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Mathematical modeling and stochastic stability analysis of viscoelastic nanobeams using higher-order nonlocal strain gradient theory

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This paper analyzes stochastic vibrations of a viscoelastic nanobeam under axial loadings. Based on the higher-order nonlocal strain gradient theory and the Liapunov functional method, bounds of the almost sure asymptotic stability of a nanobeam are obtained as a function of retardation time, variance of the stochastic force, higher-order and lower-order scale coefficients, strain gradient length scale, and intensity of the deterministic component of axial loading. Analytical results from this study are first compared with those obtained from the Monte Carlo simulation. Numerical calculations are performed for the Gaussian and harmonic non-white processes as models of axial forces.

Key words: stochastic vibrations, strain gradient theory, Liapunov method, Gaussian and harmonic process, Monte Carlo simulation.

1. Introduction

The behavior of nanosystems is a major subject of scientific research in various fields. Vibration and stability of a single-walled carbon nanotube conveying nanoflow was analyzed in [1]. According to the viscoelastic behavior of biological soft tissues, the single-walled carbon nanotube was assumed to be embedded in a Kelvin–Voigt foundation. The critical point instability of micro or nanobeams under a distributed variable-pressure force was studied in [2]. For this purpose, the inhomogeneous nonlocal theory was introduced based on conformable fractional derivatives.

Within the framework of integral formulation of Eringen's theory, Oskouie *et al.* [3] developed a novel numerical approach for the bending analysis of Euler–Bernoulli nanobeams in the context of strain- and stress-driven integral nonlocal models. The dynamic problems of single beams based on various theories have been studied by many researchers. A unified approach to field theories for elastic solids, viscous fluids, and heat-conducting electromagnetic solids and fluids that include nonlocal effects in both space and time is presented in [4]. Atanasov *et al.* [5] investigated the free vibration and buckling problem of the

Euler-Bernoulli double-microbeam system under compressive axial loading with a temperature change effect. By using various nonlocal beam theories, analytical solutions of bending, vibration and buckling were presented by Reddy [6]. The natural frequencies of the bending vibrations of a nanocantilever with a linearly changed cross-section were obtained by Aranda-Ruiz *et al.* [7]. Based on the nonlinear Timoshenko beam theory, the governing equations for functionally graded porous (FGP) beams are presented in [8]. Based on Euler–Bernoulli and Timoshenko beam theories in conjunction with the nonlocal elasticity theory of Eringen, the static analysis of nanobeams was performed in the work of Behera and Chakraverty [9]. In this work, the Rayleigh–Ritz method is used to convert the problem into a system of linear equations. Mobki *et al.* [10] investigated the size-dependent behavior of an electrostatically-actuated nanobeam considering VdW and Casimir forces. The comparison of the pull-in voltage, detachment length and natural frequency is presented in both classic and modified couple stress theories for aluminum nano-beams.

By using the Liapunov functional method the almost sure stability of the symmetrically laminated cross-ply viscoelastic plates was investigated in [11]. The dynamic stability of a viscoelastic nanobeam subjected to compressive stochastic loading, where rotary inertia is taken into account is investigated by Pavlović *et al.* [12]. The same author in works of [13] and [14] studied dynamic stability and instability of coupled nanobeams and multi-nanobeam systems.

In [15], using the nonlocal elasticity and strain gradient theories, the higher-order nonlocal strain gradient theory was presented. The author established the fact that the length scales presented in the nonlocal elasticity and strain gradient theory describe two entirely different physical characteristics of materials and structures at the nanoscale. The nonlocal elasticity theory does not include nonlocality of higher-order stresses while the common strain gradient theory only considers local higher-order strain gradients without nonlocal effects in a global sense.

Based on the higher-order nonlocal strain gradient theory a new size-dependent plate model was developed in [16]. Using the nonlocal strain gradient elasticity theory and the Euler–Bernoulli beam model, surface and thermal effects on the vibration characteristics of viscoelastic functionally graded (FG) nanobeams embedded in a viscoelastic foundation were investigated by Ebrahimi and Barati [17]. This paper shows the effects of surface stress, length scale parameter, nonlocal parameter, viscoelastic medium, internal damping constant, thermal loading, power-law index, and boundary conditions on the vibration frequencies of viscoelastic FGM nanobeams.

To avoid localization problems, a nonlocal model of strain damage was formulated in [18], in which the stress decomposition consistently follows from the thermodynamic analysis. Starting from the nonlocal elastic constitutive model proposed by Eringen and co-workers, Sciarra [19] formulated the thermodynamic framework and the boundary-value problem for nonlocal elasticity by the recourse to convex analysis and provided the complete set of nonlocal mixed variational principles. An innovative stress-driven integral model for size-dependent structural behavior of inflected elastic Timoshenko nanobeams was proposed by Baretta *et al.* [20] to overcome the inconsistencies of Eringen's strain-driven theory.

The present paper deals with stochastic vibrations of the Euler–Bernoulli viscoelastic nanobeam. Using the higher-order strain gradient theory, stability analysis was performed in the function of the retardation time, variance of the stochastic force, higher-order and lower-order scale coefficients, strain gradient length scale, and intensity of the deterministic component of axial loading. According to the tensor notation, the nonlocal constitutive relations are given in Section 2 by using the higher-order strain gradient theory. A partial differential equation of transverse motion of the Euler–Bernoulli nanobeam based on Eringen's nonlocal elasticity theory and higher-order strain gradient theory is derived in Section 3. For the governing differential equation of the nanobeam, the definition of the almost-sure stability problem is given in Section 4. For non-white excitation by using the Liapunov functional method, the conditions of almost-sure stability are obtained in Section 5. The numerical procedure of determining the boundaries of stability, as well as the analysis of obtained results, is given in Section 6. In this section analytical results are firstly validated by comparing them with the numerical results obtained from the Monte Carlo simulation. Finally, the conclusion is given in Section 7.

2. Higher-order nonlocal strain gradient theory

In [12] the case of a viscoelastic nanobeam was considered where the nonlocal stress in a certain point depends on the deformations of all points of the body. Thus, the nonlocal internal energy density potential is

$$U_0 = (\varepsilon_{ij}, \varepsilon'_{ij}\alpha_0) = \frac{1}{2} C_{ijkl}\varepsilon_{ij} \int\limits_V \alpha_0(|\mathbf{x} - \mathbf{x}'|, e_0 a)\varepsilon'_{kl}\, dV', \tag{2.1}$$

where ε_{ij} and ε'_{ij} are the Cartesian components of the strain tensors at point $\mathbf{x}$ and point $\mathbf{x}'$, respectively, C_{ijkl} is the elastic modulus tensor of classical elasticity, e_0 is the nonlocal material constant, a is the internal characteristic length, and is the principal attenuation kernel related to the nonlocality effect in terms of the Euclidean distance between point $\mathbf{x}$ and neighbouring points $\mathbf{x}'$ within a domain V.

Unlike the classical nonlocal theory used in reference [12], in this study we use the higher-order strain gradient theory where the internal energy density potential U_0 can be written as [15]

$$\begin{aligned}(2.2)\qquad & U_0(\varepsilon_{ij},\varepsilon'_{ij},\varepsilon_{ij,m},\varepsilon'_{ij,m},\alpha_1)\\ &= \frac{1}{2}C_{ijkl}\varepsilon_{ij}\int\limits_V \alpha_0(|\mathbf{x}-\mathbf{x}'|,e_0a)\varepsilon'_{kl}\,dV' + \frac{l^2}{2}C_{ijkl}\varepsilon_{ij,m}\int\limits_V \alpha_1(|\mathbf{x}-\mathbf{x}'|,e_1a)\varepsilon'_{kl,m}\,dV',\end{aligned}$$

where l is the material length scale introduced to determine the significance of the higher-order strain gradient field, $\alpha_1(|\mathbf{x}-\mathbf{x}'|,e_1a)$ is the additional attenuation kernel function introduced to describe the nonlocal effect of the first-order strain gradient field, and e_1 is the related material constant.

Based on Eq. (2.2), the classical stress tensor $\boldsymbol{\sigma}$, the higher-order stress tensor $\boldsymbol{\sigma}^{(1)}$ and the total stress of the nonlocal strain gradient theory $\boldsymbol{t}$ are [15]

$$(2.3)\qquad \boldsymbol{\sigma} = \int\limits_{V'} \alpha_0(\mathbf{x}',\mathbf{x},e_0a)\mathbf{C} : \boldsymbol{\varepsilon}'\,dV',$$

$$(2.4)\qquad \boldsymbol{\sigma}^{(1)} = l^2\int\limits_{V'} \alpha_1(\mathbf{x}',\mathbf{x},e_1a)\mathbf{C} : \nabla\boldsymbol{\varepsilon}'\,dV',$$

$$(2.5)\qquad \boldsymbol{t} = \boldsymbol{\sigma} - \nabla\boldsymbol{\sigma}^{(1)}.$$

In Eqs. (2.3) and (2.4) the symbol ":" is used to denote the double-dot product. Romano *et al.* [21] showed that the nonlocal integral elastic low is equivalent to a problem composed of constitutive differential and boundary conditions. According to [22], the constitutive boundary conditions do not conflict with the equilibrium and provide a viable approach to study size-dependent phenomena in nanobeams.

According to the higher-order nonlocal strain gradient theory, the general and extended constitutive equation in a differential form is proposed in the following form [15]

$$\begin{aligned}(2.6)\qquad & [1-(e_1a)^2\nabla^2][1-(e_0a)^2\nabla^2]t_{ij}\\ &= C_{ijkl}[1-(e_1a)^2\nabla^2]\varepsilon_{kl} - C_{ijkl}l_0^2[1-(e_0a)^2\nabla^2]\nabla^2\varepsilon_{kl},\end{aligned}$$

where ∇^2 is the Laplacian operator. For $e_0 = e_1 = e$ Eq. (2.6) obtains a simpler form where the lower-order nonlocal strain gradient constitutive equation can be written as

$$(2.7)\qquad [1-(ea)^2\nabla^2]t_{ij} = C_{ijkl}[1-l_0^2\nabla^2]\varepsilon_{kl},$$

and for $e = 0$, the constitutive equation of the pure strain gradient model is obtained

$$t_{ij} = C_{ijkl}[1 - l_0^2\nabla^2]\varepsilon_{kl}. \tag{2.8}$$

Also, for $l_0 = e_1 = 0$ in Eq. (2.6) the constitutive equation of Eringen's nonlocal theory [4] is

$$[1 - (e_0a)^2\nabla^2]t_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{2.9}$$

3. Governing equations for a viscoelastic Euler–Bernoulli nanobeam

Let us consider an axially compressed beam of the length and transverse loading per unit length q. Dynamic equations according to the typical beam element are

$$\begin{aligned} \rho A\frac{\partial^2 W}{\partial T^2} &= \frac{\partial V}{\partial X} + q, \\ \frac{\partial M}{\partial X} &= V + H\frac{\partial W}{\partial X}, \end{aligned} \tag{3.1}$$

where ρ is the mass density, A is the cross-sectional area, $W = W(X,T)$ is the transverse displacement, T is the time, V is the shear force due to bending, and $M$is the bending moment.

According to the Euler–Benoulli beam theory and higher-order nonlocal strain gradient theory

$$\varepsilon_{XX} = -Z\frac{\partial^2 W}{\partial X^2}, \qquad M = \int_A Zt_{xx}\,dA, \tag{3.2}$$

$$\left[1 - (e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left[1 - (e_0a)^2\frac{\partial^2}{\partial X^2}\right]t_{XX} = \tag{3.3}$$
$$EI\left[1-(e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left(\varepsilon_{XX}+\tau_d\frac{\partial\varepsilon_{XX}}{\partial T}\right)-El_0^2\left[1-(e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left(\frac{\partial^2\varepsilon_{XX}}{\partial X^2}+\tau_d\frac{\partial^3\varepsilon_{XX}}{\partial X^2\partial T}\right),$$

where ε_{XX} is the normal strain and E is the Young modulus.

A combination of Eqs. (3.2) and (3.3) further gives

$$\begin{aligned} &\left[1 - (e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left[1 - (e_0a)^2\frac{\partial^2}{\partial X^2}\right]M \\ &= -EI\left\{\left[1 - (e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left(1 + \tau_d\frac{\partial}{\partial T}\right)\right. \\ &\qquad \left. - l_0^2\left[1 - (e_0a)^2\frac{\partial^2}{\partial X^2}\right]\left(\frac{\partial^2}{\partial X^2} + \tau_d\frac{\partial^3}{\partial X^2\partial T}\right)\right\}\frac{\partial^2 W}{\partial X^2}. \end{aligned} \tag{3.4}$$

Finally, after eliminating V and M from Eqs. (3.1) and (3.4), one obtains the partial differential equation for transverse displacement W

$$\tag{3.5} \left[1-(e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left[1-(e_0a)^2\frac{\partial^2}{\partial X^2}\right]\left[\rho A\frac{\partial^2 W}{\partial T^2}+\frac{\partial}{\partial X}\left(\frac{\partial H}{\partial X}\right)-q\right] + EI\left\{\left[1-(e_1a)^2\frac{\partial^2}{\partial X^2}\right]-l_0^2\left[1-(e_0a)^2\frac{\partial^2}{\partial X^2}\right]\frac{\partial^2}{\partial X^2}\right\}\left(1+\tau_d\frac{\partial}{\partial T}\right)\frac{\partial^4 W}{\partial X^4}=0.$$

4. Problem formulation

According to relations (3.5), taking $q = 0$, the eighth order differential equation for transverse vibrations of the nanobeam in terms of the spatial variable can be written as

$$\tag{4.1} \left[1-(e_1a)^2\frac{\partial^2}{\partial X^2}\right]\left[1-(e_0a)^2\frac{\partial^2}{\partial X^2}\right]\left[\rho A\frac{\partial^2 W}{\partial T^2}+F(T)\frac{\partial^2 W}{\partial X^2}\right] + EI\left\{\left[1-(e_1a)^2\frac{\partial^2}{\partial X^2}\right]-l_0^2\left[1-(e_0a)^2\frac{\partial^2}{\partial X^2}\right]\frac{\partial^2}{\partial X^2}\right\}\left(1+\tau_d\frac{\partial}{\partial T}\right)\frac{\partial^4 W}{\partial X^4}=0,$$

where $F(T)$ presents the beam axial load.

For the simply supported edges, by using classical and non-classical boundary conditions, and the procedure given by Li *et al.* [23], we can formulate our boundary conditions

$$\tag{4.2} \left.\begin{matrix}X=0\\X=L\end{matrix}\right\}\; W=0,\ M=0\ \Rightarrow\ W=0,\ \frac{\partial^2 W}{\partial X^2}=0,\ \frac{\partial^4 W}{\partial X^4}=0,\ \frac{\partial^6 W}{\partial X^6}=0.$$

Now, the following parameters are used to non-dimensionalize Eq. (4.1)

$$\tag{4.3} \begin{gathered} T=k_t t,\quad W=Lw,\quad X=Lx,\\ k_t=L^2\sqrt{\frac{\rho A}{EI}},\quad 2\eta=\frac{\tau_d}{k_t},\quad f_0+f(t)=\frac{F(k_t t)L^2}{EI},\quad l=\frac{l_0}{L}, \end{gathered}$$

where η is the reduced retardation time, f_0 and $f(t)$ are the reduced constant and stochastic component of the axial force. After replacing (4.3) in (4.1) the following non-dimensionalized form of Eq. (4.1) is obtained

$$\tag{4.4} \mathcal{L}_\sigma\left[\frac{\partial^2 w}{\partial t^2}+(f_0+f(t))\frac{\partial^2 w}{\partial x^2}\right]+\mathcal{L}_\varepsilon\left(\frac{\partial^4 w}{\partial x^4}+2\eta\frac{\partial^5 w}{\partial x^4\partial t}\right)=0,$$

where

$$\tag{4.5} \begin{aligned} \mathcal{L}_\sigma&=\left(1-\mu_1^2\frac{\partial^2}{\partial x^2}\right)\left(1-\mu_0^2\frac{\partial^2}{\partial x^2}\right),\\ \mathcal{L}_\varepsilon&=\left(1-\mu_1^2\frac{\partial^2}{\partial x^2}\right)-l^2\left(1-\mu_0^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2}{\partial x^2},\qquad \mu_i=\frac{e_i a}{L},\ i=0,1. \end{aligned}$$

5. Liapunov functional method

The Liapunov functional method is an efficient tool for the analysis of the behavior of stochastic systems. This method is explained in detail in the authors' previous works [11–13]. According to the procedure performed in these studies, for the viscoelastic nanobeam given by (4.4) the appropriate expressions for the functional and its time derivative are determined.

So, the functional $\mathbf{V}$ gets the following form

$$\begin{aligned}(5.1)\quad \mathbf{V} = \int_0^1 \Bigg\{ & \left(\mathcal{L}_\sigma \frac{\partial w}{\partial t} + \eta \mathcal{L}_\varepsilon \frac{\partial^4 w}{\partial x^4}\right)^2 + \eta^2 \left(\mathcal{L}_\varepsilon \frac{\partial^4 w}{\partial x^4}\right)^2 \\ & - f_0 \left(\mathcal{L}_\varepsilon \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + (\mu_0^2 + 2\mu_1^2 + l^2)\left(\frac{\partial^3 w}{\partial x^3}\right)^2 \\ & + [\mu_0^2\mu_1^2 + (\mu_0^2 + \mu_1^2)(\mu_1^2 + l^2) + \mu_0^2 l^2]\left(\frac{\partial^4 w}{\partial x^4}\right)^2 \\ & + \mu_0^2[\mu_1^2(\mu_1^2 + l^2) + l^2(\mu_0^2 + \mu_1^2)]\left(\frac{\partial^5 w}{\partial x^5}\right)^2 + \mu_0^4\mu_1^2 l^2 \left(\frac{\partial^6 w}{\partial x^6}\right)^2 \Bigg\} dx,\end{aligned}$$

and the time derivative of the functional is

$$\begin{aligned}(5.2)\quad \frac{d\mathbf{V}}{dt} = 2\int_0^1 \Bigg\{ & \eta \mathcal{L}_\sigma \left(\frac{\partial w}{\partial t}\right) \mathcal{L}_\varepsilon \left(\frac{\partial^5 w}{\partial x^4 \partial t}\right) + \eta f_0 \mathcal{L}_\sigma \left(\frac{\partial^2 w}{\partial x^2}\right) \mathcal{L}_\varepsilon \frac{\partial^4 w}{\partial x^4} \\ & + f(t) \mathcal{L}_\sigma \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\mathcal{L}_\sigma \frac{\partial w}{\partial t} + \eta \mathcal{L}_\varepsilon \frac{\partial^4 w}{\partial x^4}\right) + \eta \left(\mathcal{L}_\varepsilon \frac{\partial^4 w}{\partial x^4}\right)^2 \Bigg\} dx.\end{aligned}$$

The measure of the solution may be taken as $\|w\| = \sqrt{\mathbf{V}}$ and the functional $\mathbf{V}$ must be positive definite. It will be fulfilled if

$$\begin{aligned}(5.3)\quad \int_0^1 \Bigg\{ & -f_0 \left(\mathcal{L}_\sigma \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + (\mu_0^2 + 2\mu_1^2 + l^2)\left(\frac{\partial^3 w}{\partial x^3}\right)^2 \\ & + [\mu_0^2\mu_1^2 + (\mu_0^2 + \mu_1^2)(\mu_1^2 + l^2) + \mu_0^2 l^2]\left(\frac{\partial^4 w}{\partial x^4}\right)^2 \\ & + \mu_0^2[\mu_1^2(\mu_1^2 + l^2) + l^2(\mu_o^2 + \mu_1^2)]\left(\frac{\partial^5 w}{\partial x^5}\right)^2 + \mu_0^4\mu_1^2 l^2 \left(\frac{\partial^6 w}{\partial x^6}\right)^2 \Bigg\} dx \geq 0.\end{aligned}$$

According to (4.2), the solutions can be written in the following form

$$(5.4)\quad w(x,t) = \sum_{m=1}^{\infty} T_m(t) \sin \alpha_m x,$$

where $\alpha_m = m\pi$, and relation (5.3) is reduced to

$$f_0 \leq \alpha_m^2 \frac{1 + \mu_1^2\alpha_m^2 + l^2\alpha_m^2(1 + \mu_0^2\alpha_m^2)}{(1 + \mu_1^2\alpha_m^2)(1 + \mu_0^2\alpha_m^2)}, \tag{5.5}$$

which presents the static stability condition of a nanobeam.

For the lower-order nonlocal strain gradient theory, $e_0 = e_1 = e$, $(\mu_0 = \mu_1 = \mu)$, the condition of static stability of a nanobeam is

$$f_0 \leq \alpha_m^2 \frac{1 + l^2\alpha_m^2}{1 + \mu^2\alpha_m^2}, \tag{5.6}$$

and for the pure strain gradient model $e_0 = e_1 = 0$, $(\mu_0 = \mu_1 = 0)$

$$f_0 \leq \alpha_m^2(1 + l^2\alpha_m^2). \tag{5.7}$$

The Liapunov functional is positive definite if the deterministic component of axial loading is smaller than the critical static buckling load which is equivalent to relation (5.5).

Now, let define us a scalar function $\lambda(t)$ as

$$\frac{1}{\mathbf{V}} \frac{\mathbf{V}}{dt} \leq \lambda(t). \tag{5.8}$$

Since the maximum point is a particular case of the stationary point, we may write

$$\delta(\dot{\mathbf{V}} - \lambda\mathbf{V}) = 0. \tag{5.9}$$

Using the associated Euler's equations leads to the following expression

$$\begin{aligned}
&\left(\lambda\mathcal{L}_\sigma^{(2)} + 2\eta\mathcal{L}_\sigma\mathcal{L}_\varepsilon\frac{\partial^4}{\partial x^4}\right)\nu + \left(\lambda\eta\mathcal{L}_\sigma\mathcal{L}_\varepsilon\frac{\partial^4}{\partial x^4} + f(t)\mathcal{L}_\sigma^{(2)}\frac{\partial^2}{\partial x^2}\right)w = 0,\\
&\left(\lambda\eta\mathcal{L}_\sigma\mathcal{L}_\varepsilon\frac{\partial^4}{\partial x^4} + f(t)\mathcal{L}_\sigma^{(2)}\frac{\partial^2}{\partial x^2}\right)\nu + \left\{\lambda\left[2\eta^2\mathcal{L}_\varepsilon^{(2)}\frac{\partial^8}{\partial x^8} + f_0\mathcal{L}_\sigma^{(2)}\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right.\right.\\
&\quad - (\mu_0^2 + 2\mu_1^2 + l^2)\frac{\partial^6}{\partial x^6} + \left.[\mu_0^2\mu_1^2 + (\mu_0^2 + \mu_1^2)(\mu_1^2 + l^2) + \mu_0^2 l^2]\right]\frac{\partial^8}{\partial x^8}\\
&\quad - \mu_0^2[\mu_1^2(\mu_1^2 + l^2) + l^2(\mu_0^2 + \mu_1^2)]\frac{\partial^{10}}{\partial x^{10}} + \mu_0^4\mu_1^2 l^2\frac{\partial^{12}}{\partial x^{12}} + 2\eta f_0\mathcal{L}_\sigma\mathcal{L}_\varepsilon\frac{\partial^6}{\partial x^6}\\
&\quad \left. + 2\eta f(t)\mathcal{L}_\sigma\mathcal{L}_\varepsilon\frac{\partial^6}{\partial x^6} + 2\eta\mathcal{L}_\varepsilon^{(2)}\frac{\partial^8}{\partial x^8}\right\}w = 0,
\end{aligned} \tag{5.10}$$

where $\nu = \partial w/\partial t$.

According to the boundary condition (4.2), and solution (5.4), from (5.10) we obtain

$$\lambda_m^2 + b_m\lambda_m - c_m = 0, \tag{5.11}$$

where

$$\begin{aligned} b_m &= 4\eta\frac{\alpha_m^4 L_{2m}}{L_{1m}},\\ c_m &= \alpha_m^4\frac{L_{1m}^4 f(t)^2 + 4\eta^2\alpha_m^4 L_{1m}L_{2m}^2[f_0\alpha_m^2 L_{1m} + f(t)\alpha_m^2 L_{1m} - \alpha_m^4 L_{2m}]}{L_{1m}^2(\eta^2\alpha_m^8 L_{2m}^2 - f_0\alpha_m^2 L_{1m}^2 + \alpha_m^4 L_{1m}L_{2m})},\\ L_{1m} &= (1+\mu_1^2\alpha_m^2)(1+\mu_0^2\alpha_m^2), \quad L_{2m} = 1+\mu_1^2\alpha_m^2 + l^2\alpha_m^2(1+\mu_0^2\alpha_m^2). \end{aligned} \tag{5.12}$$

Now, solving Eq. (5.11) gives the function in the form

$$\lambda_m = -2\eta\frac{\alpha_m^4 L_{2m}}{L_{1m}} + \frac{\alpha_m^2}{L_{1m}}\,\frac{|2\eta^2\alpha_m^6 L_{2m}^2 + f(t)L_{1m}^2|}{\sqrt{\eta^2\alpha_m^8 L_{2m}^2 - f_0\alpha_m^2 L_{1m}^2 + \alpha_m^4 L_{1m}L_{2m}}}. \tag{5.13}$$

According to the Schwarz inequality and Eq. (5.13), the following expression is obtained

$$\sigma^2 \le 4\eta^2\alpha_m^2\frac{L_{2m}^2}{L_{1m}^2}(\alpha_m^2 L_{1m}L_{2m} - f_0\alpha_m^2). \tag{5.14}$$

By solving the differential inequality (5.8), we can estimate the values of the functional $\mathbf{V}$

$$\frac{d\mathbf{V}}{dt} \le (\max_m \lambda_m)\mathbf{V}, \tag{5.15}$$

and after integrating relation (5.15) with respect to time, we obtain

$$\mathbf{V} \le \mathbf{V}_0 \exp\left[\frac{1}{t}\int_0^t \max_m \lambda_m(\tau)\,d\tau\right]t. \tag{5.16}$$

When the process $f(t)$ is ergodic and stationary, it can be concluded that the trivial solution of Eq. (4.1) is almost surely asymptotically stable if

$$E\{\max_m \lambda_m(t)\} < 0, \tag{5.17}$$

where $E\{\cdot\}$ is the mathematical expectation operator.

6. Numerical results and discussion

Firstly, the comparison between analytical and numerical results is given with the aim of approving the Liapunov functional method for this system. For that purpose, the analytically obtained results using the Liapunov functional method, the approximate results given by (5.14) and the numerical results obtained from the Monte Carlo simulation [24, 25] are compared. The numerically determined results are very important in assessing the validity and the ranges of applicability of the approximate analytical results.

In order to perform the Monte Carlo simulation for the observed system, Eq. (4.4) is first discreetizated. Thus, by substituting relation (5.4) in (4.4), the following discreetizated form of Eq. (4.4) is obtained and schematically presented in Fig. 1.

$$\ddot{T}_m + 2\eta \frac{\alpha_m^4 L_{2m}}{L_{1m}} \dot{T}_m + \left[\frac{\alpha_m^4 L_{2m}}{L_{1m}} - \alpha_m^2 (f_0 + f(t)) \right] T_m = 0. \tag{6.1}$$

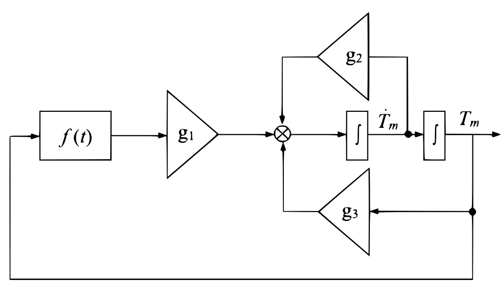

Fig. 1. Simulation scheme according to Eq. (6.1).

According to Eq. (6.1), the gains g_1, g_2 and g_3 in Fig. 1 are α_m^2, $2\eta \frac{\alpha_m^4 L_{2m}}{L_{1m}}$ and $\frac{\alpha_m^4 L_{2m}}{L_{1m}}$, respectively.

Similarly, by substituting (5.4) in (5.2), the time derivative of the functional becomes

$$\begin{aligned} \eta L_{1m} L_{2m} \alpha_m^4 \dot{T}_m^2 - f(t) \alpha_m^2 L_{1m}^2 \dot{T}_m T_m \\ + \eta \alpha_m^6 L_{2m} [\alpha_m^2 L_{2m} - L_{1m}(f_0 + f(t))] T_m^2 \geq 0. \end{aligned} \tag{6.2}$$

Now, according to the simulation scheme presented in Fig. 1, the Monte Carlo simulation was performed where the number of simulations and step size were $N = 10000$ and $\Delta t = 0.01$ s. After replacing the estimated states obtained from the simulation in Eq. (6.2) the new pairs of variances and damping coefficients were numerically obtained and compared with the analytical results.

The comparison between the analytical results (solid line), the approximated results (dash line) and the results obtained by the Monte Carlo simulation (dots) is presented in Fig. 2. As it is shown in this figure, the results from the Monte Carlo simulation properly match the analytical results, which justifies the use of the direct Liapunov method for the stochastic stability analysis of a viscoelastic nanobeam. Figure 2 also presents the approximated results obtained from the Schwarz inequality and (5.14) and provides an initial approximate insight into the boundary of almost sure stability, being valid for any stochastic process. The knowledge of the probability density function of the stochastic process gives larger stability regions (e.g. solid and dotted lines in Fig. 2 are calculated for the Gaussian process). Although there is a great distance between the approximate, on the one hand, and the analytic and numerical results, on the other, the results from relation (5.14) can be very helpful when numerical computation cannot be performed.

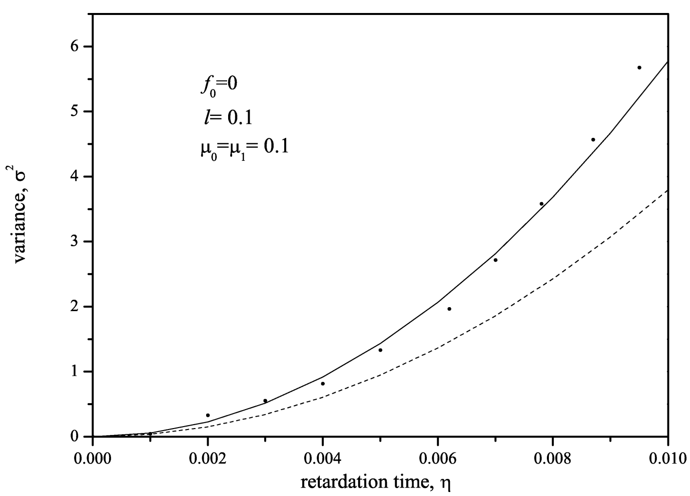

FIG. 2. A comparison of analytical, numerical and approximated results.

Relation (5.16) yields the possibility to obtain the almost sure stability domains of the nanobeam by using the higher-order strain gradient theory. The stability domains of the nanobeam are obtained for the Gaussian process and the harmonic process given by

$$f_h(t) = A_h \cos(\omega t + \theta), \tag{6.3}$$

where the phase θ is a random variable and the amplitude A_h is used to calculate the process variance $A_h^2/2$.

The solid line present the almost sure asymptotic stability boundaries for the Gaussian, while the dash line presents the boundaries for the harmonic process. Calculations are made by using Gauss-Christoffel quadratures; the parameters of Gauss–Hermitequadratures are used for the Gaussian process, and Gauss–Chebyshev quadratures for the harmonic process.

Numerical determination of stability regions according to relation (5.17) is not a simple task. The algebraic equation in which the unknown is in a subintegral expression should be solved, and at the same time one should calculate the maximum of the function $\lambda_m(\tau)$ for each iteration on the integer values of the mode m. To rapidly converge integrals to the accurate solution, the Gauss–Christoffel quadratures were used, and they are known to be fully accurate for all polynomials to a level equal to twice the number of the used nodes. In addition, for all functions that have limited derivatives on the observed interval, the Gauss–Christoffel quadratures give the most approximate integral value of all the quadrature formulas. The algebraic equation is solved with the secant method, whose convergence order is $(1+\sqrt{5})2$, where convergence to the correct solution is guaranteed for all continuous functions.

In Fig. 3 stability regions are plotted in the plane of variance and the damping coefficient for identical scale coefficients.

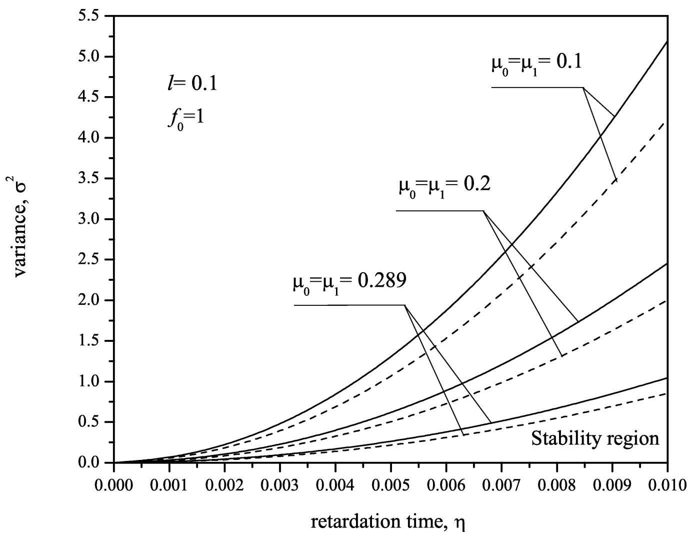

Fig. 3. Stability regions of a nanobeam for the Gaussian and harmonic process as a function of scale coefficients.

It is clear that the regions of the almost sure stability are greater when the scale parameter decreases. In general, nonlocal effects significantly reduce instability regions of the viscoelastic nanobeam.

Figures 4 and 5 present the stability surfaces for the Gaussian and harmonic process, respectively. The plot is given for the constant scale coefficient $\mu_0 = \mu_1 = 0.289$.

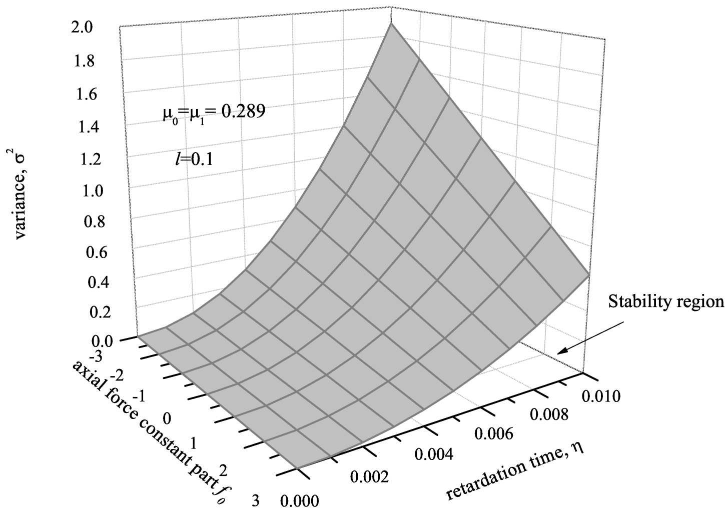

Fig. 4. Stability surface of a nanobeam for the Gaussian process in the function of the constant component of axial force.

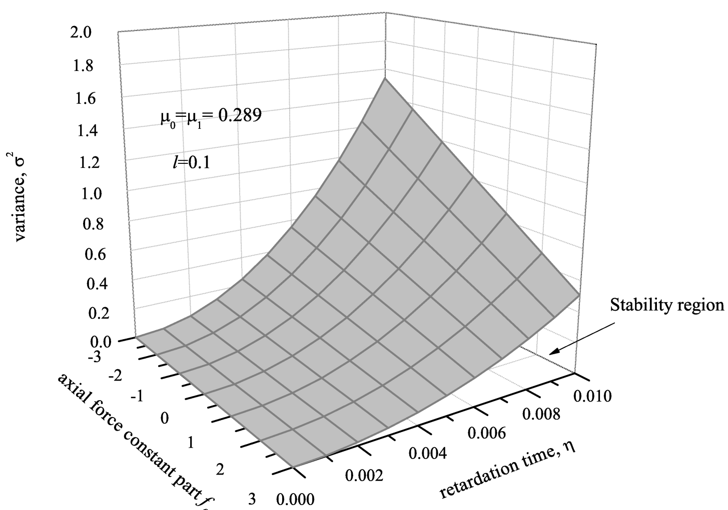

Fig. 5. Stability surface of a nanobeam for the harmonic process in the function of the constant component of axial force.

In both of these figures, stability areas are given in the function of the constant component of an axial force. As it can be seen, stability regions increase when the constant component of axial loading is changed from compression ($f_0 = 3$) to tension ($f_0 = -3$).

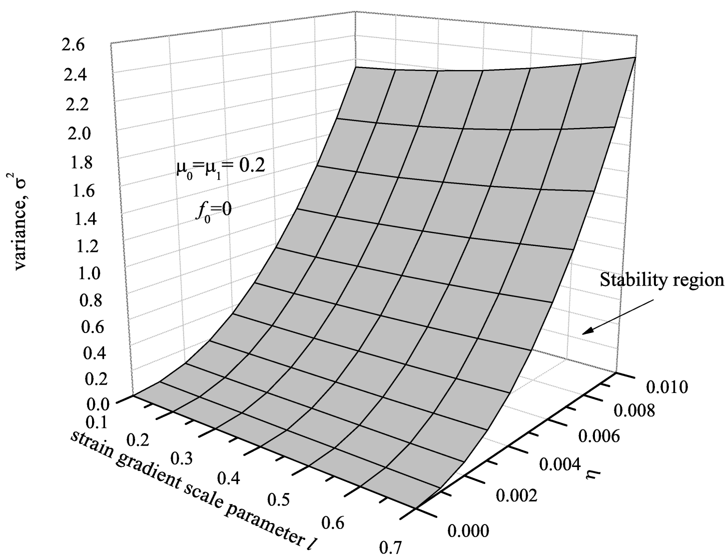

Fig. 6. Stability surface of a nanobeam for the Gaussian process in the function of the strain gradient parameter.

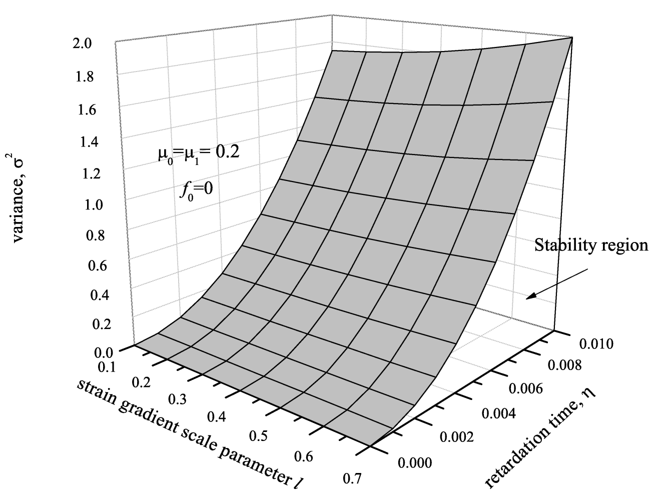

Fig. 7. Stability surface of a nanobeam for the harmonic process in the function of the strain gradient parameter.

Similarly, Figs. 6 and 7 present the stability surfaces for the Gaussian and harmonic process in the function of the strain gradient parameter l. As it is shown in these figures, this parameter has a great impact on the almost sure stability regions. It is notable that the growth of this parameter reduces the instability area.

7. Conclusions

By means of the direct Liapunov method, based on the higher-order strain gradient elasticity theory, the almost sure stability of a viscoelastic nanobeam subjected to compressive axial loadings is studied. The axial forces acting on its ends consist of a constant part and a time-dependent stochastic function. For "non-white" excitation modeled as Gausian and harmonic processes, the regions of almost sure stability are derived using the direct Liapunov method which is previously verified with the results obtained from the Monte Carlo simulation.

Bounds of almost sure stability are given in the function of the most important system parameters such as the deterministic components of axial loading, variance of the stochastic force, nanoscale coefficients and the strain gradient scale parameter.

It is shown that the scale parameter significantly enlarges instability regions for both of the observed processes, which can be reduced with the growth of the viscoelastic parameter. The almost sure stability regions increase when the constant component of axial loading is changed from compression to tension. Finally, the viscous damping coefficient and the strain gradient scale parameter increase the stability regions remarkably.

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