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Exact response probability density functions of some uncertain structural systems

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This paper has the goal of defining a class of uncertain structural systems for which it is possible to consider an approach able to give the exact response in terms of the probability density function (PDF). The uncertain structures have been identified in the discretized statically determined ones and the approach has been identified in the coupling of the approximated principal deformation modes method (APDM) and of the probability transformation method (PTM). The first one gives the explicit relationships between the response variables and the uncertainty ones, that are exact when the structures are statically determined. The second method allows to determine the explicit relationship between the PDFs of the response and of the uncertainty variables. The results of some applications have confirmed the goodness of these choices and that the proposed approach gives always exact results for both correlated and uncorrelated uncertainty random variables.

Key words: uncertain systems, exact response, probability density function, principal deformation modes, probability transformation method.

1. Introduction

It is well known that the analysis of structural systems is always affected by the uncertainties due to the characterization of materials and of geometric quantities, in addition to the external actions. In many cases the level of these uncertainties is so crucial that the use of deterministic methods for the structural response analyses may lead to unacceptable approximations. In these cases the uncertainties, and consequently, the structural response, must be adequately represented as random quantities. Moreover, an assessment of the response in terms of PDF is highly recommended, above all if reliability analysis is a required field, as well as the use of advanced specific analysis approaches, as the probabilistic methods.

In the literature there are several papers related to the application of probabilistic methods and, in the last fifty years, many significant results have been obtained in this field. An useful overview of the various probabilistic methods in the literature treating systems with uncertainties can be found in some books and review papers [1–8].

Some of the oldest methods for the evaluation of the response PDF of systems subjected to uncertainties are based on truncating the series expansions of the response PDF, its characteristic function, which is the Fourier transform of the PDF, and its log-characteristic function [7]. These truncations are carried out by neglecting the response moments, cumulants or quasi-moments [7, 9]. These approaches can provide sufficiently good results if the response is characterized by a relatively low level of non-Gaussianity. When the response is strongly non-Gaussian, the number of terms of the series may be particularly high and the convergence, that is not guaranteed, can be particularly slow. In addition, the direct evaluation of the terms of the series may not be simple. For this reason, this method is often associated with the Monte Carlo Simulation (MCS) method [2, 10, 11].

Widely used methods are those based on the perturbation approaches, based on a Taylor series expansion in terms of a set of zero mean random variables. The perturbation approaches provide accurate results for relatively low levels of uncertainty, for which only few terms of the series are used (usually the first or the first and second order are considered). On the contrary, when the level of uncertainty of the structural parameters increases the approach loses its precision and, if a high number of terms of the series is taken into account, the computational effort increases remarkably. In any case, the convergence of the approach is not guaranteed by the augmented order of the retained series terms. Major details on this method can be found in [9, 12–16].

Recently, the so-called projection approaches have been largely used for solving uncertain structural systems. They are essentially based on the projection of the structural response solution on a complete stochastic basis. Two of the most used projection approaches are those based on the Karhunen–Loève expansion [3] and on the polynomial chaos expansion. This last one is a Galerkin projection scheme based on the Wiener integral representation [3, 8]. It requires the numerical evaluation of the series expansion terms [17, 18] and can be particularly onerous if the terms of the series are not limited to a relatively small number. For this reason recently, several efforts have been made to improve the approach [19, 20]. A comparison of different projection schemes for the stochastic finite element analysis is given in [21].

Another class of methods dealing with an uncertain system is that related to the use of the random matrix expansion of the structural stiffness matrix in order to perform explicitly its inversion through an iterative approach; for example the Neumann approach is one of them [22, 23]. Once that the explicit inverse stiffness matrix is obtained, it is possible to evaluate the statistics of the response, or to perform a MCS to obtain the response PDF. The advantage of these procedures of giving the explicit relationship between the structural uncertainties and the output response is balanced by the fact that their accuracy is satisfactory only for relatively low levels of uncertainties.

The only universal instrument for the analysis of systems with uncertainties is the direct MCS [24, 25]. Improperly it can be classified as a probabilistic approach, even if it has the advantage of enabling the analysis of any system having any type of randomness. The great drawback related to the use of the MCS is the high computational effort, so that it is often limited to structures with a small number of DOFs. This has led several authors to propose more versatile MCS-based methods, characterized by a lower computational cost than that required by the direct MCS method [10, 11, 26, 27].

It is important to remark that all the aforementioned approaches give more or less approximated response results and none of them is able to give exact results in a closed form, regardless of the characteristics of the structural system under consideration. Instead, owning some exact results, even if for particular uncertain structures, would be very important because they would represent the benchmark results for the above cited approximated approaches.

The aim of this paper is defining an approach able to give exact results in terms of the response PDF for particular uncertain structures. At this purpose, in 2002 Falsone and Impollonia [26, 27] proposed the APDM, that belongs to the class of perturbation and MCS-based methods. It consists in breaking up the structural response in the base of the main deformation modes of the structure: this allows obtaining an approximation of the response, without the cost to invert the stiffness matrix of the system and enabling to reduce strongly the computational effort, the statistics of the response being obtained by the MCS directly applied to the explicit expressions of the response. Nevertheless, the APDM can be considered also as a projection method, because it consists essentially in the expansion of the structural response on a particular base through a finite number of functions, depending on the uncertain parameters, strictly related to the principal deformation modes of the structural system. In any case, the coefficient of the series can be evaluated explicitly in terms of the uncertain parameters. This method is remarkable for the purpose of this paper because the same authors evidenced that in the case of statically determinate structures it gives the exact explicit relationships between the response components and the random variables defining the structural uncertainties [27]. For these structures the APDM becomes the EPDM (approximated $\rightarrow$ exact). In this way a class of structures for which it is possible to find exact close relationships between response quantities and uncertainty quantities has been identified.

The next step is the evaluation of the exact response PDF. At this purpose, recently, an approach, based on a new version of the PTM, has been proposed for the study of some stochastic problems [28, 29]. The method provides the basis for a new philosophy in the study of stochastic structural systems, working directly in terms of input (uncertainties) and output (responses) PDFs and providing exact solutions in some cases.

Consequently, the matching of the EPDM with the PTM for the statically determinate uncertain structures appears to be a good way for reaching the above-cited goals of this paper.

For the sake of clarity, in the following Sections 2 and 3, the basics of APDM and of PTM are given, respectively, evidencing how for statically determined uncertain structures the APDM becomes an EPDM. In Section 4, the way for matching these methods in order to create an approach able to evaluate the response PDF of an uncertain structural system is shown. In Section 5, some numerical examples are reported with the aim of testing the efficiency of the approach through the comparisons with the results obtained by the direct MCS. At last, some conclusions and remarks close the paper in Section 6.

2. Fundamentals of APDM [26, 27]

The response of a discretized structural linear system having uncertain properties is governed by an equilibrium equation that can be expressed in the following form:

$$\mathbf{K}(\boldsymbol{\alpha})\mathbf{u}(\boldsymbol{\alpha}) = \mathbf{F} \tag{2.1}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T$ is the m-vector collecting the random variables modelling the uncertainties of the structural system; these variables are supposed to be defined through the knowledge of their joint PDF, $p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$; $\mathbf{K}(\boldsymbol{\alpha})$ is the structural stiffness $n \times n$ matrix depending on the uncertain parameters; $\mathbf{F}$ is the n-vector of the external actions, here considered deterministic, and $\mathbf{u}(\boldsymbol{\alpha})$ is the n-vector of the response displacements, depending on the structural parameters, and hence on $\boldsymbol{\alpha}$, besides of on the external actions. By following [26], it is always possible to express the stiffness matrix as follows:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{i=1}^{m} \mathbf{K}_i \alpha_i \tag{2.2}$$

where $\mathbf{K}_0$ is the deterministic stiffness matrix obtained setting $\alpha_i = 0$, with $i = 1, \ldots, m$, and $\mathbf{K}_i$ are deterministic matrices extracted from $\mathbf{K}(\boldsymbol{\alpha})$ by using, for example, the first order Taylor expansion:

$$\mathbf{K}_0 = \mathbf{K}(\mathbf{0}); \qquad \mathbf{K}_i = \left[\frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_i}\right]_{\boldsymbol{\alpha}=\mathbf{0}}. \tag{2.3}$$

The basic idea of the method lies on the following approximation of the response vector $\mathbf{u}(\boldsymbol{\alpha})$:

$$\mathbf{u}(\boldsymbol{\alpha}) \approx \mathbf{u}_0 + \sum_{i=1}^{m} \mathbf{u}_i(\alpha_i) = \mathbf{u}_0 + \mathbf{u}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) \tag{2.4}$$

where $\mathbf{u}_0$ is the deterministic response obtained setting $\alpha_i = 0$ in Eq. (2.1), while $\mathbf{u}_i(\alpha_i)$ are the response vectors obtained supposing that only the random variable α_i characterizes the structure uncertainties. Hence, they are the solutions of the following equations:

$$\mathbf{K}_0\mathbf{u}_0 = \mathbf{F}; \qquad (\mathbf{K}_0 + \alpha_i\mathbf{K}_i)\mathbf{u}_i(\alpha_i) = -\alpha_i\mathbf{K}_i\mathbf{u}_0. \tag{2.5}$$

In [26] the explicit closed-form expressions of the partial response vectors $\mathbf{u}_i(\alpha_i)$ have been given; they are:

$$\mathbf{u}_i(\alpha_i) = -\alpha_i\mathbf{\Phi}_i[\mathbf{I}_n + \alpha_i\mathbf{\Lambda}_i]^{-1}\mathbf{\Lambda}_i\mathbf{\Phi}_i^T\mathbf{F} \tag{2.6}$$

where $\mathbf{\Lambda}_i$ and $\mathbf{\Phi}_i$ are the eigenvalue and eigenvector matrices, respectively, of the matrix $\mathbf{K}_0^{-1}\mathbf{K}_i$, while $\mathbf{I}_n$ is the identity matrix of the order n . This implies that the matrix in square brackets in Eq. (2.6) is diagonal, making very simple its inversion. Moreover, always in [26], it has been shown that the number of the non-zero eigenvalues of the matrix $\mathbf{K}_0^{-1}\mathbf{K}_i$ is equal to the number of the structural principal deformation modes directly affected by α_i; this number, n_i, is very small with respect to the number of degrees of freedom n of the system, making very simple the evaluation of $\mathbf{u}_i(\alpha_i)$. The name of the approach (APDM) just originates from the important role of the structural principal deformation modes in the application of this method. In particular, if each random variable α_i influences only one FE (as usually happens), then the number of the significant eigenvalues cannot be greater than the number of natural modes of the element that, in turn, depends on the type of FE chosen for discretizing the structure. For example, a bar-type FE is characterized only by one natural deformation mode ($n_i = 1$); a beam-type FE (in a plane analysis) by two natural deformation modes ($n_i = 2$); a frame-type FE (in a plane analysis) by three natural deformation modes ($n_i = 3$).

When $n_i = 1$, the straightforward particularization of Eq. (2.6) gives:

$$u_{i_j}(\alpha_i) = -\mathit{\Phi}_{i_{jk}}\frac{\alpha_i\mathit{\Lambda}_{i_k}q_{0,i_k}}{1 + \alpha_i\mathit{\Lambda}_{i_k}} = \frac{a_{i_j}\alpha_i}{1 + b_{i_j}\alpha_i} \tag{2.7}$$

where u_{i_j} is the j-th element of $\mathbf{u}_i$, $\mathit{\Lambda}_{i_k}$ is the only non-zero eigenvalue of $\mathbf{K}_0^{-1}\mathbf{K}_i$ (the $k-$th) and $\Phi_{i_{jk}}$ the j-th element of the k-th eigenvector. At last, q_{0,i_k} is the $k-$th element of the modal response vector $\mathbf{q}_{0,i} = \mathbf{\Phi}_i^{-1}\mathbf{u}_0$. Obviously, the quantities a_{i_j} and b_{i_j} appearing in the last term of Eq. (2.7) can be obtained once that the eigenvalue problem of $\mathbf{K}_0^{-1}\mathbf{K}_i$ is solved.

When $n_i = 2$, as, for example, happens in the beam-type FE discretized planar structures, the non-zero eigenvalues of the matrix $\mathbf{K}_0^{-1}\mathbf{K}_i$ are not more than two. In this case the generic element of $\mathbf{u}_i$ can be obtained by the following relation:

$$u_{i_j}(\alpha_i) = -\Phi_{i_{jk}} \frac{\alpha_i \Lambda_{i_k} q_{0,i_k}}{1+\alpha_i \Lambda_{i_k}} - \Phi_{i_{jl}} \frac{\alpha_i \Lambda_{i_l} q_{0,i_l}}{1+\alpha_i \Lambda_{i_l}} \tag{2.8}$$

where Λ_{i_k} and Λ_{i_l} are the two non-zero eigenvalues of $\mathbf{K}_0^{-1}\mathbf{K}_i$ (the k-th and the l-th), $\Phi_{i_{jk}}$ and $\Phi_{i_{jl}}$ the corresponding element of the k-th and the l-th eigenvectors and, lastly, q_{0,i_k} and q_{0,i_l} are the k-th and the l-th elements of $\mathbf{q}_{0,i}$. Even in this case, it is possible to consider an alternative expression of $u_{i_j}(\alpha_i)$, that is:

$$u_{i_j}(\alpha_i) = \frac{a_{i_j}\alpha_i + b_{i_j}\alpha_i^2}{1 + c_{i_j}\alpha_i + d_{i_j}\alpha_i^2} \tag{2.9}$$

where the four coefficients appearing here can be easily obtained starting from Eq. (2.8).

At last, the generalization to the case $n_i = p$, p being the generic number of the structural principal deformation modes influenced by the uncertain parameter α_i, is quite simple. In fact, Eqs. (2.7), (2.8) and (2.9) can be generalized in:

$$u_{i_j}(\alpha_i) = -\sum_{k=1}^{p} \Phi_{i_{jk}} \frac{\alpha_i \Lambda_{i_k} q_{0,i_k}}{1+\alpha_i \Lambda_{i_k}} = \frac{\sum\limits_{k=1}^{p} b_{ik_j}\alpha_i^k}{1 + \sum\limits_{k=1}^{p} d_{ik_j}\alpha_i^k}. \tag{2.10}$$

In [27] it has been evidenced that the APDM is affected by an error $\mathbf{e}(\boldsymbol{\alpha})$ having the following expression:

$$\mathbf{e}(\boldsymbol{\alpha}) = -\sum_{i=1}^{m} \sum_{i\neq j=1}^{m} \alpha_i \mathbf{K}_i \mathbf{u}_j(\alpha_j) \tag{2.11}$$

that shows of being strictly related to the presence of the cross-terms $\mathbf{K}_i\mathbf{u}_j$, neglected in the APDM. These cross-terms may assume an important physical significance remembering that $\mathbf{u}_j$ is the structural displacement when only the random variable α_j affects the structure. Consequently, they represent the nodal forces arising in a structure characterized by the stiffness matrix $\mathbf{K}_i$ and subjected to the nodal displacements $\mathbf{u}_j$. Then, if the discretized structure is statically determinate, these terms are rigorously zero, for $i \neq j$, and no error is related to the use of Eq. (2.4). Hence, for statically determinate discretized structures, the APDM becomes the EPDM (Exact Principal Deformation Mode) approach, giving the exact relationships between the structural response and the random variables describing the uncertainties.

3. Fundamentals of the PTM [28, 29]

The basic aspects of the PTM must be looked for in the theory of the space transformation of vector-valued random variables, briefly discussed below.

If $\mathbf{x}$ is a n-dimensional random vector with joint PDF $p_{\mathbf{x}}(\mathbf{x})$ and $\mathbf{h}(\cdot)$ is a n-dimensional invertible application, such that $\mathbf{h}^{-1}(\cdot) = \mathbf{g}(\cdot)$ exists, that is:

$$\mathbf{z} = \mathbf{h}(\mathbf{x}); \qquad \mathbf{x} = \mathbf{g}(\mathbf{z}) \tag{3.1a, b}$$

then, it is well known that the joint PDFs of the random vectors $\mathbf{x}$ and $\mathbf{z}$ are related by the probability transformation law as follows [30]:

$$p_{\mathbf{z}}(\mathbf{z}) = \frac{1}{|\det[\mathbf{J}_{\mathbf{h}}(\mathbf{g}(\mathbf{z}))]|} p_{\mathbf{x}}(\mathbf{g}(\mathbf{z})) = |\det[\mathbf{J}_{\mathbf{g}}(\mathbf{z})]| p_{\mathbf{x}}(\mathbf{g}(\mathbf{z})) \tag{3.2}$$

where $\mathbf{J}_{\mathbf{h}}(\cdot)$ and $\mathbf{J}_{\mathbf{g}}(\cdot) = \mathbf{J}_{\mathbf{h}}^{-1}(\cdot)$ are the Jacobian matrices related to the transformations in Eqs.(12). Equation (3.2) allows to determine the joint PDF $p_{\mathbf{z}}(\mathbf{z})$, once that the joint PDF $p_{\mathbf{x}}(\mathbf{x})$ is known, together with the inverse transformation law, $\mathbf{g}(\cdot)$. It represents the fundamental relationship of the PTM.

It may happen that the numbers of elements of $\mathbf{x}$ and $\mathbf{z}$ are different. In this case, the PTM is still relevant, if some expedients are performed for its application. For example, if n and m are the orders of $\mathbf{x}$ and $\mathbf{z}$, respectively, and $n > m$, an efficient expedient may be the adding of new output elements through $(n - m)$ generic variables, in such a way that the augmented vector $\bar{\mathbf{z}}^T = (\mathbf{z}^T \ \hat{\mathbf{z}}^T)$, $\hat{\mathbf{z}}$ being the $(n - m)$-vector of the added generic variables, has the same number of elements of the input $\mathbf{x}$. Hence, the law expressed in Eq. (3.2) can be applied in the form:

$$p_{\bar{\mathbf{z}}}(\bar{\mathbf{z}}) = \frac{1}{|\det[\mathbf{J}_{\bar{\mathbf{h}}}(\bar{\mathbf{g}}(\bar{\mathbf{z}}))]|} p_{\mathbf{x}}(\bar{\mathbf{g}}(\bar{\mathbf{z}})) = |\det[\mathbf{J}_{\bar{\mathbf{g}}}(\bar{\mathbf{z}})]| p_{\mathbf{x}}(\bar{\mathbf{g}}(\bar{\mathbf{z}})) \tag{3.3}$$

where:

$$\bar{\mathbf{h}}(\cdot) = \begin{pmatrix} \mathbf{h}(\cdot) \\ \hat{\mathbf{h}}(\cdot) \end{pmatrix}; \qquad \bar{\mathbf{g}}(\cdot) = \bar{\mathbf{h}}^{-1}(\cdot). \tag{3.4}$$

Once that the joint PDF $p_{\bar{\mathbf{z}}}(\bar{\mathbf{z}})$ has been evaluated by Eq. (3.3), the effective joint PDF $p_{\mathbf{z}}(\mathbf{z})$ is obtained by the saturation of the $(n - m)$ added generic variables, that is:

$$p_{\mathbf{z}}(\mathbf{z}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\bar{\mathbf{z}}}(\bar{\mathbf{z}}) \, \mathrm{d}\hat{z}_1 \cdots \mathrm{d}\hat{z}_{n-m}. \tag{3.5}$$

On the other hand, when $n < m$, an effective expedient may be the enlargement of the input vector $\mathbf{x}$ by adding a deterministic zero $(m-n)$-vector. In this way, Eq. (3.1a) can be rewritten as:

$$\mathbf{z} = \mathbf{h}(\bar{\mathbf{x}}) = \begin{pmatrix} \tilde{\mathbf{h}}(\mathbf{x}) \\ \hat{\mathbf{h}}(\mathbf{x}) + \hat{\mathbf{x}} \end{pmatrix}; \qquad \bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{pmatrix} \tag{3.6}$$

where $\hat{\mathbf{x}}$ is the deterministically zero vector, while $\tilde{\mathbf{h}}(\cdot)$ and $\hat{\mathbf{h}}(\cdot)$ are the sub-vector functions, of order n and $(m-n)$, respectively, composing $\mathbf{h}(\cdot)$. Hence, the joint PDF of the augmented vector $\bar{\mathbf{x}}$ is given by:

$$p_{\bar{\mathbf{x}}}(\bar{\mathbf{x}}) = p_{\mathbf{x}}(\mathbf{x})\delta(\hat{\mathbf{x}}), \tag{3.7}$$

$\delta(\hat{\mathbf{x}})$ being the $(m-n)$-dimensional Dirac delta function placed at $\hat{\mathbf{x}} = \mathbf{0}$. The inverse relationships given into Eq. (3.1b) can be rewritten in the form:

$$\bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{g}}(\tilde{\mathbf{z}}) \\ \hat{\mathbf{g}}(\mathbf{z}) \end{pmatrix}; \qquad \tilde{\mathbf{g}}(\cdot) = \tilde{\mathbf{h}}^{-1}(\cdot); \qquad \hat{\mathbf{g}}(\mathbf{z}) = \hat{\mathbf{z}} - \hat{\mathbf{h}}(\tilde{\mathbf{g}}(\tilde{\mathbf{z}})) \tag{3.8}$$

where the expression of the direct transformation given in Eq.(17a) has been considered and where the inverse expression of Eq.(19b) is assumed to exist. At this point, following the same procedure as that used for finding the classical PTM expression, it is not difficult to find the following relationship between the joint PDF of $\mathbf{z}$ and the joint PDF of $\mathbf{x}$:

$$p_{\mathbf{z}}(\mathbf{z}) = |\det[\mathbf{J}_{\tilde{\mathbf{g}}}(\tilde{\mathbf{z}})]| p_{\mathbf{x}}(\tilde{\mathbf{g}}(\tilde{\mathbf{z}}))\delta(\hat{\mathbf{z}} - \hat{\mathbf{h}}(\tilde{\mathbf{g}}(\tilde{\mathbf{z}}))). \tag{3.9}$$

Other useful expressions of the PTM can be found in the literature [28, 29]. A very interesting one is that based on the properties of the multidimensional Dirac delta function and that has the following form:

$$p_{\mathbf{z}}(\mathbf{z}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y})\delta(\mathbf{z} - \mathbf{h}(\mathbf{y}))\mathrm{d}y_1 \cdots \mathrm{d}y_n. \tag{3.10}$$

This relationship is particularly useful when the random response variables of interest are considerably less numerous than the elements of $\mathbf{z}$. In particular, if only the element $z_j = h_j(\mathbf{x})$ must be probabilistically characterized, then Eq. (3.10) is reduced as follows:

$$p_{z_j}(z_j) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y})\delta(z_j - h_j(\mathbf{y}))\mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{3.11}$$

while, if the joint PDF of the elements $z_j = h_j(\mathbf{x})$ and $z_k = h_k(\mathbf{x})$ is required, the following relationship can be used:

$$p_{z_j z_k}(z_j, z_k) = \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y})\delta(z_j - h_j(\mathbf{y}))\delta(z_k - h_k(\mathbf{y}))\mathrm{d}y_1 \cdots \mathrm{d}y_n. \tag{3.12}$$

It is important to note that Eqs. (3.10)–3.12) show the useful property of not requiring the knowledge of the inverse transformation $\mathbf{g}(\cdot) = \mathbf{h}^{-1}(\cdot)$, that, in some cases, can represent a very hard task. Moreover, they do not depend on the number of elements of $\mathbf{z}$. On the contrary, they have the drawback of requiring n integrations with respect to the component of $\mathbf{x}$. This last drawback is overcome when $h_j(\mathbf{x})$ is a linear combination of the components of $\mathbf{x}$, that is when it is possible to write $h_j(\mathbf{x}) = \mathbf{h}_j^T\mathbf{x}$, $\mathbf{h}_j$ being the n-vector collecting the coefficients of the combination. In fact, in this case, the characteristic function of z_j can be expressed as:

$$\begin{aligned} M_{z_j}(\omega) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} p_{z_j}(z_j)\exp(-\omega z_j)dz_j \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y})\delta(z_j - \mathbf{h}_j^T\mathbf{y})\mathrm{d}y_1\cdots\mathrm{d}y_n\right]\exp(-\omega z_j)dz_j \end{aligned} \tag{3.13}$$

that, by using the properties of the Dirac delta function, becomes:

$$\begin{aligned} &M_{z_j}(\omega) \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y})\exp(-\omega\mathbf{h}_j^T\mathbf{y})\mathrm{d}y_1\cdots\mathrm{d}y_n = (2\pi)^{n-1}M_{\mathbf{x}}(\boldsymbol{\theta})_{\boldsymbol{\theta}=\omega\mathbf{h}_j}. \end{aligned} \tag{3.14}$$

This last equation is important because directly relating the characteristic functions of the input $\mathbf{x}$ and of the output z_j, without the necessity of any integration.

If the joint characteristic function of the two response variables z_j and z_k is required, it is easy to show that the following relationship holds:

$$M_{z_j z_k}(\omega_j, \omega_k) = (2\pi)^{n-2}M_{\mathbf{x}}(\boldsymbol{\theta})_{\boldsymbol{\theta}=\omega_j\mathbf{h}_j+\omega_k\mathbf{h}_k}. \tag{3.15}$$

The extension to more variables is straightforward.

Once that the characteristic functions are evaluated the corresponding PDF can be obtained by Fourier anti-transform operations.

4. Proposed approach

In this section the EPDM and the PTM approaches are combined in order to find the exact response PDFs of statically determinate structural systems characterized by mechanical and/or geometrical uncertainties.

In Section 2 it has been evidenced that the APDM method, if applied to statically determinate structures, really is an EPDM, in the sense that the following expansion is exact:

$$\mathbf{u}(\boldsymbol{\alpha}) = \mathbf{u}_0 + \sum_{i}^{m} \mathbf{u}_i(\alpha_i) = \mathbf{u}_0 + \mathbf{u}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}). \tag{4.1}$$

Hence, the objective of this section is the evaluation of the joint PDF $p_{\mathbf{u}_{\boldsymbol{\alpha}}}(\mathbf{u}_{\boldsymbol{\alpha}})$, once that the joint PDF $p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ is assigned. As seen in Section 2, it is possible to express the j-th element of $\mathbf{u}_i(\alpha_i)$ as $u_{i_j} = h_{i_j}(\alpha_i)$, where the form of the function $h_{i_j}(\cdot)$ essentially depends on the number of principal deformation modes related to the FE-type used for the structural discretization (Eqs. (2.7)–(2.10)). In any case, the required inverse function $g_{i_j}(\cdot) \equiv h_{i_j}^{-1}(\cdot)$, can be always obtained in a closed form. For example, for $n_p = 1$ and $n_p = 2$ the inverse relationships are given by:

$$\begin{aligned}
\alpha_i &= \frac{u_{i_j}}{a_{i_j} - b_{i_j} u_{i_j}}; \\
\alpha_i &= \frac{a_{i_j} - c_{i_j} u_{i_j} \pm \sqrt{\Delta_{i_j}}}{2(b_{i_j} - d_{i_j} u_{i_j})}; \\
\Delta_{i_j} &= (c_{i_j}^2 - 4d_{i_j}) u_{i_j}^2 + (4b_{i_j} - 2a_{i_j} c_{i_j}) u_{i_j} + a_{i_j}^2.
\end{aligned} \tag{4.2}$$

Equation (4.2b) shows that the inverse function has two values. In all the cases in which the inverse shows n_p solutions, the PTM can be still applied, but the sum of the PDFs corresponding to the various solutions must be considered, that is:

$$p_{u_{i_j}}(u_{i_j}) = \sum_{k=1}^{n_p} |J_{g_{i_j}^{(k)}}(u_{i_j})| p_{\alpha_i}(g_{i_j}^{(k)}(u_{i_j})). \tag{4.3}$$

If the probabilistic characterization of the structural response component u_j is required, the application of the EPDM method implies that:

$$u_j = u_{0_j} + \sum_{i=1}^{m} u_{i_j}(\alpha_i) = u_{0_j} + \mathbf{1}^T \mathbf{u}_j(\boldsymbol{\alpha}) = u_{0_j} + u_{\alpha_j}(\boldsymbol{\alpha}) \tag{4.4}$$

where $\mathbf{1}$ is the m-vector whose components are all equal to one and $\mathbf{u}_j(\boldsymbol{\alpha})$ is the m-vector whose i-th element is $u_{i_j}(\alpha_i)$. The joint PDF of $\mathbf{u}_j(\boldsymbol{\alpha})$ is obtained by the application of the PTM and it has the following form:

$$p_{\mathbf{u}_j}(\mathbf{u}_j) = |\det[\mathbf{J}_{\mathbf{g}_j}(\mathbf{u}_j)]| p_\alpha(\mathbf{g}_j(\mathbf{u}_j)). \tag{4.5}$$

It is easy to verify that the Jacobian $\mathbf{J}_{\mathbf{g}_j}$ is diagonal because the inverse functions $g_{i_j}(\cdot)$ depends only on the response component u_{i_j}. Hence, the following expression is true:

$$|\det[\mathbf{J}_{\mathbf{g}_j}(\mathbf{u}_j)]| = \prod_{i=1}^{m} \left| \frac{dg_{i_j}(u_{i_j})}{du_{i_j}} \right|. \tag{4.6}$$

Equations (4.5) and (4.6) give the joint PDF of the elements of the vector $\mathbf{u}_j(\boldsymbol{\alpha})$. In order to determine the joint PDF of $u_{\alpha_j}(\boldsymbol{\alpha})$, Eq. (4.4) must be taken into account. It establishes a relation between the m-vector $\mathbf{u}_j(\boldsymbol{\alpha})$, whose joint PDF is given into Eq. (4.5), and the response component $u_{\alpha_j}(\boldsymbol{\alpha})$. For evaluating the PDF of this last quantity the most efficient approach is the version of the PTM based on the use of the characteristic functions. In particular, the application of Eqs. (3.14) and (4.4) gives the following result:

$$M_{u_{\boldsymbol{\alpha}_j}}(\omega) = (2\pi)^{m-1} M_{\mathbf{u}_j}(\boldsymbol{\theta})_{\boldsymbol{\theta}=\omega\mathbf{1}}. \tag{4.7}$$

$M_{\mathbf{u}_j}(\boldsymbol{\theta})$ being the joint characteristic function of the vector $\mathbf{u}_j$ that can be obtained by the Fourier transform of $p_{\mathbf{u}_j}(\mathbf{u}_j)$. In this way, the probabilistic characterization of the response quantity u_{α_j}, and, hence, of u_j, is complete.

If the joint probabilistic characterization of the two response components u_j and u_k is required, the application of the approach above described requires the evaluation of the joint characteristic function $M_{u_{\alpha_j} u_{\alpha_k}}(\omega_1, \omega_2)$, that it is easy to express as follows:

$$M_{u_{\alpha_j} u_{\alpha_k}}(\omega_1, \omega_2) = (2\pi)^{m-2} M_{\mathbf{u}_j \mathbf{u}_k}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)_{\boldsymbol{\theta}_1=\omega_1\mathbf{1},\, \boldsymbol{\theta}_2=\omega_2\mathbf{1}}. \tag{4.8}$$

The joint characteristic function $M_{\mathbf{u}_j \mathbf{u}_k}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ is the double Fourier transform of the joint PDF $p_{\mathbf{u}_j \mathbf{u}_k}(\mathbf{u}_j, \mathbf{u}_k)$ that can be obtained by applying the PTM to a transformation law in which the m-vector $\boldsymbol{\alpha}$ is the input and the two m-vectors $\mathbf{u}_j$ and $\mathbf{u}_k$ represent the output. Hence, in this case, the number of input elements is smaller than the number of output elements and the version of PTM expressed into Eqs. (3.6)–(3.9) can be used. This implies that the expression of $p_{\mathbf{u}_j \mathbf{u}_k}(\mathbf{u}_j, \mathbf{u}_k)$, particularizing Eq. (3.9) to this case, is given by:

$$\begin{aligned} p_{\mathbf{u}_j \mathbf{u}_k}(\mathbf{u}_j, \mathbf{u}_k) &= |\det[\mathbf{J}_{\mathbf{g}_j}(\mathbf{u}_j)]| p_{\boldsymbol{\alpha}}(\mathbf{g}_j(\mathbf{u}_j)) \delta(\mathbf{u}_k - \mathbf{h}_k(\mathbf{g}_j(\mathbf{u}_j))) \\ &= \prod_{i=1}^{m} \left| \frac{dg_{i_j}(u_{i_j})}{du_{i_j}} \right| p_{\boldsymbol{\alpha}}(\mathbf{g}_j(\mathbf{u}_j)) \delta(\mathbf{u}_k - \mathbf{h}_k(\mathbf{g}_j(\mathbf{u}_j))). \end{aligned} \tag{4.9}$$

This approach can be generalized to response higher order probabilistic characterizations.

5. Numerical examples

The numerical examples reported in this section aim to verify and highlight the fundamental statement of the present work showing that, for statically determinate discretized uncertain structures, the joint use of the EPDM and of the PTM allows to obtain the exact response PDFs. It is assumed that, in all the considered examples, the structural uncertain parameter is represented by the Young modulus of each FE in which the structural system has been discretized. In particular, the Young modulus of the generic FE is modelled as a random variable with the constant mean value E_0 and fluctuation α_i, according to the expression:

$$E_i = E_0(1 + \alpha_i), \qquad i = 1, 2, \ldots, m, \tag{5.1}$$

m being the number of the FE used in the discretization. The examples reported have been chosen in such a way that the corresponding FE typology are characterized by a different number of principal deformation modes, n_p.

5.1. Bar type FE

For this FE typology, two examples of statically determinate structural systems are taken into account. The truss-structure represented in Fig. 1 is first considered. It is characterized by the following geometrical and mechanical deterministic parameters: $L = 5\,\text{m}$, $H = 4\,\text{m}$; all the bars have the same cross-sections area ($4 \times 10^{-2}\,\text{m}^2$) and a random Young modulus defined as in Eq. (4.8), where $i = 1, \ldots, 9$ and $E_0 = 2.10 \times 10^8\,\text{kN/m}^2$. The random variables α_i are assumed to

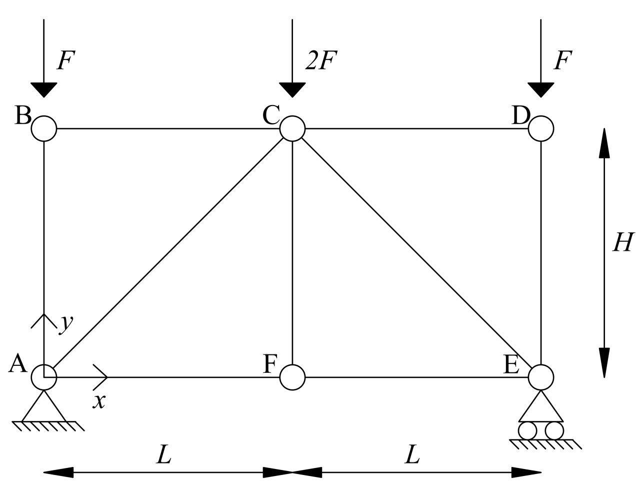


FIG. 1. The truss-structure.

be uniformly distributed in the range $[-0.30, 0.30]$. The truss-structure is forced by static deterministic forces: a force $F = 10\,\mathrm{kN}$ applied to the nodes B and D, and a force $2F$ applied to the node C.

The results presented here confirm that the proposed approach leads to the exact expression of $\mathbf{u}(\boldsymbol{\alpha})$, and, hence, by applying Eq. (4.7), to the exact PDF of any responses of interest. The PDFs of some nodal displacements have been obtained, comparing the results obtained by the proposed approach and those by Monte Carlo simulations (for these last ones 5×10^5 samples have been considered). In Fig. 2, the PDF of the vertical displacement of node C, $p_{u_y^C}(u_y^C)$, and the PDF of the horizontal displacement of node E, $p_{u_x^E}(u_x^E)$, are, respectively, reported.

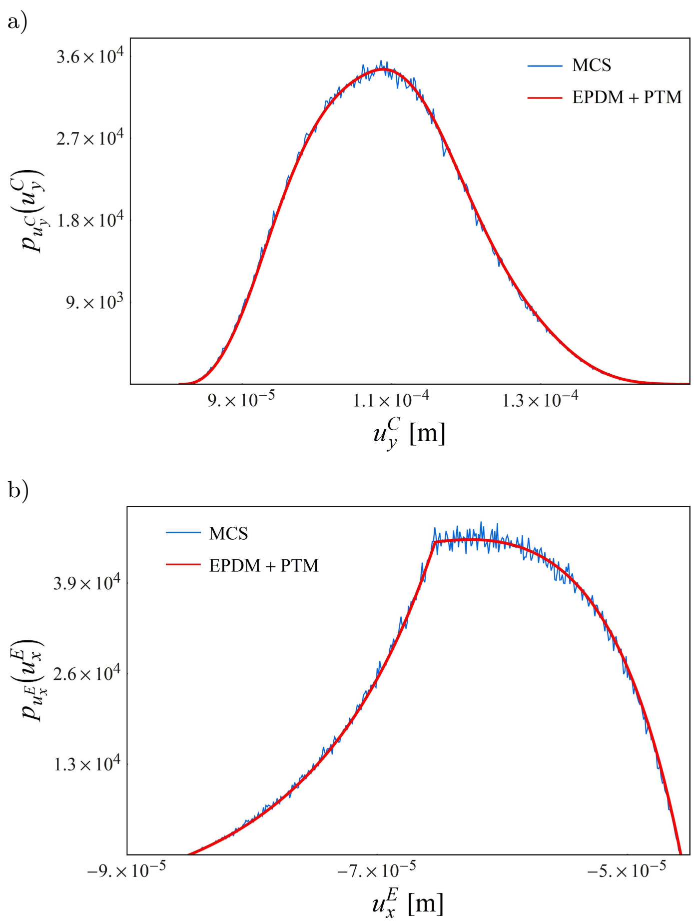


Fig. 2. PDF of the vertical displacement of node C (a) and of the horizontal displacement of node E (b).

Successively, the statically determined beam represented in Fig. 3 is taken into account. Its length is $L = 8\,\mathrm{m}$, while its cross section is rectangular with area equal to $1.5 \times 10^{-3}\,\mathrm{m}^2$. The external actions are a uniformly distributed axial load with intensity $q = 150\,\mathrm{kN/m}$ and a static deterministic axial force applied to the free end with intensity $F = qL$. Due to the load characteristics, the discretization can be made by means of bar-type FEs. In particular, four FE of equal length have been used. The random Young modulus is defined as in Eq. (4.8), with $i = 1, \ldots, 4$ and $E_0 = 3 \times 10^7\,\mathrm{kN/m}^2$.

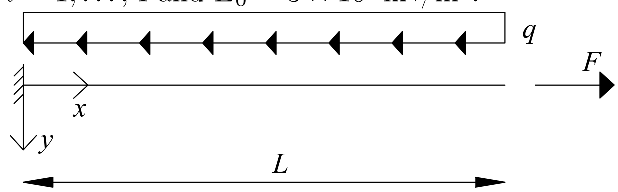


Fig. 3. Cantilever beam (bar type FE).

In Fig. 4 the PDF of the horizontal displacement of the free end is reported for the case that the random variables α_i are assumed to be uniformly distributed in the range $[-0.40, 0.40]$. By inspection of Fig. 4 the goodness of the comparison is clear, even if a high level of uncertainty is present in the beam Young modulus.

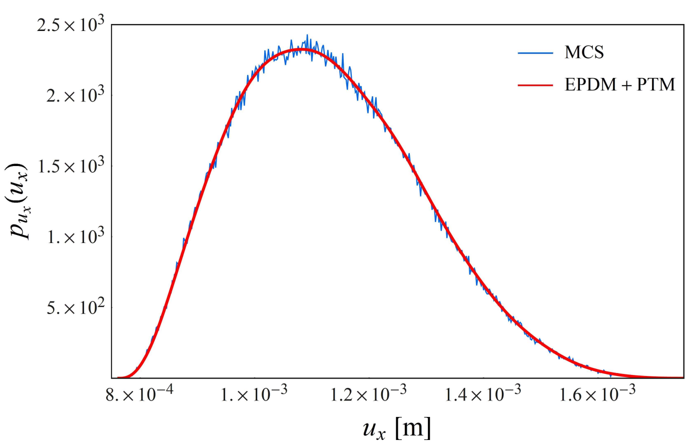


Fig. 4. PDF of the horizontal displacement of the free end.

5.2. Beam-type FE

The statically determinated cantilever beam under the deterministic transversal load $q = 50\,\mathrm{kN/m}$ is now considered. The beam (Fig. 5) differs from the previous one only for the condition of load and it is characterized by an inertia

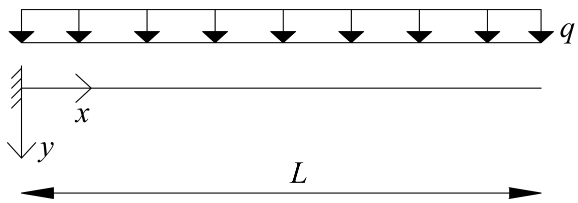


Fig. 5. Cantilever beam (beam-type FE).

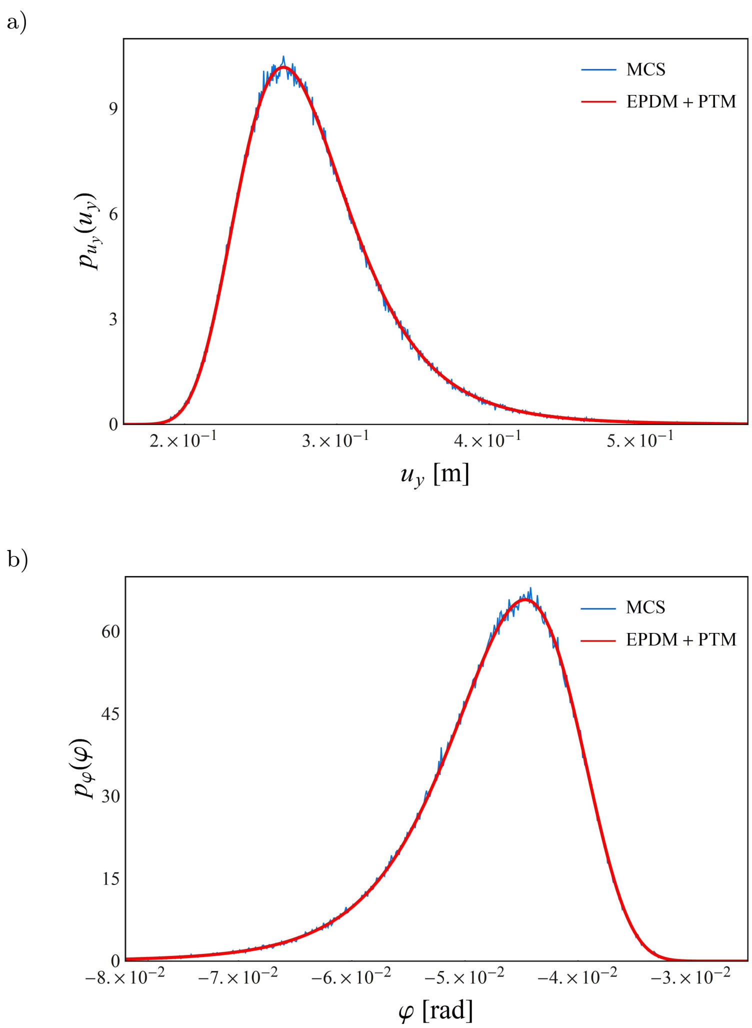


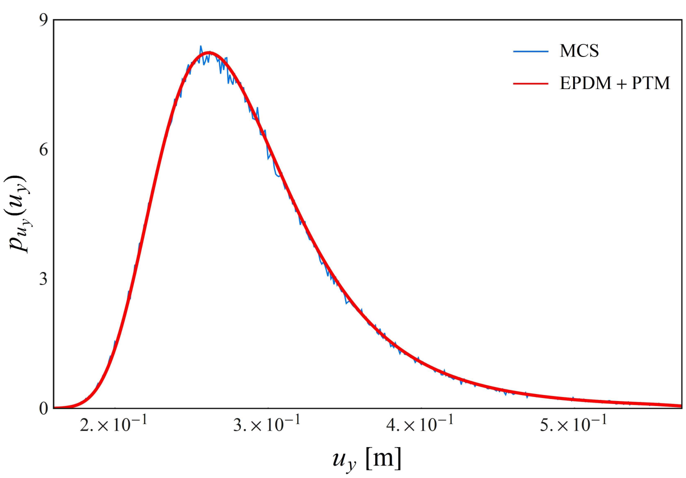
Fig. 6. PDF of the vertical displacement (a) and rotation (b) of the cantilever free end; uncorrelated random variables α_i.

moment equal to $I = 3.125 \times 10^{-3}\,\mathrm{m}^4$. Two different distributions of random variables α_i are examined: firstly, they have been assumed to be zero-mean, Gaussian, independent and defined by a variance $\sigma^2 = (0.20)^2$; in the second case, the same random variables are considered as correlated following the given correlation law:

$$\rho = \exp\left(-\frac{\Delta x}{\lambda}\right) \tag{5.2}$$

where Δx is the distance between two points along the beam axes and λ is the correlation length, assumed in this example equal to $\lambda = 0.8L$. The dis-

a)



b)

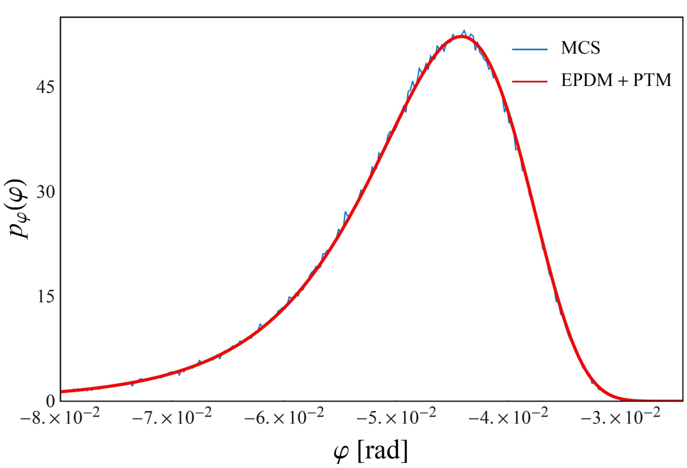


FIG. 7. PDF of the vertical displacement (a) and rotation (b) of the cantilever free end; correlated random variables α_i.

cretization is made by means of four beam-type FEs of equal length. The mid-point method is adopted to discretise the random field by four random variables α_i, so that a random variable is representative of the fluctuation of the Young modulus in each element.

By the application of the EPDM + PTM approach and paying attention to the fact that the beam-type FE is characterized by $n_p = 2$, it is possible obtain the exact PDF of any transversal displacement. The PDFs of the vertical displacement and of the rotation of the free end section are given in Figs. 6 and 7 for both the cases of uncorrelated and correlated random variables α_i. Once again the EPDM + PTM approach is compared with the MCS applied in Eq. (2.1), performed by 5×10^5 samples. Even in this example, the results are practically overlapped.

It is important to note that the proposed gives the exact results even when the uncertainties are strongly correlated, as it must be, due to the fact that the correlation properties of the uncertainties have no influence on the fundamental steps of the EPDM + PTM approach, but only on the definition of the input joint PDF.

5.3. Two-dimensional FE

As the last example, the two-dimensional panel under plane stress of Fig. 8 has been analysed utilising the two-dimensional FE. The two-dimensional element considered herein is the simple triangular element with 6 DOF (each node has two degrees of freedom, the displacements u_x and u_y). The following data are assumed as known (deterministic) input parameters: the length is $L = 6\,\mathrm{m}$, the height is $H = 3\,\mathrm{m}$, the Poisson coefficient is equal to 0.2 and the external actions are two uniformly distributed loads with intensity $p = q = 1000\,\mathrm{kN/m}$. The Young modulus is uncertain and modelled by a two-dimensional stochastic

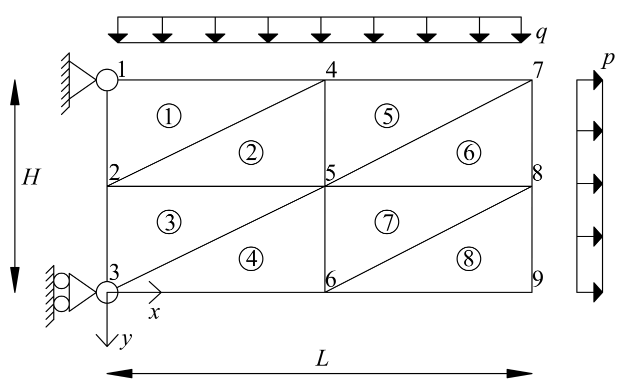


Fig. 8. Two-dimensional panel.

field with constant mean value $E_0 = 3 \times 10^7\,\mathrm{kN/m^2}$ and expressed as:

$$E = E_0(1 + \alpha(x, y)). \tag{5.3}$$

The zero mean two-dimensional stochastic field $\alpha(x, y)$ is assumed as Gaussian with the squared exponential covariance function:

$$\mathbf{\Sigma}_\alpha(|\Delta\mathbf{x}|) = \sigma^2\rho(|\Delta\mathbf{x}|) = \sigma^2 \exp\left(-\frac{|\Delta\mathbf{x}|}{\lambda}\right)^2 \tag{5.4}$$

where $|\Delta\mathbf{x}|$ is the distance between two points of the field, σ^2 is the variance and $\lambda = 0.2\mathrm{H}$ is the correlation coefficient. The midpoint method is adopted to discretise the random field by eight random variables α_i, so that a random variable

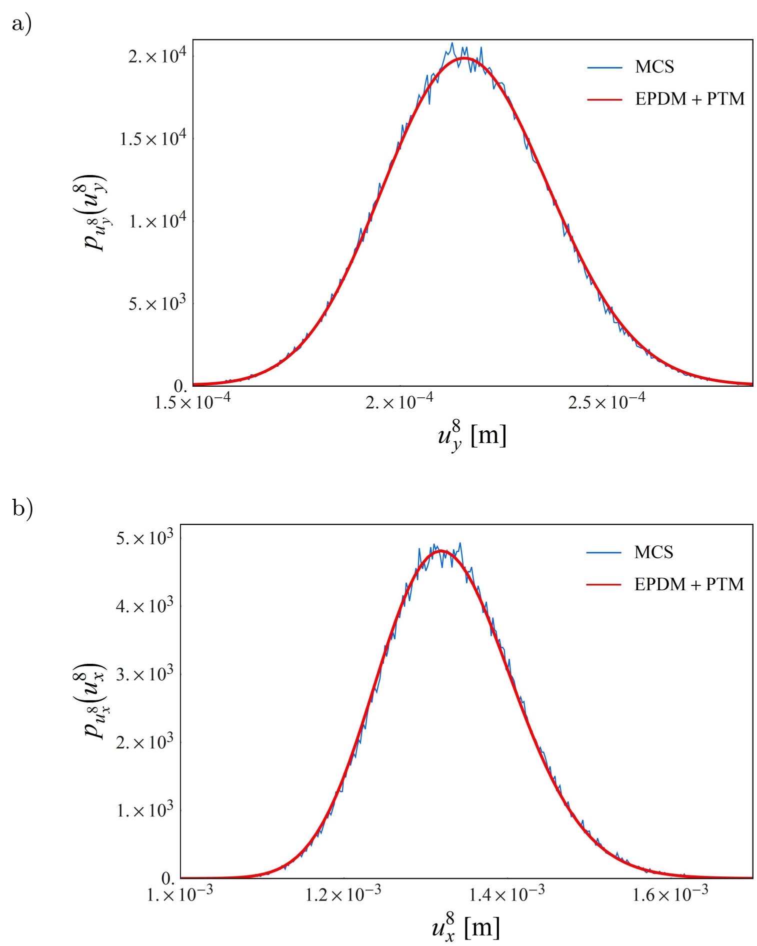


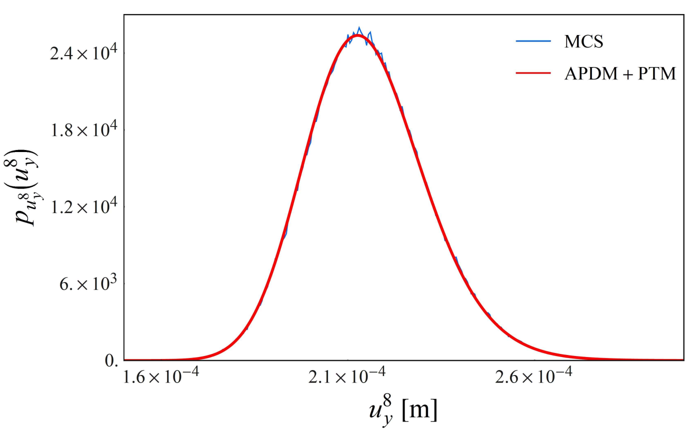
Fig. 9. PDFs of the vertical (a) and horizontal (b) displacement of node 8; correlated random variables α_i.

is representative of the fluctuation of the Young modulus in each element. Three natural deformations are present in the generic element, hence, for this example $n_p = 3$.

The PDFs of the horizontal and vertical displacement of the node 8, respectively, are depicted in Figs. 9, according to the proposed method and for $\sigma^2 = (0.1)^2$. The comparison has been made with respect to the classical Monte Carlo simulation, performed by 2×10^5 samples.

In order to confirm that the correlation assumptions have no influence on the capability of the proposed approach to give the exact response PDF for statically determined uncertain structures, the same panel before studied is considered under the assumption of uncorrelated random variables α_i. The confirming results are represented in Figs. 10.

a)



b)

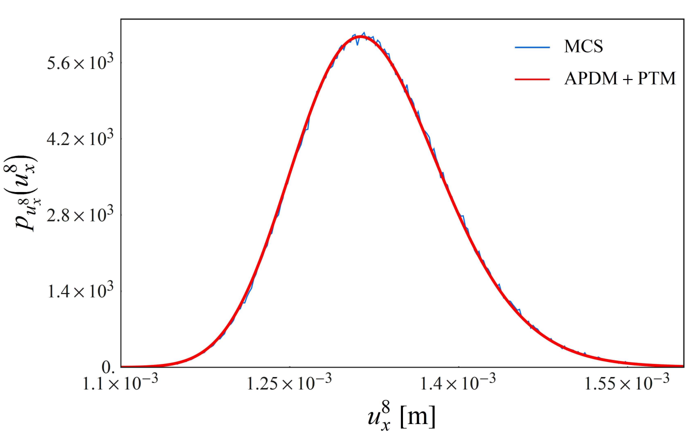


Fig. 10. PDF of the vertical (a) and horizontal (b) displacement of node 8. Uncorrelated random variables α_i.

6. Conclusions

The goal of identifying a strategy for the evaluation of the exact response PDF of discretized statically determinate uncertain structures has been reached. This thanks to the application of the EPDM, which gives exact results if applied to the statically determinate structures, coupled with the application of the PTM able to give explicitly the response PDF once that the relationships between the response random variables and the uncertainty random variables are explicitly given. When the uncertain structure is discretized by a FE approach, attention must be paid to the FE type used because it determines the number n_p of principal deformation modes influenced by each uncertainty random variable and this number influences the form of the expressions to be used in the analysis. It is important to note that this is a peculiar property of the APDM approach that cannot be stressed if other projection methods are used for discretizing the uncertainty random fields.

At last, the applications have confirmed the prevised results, regardless of the level of correlation of the uncertainty random variables.

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