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Two imperfectly bonded half-planes with an arbitrary inclusion subject to linear eigenstrains in anti-plane shear

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An analytic solution to the anti-plane problem of an arbitrary inclusion within an elastic bimaterial under the premise of linear eigenstrains is developed. The bonding along the bimaterial interface is considered to be homogeneously imperfect. The boundary value problem is reduced to a single nonhomogeneous first order differential equation for an analytic function prescribed in the lower half-plane where the inclusion is located. The general solution is given in terms of the imperfect interface parameter and an auxiliary function constructed from the conformal mapping function. In particular, the solution obtained for a circular inclusion demonstrates that the imperfect interface together with the prescribed linear eigenstrains have a pronounced effect on the induced stress field within the inclusion and show a strong non-uniform behaviour especially when the inclusion is near the imperfect interface. Specific solutions are derived in a closed form and verified with existing solutions.

Key words: elastic fields, arbitrary inclusion, mperfect interface, antiplane elasticity, linear eigenstrain.

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1. Introduction

The notion of an inclusion is that it represents a subdomain of the host material body. The elastic analysis of said inclusion undergoing a prescribed uniform stress free strain is a classic topic and is commonly referred to as the Eshelby problem. Eshelby [5], in his pioneering work, solved the problem of an ellipsoidal subdomain within an infinite isotropic elastic material subjected to uniform stress free transformations and he proved that the stresses within the inclusion must be constant. Since his celebrated work, a tremendous amount of effort has been devoted to the study of the Eshelby problem (see, for example, [30] for a detailed literature review). Other inclusion shapes that have been investigated include cuboidal inclusions [3] and polygonal inclusions [15, 7, 25]. While various methods have been developed to study Eshelby's problem, the Green's function approach appears to be the most common. However, due to the non-trivial integration, the Green's function approach cannot be readily applied to solve problems associated with more complex inclusion shapes. As a result, Ru [17] proposed a method based on conformal mapping and analytic continuation to study arbitrary shaped inclusions in a plane or half-plane.

Exciting real world examples that lead to the Eshelby problem are numerous. For instance, passivated interconnect lines, isolation trenches (see [17]) and in the construction of tunnels in geologic strata (see [21]). Over the past years, extensive studies have been undertaken to model such problems by adopting a simplified approach where the inclusion is located in a half-plane [9, 8, 10, 24, 28, 13, 12]. However, this approach is not general enough and in some practical cases such as those encountered in electronic packaging [6] it is relevant to consider an inclusion within one of two jointed elastic half-planes [1, 29, 18, 27]. It should be mentioned that in all of the aforementioned studies the eigenstrains within the inclusion are assumed to be uniform. Such a restriction is limiting because most electronic devices (such as electronic chips, field effect transistors) the eigenstrains exhibit non-uniform behaviour. Hence, the impact of non-uniform eigenstrains on the induced stress fields is of significant interest.

Sharma *et al.* [19] considered dilatational Gaussian and exponential eigenstrains over a single ellipsoidal domain. Shodja *et al.* [20] studied the effects of nonuniform eigenstrains over a nested domain of ellipsoidal inclusions. Kamali *et al.* [11] developed a semi-analytical method to study the interaction between a screw dislocation with embedded multi-inhomogeneities of arbitrary shape incorporating imperfect bonding. The work is based on extremization of the total potential energy of the medium and incorporates the singularity as well as the surface/interface geometry. Rahman [16] and Nie *et al.* [14] investigated an ellipsoidal and elliptic inclusion subject to polynomial and linear eigenstrains, respectively. Recently, Chen [2] has obtained closed form solutions for an elliptic inclusion in an entire plane subject to linear eigenstrains in antiplane elasticity. In all the aforementioned works, even though the eigenstrains are non-uniform, the limiting assumption of perfect bonding across material boundaries is utilized. Furthermore, only simple shapes are investigated and in the majority of the works the Green's function approach is applied to the solution process. It is worth mentioning that within this area of research there exists a plethora of other contributions but for the sake of brevity they are not cited.

The assumption of perfect bonding between material boundaries is a convenient idealization to significantly simplify the analysis. This limiting approach prevents the study of more realistic imperfect interface which is capable of describing interface damage (such as microcracks, impurities and debonding). While this model has been used extensively in the area of composite mechanics [22, 23], it has received little attention in the modeling of jointed half-planes. Recently, Wang *et al.* [26] utilized Ru's approach and explored the role of an imperfect interface between two dissimilar media on the internal stress field within an arbitrary inclusion subject to a uniform eigenstrain (i.e. a thermal inclusion). The results showed that the imperfect interface description does have an impact on the elastic fields. Hence, it would be of great practical and theoretical interest to investigate the problem of an inclusion of arbitrary shape with linear eigenstrains located in one of two imperfectly bonded dissimilar materials in antiplane elasticity. The results from this work will have great benefits to the design of electronic devices.

The formulation of the basic boundary value problem for an arbitrary shaped inclusion within one of the two imperfectly bonded dissimilar half-planes subject to linear eighenstrains in antiplane elasticity is presented in Section 2. In Section 3 the boundary value problem is reduced to a single first order nonhomogenous linear differential equation for an analytic function in the lower half-plane. In Section 4 we discuss a few limiting cases where the exact solution to the stress field is derived in terms of the auxiliary function. Also we reproduce the results given by [2] when the upper and lower half planes are identical. Section 5 outlines detailed results to illustrate the significant effects of the imperfect interface and the linear eigenstrains on the induced elastic fields within an inclusion of circular shape.

2. Basic equations and formulation

2.1. Basic equations

In a three dimensional Cartesian system, consider an isotropic, homogeneous elastic domain undergoing anti-plane deformation determined by the out-of-plane displacement along the x_3 direction. The state of antiplane strain is defined as one in which the Cartesian scaler component of the displacement field takes the form

$$u_1(x_1, x_2, x_3) = u_2(x_1, x_2, x_3) = 0, \qquad u_3 = u_3(x_1, x_2). \tag{2.1}$$

It follows from this that the only non-zero stress components are determined via $\sigma_{3\beta} = \mu u_{3,\beta}$ where μ is the shear modulus, the $','$ implies differentiation with respect to x_1 and x_2, respectively and $(\beta = 1, 2)$. Thus, at all points in the body, equilibrium is satisfied provided the displacement field is harmonic. Hence, one can always represent the anti-plane stresses and the out-of-plane displacement by a complex potential $f(z)$ where $z = x_1 + ix_2$ is the complex coordinate as

$$\sigma_{31} - i\sigma_{32} = f'(z), \qquad u_3(x_1, x_2) = \frac{1}{\mu}\,\mathrm{Re}[f(z)], \tag{2.2}$$

where μ represents the shear modulus and $'$ denotes the derivative with respect to the complex variable. In addition, the shear traction on a directed curve from point A to B in the complex plane can be written in terms of $f(z)$ as

$$T = \int_A^B (\sigma_{31}dx_2 - \sigma_{32}dx_1) = \mathrm{Im}[f(z)]. \tag{2.3}$$

2.2. Formulation

Consider two dissimilar isotropic and homogeneous half-planes imperfectly bonded along the real axis. Let the lower half-plane contain an internal subdomain of arbitrary shape which undergoes linear anti-plane stress free shear eigenstrains (ϵ_{13}^*, ϵ_{23}^*). Let Ω_2 denote the upper half-plane, Ω_o and Ω_1 denotes the subdomain and its supplement to the lower half-plane, respectively and Γ be the interface separating Ω_o and Ω_1. Throughout the paper, the subscripts 0, 1, 2 are used to identify the respective quantities in Ω_o, Ω_1 and Ω_2.

In the following analysis, we= assume the eigenstrains, ϵ_{13}^* and ϵ_{23}^*, to be linear functions in x_1 and x_2 and given as

$$\epsilon_{13}^* = a_0 + a_1 x_1 + a_2 x_2, \qquad \epsilon_{23}^* = b_0 + b_1 x_1 + b_2 x_2, \tag{2.4}$$

where a_k and b_k, $k = 0, 1, 2$ are six arbitrary and real coefficients that can be determined from experiments.

In anti-plane elasticity, the eigenstrains, ϵ_{13}^* and ϵ_{23}^*, are related to the out-of-plane eigendisplacement, $u_3^*(x_1, x_2)$, through the kinematic relations

$$2\epsilon_{13}^* = u_{3,1}^*, \qquad 2\epsilon_{23}^* = u_{3,2}^*, \tag{2.5}$$

where the $','$ implies differentiation with respect to x_1 and x_2, respectively. It should be noted that although the total strain is related to the total displacement gradient field, in the more general case, the displacement $u_3^*(x_1, x_2)$ may not satisfy Laplace's equation exactly. Hence, we only consider a compatible linear eigenstrain field.

The eigenstrains given in (2.4) must satisfy a comptability equation to ensure that a unique out-of-plane displacement is obtained. Using (2.5) the compatibility equation is

$$\epsilon_{13,2}^* = \epsilon_{23,1}^*, \tag{2.6}$$

from which we find that $a_2 = b_1$ ensuring that (2.4) is compatible. In view of the aforementioned, it can be readily shown that the out-of-plane eigendisplacment can be expressed in terms of the complex variable, z, and its conjugate as

$$u_3^*(z, \overline{z}) = \overline{c_0} z + c_0 \overline{z} + \overline{c_1} z^2 + c_1 \overline{z}^2 + c_2 z \overline{z}, \tag{2.7}$$

where

$$c_0 = a_0 + i b_0, c_1 = \left(\frac{a_1 - b_2 + 2 b_1 i}{4} \right), \qquad c_2 = \left(\frac{a_1 + b_1}{2} \right). \tag{2.8}$$

In this study, the two half-planes are assumed to be bonded imperfectly through the real axis and perfect bonding conditions are prescribed at the boundary of the arbitrary inclusion. Using equations (2.2), (2.3) the boundary value problem for the imperfectly bonded jointed half-planes takes the form

$$\begin{aligned}
(f_1(z) - \overline{f_1(z)}) &= (f_2(z) - \overline{f_2(z)}),\\
(\overline{f_1'(z)} - f_1'(z)) &= \alpha i\left[\frac{1}{\mu_2}(f_2(z) + \overline{f_2(z)})\right.\\
&\qquad \left. - \frac{1}{\mu_1}(f_1(z) + \overline{f_1(z)})\right], \quad x_2 = 0,\\
(f_1(z) - \overline{f_1(z)}) &= (f_0(z) - \overline{f_0(z)}),\\
(f_1(z) + \overline{f_1(z)}) &= (f_0(z) + \overline{f_0(z)}) + 2\mu_1 u_3^*(z,\overline{z}), \quad z \in \Gamma,\\
f_1(z) &= O(1), \quad |z| \to \infty\ (x_2 \le 0),\\
f_2(z) &= O(1), \quad |z| \to \infty\ (x_2 \ge 0),
\end{aligned} \tag{2.9}$$

where α is an interface constant determined by the geometric and material properties of the interphase layer. Note that the relations along the real axis represent a continuity of tractions and a jump in the out-of-plane displacement whereas the conditions at the interface Γ represent the continuity of tractions and displacements. Thus, three analytic functions, $f_k(z)$, $(k = 0, 1, 2)$, are determined by conditions (2.9).

To accommodate an inclusion of arbitrary shape the technique of conformal mapping is employed. According to the Riemann mapping theorem, there exists a function [4]

$$z = \omega(\xi) = \Big(R\xi + \sum_{k=0}^{\infty} d_k \xi^{-k}\Big) \tag{2.10}$$

where R is a real constant and $\omega(\xi)$ is univalent and analytic for $|\xi| \ge 1$ and $\omega'(\xi) \ne 0$ for $|\xi| \ge 1$ which maps the exterior of the inclusion conformally onto the parametric plane $(\xi : |\xi| \ge 1)$. In many practical examples, the infinite series appearing above can be truncated and replaced with reasonable accuracy by a polynomial in $1/\xi$ which includes only a finite number of terms (say M).

Ru [17] suggested a procedure that utilizes conformal mapping and analytic continuation to derive exact solutions for inclusions of arbitrary shape. The idea is that there exists a function, $D(z)$, which satisfies the condition

$$\overline{z} = D(z), \qquad z \in \Gamma, \tag{2.11}$$

which is analytic in the exterior of Γ except at infinity where it has a pole of finite degree determined by the asymptotic behaviour

$$D(z) \to P(z) + O(1), \qquad |z| \to \infty. \tag{2.12}$$

3. General solution

Let us now proceed with the derivation of the general solution to the problem in terms of function $D(z)$ and the associated polynomials $P(z)$ and $H(z)$. To begin, let us first consider the interface condition in (2.9) along Γ which can be written into the equivalent form

$$f_0(z) = f_1(z) - \mu_1[\overline{c_0}z + c_0 D(z) + c_2 z D(z) + \overline{c_1}z^2 + c_1 D(z)D(z)], \quad z \in \Gamma. \tag{3.1}$$

Since the left and right hand sides of (3.1) are analytic in the lower half-plane inside and outside the curve Γ, respectively, the continuity condition (3.1) ensures that any one of terms can be extended analytically across Γ. Hence, we can define the function $\Delta(z)$ as follows

$$\Delta(z) = \begin{cases} f_1(z) - \mu_1[\overline{c_0}z + c_0 D(z) + c_2 z D(z) \\ \qquad +\overline{c_1}z^2 + c_1 D(z)D(z)], \quad z \in \Omega_1, \\ f_0(z), \quad z \in \Omega_0. \end{cases} \tag{3.2}$$

It is clear that $\Delta(z)$ is continuous across Γ and analytic in the whole lower half plane. In particular, its asymptotic behaviour at infinity is described by

$$\Delta(z) \to -\mu_1[\overline{c_0}z + c_0 P(z) + c_2 z P(z) + \overline{c_1}z^2 + c_1 H(z)], \quad |z| \to \infty (x_2 \le 0), \tag{3.3}$$

where $H(z)$, a polynomial in z, is the principle part of the product $D(z)D(z)$ at infinity, namely

$$D(z)D(z) \to H(z) + O(1), \quad |z| \to \infty. \tag{3.4}$$

It follows from (3.2) that

$$f_1(z) = \Delta(z) + \mu_1[\overline{c_0}z + c_0 D(z) + c_2 z D(z) + \overline{c_1}z^2 + c_1 D(z)D(z)], \quad z \in \Omega_1. \tag{3.5}$$

Now the two remaining interface conditions along the real axis can be written as

$$\frac{2\alpha i}{\mu_2} f_2(z) + \alpha i\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)\overline{f_1}(z) - \overline{f_1'}(z) = \alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) f_1(z) - f_1'(z), \quad x_2 = 0. \tag{3.6}$$

Substitution of (3.5) into (3.6) and noting that $D(z)$ and its derivative are analytic in the upper half plane yields

$$
\begin{aligned}
(3.7)\qquad & \frac{2\alpha i}{\mu_2} f_2(z) - \overline{\Delta'}(z) + \alpha i\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)\overline{\Delta}(z) \\
&+ \mu_1[\overline{c_0} + c_0 D'(z) + c_2 D(z) + c_2 z D'(z) + 2\overline{c_1} z + 2c_1 D(z) D'(z)] \\
&- \alpha i \mu_1 \left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)[\overline{c_0} z + c_0 D(z) + c_2 z D(z) + \overline{c_1} z^2 + c_1 D(z) D(z)] \\
= & -\Delta'(z) + \alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)\Delta(z) \\
&+ \mu_1[c_0 + \overline{c_0}\overline{D'}(z) + c_2\overline{D}(z) + c_2 z\overline{D'}(z) + 2c_1 z + 2\overline{c_1}\overline{D}(z)\overline{D'}(z)] \\
&- \alpha i \mu_1\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)[c_0 z + \overline{c_0}\overline{D}(z) + c_2 z\overline{D}(z) + c_1 z^2 + \overline{c_1}\overline{D}(z)\overline{D}(z)], \quad x_2 = 0.
\end{aligned}
$$

The left and right hand sides of (3.7) are analytic in the upper and lower half planes, respectively, and approach the same polynomial

$$
\begin{aligned}
(3.8)\qquad & \mu_1[c_0 + \overline{c_0}\overline{P'}(z) + c_2\overline{P}(z) + c_2 z\overline{P'}(z) + 2c_1 z + \overline{c_1}\overline{H'}(z)] \\
&- \alpha i \mu_1\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)[c_0 z + \overline{c_0}\overline{P}(z) + c_2 z\overline{P}(z) + c_1 z^2 + \overline{c_1}\overline{H}(z)] \\
&+ \mu_1[\overline{c_0} + c_0 P'(z) + c_2 P(z) + c_2 z P'(z) + 2\overline{c_1} z + c_1 H'(z)] \\
&- \alpha i \mu_1\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)[\overline{c_0} z + c_0 P(z) + c_2 z P(z) + \overline{c_1} z^2 + c_1 H(z)], \quad |z| \to \infty.
\end{aligned}
$$

Thus, it is concluded that the left and right hand sides are equal to the above polynomial in the upper and lower half planes, respectively. It follows that

$$
\begin{aligned}
(3.9)\qquad & \frac{2\alpha i}{\mu_2} f_2(z) = \overline{\Delta'}(z) - \alpha i\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)\overline{\Delta}(z) + \alpha i \mu_1\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) \\
&\times [c_0(D(z) - P(z)) + c_2 z(D(z) - P(z)) + c_1(D(z)D(z) - H(z))] \\
&- \mu_1[c_0(D'(z) - P'(z)) + c_2(D(z) - P(z)) \\
&+ c_2 z(D'(z) - P'(z)) + c_1(2D(z)D'(z) - H'(z))] \\
&+ \mu_1[c_0 + \overline{c_0}\overline{P'}(z) + c_2\overline{P}(z) + c_2 z\overline{P'}(z) + 2c_1 z + \overline{c_1}\overline{H'}(z)] \\
&- \alpha i \mu_1\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)[c_0 z + \overline{c_0}\overline{P}(z) + c_2 z\overline{P}(z) + c_1 z^2 + \overline{c_1}\overline{H}(z)], \quad x_2 > 0,
\end{aligned}
$$

in the upper half plane and

$$
(3.10)\qquad \Delta'(z) - \alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)\Delta(z) = G(z), \quad x_2 < 0,
$$

in the lower half plane, where

$$
\begin{aligned}
G(z) = {} & \mu_1\big\{\overline{c_0}[\overline{D'}(z) - \overline{P'}(z)] + c_2[\overline{D}(z) - \overline{P}(z)] + c_2 z[\overline{D'}(z) - \overline{P'}(z)] \\
& + \overline{c_1}[2\overline{D}(z)\overline{D'}(z) - \overline{H'}(z)]\big\} \\
& - \alpha i \mu_1\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \\
& \times \big\{\overline{c_0}[\overline{D}(z) - \overline{P}(z)] + c_2 z[\overline{D}(z) - \overline{P}(z)] + \overline{c_1}[\overline{D}(z)\overline{D}(z) - \overline{H}(z)]\big\} \\
& - \mu_1[\overline{c_0} + c_0 P'(z) + c_2 P(z) + c_2 z P'(z) + 2\overline{c_1} z + c_1 H'(z)] \\
& + \alpha i \mu_1\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)[\overline{c_0} z + c_0 P(z) + c_2 z P(z) + \overline{c_1} z^2 + c_1 H(z)].
\end{aligned}
\tag{3.11}
$$

The condition (3.10) gives a simple nonhomogeneous first-order differential equation for the unknown function $\Delta(z)$ in the entire lower half plane. The general solution of (3.10) is given explicitly by

$$
\begin{aligned}
\Delta(z) = {} & \exp\left[\alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) z\right] \\
& \times \left[\int_{z_0}^{z} G(t)\exp\left[-\alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) t\right] \mathrm{d}t + C_0\right], \quad x_2 < 0,
\end{aligned}
\tag{3.12}
$$

where z_0 is an arbitrary selected point in the lower half plane. The path of integration is taken along an arbitrary curve within the lower half plane and C_0 is an arbitrary constant of integration determined by z_0 and the asymptotic behaviour at infinity. For example, one can choose the point z_0 to be $-i\infty$ and then C_0 should be determined so that

$$
\begin{aligned}
\Delta(z) = {} & -\mu_1[\overline{c_0} z + c_0 P(z) + c_2 z P(z) + \overline{c_1} z^2 + c_1 H(z)] \\
& + \exp\left[\alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) z\right] \\
& \times \left[\int_{-i\infty}^{z} G_0(t)\exp\left[-\alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) t\right] \mathrm{d}t\right], \quad x_2 < 0, \\
G_0(t) = {} & \mu_1\big\{\overline{c_0}[\overline{D'}(t) - \overline{P'}(t)] + c_2[\overline{D}(t) - \overline{P}(t)] \\
& + c_2 t[\overline{D'}(t) - \overline{P'}(t)] + \overline{c_1}[2\overline{D}(t)\overline{D'}(t) - \overline{H'}(t)]\big\} \\
& - \alpha i \mu_1\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)\big\{\overline{c_0}[\overline{D}(t) - \overline{P}(t)] + c_2 t[\overline{D}(t) - \overline{P}(t)] \\
& + \overline{c_1}[\overline{D}(t)\overline{D}(t) - \overline{H}(t)]\big\}, \quad x_2 < 0.
\end{aligned}
\tag{3.13}
$$

Hence, the solution for the inclusion problem depends on the imperfect interface parameter, the prescribed linear eigenstrains as well as the auxiliary function $D(z)$ and the associated polynomials $P(z)$ and $H(z)$. Once $\Delta(z)$ has been determined from (3.13) then the entire elastic fields in the whole plane can be calculated.

In the next section, we discuss some special cases of particular interest.

4. Limiting cases

4.1. Perfect bonding

First, we consider the case when the two elastic half-planes are perfectly bonded together (ie $\alpha=\infty$). It immediately follows from (3.5), (3.9) and (3.10) that

$$\begin{aligned}\Delta(z) =&\mu_1\frac{(\frac{1}{\mu_2}-\frac{1}{\mu_1})}{(\frac{1}{\mu_2}+\frac{1}{\mu_1})}\{\overline{c_0}[\overline{D}(z)-\overline{P}(z)]+c_2z[\overline{D}(z)-\overline{P}(z)]\\ &+\overline{c_1}[\overline{D}(z)\overline{D}(z)-\overline{H}(z)]\}\\ &-\mu_1[\overline{c_0}z+c_0P(z)+c_2zP(z)+\overline{c_1}z^2+c_1H(z)], \quad x_2<0,\end{aligned} \tag{4.1}$$

$$\begin{aligned}f_1(z) =& \Delta(z)+\mu_1[\overline{c_0}z+c_0D(z)+c_2zD(z)+\overline{c_1}z^2\\ &+c_1D(z)D(z)], \quad x_2<0,\end{aligned}$$

$$\begin{aligned}f_2 =& -\frac{\mu_2}{2}\Big(\frac{1}{\mu_2}-\frac{1}{\mu_1}\Big)\overline{\Delta}(z)+\frac{\mu_1\mu_2}{2}\Big(\frac{1}{\mu_2}+\frac{1}{\mu_1}\Big)\\ &\times[c_0(D(z)-P(z))+c_2z(D(z)-P(z))+c_1(D(z)D(z)-H(z))]\\ &-\frac{\mu_1\mu_2}{2}\Big(\frac{1}{\mu_2}-\frac{1}{\mu_1}\Big)\\ &\times[c_0z+\overline{c_0}\overline{P}(z)+c_2z\overline{P}(z)+c_1z^2+\overline{c_1}\overline{H}(z)], \quad x_2>0.\end{aligned} \tag{4.2}$$

In particular, within the inclusion Ω_0 we have

$$\begin{aligned}f_0 = \mu_1\frac{(\frac{1}{\mu_2}-\frac{1}{\mu_1})}{(\frac{1}{\mu_2}+\frac{1}{\mu_1})}&\{\overline{c_0}[\overline{D}(z)-\overline{P}(z)]+c_2z[\overline{D}(z)-\overline{P}(z)]\\ &+\overline{c_1}[\overline{D}(z)\overline{D}(z)-\overline{H}(z)]\}-\mu_1[\overline{c_0}z+c_0P(z)+c_2zP(z)\\ &+\overline{c_1}z^2+c_1H(z)], \quad z\in\Omega_0.\end{aligned} \tag{4.3}$$

Expressions (4.1-4.3) give the explicit solution for not only the internal stress field in the inclusion but also for both half planes. The solutions for the elastic fields depend explicitly on the functions $D(z)$, $P(z)$ and $H(z)$. To the authors knowledge, no such complete solution for an inclusion of arbitrary shape subjected to linear eigenstrains in a bimaterial has been reported in the literature for antiplane elasticity.

4.2. Both half-planes are identical

Now, let us consider the case when the upper and lower half planes are identical and perfect bonding between the half planes is assumed (i.e. $\mu_1 = \mu_2$). In this case, the expressions given by (4.1-4.3) reduce to

$$f_1 = \mu_1[c_0(D(z) - P(z)) + c_2 z(D(z) - P(z)) + c_1(D(z)D(z) - H(z))], \quad x_2 < 0, \tag{4.4}$$

and

$$f_0 = -\mu_1[\overline{c_0} z + c_0 P(z) + c_2 z P(z) + \overline{c_1} z^2 + c_1 H(z)], \quad z \in \Omega_0. \tag{4.5}$$

The exact solution for the internal stress field within the arbitrary inclusion in the entire plane is given by expression (4.5).This expression depends only on the polynomials $P(z)$ and $H(z)$ and not the function $D(z)$. This result is a huge benefit since both $P(z)$ and $H(z)$ admit simple forms whereas $D(z)$ requires the evaluation of the inverse of the mapping function which is often difficult to handle [18].

To validate our approach consider an elliptic inclusion embedded in the lower half plane at some point $x_0 = 0$ and y_0, $(y_0 < 0)$.The functions $P(z)$, $H(z)$ as well as for completeness the auxiliary function $D(z)$ is given by

$$\begin{aligned} D(z) &= R^2(z - iy_0) - iy_0 + (\frac{1 - R^4}{2R^2})(z - iy_0)\left[1 + \sqrt{1 - \left(\frac{2d}{z - iy_0}\right)^2}\right] \\ P(z) &= \frac{1}{R^2}(z - iy_0) - iy_0, \qquad H(z) = \frac{(z - iy_0)^2}{R^4} - 2\frac{iy_0}{R^2} - y_0^2. \end{aligned} \tag{4.6}$$

Assuming the parameters $y_0 = 0$ and $c_0 = 0$, expression (4.5) takes the form

$$f_0(z) = -\mu_1[a_1(1 + m)^2 - 2b_1 i(1 - m)^2 - b_2(1 - m)^2]\frac{z^2}{4}, \quad z \in S_0, \tag{4.7}$$

which is identical to the work of [2] by noting that the coefficients a_1, b_1, b_2 can be related to [2] through

$$a_1 = \frac{c_1^*}{2a}, \quad b_1 = \frac{c_2^*}{2a}, \quad b_2 = \frac{c_3^*}{2a}.$$

4.3. Traction free upper half-plane

Now consider the case when the elastic lower half plane contains an inclusion of arbitrary shape and a traction free condition along the surface of the lower half plane is assumed (i.e. $\alpha = 0$). It immediately follows from (3.5) and (3.10) that in the lower half plane the stress field is given by

$$
\begin{aligned}
(4.8)\qquad f_1'(z) = \mu_1\big\{&c_0[D'(z) - P'(z)] + c_2[D(z) - P(z)]\\
&+ c_2 z[D'(z) - P'(z)] + c_1[2D(z)D'(z) - H'(z)]\big\}\\
&+ \mu_1\big\{\overline{c_0}[\overline{D'}(z) - \overline{P'}(z)] + c_2[\overline{D}(z) - \overline{P}(z)]\\
&+ c_2 z[\overline{D'}(z) - \overline{P'}(z)] + \overline{c_1}[2\overline{D}(z)\overline{D'}(z) - \overline{H'}(z)]\big\}, \quad x_2 < 0.
\end{aligned}
$$

and the internal stress within the arbitrary inclusion is given by

$$
\begin{aligned}
(4.9)\qquad f_0'(z) = \mu_1\big\{&\overline{c_0}[\overline{D'}(z) - \overline{P'}(z)] + c_2[\overline{D}(z) - \overline{P}(z)]\\
&+ c_2 z[\overline{D'}(z) - \overline{P'}(z)] + \overline{c_1}[2\overline{D}(z)\overline{D'}(z) - \overline{H'}(z)]\big\}\\
&- \mu_1\{\overline{c_0} + c_0 P'(z) + c_2 P(z) + c_2 z P'(z) + 2\overline{c_1} z + c_1 H'(z)\}, \quad z \in \Omega_0.
\end{aligned}
$$

Expressions (4.8)–(4.9) give the exact solution for the elastic fields of an arbitrary inclusion in an elastic half-plane subjected to linear eignstrains. It is apparent that in contrast to the results given in Section 4.2, the internal stresses within the arbitrary inclusion in a half-plane depends on $P(z)$, $H(z)$ and $D(z)$. Moreover, comparing (4.8)–(4.9) with (4.4)–(4.5) reveals that the last term on the right hand side of (4.8) and the first term on the right hand side of (4.9) represent the effect of the free surface on the lower half plane.

4.4. Rigid upper half-plane

Finally, if the upper half-plane is rigid (i.e. $\mu_2 = \infty$)the expressions for the elastic field within the inclusion and the surrounding material subject to linear eigenstrains take the form

$$
\begin{aligned}
(4.10)\qquad f_0(z) = &-\mu_1[\overline{c_0}z + c_0 P(z) + c_2 z P(z) + \overline{c_1}z^2 + c_1 H(z)]\\
&- \mu_1\{\overline{c_0}[\overline{D}(z) - \overline{P}(z)] + c_2 z[\overline{D}(z) - \overline{P}(z)] + \overline{c_1}[\overline{D}(z)\overline{D}(z) - \overline{H}(z)]\}, \quad z \in \Omega_0
\end{aligned}
$$

and

$$
\begin{aligned}
(4.11)\qquad f_1(z) = \mu_1\{&c_0[D(z) - P(z)] + c_2 z[D(z) - P(z)]\\
&+ c_1[D(z)D(z) - H(z)]\} - \mu_1\{\overline{c_0}[\overline{D}(z) - \overline{P}(z)]\\
&+ c_2 z[\overline{D}(z) - \overline{P}(z)] + \overline{c_1}[\overline{D}(z)\overline{D}(z) - \overline{H}(z)]\}, \quad z \in \Omega_1
\end{aligned}
$$

where the last terms on the right hand side represent the effect of the rigid upper half-plane. The solutions are based on the auxiliary function $D(z)$ as well as the associated polynomials $P(z)$ and $H(z)$.

5. Effect of imperfect bonding: an example

The aim of the present study is to investigate the effects of imperfect bonding and linear eigenstrains on the stress fields in two imperfectly bonded elastic half-planes containing an arbitrary shaped inclusion. For the sake of illustration let us consider a circular inclusion of radius R centred at the point $y = y_0 < 0$, $(x_1 = 0)$ and is subjected to the following linear eigenstrains

$$\begin{aligned} \epsilon_{13}^* &= 0.001 + 0.002x_1 + 0.0005x_2, \\ \epsilon_{23}^* &= 0.00067 + 0.0005x_1 + 0.001x_2. \end{aligned} \tag{5.1}$$

In addition, the required functions to describe the circular inclusion are given by:

$$D(z) = \frac{R^2}{z - iyo} - iyo, \qquad P(z) = -iyo, \qquad H(z) = -yo^2. \tag{5.2}$$

Since the stress fields are directly related to the derivative of $\Delta(z)$ let us express the latter directly. To this end, first write (3.10) in the form

$$\begin{aligned} &\big[\Delta(z) + \mu_1[\overline{c_0}z + c_0P(z) + c_2zP(z) + \overline{c_1}z^2 + c_1H(z)]\big]'' \\ &\quad - \alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)\big[\Delta(z) + \mu_1[\overline{c_0}z + c_0P(z) \\ &\quad + c_2zP(z) + \overline{c_1}z^2 + c_1H(z)]\big]' = G_o'(z), \quad x_2 < 0. \end{aligned} \tag{5.3}$$

In a similar way to what has been shown before the solution to (5.3) is given by

$$\begin{aligned} \Delta'(z) &= -\mu_1[\overline{c_0}z + c_0P(z) + c_2zP(z) + \overline{c_1}z^2 + c_1H(z)]' \\ &\quad + \exp\left[\alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)z\right] \\ &\quad \times \left[\int_{-i\infty}^{z} G_0'(t)\exp\left[-\alpha i\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right)t\right]\mathrm{d}t\right], \quad x_2 < 0, \end{aligned} \tag{5.4}$$

where $G_0(t)$ is defined as before. In the particular case of a circular inclusion $G_0(t)$ takes the following form

$$\begin{aligned} G_0(t) &= \mu_1\left\{-\frac{R^2}{(t + iyo)^2}[\overline{c_0} + c_2t + 2iyo\overline{c_1}] + \frac{c_2R^2}{(t + iyo)} - \frac{2\overline{c_1}R^4}{(t + iyo)^3}\right\} \\ &\quad - i\alpha\mu_1\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)\left\{\frac{R^2}{(t + iyo)}[\overline{c_0} + c_2t + 2iyo\overline{c_1}] + \frac{\overline{c_1}R^4}{(t + iyo)^2}\right\}. \end{aligned} \tag{5.5}$$

In many practical situations such as passivated interconnect lines it is the internal stress in the inclusion that is important. Thus, to judge the impact of the imperfect interface on the stress fields within the inclusion let us consider an upper layer that is both soft and hard relative to the bottom layer.

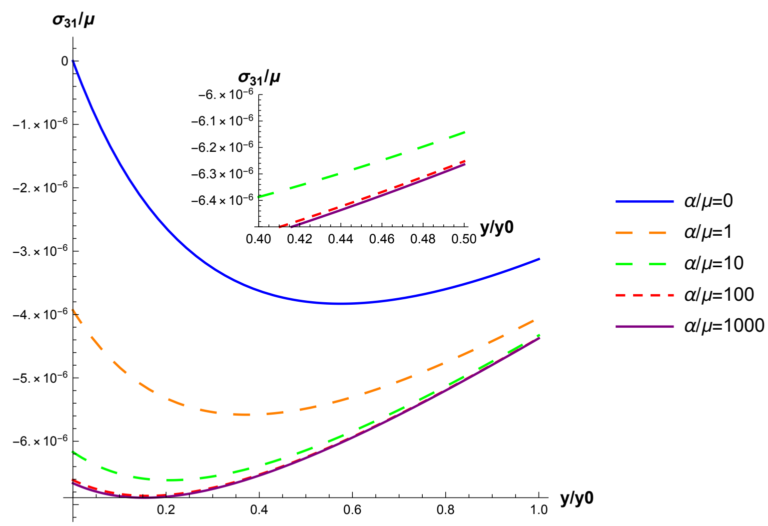

Fig. 1. Distribution of the shear stress σ_{31} along the x_2 axis for various imperfect interfaces with $\mu_2 = 0.50\mu_1$ and $R = -y_0 (y_0 < 0)$.

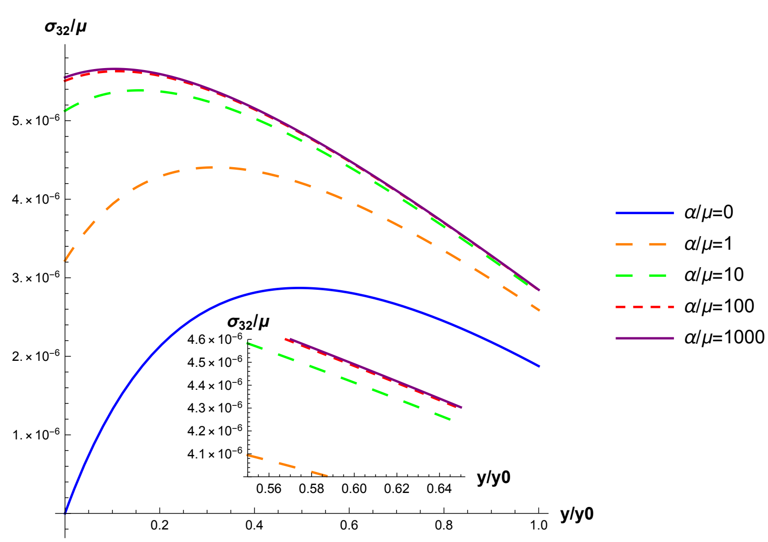

Fig. 2. Distribution of the shear stress σ_{32} along the x_2 axis for various imperfect interfaces with $\mu_2 = 0.50\mu_1$ and $R = -y_0 (y_0 < 0)$.

Solving Eq. (5.4) for the shear stresses distributed along the x_2 axis for various imperfect interface combinations corresponding to a soft and stiff upper layer, respectively with $R = -y_0$ ($y_0 < 0$) (i.e. the circular inclusion is just in contact with the real axis) is illustrated in Figs. 1–4. The results clearly demonstrate

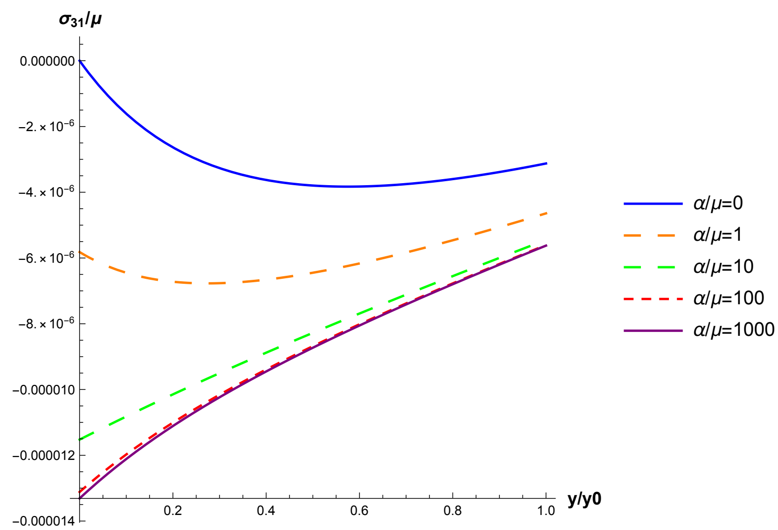

Fig. 3. Distribution of the shear stress σ_{31} along the x_2 axis for various imperfect interfaces with $\mu_2 = 2\mu_1$ and $R = -y_0(y_0 < 0)$.

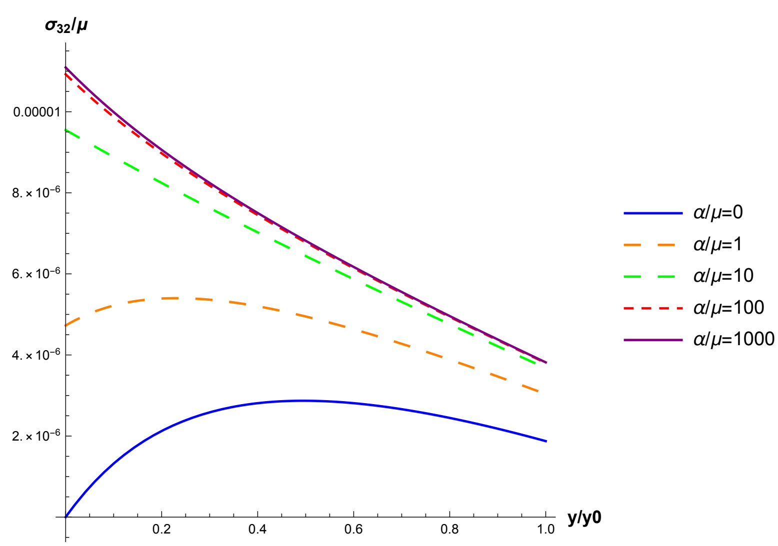

Fig. 4. Distribution of the shear stress σ_{32} along the x_2 axis for various imperfect interfaces with $\mu_2 = 2\mu_1$ and $R = -y_0(y_0 < 0)$.

that the imperfect interface in conjunction with a linear eigenstrain distribution has a significant effect on the induced stress field within the inclusion and show a strong and non-uniform behaviour. Moreover, the limiting cases of perfect bonding (i.e. $\alpha/\mu = \infty$) and the traction free surface (i.e. $\alpha/\mu = 0$), a pronounced non-uniform effect on the individual stress components is also observed in all four figures. These results are of major interest to the design of electronic devices.

6. Conclusions

An analytic solution for the antiplane elastic fields of an arbitrary shaped inclusion in one of two imperfectly joined dissimilar elastic half-planes is developed. The boundary value problem is reduced to a nonhomogeneous first order linear ordinary differential equation for an analytic function defined in the entire lower half-plane including the inclusion. The exact general solution is derived in terms of the homogeneous imperfect interface parameter as well as the auxiliary function which is constructed from the mapping function associated with the boundary curve of the inclusion. In particular, for the case of a circular inclusion, the general solution admits a very simple form and the results convincingly demonstrate that the imperfect interface in conjunction with a linear eigenstrain distribution has a significant effect on the induced stress field within the inclusion and show a strong and non-uniform behaviour especially when the inclusion is near the imperfect interface. Existing solutions for a homogeneous perfectly bonded bimaterial, two identical perfectly bonded half-planes, a traction free half-plane and a rigid upper half-plane are obtained as limiting cases and verified with existing solutions.

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References

1. K. Aderogeba, *On eigenstresses in dissimilar media*, The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics, **35**, 281–292, 1977.
2. Y. Chen, *Closed-form solution for eshelby's elliptic inclusion in antiplane elasticity*, Zeitschrift fur angewandte Mathematik und Physik ZAMP, **64**, 1797–1805, 2013.
3. Y. Chiu, *On the stress field due to initial stresses in a cuboid surrounded by an infinite elastic space*, ASME Journal of Applied Mechanics, **44**, 4, 587–590, 1977.
4. A. England, *Complex Variable Methods in Elasticity*, Wiley-Interscience, London, 1971.

5. J.D. ESHELBY, *The determination of the elastic field of an ellipsoidal inclusion, and related problems*, Proceedings of the Royal Society, London, A **241**, 5, 376–396, 1957.

6. L. FREUND, *The mechanics of electronic materials*, Zeitschrift fur angewandte Mathematik and Physik ZAMP, **37**, 185–196, 2000.

7. X. GAO, M. LIU, *Strain gradient solution for the eshelby-type polyhedral inclusion problem*, Journal of Mechanics and Physics of Solids, **60**, 2, 261–276, 2012.

8. F. GLAS, *Analytical calculation of the strain field of single and periodic misfitting polygonal wires in a half-space*, The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics, A **82**, 13, 2591–2608, 2001.

9. F. GLAS, *Elastic relaxation of truncated pyramidal quantum dots and quantum wires in a half space: an analytical calculation*, Journal of Applied Physics, **90**, 7, 3232, 2001.

10. F. GLAS, *Elastic relaxation of a truncated circular cylinder with uniform dilatational eigenstrains in a half space*, Physica Status Solidi, (b) **237**, 2, 599–610, 2003.

11. M. KAMALI, H. SHODJA, N. MASOUDVAZIRI, *A screw dislocation near a damaged arbitrary inhomogeneity-matrix interface*, International Journal of Damage Mechanics, https://doi.org/10.1177/1056789519842371, 2019.

12. Y. LEE, W. ZOU, H. REN, *Eshelby's problem of inclusion with arbitrary shape in an isotropic elastic hafl-plane*, International Journal of Solids and Structures, **81**, 399–410, 2016.

13. S. LIU, X. JIN, Z. WANG, L. KEER, Q. WANG, *Analytical solution for elastic fields caused by eigenstrains in a half-space and numerical implementation based on fft*, International Journal of Plasticity, **35**, 135–154, 2012.

14. G. NIE, C. CHAN, F. SHIN, *Non-unifom eigenstrain induced stress field in an elliptic inhomogeneity embedded in orthotropic media with complex roots*, International Journal of Solids and Structures, **44**, 3573–3593, 2007.

15. H. NOZAKI, M. TAYA, *Elastic fields in a polyhedral inclusion with uniform eigenstrain and related problems*, ASME Journal of Applied Mechanics, **68**, 3, 441–452, 2001.

16. M. RAHMAN, *The isotropic ellipsoidal inclusion with a polynomial distribution of eigenstrain*, ASME Journal of Applied Mechanics, **69**, 593–601, 2002.

17. C. RU, *Analytic solution for eshelby's problem of an incluison of arbitrary shape in an plane or half-plane*, ASME Journal of Applied Mechanics, **66**, 315–322, 1999.

18. C. RU, P. SCHIAVONE, A. MIODUCHOWSKI, *Elastic fields in two jointed half-planes with an inclusion of arbitrary shape*, Zeitschrift fur angewandte Mathematik and Physik ZAMP, **52**, 18–32, 2001.

19. P. SHARMA, R. SHARMA, *On the Eshelby's inclusion problem for ellipsoids with nonuniform dilatational gaussian and exponential eignestrains*, ASME Journal of Applied Mechanics, **70**, 418–425, 2003.

20. H. SHODJA, B. SHOKROLAHI-ZADEH, *Ellipsoidal domains: Piecewise nonuniform and impotent eigenstrain fields*, Journal of Elasticity, **86**, 1–18, 2007.

21. O. STRACK, A. VERRUIJT, *A complex variable solution for a deforming buoyant tunnel in a heavy elastic half-plane*, International Journal for Numerical and Analitical Methods in Geomechanics, **26**, 12, 1235–1252, 2002.

22. L. Sudak, C. Ru, P. Schiavone, A. Mioduchowski, *A circular inclusion with inhomogeneously imperfect inteface in plane elasticity*, Journal of Elasticity, **55**, 19–41, 1999.

23. L. Sudak, X. Wang, *An irregular-shaped inclusion with imperfect interface in antiplane elasticity*, Acta Mechanica, **224**, 9, 2009–2023, 2013.

24. Y. Sun, Y. Peng, *Analytic solution for the problems of an inclusion of arbitrary shape embedded in half plane*, Applied Mathematics of Computation, **140**, 1, 105–113, 2003.

25. S. Trotta, F. Marmo, L. Rosati, *Evaluation of the Eshelby tensor for polygonal inclusions*, Composite Part B: Engineering, **115**, 170–185, 2017.

26. X. Wang, L. Sudak, C. Ru, *Elastic fields in two imperfectly bonded half-planes with a thermal inclusion of arbitrary shape*, Zeitschrift fur angewandte Mathematik and Physik ZAMP, **58**, 488–509, 2007.

27. Z. Wang, H. Yu, Q. Wang, *Analytical solutions for elastic fields caused by eigenstrains in two jointed and perfectly bonded half-spaces and related problems*, International Journal of Plasticity, **76**, 1–28, 2016.

28. L. Wu, *The elastic field induced by a hemisphere inclusion in the half space*, Acta Mechanica Sinica, **19**, 3, 925–932, 2003.

29. H. Yu, S. Sanday, *Elastic field in jointed semi-infinite solids with an inclusion*, Proceedings of the Royal Society of London, A. **434**, 521–530, 1991.

30. K. Zhou, K. Hoh, X. Wang, L.M. Keer, J.H.L. Pang, B. Song, Q.J. Wang, *A review of recent works on inclusions*, Mechanics of Materials, **60**, 144–158, 2013.

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