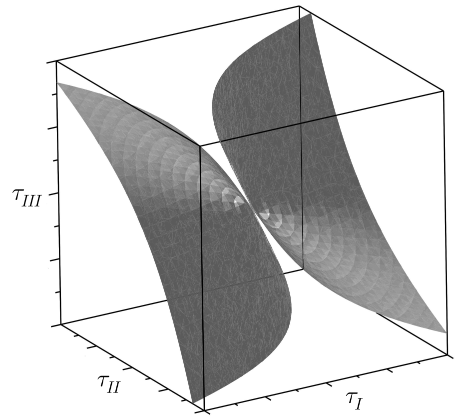

FIG. 2. If observed within the 3D framework of principal stresses, the regime of auxeticity lies between the two surfaces displayed.

B. The 2D setting

The 2D counterpart of the formula (3.29) is $\nu = \frac{k-\mu}{k+\mu}$. The following subdomains of the design domain are worth of considering:

(a) $\hat{k}(x) = 0$, $\hat{\mu}(x) > 0$. Then the minimizer $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ is such that $\mathrm{tr}\, \hat{\boldsymbol{\tau}}^{(\alpha)}(x) = 0$, $\alpha = 1, \ldots, n$ and there $\hat{\nu}(x) = -1$.

(b) $\hat{k}(x) > 0$, $\hat{\mu}(x) = 0$. Then the minimizer $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ is such that $\mathrm{dev}\, \hat{\boldsymbol{\tau}}^{(\alpha)}(x) = \mathbf{0}$, $\alpha = 1, \ldots, n$ and there $\hat{\nu}(x) = 1$.

(c) $\hat{k}(x) - \hat{\mu}(x) < 0$. Then $\hat{\nu}(x) < 0$ and just this subdomain enjoys the auxetic property at a point x, if the solution $(\hat{\boldsymbol{\tau}}^{(1)}(x), \ldots, \hat{\boldsymbol{\tau}}^{(n)}(x))$ of the problem (3.13) satisfies the property:

$$\sum_{\alpha=1}^{n} \eta_\alpha (\mathrm{tr}(\hat{\boldsymbol{\tau}}^{(\alpha)}(x)))^2 < \sum_{\alpha=1}^{n} \eta_\alpha \|\mathrm{dev}(\hat{\boldsymbol{\tau}}^{(\alpha)}(x))\|^2. \tag{3.34}$$

If $n = 1$ then the condition above reduces to

$$|\mathrm{tr}\, \hat{\boldsymbol{\tau}}(x)| < \|\mathrm{dev}\, \hat{\boldsymbol{\tau}}(x)\| \tag{3.35}$$

or, if expressed in terms of principal stresses, this condition assumes the form

$$(\hat{\tau}_I(x))^2 + (\hat{\tau}_{II}(x))^2 + 6\hat{\tau}_I(x)\hat{\tau}_{II}(x) < 0. \tag{3.36}$$

Alternatively,

$$(\hat{\tau}_I(x) + (3 + 2\sqrt{2}\,)\hat{\tau}_{II}(x))((3 + 2\sqrt{2})\hat{\tau}_I(x) + \hat{\tau}_{II}(x)) < 0. \tag{3.37}$$

The regime (3.37) is composed of two cones in the plane, see Fig. 3.

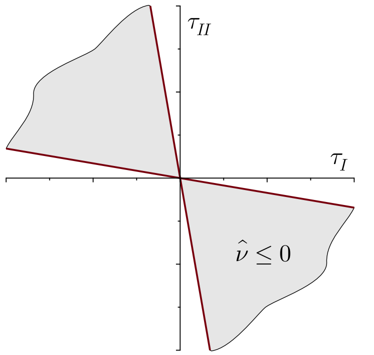

FIG. 3. If observed within the framework of principal stresses, the regime of auxeticity for the 2D setting assumes the form of a cone displayed above.

REMARK 3.1. The assumption of isotropy is kept throughout the optimization process, hence the final design cannot become non-isotropic. No jump to a new class of anisotropy is possible. The optimization may introduce instabilities mentioned above, but, even unstable, the material remains isotropic. The same remark applies to the YMD method discussed in the next section.

3.5. The stress fields in the optimal structure

Assume that the 3D optimized structure made of the optimal material of moduli $\hat{k}(x)$, $\hat{\mu}(x)$ is subjected to the α-th load. This load induces a stress field $\tilde{\boldsymbol{\tau}}^{(\alpha)}$. We shall prove that this stress field coincides with the stress field $\hat{\boldsymbol{\tau}}^{(\alpha)}$ being the α-th component of the solution to the problem (P_{IMD}), cf. (3.13).

The α-th compliance of the optimal structure is equal to

$$\wp^{(\alpha)} = \int_\Omega (\tilde{\boldsymbol{\tau}}^{(\alpha)} \cdot (\hat{\mathbf{C}}^{-1}\tilde{\boldsymbol{\tau}}^{(\alpha)}))\,dx = \min\left\{ \int_\Omega (\boldsymbol{\tau} \cdot (\hat{\mathbf{C}}^{-1}\boldsymbol{\tau}))\,dx \;\middle|\; \text{over } \boldsymbol{\tau} \in \Sigma_\alpha(\Omega) \right\} \tag{3.38}$$

where

$$\boldsymbol{\tau} \cdot \hat{\mathbf{C}}^{-1}\boldsymbol{\tau} = \frac{1}{3\hat{k}}(\mathrm{Tr}\,\boldsymbol{\tau})^2 + \frac{1}{10\hat{\mu}}\|\beta\,\mathrm{dev}\,\boldsymbol{\tau}\|^2 \tag{3.39}$$

and $\beta = \sqrt{5}$. Substitution of (3.14) leads to the formula for the optimal compliance as below

$$\wp^{(\alpha)} = \frac{\hat{Z}_\eta}{E_0} \min_{\boldsymbol{\tau}\in\Sigma_\alpha(\Omega)} \int_\Omega \left[\frac{(\mathrm{Tr}(\boldsymbol{\tau}))^2}{\sqrt{\sum_{\gamma=1}^n (\mathrm{Tr}(\sqrt{\eta_\gamma}\,\hat{\boldsymbol{\tau}}^{(\gamma)}))^2}} + \beta \frac{\|\mathrm{dev}(\boldsymbol{\tau})\|^2}{\sqrt{\sum_{\gamma=1}^n \|\mathrm{dev}(\sqrt{\eta_\gamma}\,\hat{\boldsymbol{\tau}}^{(\gamma)})\|^2}} \right] dx. \tag{3.40}$$

Before computing the weighted compliance we insert $\boldsymbol{\tau}^{(\alpha)} = \tilde{\boldsymbol{\tau}}^{(\alpha)}$ and introduce the notation

$$
\begin{aligned}
\tilde{\rho}_1 &= \sqrt{\sum_{\alpha=1}^{n} (\mathrm{Tr}(\sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)}))^2}, \qquad \tilde{\rho}_2 = \sqrt{\sum_{\alpha=1}^{n} \|\mathrm{dev}(\sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)})\|^2} \\
\hat{\rho}_1 &= \sqrt{\sum_{\alpha=1}^{n} (\mathrm{Tr}(\sqrt{\eta_\alpha}\,\hat{\boldsymbol{\tau}}^{(\alpha)}))^2}, \qquad \hat{\rho}_2 = \sqrt{\sum_{\alpha=1}^{n} \|\mathrm{dev}(\sqrt{\eta_\alpha}\,\hat{\boldsymbol{\tau}}^{(\alpha)})\|^2}.
\end{aligned} \tag{3.41}
$$

The weighted compliance equals

$$
F_\eta = \frac{\hat{Z}_\eta}{E_0} \int_\Omega \left[\frac{(\tilde{\rho}_1)^2}{\hat{\rho}_1} + \beta \frac{(\tilde{\rho}_2)^2}{\hat{\rho}_2} \right] dx. \tag{3.42}
$$

According to (1.22) the following estimates are satisfied

$$
\int_\Omega \frac{(\tilde{\rho}_i)^2}{\hat{\rho}_i}\, dx \geq \frac{\left(\int_\Omega \tilde{\rho}_i\, dx\right)^2}{\int_\Omega \hat{\rho}_i dx}, \qquad i = 1, 2 \tag{3.43}
$$

and because the collection $(\hat{\boldsymbol{\tau}}^{(1)}, \ldots, \hat{\boldsymbol{\tau}}^{(n)})$ is the minimizer of (3.13), the estimate

$$
\int_\Omega (\tilde{\rho}_1 + \beta \tilde{\rho}_2)\, dx \geq \int_\Omega (\hat{\rho}_1 + \beta \hat{\rho}_2)\, dx \tag{3.44}
$$

holds. Now we shall prove that

$$
\frac{\left(\int_\Omega \tilde{\rho}_1\, dx\right)^2}{\int_\Omega \hat{\rho}_1\, dx} + \beta \frac{\left(\int_\Omega \tilde{\rho}_2\, dx\right)^2}{\int_\Omega \hat{\rho}_2\, dx} \geq \int_\Omega (\hat{\rho}_1 + \beta \hat{\rho}_2)\, dx. \tag{3.45}
$$

To prove the above estimate let us introduce the notation

$$
S_i = \int_\Omega \hat{\rho}_i\, dx, \qquad E_i = \int_\Omega (\tilde{\rho}_i - \hat{\rho}_i)\, dx. \tag{3.46}
$$

The estimate (3.44) is equivalent to

$$
E_1 + \beta E_2 \geq 0, \qquad \beta > 0, \tag{3.47}
$$

while due to $S_i > 0$, the above inequalities lead to

$$
\frac{(S_1 + E_1)^2}{S_1} + \beta \frac{(S_2 + E_2)^2}{S_2} \geq S_1 + \beta S_2, \tag{3.48}
$$

which confirms (3.45). Let us come back to (3.42). By using (3.43), (3.44), (3.45) we estimate F_η as below

$$F_\eta \geq \frac{\hat{Z}_\eta}{E_0} \int\limits_\Omega (\hat{\rho}_1 + \beta \hat{\rho}_2)\, dx = \frac{1}{\Lambda} (\hat{Z}_\eta)^2 \tag{3.49}$$

and the equality is attained only if $\tilde{\boldsymbol{\tau}}^{(\alpha)} = \hat{\boldsymbol{\tau}}^{(\alpha)}$. Since we know that $F_\eta = (\hat{Z}_\eta)^2/\Lambda$, the equality $\tilde{\boldsymbol{\tau}}^{(\alpha)} = \hat{\boldsymbol{\tau}}^{(\alpha)}$ holds. Thus, the solution to the problem (3.13) determines the stress fields in the optimal structure subjected to subsequent loads $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(n)}$.

4. Young's Modulus Design (YMD)

Skeletal microstructures of given layout of ligaments of shapes of bars are characterized by effective Poisson's ratios being almost independent of the ligaments' areas of cross sections. Controlling the cross-sections means controlling the effective Young modulus $E(x)$, see [23]. This is one of the reasons to develop such a version of the free material design (called YMD) in which Poisson's ratio is fixed and Young's modulus is the only design variable. The hitherto published results on YMD referred to 3D case. The present section is aimed at constructing the pairs of mutually dual problems for the YMD method for both cases: $d = 3$, $d = 2$. Moreover, the novelty of the present exposition lies in disclosing that the stress fields in the optimal structure induced by subsequent loads coincide with the components of the minimizer of the auxiliary problem (4.6) formulated below.

4.1. The 3D setting

In case of isotropy, if $\nu = \text{const}$, the bulk and shear moduli are expressed by Young's modulus and Poisson's ratio according to

$$k(x) = \frac{E(x)}{3(1-2\nu)}, \qquad \mu(x) = \frac{E(x)}{2(1+\nu)}. \tag{4.1}$$

Hence

$$\operatorname{tr} \mathbf{C} = aE(x), \qquad a = \frac{6-9\nu}{(1+\nu)(1-2\nu)}, \tag{4.2}$$

where a is treated as a given parameter, since the Poisson ratio ν is treated as a fixed one. The Hooke tensor may be represented by

$$\mathbf{C} = E_1(x)\mathbf{G}, \qquad \mathbf{G} = \frac{1+\nu}{6-9\nu}\boldsymbol{\Lambda}_1 + \frac{1-2\nu}{6-9\nu}\boldsymbol{\Lambda}_2, \tag{4.3}$$

where $E_1(x) = aE(x)$ is the unit cost. The cost condition (2.7) assumes the form $\langle E_1 \rangle \le E_0$, $E_0 = \Lambda/|\Omega|$. Note that $\operatorname{tr} \mathbf{G} = 1$. The inverse of tensor $\mathbf{G}$ has the form

$$\mathbf{G}^{-1} = \frac{6-9\nu}{1+\nu}\boldsymbol{\Lambda}_1 + \frac{6-9\nu}{1-2\nu}\boldsymbol{\Lambda}_2. \tag{4.4}$$

The problem of minimizing the weighted compliance (2.6) reads

$$J_{\Lambda,\eta} = \min\Big\{ \int\limits_{\Omega} \frac{1}{E_1}\Big(\sum_{\alpha=1}^{n} \eta_\alpha \boldsymbol{\tau}^{(\alpha)} \cdot (\mathbf{G}^{-1}\boldsymbol{\tau}^{(\alpha)})\Big)\, dx \;\Big|\; \boldsymbol{\tau}^{(\alpha)} \in \Sigma_\alpha(\Omega),\ \alpha = 1,\ldots,n;\ E_1 \ge 0,\ \langle E_1\rangle \le E_0 \Big\}. \tag{4.5}$$

Upon performing minimization over E_1 one finds $J_{\Lambda,\eta} = \frac{1}{\Lambda}(\hat{Z}_\eta)^2$, where $\hat{Z}_\eta$ is given by the problem: find $(\hat{\boldsymbol{\tau}}^{(1)},\ldots,\hat{\boldsymbol{\tau}}^{(n)})$, the minimizer of the problem

$$\hat{Z}_\eta = \min_{\substack{\boldsymbol{\tau}^{(\alpha)} \in \Sigma_\alpha(\Omega)\\ \alpha=1,\ldots,n}} \int\limits_{\Omega} \rho_\#([\sqrt{\eta_1}\,\boldsymbol{\tau}^{(1)} \ldots \sqrt{\eta_n}\,\boldsymbol{\tau}^{(n)}])dx \qquad (P_{YMD}) \tag{4.6}$$

the integrand being the norm given by

$$\rho_\#([\boldsymbol{\sigma}^{(1)} \ldots \boldsymbol{\sigma}^{(n)}]) = \sqrt{\boldsymbol{\sigma}^{(1)} \cdot \mathbf{G}^{-1}\boldsymbol{\sigma}^{(1)} + \cdots + \boldsymbol{\sigma}^{(n)} \cdot \mathbf{G}^{-1}\boldsymbol{\sigma}^{(n)}} \tag{4.7}$$

or, equivalently

$$\rho_\#([\boldsymbol{\sigma}^{(1)} \cdots \boldsymbol{\sigma}^{(n)}]) = \sqrt{\frac{6-9\nu}{1+\nu}\sum_{\alpha=1}^{n}(\operatorname{Tr}\boldsymbol{\sigma}^{(\alpha)})^2 + \frac{6-9\nu}{1-2\nu}\sum_{\alpha=1}^{n}\|\operatorname{dev}\boldsymbol{\sigma}^{(\alpha)}\|^2}. \tag{4.8}$$

On finding the minimizer $(\hat{\boldsymbol{\tau}}^{(1)},\ldots,\hat{\boldsymbol{\tau}}^{(n)})$ of (4.6) we can compute the optimal modulus:

$$\hat{E}_1(x) = E_0 \frac{\rho_\#([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}(x) \ldots \sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}(x)])}{\langle \rho_\#([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\rangle}. \tag{4.9}$$

Thus, the optimal Young modulus equals: $\hat{E}(x) = \hat{E}_1(x)/a$. We see that $\langle \hat{E}_1\rangle = E_0$, hence the cost condition (2.7) is satisfied sharply.

To formulate the problem dual to (4.6) one should construct the function dual to (4.7), or

$$\rho^o_\#([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \max_{\substack{\boldsymbol{\sigma}^{(\alpha)} \in E^2_s\\ \boldsymbol{\sigma}^{(\alpha)} \neq \mathbf{0}\\ \alpha=1,\ldots,n}} \frac{\sum_{\alpha=1}^{n} \boldsymbol{\varepsilon}^{(\alpha)} \cdot \boldsymbol{\sigma}^{(\alpha)}}{\rho_\#([\boldsymbol{\sigma}^{(1)} \ldots \boldsymbol{\sigma}^{(n)}])}. \tag{4.10}$$

The maximization operation above can be performed analytically. We find

$$\rho^0_\#([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \sqrt{\boldsymbol{\varepsilon}^{(1)} \cdot (\mathbf{G}\boldsymbol{\varepsilon}^{(1)}) + \cdots + \boldsymbol{\varepsilon}^{(n)} \cdot (\mathbf{G}\boldsymbol{\varepsilon}^{(n)})}. \tag{4.11}$$

Thus, the norm dual to (4.8) has the form

$$\rho^0_\#([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \sqrt{\frac{1+\nu}{6-9\nu} \sum_{\alpha=1}^{n} (\mathrm{Tr}\, \boldsymbol{\varepsilon}^{(\alpha)})^2 + \frac{1-2\nu}{6-9\nu} \sum_{\alpha=1}^{n} \|\mathrm{dev}\, \boldsymbol{\varepsilon}^{(\alpha)}\|^2}. \tag{4.12}$$

Now, we are ready to form the problem dual to (4.6): find $(\hat{\mathbf{v}}^{(1)}, \ldots, \hat{\mathbf{v}}^{(n)})$, the maximizer of the problem:

$$\begin{aligned} \hat{Z}_\eta = \max\Bigg\{ \sum_{\alpha=1}^{n} \sqrt{\eta_\alpha}\, f^{(\alpha)}(\mathbf{v}^{(\alpha)}) \;\Bigg|\; &\text{over } \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)} \in V(\Omega) \\ &\text{such that } \rho^o_\#([\boldsymbol{\varepsilon}(\mathbf{v}^{(1)}(x)) \ldots \boldsymbol{\varepsilon}(\mathbf{v}^{(n)}(x))]) \le 1 \text{ a.e. in } \Omega \Bigg\} \quad (P^*_{YMD}). \end{aligned} \tag{4.13}$$

Thus, the YMD method reduces to solving the pair : (P_{YMD}), (P^*_{YMD}) (see (4.6), (4.13)) of mutually dual problems.

4.2. The 2D setting

The counterparts of the formulae (4.1) read

$$k(x) = \frac{E(x)}{2(1-\nu)}, \qquad \mu(x) = \frac{E(x)}{2(1+\nu)}. \tag{4.14}$$

The unit cost of the design is given by

$$\mathrm{tr}\, \mathbf{C} = aE(x), \qquad a = \frac{3-\nu}{1-\nu^2}. \tag{4.15}$$

The main design variable is now $E_1(x) = aE(x)$ and is subject to the cost condition $\langle E_1 \rangle \le E_0$.

The problem of minimizing the weighted compliance has the form (4.5) with

$$\mathbf{G} = \frac{1+\nu}{3-\nu}\boldsymbol{\Lambda}_1 + \frac{1-\nu}{3-\nu}\boldsymbol{\Lambda}_2, \qquad \mathbf{G}^{-1} = \frac{3-\nu}{1+\nu}\boldsymbol{\Lambda}_1 + \frac{3-\nu}{1-\nu}\boldsymbol{\Lambda}_2. \tag{4.16}$$

The reduced problem (4.6) involves now the integrand given by

$$\rho_\#([\boldsymbol{\tau}^{(1)} \ldots \boldsymbol{\tau}^{(n)}]) = \sqrt{\frac{3-\nu}{1+\nu} \sum_{\alpha=1}^{n} (\mathrm{Tr}\, \boldsymbol{\tau}^{(\alpha)})^2 + \frac{3-\nu}{1-\nu} \sum_{\alpha=1}^{n} \|\mathrm{dev}\, \boldsymbol{\tau}^{(\alpha)}\|^2}. \tag{4.17}$$

The optimal modulus $\hat{E}_1$ is given by (4.9) and satisfies $\langle \hat{E}_1 \rangle = E_0$, hence the cost condition is fulfilled.

The problem (4.13) involves the locking condition for virtual strains being given by the function polar to (4.17) or

$$\rho^o_{\#}([\boldsymbol{\varepsilon}^{(1)} \ldots \boldsymbol{\varepsilon}^{(n)}]) = \sqrt{\frac{1+\nu}{3-\nu}\sum_{\alpha=1}^{n}(\mathrm{Tr}\,\boldsymbol{\varepsilon}^{(\alpha)})^2 + \frac{1-\nu}{3-\nu}\sum_{\alpha=1}^{n}\|\mathrm{dev}\,\boldsymbol{\varepsilon}^{(\alpha)}\|^2}. \tag{4.18}$$

The problem is reduced to the LCP scheme composed of (4.6), (4.13) with the norms $\rho_{\#}$, $\rho^o_{\#}$ given by (4.17), (4.18).

4.3. Construction of the stress fields in the optimal structure

Consider now the elastic properties of the structure made of the material of optimal Young's modulus. The state of stress induced in this structure by the load of index α is denoted by $\tilde{\boldsymbol{\tau}}^{(\alpha)}$. We shall prove that this state of stress coincides with the stress field $\hat{\boldsymbol{\tau}}^{(\alpha)}$ being the αth component of the minimizer $(\hat{\boldsymbol{\tau}}^{(1)} \ldots \hat{\boldsymbol{\tau}}^{(n)})$ of the problem (4.6). The starting point is the equality

$$\begin{aligned} \wp^{(\alpha)} &= \int_\Omega \frac{1}{\hat{E}_1(x)}(\tilde{\boldsymbol{\tau}}^{(\alpha)} \cdot (\mathbf{G}^{-1}\tilde{\boldsymbol{\tau}}^{(\alpha)}))\,dx \\ &= \min\left\{ \int_\Omega \frac{1}{\hat{E}_1(x)}(\boldsymbol{\tau} \cdot (\mathbf{G}^{-1}\boldsymbol{\tau}))\,dx \;\Big|\; \text{over } \boldsymbol{\tau} \in \Sigma_\alpha(\Omega) \right\}. \end{aligned} \tag{4.19}$$

Since $\tilde{\boldsymbol{\tau}}^{(\alpha)}$ is the minimizer of the problem above, we have

$$\begin{aligned} F_\eta &= \int_\Omega \frac{1}{\hat{E}_1(x)} \sum_{\alpha=1}^{n} (\sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)}) \cdot (\mathbf{G}^{-1}(\sqrt{\eta_\alpha}\,\tilde{\boldsymbol{\tau}}^{(\alpha)}))\,dx \\ &= \frac{1}{E_0}\big\langle \rho_{\#}([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}])\big\rangle J_1 \end{aligned} \tag{4.20}$$

where

$$J_1 = \int_\Omega \frac{(\rho_{\#}([\sqrt{\eta_1}\,\tilde{\boldsymbol{\tau}}^{(1)}(x) \ldots \sqrt{\eta_n}\,\tilde{\boldsymbol{\tau}}^{(n)}(x)]))^2}{\rho_{\#}([\sqrt{\eta_1}\hat{\boldsymbol{\tau}}^{(1)}(x) \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}(x)])}\,dx. \tag{4.21}$$

By using the inequality (3.45) and the inequality

$$\int_\Omega \rho_{\#}([\sqrt{\eta_1}\,\tilde{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\,\tilde{\boldsymbol{\tau}}^{(n)}])\,dx \geq \int_\Omega \rho_{\#}([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)} \ldots \sqrt{\eta_n}\hat{\boldsymbol{\tau}}^{(n)}])\,dx \tag{4.22}$$

we note that

$$J_1 \geq \int_\Omega \rho_\#([\sqrt{\eta_1}\,\hat{\boldsymbol{\tau}}^{(1)}(x)\ldots\sqrt{\eta_n}\,\hat{\boldsymbol{\tau}}^{(n)}(x)])\,dx. \tag{4.23}$$

This implies $F_\eta \geq (\hat{Z}_\eta)^2/\Lambda$ and the equality is attained for $\tilde{\boldsymbol{\tau}}^{(\alpha)} = \hat{\boldsymbol{\tau}}^{(\alpha)}$. We conclude that the solution to the problem (4.6) determines the stress fields occurring in the optimal structure for the subsequent load cases.

5. Conclusions

The problem of minimization of the weighted sum of compliances over the fields of elastic moduli forming the Hooke tensor, with the unit cost equal to its trace, reduces to the so-called linear constrained problem (LCP) composed of two mutually dual auxiliary problems (P), and (P^*). The former involves the stress fields corresponding to the subsequent load variants. The latter involves the displacement fields; the associated strain fields satisfy the locking pointwise conditions. In the problems of optimization of:

– anisotropy (AMD);
– distribution of the bulk and shear moduli of the isotropic media (IMD);
– distribution of the Young modulus under the condition of the Poisson ratio being predefined (YMD);

the auxiliary problems of the LCP setting have been successfully formulated, including the explicit constructions of the integrands of the primal problems and the locking loci in their dual versions. In particular, the locking locus of the problem (P^*_{AMD}) has been constructed; it is the unit ball in $E^2_s \times E^2_s$ with respect to the Schatten ∞-norm. In the 2D case, for $n = 2$, its explicit definition has been found. It occurs that this ball is naturally divided into an internal ball (where the angles between the virtual strain fields are arbitrary) surrounded by the domain, where these angles are bounded from below. Along the edge of the locking locus the virtual strains are orthogonal.

Within each FMD problem considered the cutting-off property holds: the material part of the structure is just the effective domain of the minimizer of the problems: (P_{AMD}), (P_{IMD}), (P_{YMD}) with the integrands of linear growth.

The paper proves that the subsequent stress fields of the collections of the stress fields forming the minimizers of the primal auxiliary problems determine the fields of stress in the optimum structure if subjected to the subsequent load variants. This fact underlines a key role of the stress-based auxiliary problems of the AMD, IMD and YMD methods in the process of optimum designing.

The optimum design settings: AMD, YMD and IMD do not deliver algorithms of forming the underlying microstructure of the material with given optimal effective characteristics. One of the material forming method is to design a graded microstructure of spatially varying properties by constructing the family of periodicity cells $Y(x)$ which, according to the theory of homogenization, stands for the family of representative volume elements (RVE) of the optimal composite. The simplest concept is to choose a skeletal microstructure which exhibits an almost constant value of the effective Poisson ratio despite variation of transverse dimensions of ligaments. Such a concept has been put forward in [23], where also the algorithm of 3D printing has been described, enabling manufacturing some prototypes of planar handles.

Manufacturing planar RVEs of isotropic properties is based on the known property of isotropy in the plane being generated by rotation by 120° of a one third of a hexagonal periodicity cell, as revealed in [43–45] and used in [46, 47, 25] and just very recently applied by Casalotti *et al.* [48]. Extension of this idea toward a spatial design of macroscopically isotropic RVEs is an open question. Instead, in 3D setting the cubic symmetry is a natural choice, cf. the micro-lattices of metamaterials described in [49].

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