ARCHIVES OF MECHANICS
Arch. Mech. **71** (4–5), 433–457, 2019, DOI: 10.24423/aom.3145

Probabilistic solutions of a stretched beam discretized with finite difference scheme and excited by Kanai–Tajimi ground motion

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The probabilistic solutions of the elastic stretched beam are studied under the excitation of Kanai–Tajimi ground motion. Finite difference scheme is adopted to formulate the nonlinear multi-degree-of-freedom system about the random vibration of the beam. The state-space-split is employed to make the high-dimensional Fokker–Planck–Kolmogorov equation reduced to 4-dimensional Fokker–Planck–Kolmogorov equations which are solved by the exponential polynomial closure method for the probabilistic solutions of the system responses. The rules for selecting the state variables are proposed in order to reduce the dimensionality of Fokker–Planck–Kolmogorov equation by the state-space-split method. The numerical results obtained by the state-space-split and exponential polynomial closure method, Monte Carlo simulation method, and equivalent linearization method are presented and compared to show the computational efficiency and numerical accuracy of the state-space-split and exponential polynomial closure method in analyzing the probabilistic solutions of the strongly nonlinear stretched beam systems formulated by a finite difference scheme and excited by the Kanai–Tajimi ground motion.

Key words: stretched beam, nonlinear random vibration, FPK equation, Kanai–Tajimi ground motion, finite difference scheme.

1. Introduction

There are many results about the vibrations of the hinged and elastic stretched beam because this beam can find its applications in many areas of science and engineering. However, the nonlinear random vibrations or the probabilistic solutions of the beam were seldom studied when the beam is modeled as multi-degree-of-freedom (MDOF) system. The hinged, axially restrained and elastic stretched beam was modeled as MDOF system with Galerkin's method and then an expression for the joint probability-density function (PDF) of the first N modal amplitudes was obtained under the excitation of uniformly distributed Gaussian white noise which is assumed to be uncorrelated in space [1]. If the Gaussian white noise is correlated or fully correlated in space as usual in applications, the joint PDF of the first N modal amplitudes is not obtainable because of the difficulties in solving the relevant Fokker–Planck–Kolmogorov (FPK) equation in this case. The technique of equivalent linearization was used to investigate the mean square responses of a simply supported Bernoulli–Euler beam undergoing moderately large random vibrations [2]. An improved stochastic linearization method was employed to investigate the modal amplitude mean-square of the stretched beam when the beam is modeled as MDOF system with Galerkin's method [3]. The probabilistic solutions of the deflection of the stretched beam was studied when the beam is modeled as MDOF system with Galerkin's method and excited by filtered Gaussian white noise [4]. The probabilistic solutions of the deflection of the stretched beam has been studied recently when the beam is modeled as MDOF system by a finite difference scheme and excited by Gaussian white noise [5].

Besides the beam systems, many other nonlinear systems in science and engineering can be modeled as nonlinear stochastic dynamical (NSD) systems with multiple degrees of freedom and excited by random noise [6–8]. In the case of Gaussian white noise or filtered Gaussian white noise, the probabilistic solution of the system is governed by Fokker-Planck-Kolmogorov (FPK) equation [6–10]. However, the analysis on the probabilistic solutions of MDOF-NSD systems or high-dimensional FPK equation is a challenge because of the high dimensionality [11, 12], especially for the systems with strongly coupled state variables, strong nonlinearity or many nonlinear terms.

Some methods were developed or extended for obtaining the approximate probabilistic solutions of NSD systems, such as the Wiener path integral method [13–16] and path integral method [17–19], stochastic averaging method [20, 21], perturbation method [22], A-type Gram-Charlier series or Hermite-polynomial closure method [23], C-type Gram-Charlier series method [24], equivalent nonlinear system method [25], dissipation energy balancing method [26], maximum entropy method [27], finite difference method [28], finite element method [29], and exponential polynomial closure (EPC) method [30]. It is known that most of these methods only work for analyzing the single-degree-of-freedom (SDOF) systems or solving the 2-dimensional FPK equations in steady state under some conditions. Few methods, such as the EPC method, can work for solving the 4-dimensional FPK equations corresponding to the 2DOF systems with polynomial or power-type nonlinearity [31]. Some of the methods, such as the Hermite-polynomial closure method, may also suffer from the loss of accuracy of the PDF in the tail regions which play an important role in system reliability analysis. There are three methods popularly used to study the MDOF-NSD systems. One of the them is the equivalent linearization (EQL) method which was first proposed by Booton [32] in the research on nonlinear random vibration of circuit and further investigated by many other researchers thereafter [33–35]. It is well known that the EQL method is only suitable for the weakly nonlinear systems under Gaussian excitation because the system responses are assumed to be close to Gaussian. The second is the Monte Carlo simulation (MCS) method which was first proposed by Metropolis and Ulam [36] in their research on nuclear physics and further studied by many other researchers in science, engineering, and mathematics thereafter [37–39]. There are some challenges with MCS method in analyzing the large strongly nonlinear stochastic dynamical systems, such as the problems of round-off error, numerical stability, convergence, and requirement for a huge sample size when the small probability of response is concerned. The third is the cumulant-neglect closure method. With this method, the moment equations derived from FPK equation were solved to obtain statistical moments by introducing the cumulant-neglect closure to moment equations to make the hierarchy of moment equations closed to a desired level [40–43]. The moments obtained with this method are exact for linear systems excited by additive and multiplicative white noises and appear to be accurate for weakly nonlinear systems excited by additive white noises.

In order to solve the high-dimensional FPK equations corresponding to MDOF-NSD systems, the EPC method was proposed and studied in the past two decades for solving the FPK equations, which works well for the systems with 1 or 2 degrees of freedom or for the 2 or 4-dimensional FPK equations [30, 31]. Various 2-dimensional and 4-dimensional FPK equations corresponding to SDOF and 2-DOF systems with polynomial-type strong nonlinearity were analyzed accurately with the EPC method. The accuracy of the PDFs obtained by the EPC method were verified by MCS. The EPC method was well extended for analyzing more general problems [44]. In 2011, a new method named state-space-split (SSS) method was proposed for the probabilistic solutions of some large MDOF-NSD systems by solving the relevant FPK equations in high dimensionality [45, 46]. It was extended or applied thereafter for analyzing some strongly nonlinear systems excited by Gaussian white noise, Poissonian white noise or colored noise being filtered Gaussian white noise [4, 47–52]. By the SSS method, a high-dimensional FPK equation can be reduced to low-dimensional FPK equations which are solvable by EPC method even if the system nonlinearity is strong.

The high-dimensional nonlinear beam systems analyzed by the SSS-EPC method in the past were formulated by Galerkin's method [4] or the excitation is Gaussian white noise when the nonlinear beam system is formulated by finite difference scheme [5]. In this paper, the SSS-EPC method is extended to study the probabilistic solutions of the stretched beam systems formulated by a finite difference scheme and excited by uniformly or locally distributed filtered Gaussian white noise. The equation of motion of the beam is a strongly nonlinear partial differential equation in time and space. A finite difference scheme is employed to make the nonlinear partial differential equation reduced to a 10 or 11-degree-of-freedom system though larger system can also be formulated and analyzed without difficulty. Including the equation about ground motion, the total number of degrees of freedom of the formulated strongly nonlinear system is 11 or 12. The dimensionality of relevant FPK equation corresponding to the system is 22 or 24. The results obtained by SSS-EPC method are compared with those obtained by EQL method and MCS to show the computational efficiency and numerical accuracy of SSS-EPC method in analyzing the probabilistic solutions of the MDOF-NSD systems formulated by a finite difference scheme and excited by the uniformly or locally distributed filtered Gaussian white noise describing the excitations of Kanai–Tajimi Ground Motion. The MCS is conducted on the original MDOF-NSD systems formulated by a central finite difference scheme.

2. Nonlinear stochastic dynamical system of the stretched beam

Consider the Euler–Bernoulli beam with pin supports on its two ends and excited by a distributed random force being filtered Gaussian white noise as shown in Fig. 1.

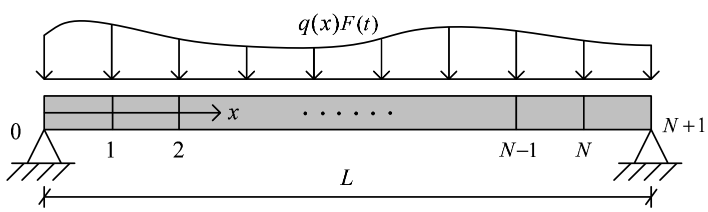


Fig. 1. Finite difference model of the stretched beam under distributed random force.

The governing equation of motion of this beam is

$$\rho A\ddot{Y}(x,t) + c\dot{Y}(x,t) + EIY^{(4)}(x,t) - \frac{EA}{2L}Y''(x,t)\int_0^L Y'^2(x,t)dx = q(x)F(t) \tag{2.1}$$

where $Y(x,t)$ is the deflection of the beam at time t and the location with distance x to the left-hand side of the beam; $\dot{Y} = \frac{dY}{dt}$; $\ddot{Y} = \frac{d^2Y}{dt^2}$; $Y' = \frac{dY}{dx}$; $Y'' = \frac{d^2Y}{dx^2}$; $Y^{(4)} = \frac{d^4Y}{dx^4}$; ρ is the mass density of the material; c is the damping constant; E is Young's modulus of the beam material; I is the moment inertia of the cross section of the beam; A is the area of the cross section of the beam; L is the length of the beam; $q(x)$ is the mass distribution on the beam and $F(t)$

is ground acceleration in the direction perpendicular to the beam axis. $F(t)$ is described by the Kanai–Tajimi model given by [53, 54]

$$F(t) = \omega_g^2 U(t) + 2\xi_g \omega_g \dot{U}(t), \tag{2.2}$$

$$\ddot{U}(t) + 2\xi_g \omega_g \dot{U}(t) + \omega_g^2 U(t) = W(t), \tag{2.3}$$

in which ξ_g is the damping constant in ground motion; ω_g represents the dominant ground frequency; $W(t)$ is Gaussian white noise with zero mean and autocorrelation $E[W(t)W(t+\tau)] = S\delta(\tau)$ in which $\delta(\tau)$ is Dirac's delta function and S is a constant representing the power spectral density (PSD) of $W(t)$ or the intensity of ground acceleration.

3. Multi-degree-of-freedom nonlinear stochastic dynamical system of the stretched beam formulated with finite difference scheme

With a central finite difference scheme as shown in Fig. 1 and the following approximations of the derivatives

$$\begin{aligned} Y_n' &= \frac{dY}{dx}\Big|_{x=x_n} = \frac{Y_{n+1} - Y_{n-1}}{2h}, \\ Y_n'' &= \frac{d^2Y}{dx^2}\Big|_{x=x_n} = \frac{Y_{n+1} - 2Y_n + Y_{n-1}}{h^2}, \\ Y_n^{(4)} &= \frac{d^4Y}{dx^4}\Big|_{x=x_n} = \frac{Y_{n+2} - 4Y_{n+1} + 6Y_n - 4Y_{n-1} + Y_{n-2}}{h^4}. \end{aligned} \tag{3.1}$$

Equation (2.1) can be discretized into the following system at the node n.

$$\begin{aligned} &\ddot{Y}_n + \frac{c}{\rho A}\dot{Y}_n + \alpha(Y_{n+2} - 4Y_{n+1} + 6Y_n - 4Y_{n-1} + Y_{n-2}) \\ &\quad - \beta(Y_{n+1} - 2Y_n + Y_{n-1}) \sum_{i=1}^{N+1} (Y_{i+1}^2 + Y_i^2 + Y_{i-1}^2 + Y_{i-2}^2 \\ &\quad + Y_{i+1}Y_i - 2Y_{i+1}Y_{i-1} - Y_{i+1}Y_{i-2} - Y_iY_{i-1} - 2Y_iY_{i-2} \\ &\qquad\qquad + Y_{i-1}Y_{i-2}) = \frac{q(x_n)}{\rho A} F(t) \qquad (n = 1, 2, \ldots, N), \end{aligned} \tag{3.2}$$

where Y_i is the deflection at the point i with a distance x_i to the left-hand side of the beam; $\alpha = \frac{EI}{h^4 \rho A}$; $\beta = \frac{E}{24Lh^3\rho}$; and $h = x_{i+1} - x_i$. The central finite difference scheme is adopted in formulating Eqs. (3.2) and handling the boundary conditions in the following.

For the beam with pin supports on its two ends, the boundary conditions shown in Fig. 1 can be expressed with a finite difference scheme by

$$Y_0 = 0, \quad Y_{-1} = -Y_1, \quad Y_{N+1} = 0, \quad Y_{N+2} = -Y_N. \tag{3.3}$$

Introducing the boundary conditions expressed by Eq. (3.3) to Eq. (3.2) and noting Eqs. (2.2) and (2.3), an MDOF-NSD system excited by filtered Gaussian white noise can be formulated as follows

$$(3.4)\qquad \begin{cases} \ddot{Y}_n + \dfrac{c}{\rho A}\dot{Y}_n + \alpha(Y_{n+2} - 4Y_{n+1} + 6Y_n - 4Y_{n-1} + Y_{n-2}) \\ \quad - \beta(Y_{n+1} - 2Y_n + Y_{n-1}) \sum_{i=1}^{N+1} (Y_{i+1}^2 + Y_i^2 + Y_{i-1}^2 + Y_{i-2}^2 \\ \quad + Y_{i+1}Y_i - 2Y_{i+1}Y_{i-1} - Y_{i+1}Y_{i-2} - Y_iY_{i-1} - 2Y_iY_{i-2} \\ \quad + Y_{i-1}Y_{i-2}) = \dfrac{q(x_n)}{\rho A}[\omega_g^2 U(t) + 2\xi_g\omega_g\dot{U}(t)] \quad (n = 1, 2, \ldots, N), \\ \ddot{U}(t) + 2\xi_g\omega_g\dot{U}(t) + \omega_g^2 U(t) = W(t). \end{cases}$$

The number of degrees of freedom of the system is $N + 1$.

4. Dimensionality reduction by state-space-split method

In the following discussion, the summation convention applies unless stated otherwise. The random state variables or vectors are denoted by a capital letter and the corresponding deterministic state variables or vectors are denoted by the same letter in lowercase.

The system governed by Eq. (3.4) under the boundary conditions given by Eq. (3.3) can be generally expressed by the following MDOF-NSD system.

$$(4.1)\qquad \begin{cases} \ddot{Y}_i + a\dot{Y}_i + g_i(\mathbf{Y}) - h_i[\omega_g^2 U(t) + 2\xi_g\omega_g\dot{U}(t)] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, N), \\ \ddot{U}(t) + 2\xi_g\omega_g\dot{U}(t) + \omega_g^2 U(t) = W(t), \end{cases}$$

where $Y_i \in \mathbb{R}$ $(i = 1, 2, \ldots, N)$, are the components of the vector process $\mathbf{Y} \in \mathbb{R}^N$; $g_i(\mathbf{Y})$ are the polynomial functions of $\mathbf{Y}$ and $g_i(\mathbf{Y}) : \mathbb{R}^{n_\mathbf{Y}} \to \mathbb{R}$; $a = \frac{c}{\rho A}$, $h_i = \frac{q(x_i)}{\rho A}$ are constants.

Setting

$$\begin{gathered} Y_i = X_{2i-1}, \qquad \dot{Y}_i = X_{2i}, \qquad f_{2i-1} = X_{2i}, \\ f_{2i} = h_i[\omega_g^2 X_{2N+1} + 2\xi_g\omega_g X_{2(N+1)}] - 2\xi\omega_1 X_{2i} - g_i(\mathbf{X}), \quad (i = 1, 2, \ldots, N), \\ U = X_{2N+1}, \qquad \dot{U} = X_{2(N+1)}, \qquad f_{2N+1} = X_{2(N+1)}, \\ f_{2(N+1)} = -\omega_g^2 X_{2N+1} - 2\xi_g\omega_g\dot{X}_{2N+2}, \qquad n_\mathbf{x} = 2(N + 1), \end{gathered}$$

then Eq. (4.1) can be expressed by the following coupled Langevin equations or

Ito differential equations.

$$\tag{4.2} \begin{cases} \dfrac{dX_i}{dt} = f_i(\mathbf{X}) \quad i = 1, 2, \ldots, n_{\mathbf{x}} - 1, \\ \dfrac{dX_{n_{\mathbf{x}}}}{dt} = f_{n_{\mathbf{x}}}(\mathbf{X}) + W(t), \end{cases}$$

where $\mathbf{X} \in \mathbb{R}^{n_{\mathbf{x}}}$; X_i $(i = 1, 2, \ldots, n_{\mathbf{x}})$, are the components of the state vector process $\mathbf{X}$; $f_i(\mathbf{X}) : \mathbb{R}^{n_{\mathbf{x}}} \to \mathbb{R}$.

The state vector process $\mathbf{X}$ is Markovian and the PDF $p(\mathbf{x}, t)$ of the Markovian vector is governed by the FPK equation. Because the white noise $W(t)$ is Gaussian, the stationary PDF $p(\mathbf{x})$ of the Markovian vector is governed by the following reduced FPK equation [6]:

$$\tag{4.3} \frac{\partial}{\partial x_j}[f_j(\mathbf{x})p(\mathbf{x})] - \frac{S}{2}\frac{\partial^2 p(\mathbf{x})}{\partial x_{n_{\mathbf{x}}}^2} = 0,$$

where $\mathbf{x}$ is the deterministic state vector and $\mathbf{x} \in \mathbb{R}^{n_{\mathbf{x}}}$.

It is assumed that the solution to Eq. (4.3) fulfills the following conditions:

$$\tag{4.4} \lim_{x_j \to \pm\infty} f_j(\mathbf{x})p(\mathbf{x}) = 0 \quad \text{and} \quad \lim_{x_j \to \pm\infty} \frac{\partial p(\mathbf{x})}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n_{\mathbf{x}},$$

which can be fulfilled by the deflection and the velocity of the beam.

Separate the state vector $\mathbf{X}$ into two parts as $\mathbf{X}_1 \in \mathbb{R}^{n_{\mathbf{x}1}}$ which is referred to as the first subspace and $\mathbf{X}_2 \in \mathbb{R}^{n_{\mathbf{x}2}}$ which is referred to as the second subspace, i.e., $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2\} \in \mathbb{R}^{n_{\mathbf{x}}} = \mathbb{R}^{n_{\mathbf{x}1}} \times \mathbb{R}^{n_{\mathbf{x}2}}$. The state variables in $\mathbf{X}_1$ are referred to as target state variables which PDF is desired. In analyzing the above beam system, define the vector $\mathbf{X}_1$ such that $\mathbf{X}_1 = \{Y_i(t), \dot{Y}_i(t), U(t), \dot{U}(t)\} = \{X_{2i-1}(t), X_{2i}(t), X_{n_{\mathbf{x}}-1}(t), \dot{X}_{n_{\mathbf{x}}}(t)\}$, $i \in [1, N]$. The PDF of $\mathbf{X}_1$ is analyzed in the following with the SSS-EPC method [45, 46].

Denote the PDF of $\mathbf{X}_1$ as $p_1(\mathbf{x}_1)$. In order to obtain $p_1(\mathbf{x}_1)$, integrating both sides of Eq. (4.3) over $\mathbb{R}^{n_{\mathbf{x}2}}$ gives

$$\tag{4.5} \int\limits_{\mathbb{R}^{n_{\mathbf{x}2}}} \frac{\partial}{\partial x_j}[f_j(\mathbf{x})p(\mathbf{x})]\, d\mathbf{x}_2 - \frac{S}{2} \int\limits_{\mathbb{R}^{n_{\mathbf{x}2}}} \frac{\partial^2 p(\mathbf{x})}{\partial x_{n_{\mathbf{x}}}^2} d\mathbf{x}_2 = 0.$$

Because of the conditions in Eq. (4.4), we have

$$\tag{4.6} \int\limits_{\mathbb{R}^{n_{\mathbf{x}2}}} \frac{\partial}{\partial x_j}[f_j(\mathbf{x})p(\mathbf{x})] d\mathbf{x}_2 = 0 \qquad x_j \in \mathbb{R}^{n_{\mathbf{x}2}}.$$

Equation (4.5) can then be written after integration by part as

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right]d\mathbf{x}_2 - \frac{S}{2}\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial^2 p(\mathbf{x})}{\partial x_{n_\mathbf{x}}^2}d\mathbf{x}_2 = 0, \quad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}, \tag{4.7}$$

which can be equivalently written as

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right]d\mathbf{x}_2 - \frac{S}{2}\frac{\partial^2}{\partial x_{n_\mathbf{x}}^2}\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} p(\mathbf{x})d\mathbf{x}_2 = 0, \quad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{4.8}$$

Since

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} p(\mathbf{x})d\mathbf{x}_2 = p(\mathbf{x}_1) \tag{4.9}$$

Eq. (4.8) can be rewritten as

$$\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right]d\mathbf{x}_2 - \frac{S}{2}\frac{\partial^2 p(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \quad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}} \tag{4.10}$$

which can be written equivalently as

$$\frac{\partial}{\partial x_j}\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} f_j(\mathbf{x})p(\mathbf{x})d\mathbf{x}_2 - \frac{S}{2}\frac{\partial^2 p(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \quad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{4.11}$$

Collecting the terms purely in $\mathbf{x}_1$ in one part and the other terms in the other part. Then $f_j(\mathbf{x})$ is expressed in terms of two parts as

$$f_j(\mathbf{x}) = f_j^I(\mathbf{x}_1) + f_j^{II}(\mathbf{x}). \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.11) gives

$$\frac{\partial}{\partial x_j}\int_{\mathbb{R}^{n_{\mathbf{x}_2}}} \left[f_j^I(\mathbf{x}_1) + f_j^{II}(\mathbf{x})\right]p(\mathbf{x})d\mathbf{x}_2 - \frac{S}{2}\frac{\partial^2 p(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \quad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{4.13}$$

Noting Eq. (4.9), Eq. (4.13) can be written as

$$\frac{\partial}{\partial x_j}\left[f_j^I(\mathbf{x}_1)p(\mathbf{x}_1) + \int_{\mathbb{R}^{n_{\mathbf{x}_2}}} f_j^{II}(\mathbf{x})p(\mathbf{x})d\mathbf{x}_2\right] - \frac{S}{2}\frac{\partial^2 p(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \quad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{4.14}$$

Express $f_j^{\mathrm{II}}(\mathbf{x})$ as $\sum_k f_j^{\mathrm{II}}(\mathbf{x}_1, \mathbf{z}_k)$ in which $\mathbf{z}_k \in \mathbb{R}^{n_{\mathbf{z}_k}} \subset \mathbb{R}^{n_{\mathbf{x}_2}}$ and $n_{\mathbf{z}_k}$ denotes the number of the state variables in $\mathbf{z}_k$. For real problems, $n_{\mathbf{z}_k} \ll n_{\mathbf{x}_2}$. For the system expressed by Eq. (3.4), (4.1) or (4.2), $n_{\mathbf{z}_k} = 3$. Then Eq. (4.14) can be written as

$$\frac{\partial}{\partial x_j}\left[f_j^I(\mathbf{x}_1)p_1(\mathbf{x}_1) + \sum_k \int\limits_{\mathbb{R}^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1, \mathbf{z}_k)p_k(\mathbf{x}_1, \mathbf{z}_k)d\mathbf{z}_k\right] - \frac{S}{2}\frac{\partial^2 p(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \qquad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}} \tag{4.15}$$

in which $p_k(\mathbf{x}_1, \mathbf{z}_k)$ denotes the joint PDF of $\{\mathbf{X}_1, \mathbf{Z}_k\}$. The summation convention does not apply for the indexes k in Eq. (4.15) and in the following discussions.

From Eq. (4.15), it is seen that the coupling of $\mathbf{X}_1$ and $\mathbf{X}_2$ comes from $f_j^{II}(\mathbf{x}_1, \mathbf{z}_k)p_k(\mathbf{x}_1, \mathbf{z}_k)$. Because

$$p_k(\mathbf{x}_1, \mathbf{z}_k) = p_1(\mathbf{x}_1)q_k(\mathbf{z}_k; \mathbf{x}_1), \tag{4.16}$$

where $q_k(\mathbf{z}_k; \mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_k$ for given $\mathbf{X}_1 = \mathbf{x}_1$, substituting Eq. (4.16) into Eq. (4.15) gives

$$\frac{\partial}{\partial x_j}\left\{\left[f_j^I(\mathbf{x}_1) + \sum_k \int\limits_{\mathbb{R}^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1, \mathbf{z}_k)q_k(\mathbf{z}_k; \mathbf{x}_1)d\mathbf{z}_k\right]p_1(\mathbf{x}_1)\right\} - \frac{S}{2}\frac{\partial^2 p(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \qquad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}. \tag{4.17}$$

The conditional PDF $q_k(\mathbf{z}_k; \mathbf{x}_1)$ is needed in Eq. (4.17), but it is not available. A large amount of numerical analysis showed that the conditional PDF $q_k(\mathbf{z}_k; \mathbf{x}_1)$ employed in dimension reduction procedure of the SSS method can be effectively replaced by that obtained by EQL and the obtained approximate PDFs of $\mathbf{X}_1$ are accurate even if the systems are strongly nonlinear [4, 5, 45, 46, 48–50]. Approximately replacing the conditional PDF $q_k(\mathbf{z}_k; \mathbf{x}_1)$ by that obtained by EQL, then Eq. (4.17) becomes

$$\frac{\partial}{\partial x_j}\left\{\left[f_j^I(\mathbf{x}_1) + \sum_k \int\limits_{\mathbb{R}^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1, \mathbf{z}_k)\overline{q}_k(\mathbf{z}_k; \mathbf{x}_1)d\mathbf{z}_k\right]\tilde{p}_1(\mathbf{x}_1)\right\} - \frac{S}{2}\frac{\partial^2 \tilde{p}(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \qquad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}}, \tag{4.18}$$

where $\overline{q}_k(\mathbf{z}_k; \mathbf{x}_1)$ is the approximate conditional PDF of $\mathbf{Z}_k$ obtained by EQL for the given $\mathbf{X}_1 = \mathbf{x}_1$ and $\tilde{p}_1(\mathbf{x}_1)$ is then the approximate PDF of $\mathbf{X}_1$. It is noted that the approximate conditional PDF $\overline{q}_k(\mathbf{z}_k; \mathbf{x}_1)$ leads to the difference between the approximate solution $\tilde{p}_1(\mathbf{x}_1)$ and exact solution $p_1(\mathbf{x}_1)$. Denote

$$\tilde{f}_j(\mathbf{x}_1) = f_j^{\mathrm{I}}(\mathbf{x}_1) + \sum_k \int\limits_{\mathbb{R}^{n_{\mathbf{z}_k}}} f_j^{\mathrm{II}}(\mathbf{x}_1, \mathbf{z}_k)\overline{q}(\mathbf{z}_k; \mathbf{x}_1)d\mathbf{z}_k, \tag{4.19}$$

then Eq. (4.18) is finally written as

$$\frac{\partial}{\partial x_j}[\tilde{f}_j(\mathbf{x}_1)\tilde{p}_1(\mathbf{x}_1)] - \frac{S}{2}\frac{\partial^2 \tilde{p}(\mathbf{x}_1)}{\partial x_{n_\mathbf{x}}^2} = 0, \qquad x_j \in \mathbb{R}^{n_{\mathbf{x}_1}} \tag{4.20}$$

which is the approximate FPK equation for the joint PDF of the state variables in the subspace $\mathbb{R}^{n_{\mathbf{x}_1}}$. It is seen that $\mathbf{X}_1$ only contains four state variables, i.e., $\{X_{2i-1}(t), X_{2i}(t), X_{n_\mathbf{x}-1}(t), \dot{X}_{n_\mathbf{x}}(t)\}$, $i \in [1, N]$ or $\{Y_i, \dot{Y}_i, U, \dot{U}\}$ where Y_i and $\dot{Y}_i$ are the displacement and velocity, respectively, at the node i. Hence, the resulting approximate FPK equation is 4-dimensional. The EPC method can be employed to solve Eq. (4.20) in the following numerical analysis [30, 31].

5. Solution procedure of exponential polynomial closure method

The EPC solution procedure is briefed in the following. Consider the following reduced low-dimensional FPK equation.

$$\frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right] - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x})p(\mathbf{x})\right] = 0, \tag{5.1}$$

where $\mathbf{X} \in \mathbb{R}^{m_\mathbf{x}}$ with the assumption that $m_\mathbf{x}$ is small or $m_\mathbf{x} = 4$ when solving the approximate dimension-reduced FPK equation (4.20) in the random vibration analysis of the stretched beam excited by the Kanai–Tajimi ground motion.

The approximate solution $\tilde{p}(\mathbf{x}; \mathbf{a})$ of Eq. (5.1) is assumed to be

$$\tilde{p}(\mathbf{x}; \mathbf{a}) = c \exp^{Q_n(\mathbf{x};\mathbf{a})}, \tag{5.2}$$

where $\mathbf{a}$ is an unknown parameter vector, $\mathbf{a} = \{a_1, a_2, \ldots, a_{N_p}\}$, N_p is the total number of unknown parameters, and $Q_n(\mathbf{x}; \mathbf{a})$ is a n-degree polynomial in $\mathbf{x} \in \mathbb{R}^{m_\mathbf{x}}$. This replacement may cause some error in the approximate solution and hence some residual errors in the FPK equation.

Eq. (5.1) can also be written in the following form:

$$\frac{\partial f_j}{\partial x_j}p + f_j\frac{\partial p}{\partial x_j} - \frac{1}{2}\left(\frac{\partial^2 G_{ij}}{\partial x_i \partial x_j}p + \frac{\partial G_{ij}}{\partial x_j}\frac{\partial p}{\partial x_i} + \frac{\partial G_{ij}}{\partial x_i}\frac{\partial p}{\partial x_j} + G_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}\right) = 0. \tag{5.3}$$

Generally, Eq. (5.3) cannot be satisfied exactly with $\tilde{p}(\mathbf{x}; \mathbf{a})$ because $\tilde{p}(\mathbf{x}; \mathbf{a})$ is only an approximation of $p(\mathbf{x})$ and the number N_p of the unknown parameters is limited in practice. Substituting $\tilde{p}(\mathbf{x};\mathbf{a})$ for $p(\mathbf{x})$ in Eq. (5.3) leads to the following residual error.

$$(5.4)\qquad \Delta(\mathbf{x};\mathbf{a}) = \frac{\partial f_j}{\partial x_j}\tilde{p} + f_j\frac{\partial \tilde{p}}{\partial x_j} - \frac{1}{2}\bigg(\frac{\partial^2 G_{ij}}{\partial x_i \partial x_j}\tilde{p} + \frac{\partial G_{ij}}{\partial x_j}\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial G_{ij}}{\partial x_i}\frac{\partial \tilde{p}}{\partial x_j} + G_{ij}\frac{\partial^2 \tilde{p}}{\partial x_i \partial x_j}\bigg).$$

Substituting $\tilde{p}(\mathbf{x};\mathbf{a}) = c\exp^{Q_n(\mathbf{x};\mathbf{a})}$ into Eq. (5.4) gives

$$(5.5)\qquad \Delta(\mathbf{x};\mathbf{a}) = \delta(\mathbf{x};\mathbf{a})\tilde{p}(\mathbf{x};\mathbf{a}),$$

where

$$(5.6)\qquad \delta(\mathbf{x};\mathbf{a}) = f_j\frac{\partial Q_n}{\partial x_j} - \frac{1}{2}\bigg(\frac{\partial G_{ij}}{\partial x_j}\frac{\partial Q_n}{\partial x_i} + \frac{\partial G_{ij}}{\partial x_i}\frac{\partial Q_n}{\partial x_j} + G_{ij}\frac{\partial^2 Q_n}{\partial x_i \partial x_j} + G_{ij}\frac{\partial Q_n}{\partial x_i}\frac{\partial Q_n}{\partial x_j}\bigg) + \frac{\partial f_j}{\partial x_j} - \frac{1}{2}\frac{\partial^2 G_{ij}}{\partial x_i \partial x_j}.$$

Because $\tilde{p}(\mathbf{x};\mathbf{a}) \neq 0$, therefore, the only possibility for $\tilde{p}(\mathbf{x};\mathbf{a})$ to satisfy Eq. (5.3) is $\delta(\mathbf{x};\mathbf{a}) = 0$. However, usually $\delta(\mathbf{x};\mathbf{a}) \neq 0$ because $\tilde{p}(\mathbf{x};\mathbf{a})$ is only an approximation of $p(\mathbf{x})$. In this case, a set of mutually independent functions $h_k(\mathbf{x})$ which span space $\mathbb{R}^{N_p}$ are introduced to make the projection of $\delta(\mathbf{x};\mathbf{a})$ on $\mathbb{R}^{N_p}$ vanish, which leads to

$$(5.7)\qquad \int\limits_{\mathbb{R}^{m_x}} \delta(\mathbf{x};\mathbf{a})\, h_k(\mathbf{x}) d\mathbf{x} = 0, \qquad k = 1, 2, \ldots, N_p$$

or

$$(5.8)\qquad \int\limits_{\mathbb{R}^{m_x}} \bigg\{ f_j\frac{\partial Q_n}{\partial x_j} - \frac{1}{2}\bigg(\frac{\partial G_{ij}}{\partial x_j}\frac{\partial Q_n}{\partial x_i} + \frac{\partial G_{ij}}{\partial x_i}\frac{\partial Q_n}{\partial x_j} + G_{ij}\frac{\partial^2 Q_n}{\partial x_i \partial x_j} + G_{ij}\frac{\partial Q_n}{\partial x_i}\frac{\partial Q_n}{\partial x_j}\bigg) + \frac{\partial f_j}{\partial x_j} - \frac{1}{2}\frac{\partial^2 G_{ij}}{\partial x_i \partial x_j}\bigg\} h_k(\mathbf{x}) d\mathbf{x} = 0, \qquad k = 1, 2, \ldots, N_p.$$

The above Eq. (5.8) means that the reduced FPK equation is satisfied with $\tilde{p}(\mathbf{x};\mathbf{a})$ in the weak sense of integration if $\delta(\mathbf{x};\mathbf{a})\, h_k(\mathbf{x})$ is integrable in $\mathbb{R}^{m_x}$.

By selecting $h_k(\mathbf{x})$ as $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} f_N(\mathbf{x})$, being $k_1, k_2, \ldots, k_n = 0, 1, \ldots, N_p$ and $k = k_1 + k_2 + \cdots + k_n$ such that $\delta(\mathbf{x};\mathbf{a})\, h_k(\mathbf{x})$ is integrable in $\mathbb{R}^{m_x}$, N_p nonlinear algebraic equations in terms of N_p undetermined parameters can be obtained from Eq. (5.8). The algebraic equations can be solved to determine the parameters. Numerical experience shows that a convenient and effective choice for the function $f_N(\mathbf{x})$ is the PDF obtained from EQL or the Gaussian closure procedure. Hence, it is a normal PDF.

6. Numerical analysis

Example 1. Consider the stretched beam with pin supports on its two ends. The uniformly distributed load is applied over the whole length of the beam as shown by Fig. 2. The beam parameters are given by $L = 7$ m, $E = 2.1 \times 10^{11}$ pa, $I = 2.17 \times 10^{-4}$ m^4, $A = 8.6112 \times 10^{-3}$ m^2, $\rho = 7.85 \times 10^3$ kg/m^3, $c = 10^3$, and $q(x)W(t) = 5 \times 10^4 W(t)$ N/m, $S = 0.05$ m^2/s^3, $\omega_g = 50$ rad/s, $\xi_g = 0.3$. The number of nodes N in a finite difference scheme is 11.

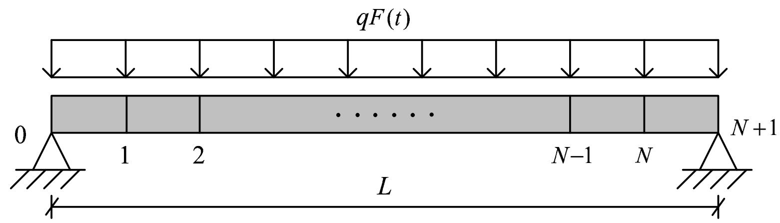


Fig. 2. Finite difference model of beam under uniformly distributed load.

The equations of motion of the system formulated by central finite difference scheme are

$$\tag{6.1} \begin{cases} \ddot{Y}_i + \dfrac{c}{\rho A}\dot{Y}_i + \alpha(Y_{i+2} - 4Y_{i+1} + 6Y_i - 4Y_{i-1} + Y_{i-2}) \\ \quad - \beta(Y_{i+1} - 2Y_i + Y_{i-1}) \sum_{m=1}^{12}(Y_{m+1}^2 + Y_m^2 + Y_{m-1}^2 + Y_{m-2}^2 \\ \quad + Y_{m+1}Y_m - 2Y_{m+1}Y_{m-1} - Y_{m+1}Y_{m-2} - Y_m Y_{m-1} - 2Y_m Y_{m-2} \\ \quad + Y_{m-1}Y_{m-2}) - \dfrac{q(x_i)}{\rho A}[\omega_g^2 U(t) + 2\xi_g\omega_g \dot{U}(t)] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, 11), \\ \ddot{U}(t) + 2\xi_g\omega_g\dot{U}(t) + \omega_g^2 U(t) = W(t), \end{cases}$$

with the boundary conditions $Y_0 = 0, Y_{-1} = -Y_1, Y_{12} = 0, Y_{13} = -Y_{11}$.

The formulated system in Eq. (6.1) is a 12-DOF system and the dimensionality of the relevant FPK equation is 24. There are filtered Gaussian white noises in all the equations of motion and the filtered Gaussian white noises are fully correlated. In order to use SSS method for dimension reduction, the state variables in the first subspace are selected to be $\mathbf{X}_1 = \{Y_i, \dot{Y}_i, U_i, \dot{U}_i\}$, $i \in [1, 11]$. After dimension reduction by the SSS method, the dimension reduced 4-dimensional FPK is solved by EPC method. Because the structure is symmetric and the loading is symmetrically distributed over the beam, the deflection is also symmetric about the middle of the beam, i.e., $Y_i = Y_{12-i}$ and $\dot{Y}_i = \dot{Y}_{12-i}$ for $i \in [1, 6]$. Therefore, only the marginal PDFs and logarithmic PDFs of Y_i for $i \in [1, 6]$ are shown in the following figures. In order to shown the accuracy of the PDFs and

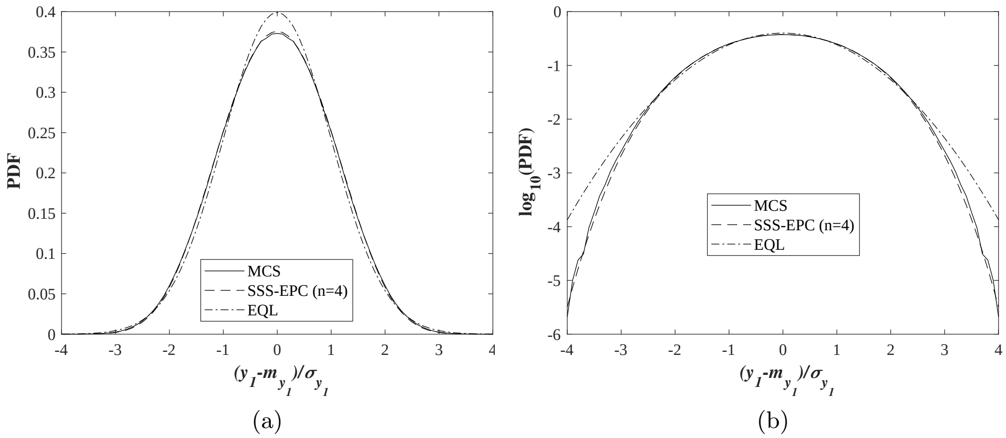


FIG. 3. (a) The PDFs of the deflection Y_1 under uniformly distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_1 under uniformly distributed load over the whole beam.

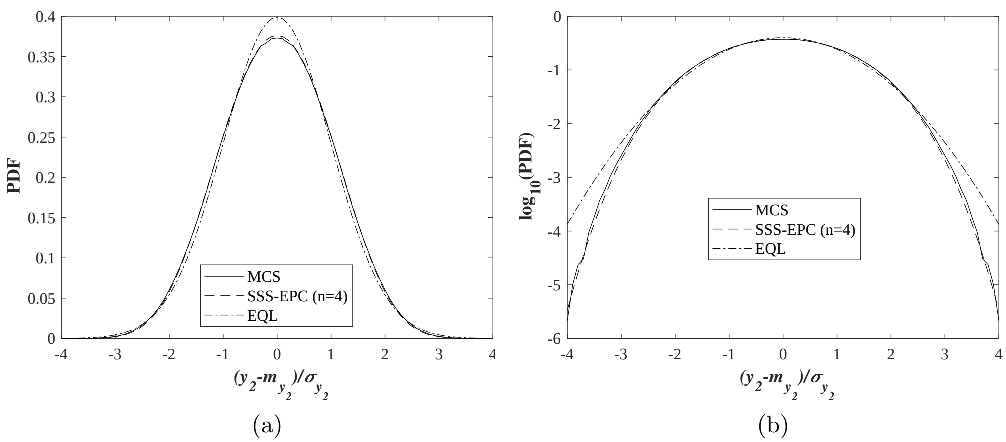
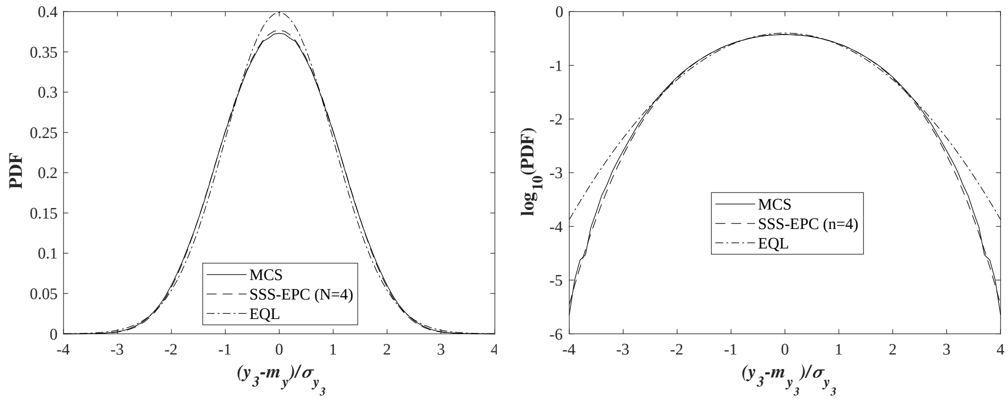


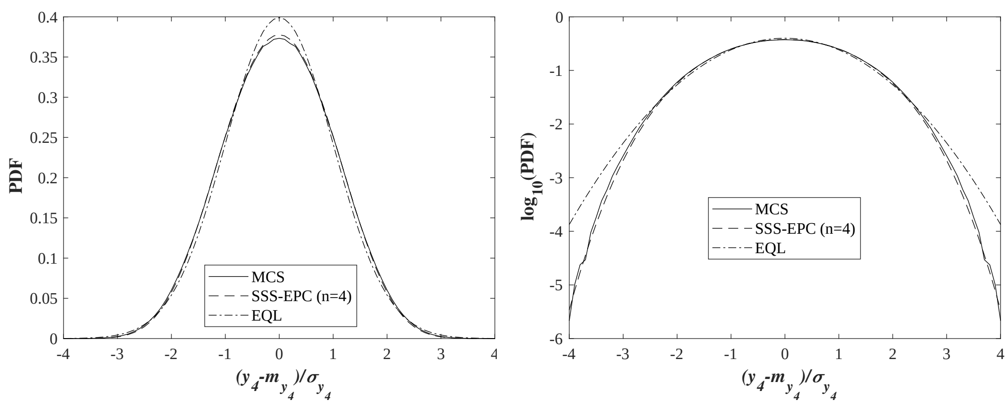
FIG. 4. (a) The PDFs of the deflection Y_2 under uniformly distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_2 under uniformly distributed load over the whole beam.

logarithmic PDFs of Y_i obtained by the SSS-EPC method, the results obtained by MCS and EQL are also shown in Figs. 3–8. For this beam system, the mean value of deflection equals zero, i.e., $m_{y_i} = 0$ and $m_{\dot{y}_i} = 0$. The standard deviation of deflection obtained by EQL in the middle of the beam is $\sigma_{y_6} = 0.0681m$. From these figures it is observed that the results obtained by the SSS-EPC method are close to those obtained by MCS. However, the computational time needed by MCS is about 500 times of that needed by the SSS-EPC method when the



(a) (b)

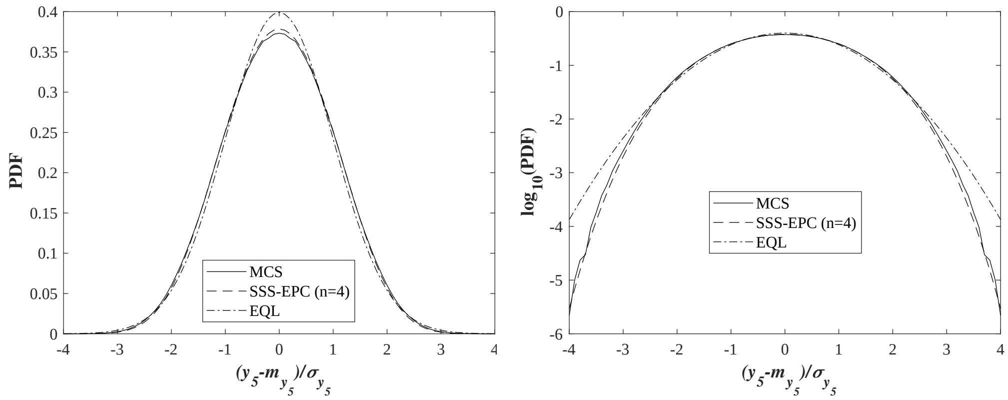
Fig. 5. (a) The PDFs of the deflection Y_3 under uniformly distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_3 under uniformly distributed load over the whole beam.



(a) (b)

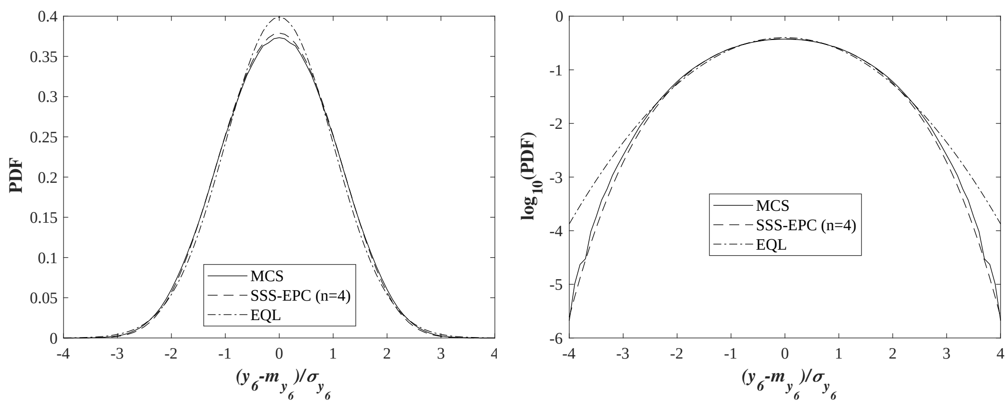
Fig. 6. (a) The PDFs of the deflection Y_4 under uniformly distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_4 under uniformly distributed load over the whole beam.

sample size used in MCS is 10^8. The computational time needed by the SSS-EPC method is mainly spent on the EQL procedure. The rate of computational time between MCS and SSS-EPC can increase quickly as the number of degrees of system freedom increases. By comparing the results obtained by the SSS-EPC method and MCS with those obtained by the EQL method, it is observed that the system nonlinearity is very strong and the results obtained by EQL is far from being acceptable.



(a) (b)

FIG. 7. (a) The PDFs of the deflection Y_5 under uniformly distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_5 under uniformly distributed load over the whole beam.



(a) (b)

FIG. 8. (a) The PDFs of the deflection Y_6 under uniformly distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_6 under uniformly distributed load over the whole beam.

All the PDFs at different nodes or beam locations are similar to each other because the rate of the nonlinear restoring force and the linear restoring force in Eq. (2.1) does not change much, which is due to the fact that $\int_0^L Y'^2(x,t)dx$ is free of x or beam location for the stretched beam. A similar behavior can also be observed in the PDF solutions of the following Example 2.

In order to check the convergence of the solution as the the nodes number N in the finite-difference gird increases or the value of h decreases, the PDFs and

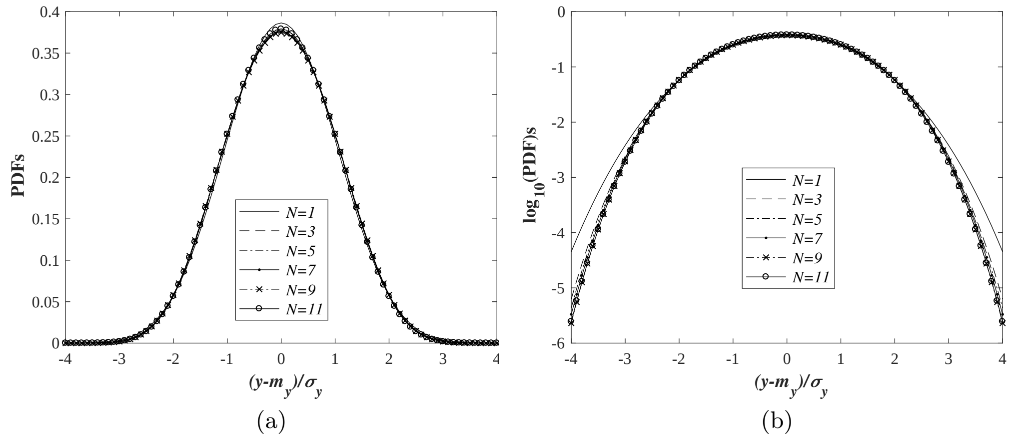


Fig. 9. (a) PDFs in the middle of the beam for different number of nodes. (b) Logarithmic PDFs in the middle of the beam for different number of nodes.

logarithmic PDFs of the deflection in the middle of beam, which are obtained by the SSS-EPC method, are shown in Fig. 9. From Fig. 9, it is seen that the PDFs and logarithmic PDFs corresponding to $N = 9$ and 11 almost overlap, which means that the solution corresponding to $N = 9$ or 11 can be considered as a converged solution for the adopted finite difference scheme. The convergence behaviors of the solutions at other nodes are similar. $N = 11$ rather than $N = 9$ is adopted in the above analysis on the PDF solutions at different nodes so that the PDF solutions at more beam locations can be obtained, shown and compared.

Example 2. Consider the stretched beam with pin supports on its two ends. The distributed load is only applied between node 5 and node 6 of the beam as shown by Fig. (10), which can be considered as a point load applied in the middle of the beam. The beam parameter values are all to be the same as those in Example 1 except $q = 2\times10^5$ is applied between middle node 5 and node 6 but $q = 0$ elsewhere. The number of unknowns N in a finite difference scheme is 10.

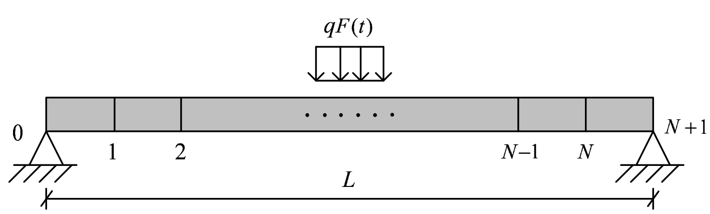


Fig. 10. Finite difference model of beam under partially distributed load in the middle of the beam.

The equations of motion of the system formulated by the central finite difference scheme are

$$
(6.2)\qquad \begin{cases}
\ddot{Y}_i + \dfrac{c}{\rho A}\dot{Y}_i + \alpha(Y_{i+2} - 4Y_{i+1} + 6Y_i - 4Y_{i-1} + Y_{i-2}) \\
\quad - \beta(Y_{i+1} - 2Y_i + Y_{i-1}) \sum_{m=1}^{11} (Y_{m+1}^2 + Y_m^2 + Y_{m-1}^2 + Y_{m-2}^2 \\
\quad + Y_{m+1}Y_m - 2Y_{m+1}Y_{m-1} - Y_{m+1}Y_{m-2} - Y_mY_{m-1} - 2Y_mY_{m-2} \\
\quad + Y_{m-1}Y_{m-2}) = 0 \quad (i = 1, 2, 3, 4, 7, 8, 9, 10), \\
\ddot{Y}_i + \dfrac{c}{\rho A}\dot{Y}_i + \alpha(Y_{i+2} - 4Y_{i+1} + 6Y_i - 4Y_{i-1} + Y_{i-2}) \\
\quad - \beta(Y_{i+1} - 2Y_i + Y_{i-1}) \sum_{m=1}^{11} (Y_{m+1}^2 + Y_m^2 + Y_{m-1}^2 + Y_{m-2}^2 \\
\quad + Y_{m+1}Y_m - 2Y_{m+1}Y_{m-1} - Y_{m+1}Y_{m-2} - Y_mY_{m-1} - 2Y_mY_{m-2} \\
\quad + Y_{m-1}Y_{m-2}) - \dfrac{q(x_i)}{\rho A}[\omega_g^2 U(t) + 2\xi_g\omega_g\dot{U}(t)] = 0 \quad (i = 5, 6), \\
\ddot{U}(t) + 2\xi_g\omega_g\dot{U}(t) + \omega_g^2 U(t) = W(t),
\end{cases}
$$

with the boundary conditions $Y_0 = 0$, $Y_{-1} = -Y_1$, $Y_{11} = 0$, $Y_{12} = -Y_{10}$.

The formulated system is a 11-DOF system and the dimensionality of the relevant FPK equation is 22. There are filtered Gaussian white noises only in the 5th and 6th equations of motion and the filtered Gaussian white noises in the 5th and 6th equations are fully correlated.

Similar to the case in Example 1, the state variables in the first subspace are selected to be $\mathbf{X}_1 = \{Y_i, \dot{Y}_i, U_i, \dot{U}_i\}$, $i \in [1, 10]$ in order to use SSS method for dimension reduction. After dimension reduction by the SSS method, the dimension reduced 4-dimensional FPK is solved by EPC method.

Because the structure is symmetric and the loading is also symmetrically distributed in the middle of the beam, the deflection is also symmetric about the middle of the beam, i.e., $Y_i = Y_{11-i}$ and $\dot{Y}_i = \dot{Y}_{11-i}$ for $i \in [1, 5]$. Therefore, only the marginal PDFs and logarithmic PDFs of Y_i for $i \in [1, 5]$ are shown in the following figures. In order to show the accuracy of the PDFs and logarithmic PDFs of Y_i obtained by SSS-EPC method, the results obtained by MCS and EQL are also shown in Figs. 11–15. For this beam system, the mean value of deflection equals zero, i.e., $m_{y_i} = 0$ and $m_{\dot{y}_i} = 0$. The standard deviation of deflection in the middle of the beam is $\sigma_{y_6} = 0.0771$ m. From these figures it is observed that the results obtained by the SSS-EPC method are close to those obtained by MCS. However, the computational time needed by MCS is about 500 times of that needed by the SSS-EPC method when the sample size used in MCS is 10^8. The computational time needed by SSS-EPC method is mainly spent on the EQL procedure. The rate of computational time between

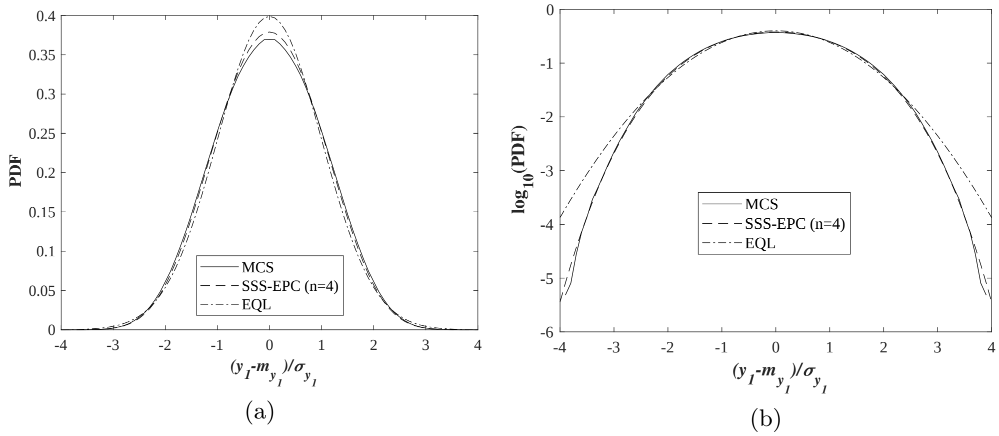


Fig. 11. (a) The PDFs of the deflection Y_1 under partially distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_1 under partially distributed load over the whole beam.

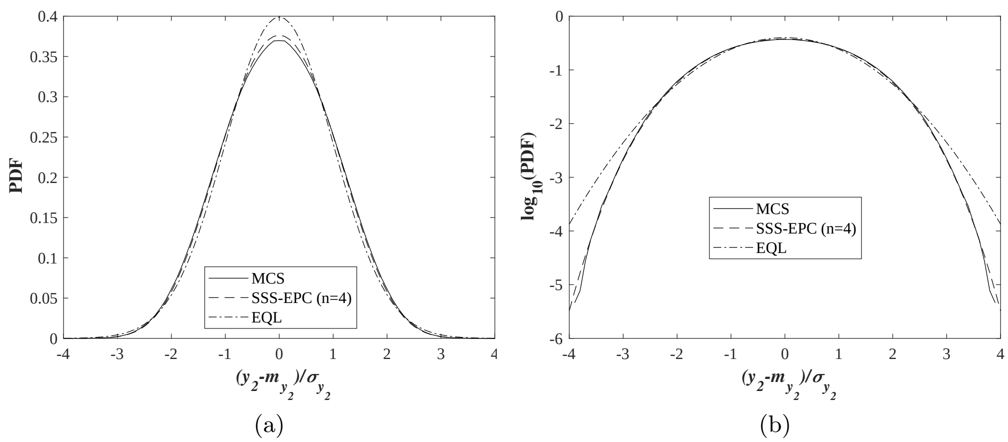
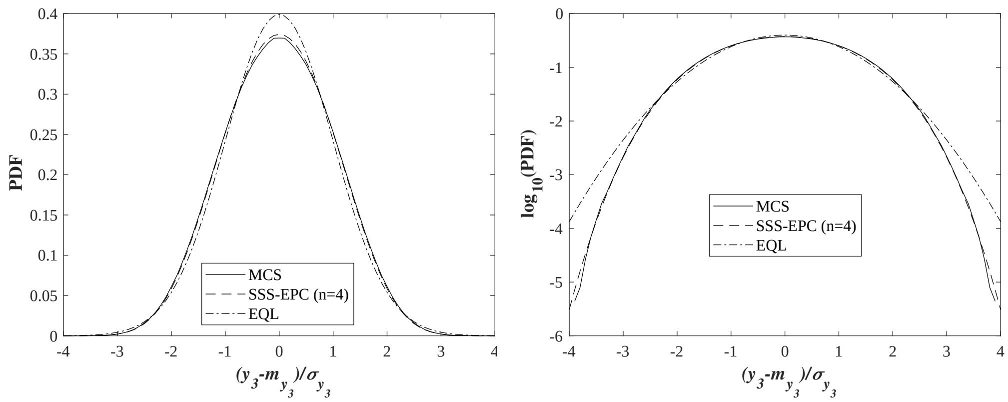


Fig. 12. (a) The PDFs of the deflection Y_2 under partially distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_2 under partially distributed load over the whole beam.

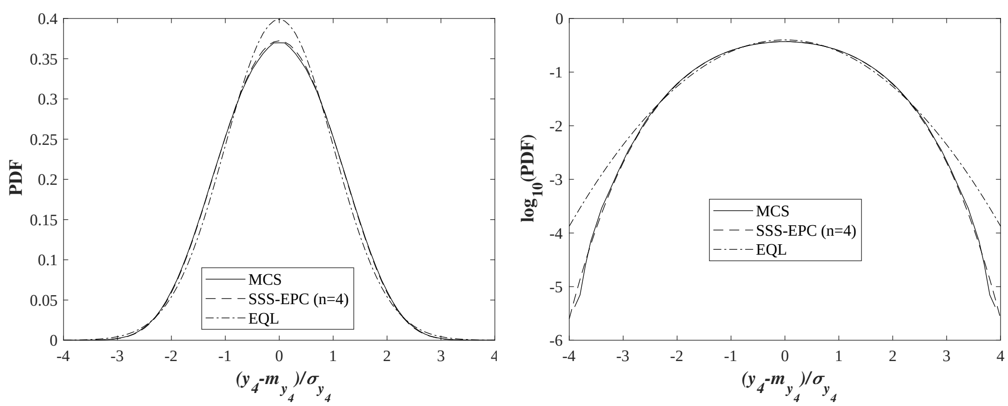
MCS and SSS-EPC can increase quickly as the number of degrees of system freedom increases. By comparing the results obtained by SSS-EPC method and MCS with those obtained by EQL method, it is observed that the system nonlinearity is very strong and the results obtained by EQL are far from being acceptable.

The convergence of the solution is also checked as the nodes number N in the finite-difference gird increases or the value of h decreases. Only the convergence



(a) (b)

FIG. 13. (a) The PDFs of the deflection Y_3 under partially distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_3 under partially distributed load over the whole beam.



(a) (b)

FIG. 14. (a) The PDFs of the deflection Y_4 under partially distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_4 under partially distributed load over the whole beam.

behavior of the PDF of the deflection in the middle of beam is presented in view that the convergence behaviors of the solutions at other nodes are similar. In view that the system is symmetric and the load is symmetrically distributed in the middle of beam, the deflection in the middle of beam equals the deflection at the node nearest to the middle of beam under the adopted finite difference scheme. The PDFs and logarithmic PDFs of the deflection in the middle of beam, which are obtained by the SSS-EPC method, are shown in Fig. 16. From

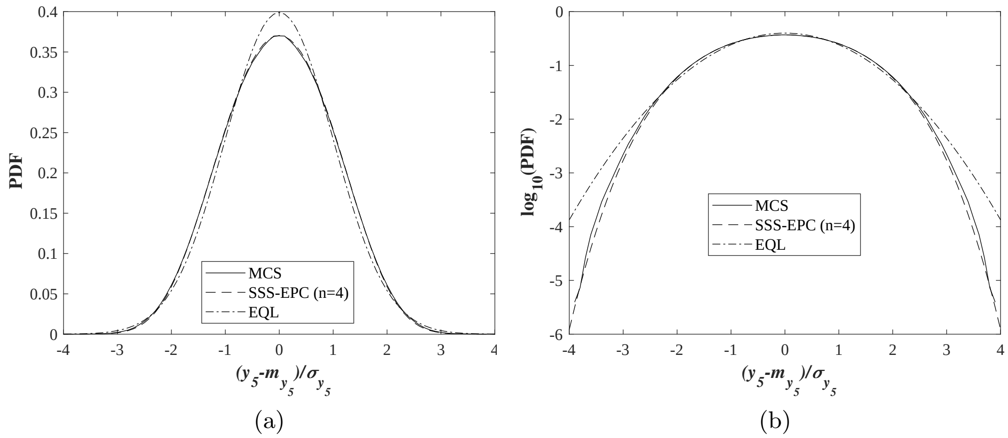


Fig. 15. (a) The PDFs of the deflection Y_5 under partially distributed load over the whole beam. (b) The logarithm of PDFs of the deflection Y_5 under partially distributed load over the whole beam.

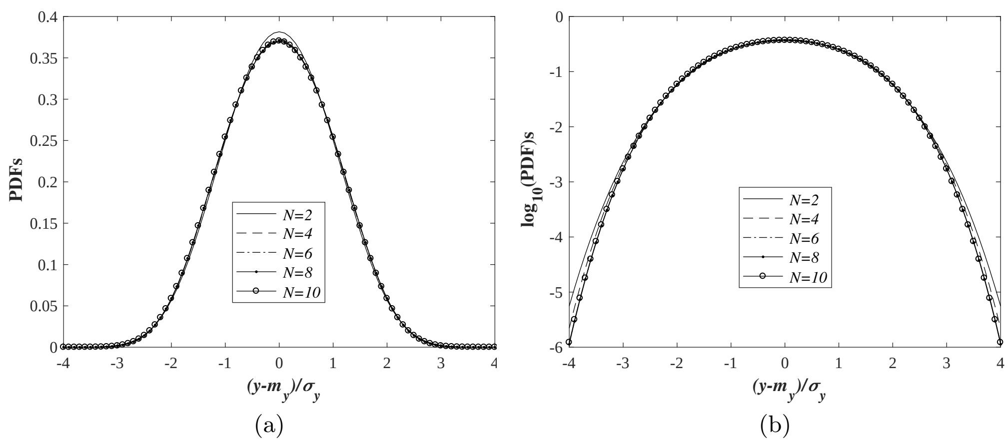


Fig. 16. (a) PDFs in the middle of the beam for different number of nodes. (b) Logarithmic PDFs in the middle of the beam for different number of nodes.

Fig. 16, it is seen that the PDFs and logarithmic PDFs corresponding to $N = 6, 8$ and 10 almost overlap, which means that the solution at $N = 6, 8$ or 10 can be considered as a converged solution for the adopted finite difference scheme. $N = 10$ rather than $N = 6$ or 8 is adopted in the above analysis on the PDF solutions at different nodes so that the PDF solutions at more beam locations can be obtained, shown and compared. Since this load is considered as a concentrated load applied in the middle of the beam, the sum of the distributed load keeps constant as nodes number N changes. In the presented case, $q = 2 \times 10^5$ for $N = 10$. Then $q = 2 \times 10^5 \times \frac{N+1}{11}$ for $N = 2, 4, 6, 8$ and 10. The distributed load applies between node $N/2$ and node $N/2 + 1$.

7. Conclusions

The equations of motion of the stretched beam with pin supports at its two ends are formulated with a central finite difference scheme when the beam is excited by filtered Gaussian white noise. The rule for selecting the state variables which PDF is desired, is that there must be diffusion terms in the dimension-reduced FPK equations for employing the SSS-EPC method. The SSS-EPC method is extended to analyze the probabilistic solutions of the beam systems formulated by a central finite difference scheme. Both beams excited by uniformly distributed load over the whole length of the beam and the concentrated load applied in the middle of the beam are analyzed. When the beam is modeled as a MDOF system, the dimension-reduction procedure of the SSS method is employed to make the high-dimensional FPK equation governing the PDF solutions of the beam reduced to some 4-dimesional FPK equations. Then the EPC method is employed to solve the 4-dimensional FPK equations. The techniques about selecting the subspace in the SSS procedure are introduced and discussed for this nonlinear stochastic dynamical beam system. The first subspace in SSS dimension-reduction procedure must be 4-dimensional for this beam system. The effectiveness and efficiency of the SSS-EPC method are verified by comparing the results obtained with MCS and EQL methods. From numerical analysis it is observed that the SSS-EPC method works well for obtaining the PDFs of the deflections of the beam at all nodes. It is observed that the computational time needed by MCS is about 500 times that needed by SSS-EPC for the 12 or 11-DOF beam systems when the sample size adopted in MCS is 10^8. The MCS is conducted on the original MDOF-NSD systems formulated by a central finite difference scheme. Though the results from EQL are far from being acceptable for the analyzed strongly nonlinear systems, but the SSS-EPC can still give accurate results when the conditional PDF from EQL is adopted both in the SSS dimension-reduction procedure and the EPC solution procedure.

Acknowledgements

The results presented in this paper were obtained under the supports of the Research Committee of University of Macau (Grant No. MYRG2018-00116-FST) and The Science and Technology Development Fund, Macau SAR (Grant No. 042/2017/A1).

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Received November 30, 2018; revised version June 14, 2019.
Published online September 18, 2019.