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Potential method in the theory of thermoelasticity for materials with triple voids

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In the present paper the linear theory of thermoelasticity for isotropic and homogeneous solids with macro-, meso- and microporosity is considered. In this theory the independent variables are the displacement vector field, the changes of the volume fractions of pore networks and the variation of temperature. The fundamental solution of the system of steady vibrations equations is constructed explicitly by means of elementary functions. The basic internal and external boundary value problems (BVPs) are formulated and the uniqueness theorems of these problems are proved. The basic properties of the surface (single-layer and double-layer) and volume potentials are established and finally, the existence theorems for regular (classical) solutions of the internal and external BVPs of steady vibrations are proved by using the potential method (boundary integral equation method) and the theory of singular integral equations.

Key words: thermoelasticity, triple voids, fundamental solution, steady vibrations, uniqueness and existence theorems.

1. Introduction

The construction of mathematical models of fluid flow through porous media and the intensive investigation of the problems of porous continua arise by the extensive use of porous materials into civil engineering, geotechnical engineering, technology, hydrology, and recent years, medicine and biology (for details see Bear [1], Coussy [2], Cowin [3], de Boer [4], Ichikawa and Selvadurai [5], Wang [6]).

There are many different approaches to a theoretical formulation of linear and nonlinear models for materials with single and multiple porosity. Wide information on the construction of mathematical models that describe phenomena of flow and transport in porous media is given in the new books of Bear [1], Das *et al.* [7] and Straughan [8]. Historically, mathematical models of single porosity materials were the first models in the theory of porous media (see e.g. de Boer [4]).

There are a number of theories which describe mechanical properties of single porosity materials (see de Boer [4], Cheng [9] and the references therein), and the most well known of them are Biot [10] consolidation theory based on Darcy's law and Nunziato-Cowin [11] theory based on the volume fraction concept.

Indeed, Nunziato and Cowin [11] introduced a theory for the behavior of single porous deformable materials in which the skeletal or matrix materials are elastic and the interstices are voids. On the basis of this model the linear theory of elastic materials with single voids is developed by the same authors in [12]. Then, Ieşan [13] presented a linear theory of thermoelastic materials with single voids. In this theory the independent variables are the displacement vector field, the change of the volume fraction of pores and the variation of temperature. The basic results on the these theories may be found in the books of Ciarletta and Ieşan [14], Ieşan [15], Straughan [16] and the references therein.

Moreover, Ieşan and Quintanilla [17] developed the theory of thermoelasticity for deformable materials with double voids by using the volume fraction concept. In this model the independent variables are the displacement vector, the changes of the volume fractions of pores and fissures and the variation of temperature. On the basis of this theory the elastodynamic problem of an infinite thermoelastic double voids body with a spherical cavity in the context of Lord–Shulman theory of thermoelasticity with one relaxation time is examined and some numerical results are obtained by Kumar and Vohra [18, 19], Kumar *et al.* [20]. The variational principle for Lord-Shulman theory of thermoelastic material with double voids is developed by Kumar *et al.* [21] and the propagation of plane waves for thermoelastic material with double voids with one relaxation time is studied. Exponential decay, existence and uniqueness of the solutions in the one-dimensional version of thermoelasticity for solids with double voids are established by Bazarra *et al.* [22].

Furthermore, plane waves, uniqueness theorems and existence of eigenfrequencies in the theory of rigid bodies with double voids are investigated by Svanadze [23]. The existence of classical solutions in the external BVPs of steady vibrations of this theory is established by the same author in [24]. The basic three-dimensional BVPs of the equilibrium theory of elasticity for materials with a double voids structure are studied by using the potential method and the theory of singular integral equations by Ieşan [25]. The existence and uniqueness of solutions of the BVPs of steady vibrations in the theories of elasticity and thermoelasticity for materials with double voids are proved by Svanadze [26, 27].

Recently, the micropolar model of thermoelasticity for solids with double voids has been presented by Marin *et al.* [28] and the existence, uniqueness and stability of the weak solutions are proved. The existence and stability results for thermoelastic dipolar bodies with double voids are obtained by Marin and NICAISE [29]. Spatial and temporal behavior of solutions of the dynamical problems in the linear theory of thermoelasticity for solids with double voids are studied by ARUSOAIE [30] and FLOREA [31]. A priori estimates for the amplitude of a harmonic vibration in the linear thermoelasticity theory of anisotropic materials with double voids are derived by FLOREA [32]. The generalized theory of thermoelastic diffusion for materials with double voids based upon the Lord-Shulman model are presented by KANSAL [33]. Stability and uniqueness in the double and triple porosity elasticity are studied by STRAUGHAN [34, 35]. The basic properties of the acceleration waves in the nonlinear double porosity elasticity are established by GENTILE and STRAUGHAN [36]. Several models of the multi-porosity elasticity are presented by STRAUGHAN [37, 38].

More recently, the linear equilibrium and quasi static models of elasticity and thermoelasticity for materials with triple voids have been developed by SVANADZE [39, 40] and the basic BVPs are investigated by using the potential method.

Most recent results in the linear theory of double porosity thermoelasticity under local thermal non-equilibrium have been obtained by FRANCHI *et al.* [41] and SVANADZE [42]. Indeed, in [41], the uniqueness and decay of solutions for anisotropic double porosity solids are established, and in [42], the basic BVPs of steady vibrations for isotropic double porosity solids are investigated by using the potential method.

The basic results in the theories of double and multiple porosity elasticity and thermoelasticity may be found in the book of STRAUGHAN [8] and the references therein.

It is well known (see e.g. NOWACKI [43] and KUPRADZE *et al.* [44]) that in solid mechanics one encounters two types of dynamical problems; on the one hand, there are the problems in which the laws of motion as functions of time are known in advance and usually have a sinusoidal character. The problems of this type describe the steady-state or the steady vibrations. On the other hand, there are the problems in which the character of the dependence of time is unknown and has to be determined from the solution itself. The problems of the second type describe the nonstationary motions, unrestricted with respect to the time.

In the present paper the linear theory of thermoelasticity for isotropic and homogeneous elastic solids with macro-, meso- and microporosity is considered and the steady vibrations problems are studied by using the potential method. In this theory the independent variables are the displacement vector field, the volume fractions of pore networks and the variation of temperature.

This work is articulated as follows. In Section 2, the governing field equations of steady vibrations of the considered theory are given. In Section 3, the fundamental solution of the system of steady vibrations equations is constructed explicitly by means of elementary functions and its basic properties are established. In Section 4, the radiation conditions are established and the basic internal and external BVPs are formulated. In Section 5, the uniqueness theorems for these problems are proved. In Section 6, the basic properties of the surface (single-layer and double-layer) and volume potentials are established and finally, the existence theorems for regular (classical) solutions of the internal and external BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$. In what follows we consider an isotropic and homogeneous thermoelastic solid with macro-, meso- and microporosity (first, second and third porosity) structure that occupies the region Ω of $\mathbb{R}^3$. Let $\mathbf{u} = (u_1, u_2, u_3)$ be the displacement vector in solid, $\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x})$ and $\varphi_3(\mathbf{x})$ are the changes of the volume fractions from the reference configuration corresponding to macro-, meso- and microporosity, respectively; θ is the temperature measured from some constant absolute temperature T_0 ($T_0 > 0$).

We assume that repeated indices are summed over the range (1,2,3) and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

Usually, as in the classical theory of thermoelasticity (see e.g. NOWACKI [43] and KUPRADZE *et al.* [44]), the steady vibrations case of the dynamic equations means, that all the independent variables (displacement vector, temperature, etc.) are postulated to have a harmonic time variation. In this connection, the governing field equations of steady vibrations in the linear theory of thermoelasticity for materials with triple voids are given by (see SVANADZE [39, 40])

- *the constitutive equations*

$$\begin{aligned} &t_{lj} = \lambda e_{rr}\delta_{lj} + 2\mu e_{lj} + (b_r\varphi_r - \gamma_0\theta)\delta_{lj}, \\ &\sigma_j^{(l)} = a_{lr}\varphi_{r,j}, \qquad \rho\eta = \gamma_0 e_{rr} + \gamma_j\varphi_j + c\,\theta; \end{aligned} \tag{2.1}$$

- *the equations of steady vibrations*

$$\begin{aligned} &t_{lj,j} + \rho\omega^2 u_l = -\rho F_l, \\ &\sigma_{j,j}^{(l)} + \xi^{(l)} + \rho_l\omega^2\varphi_l = -\rho F_{l+3} \qquad \text{(no sum by } l); \end{aligned} \tag{2.2}$$

- *Fourier's law*

$$q_l = k\theta_{,l}; \tag{2.3}$$

- *and the equation of energy*

$$-i\omega\rho T_0\eta = q_{l,l} + \rho F_7. \tag{2.4}$$

In these equations, t_{lj} is the component of total stress tensor, ρ is the reference mass density, $\rho > 0$, q_l is the component of the heat flux vector, $\sigma_j^{(l)}$ and ρ_l are the component of the equilibrated stress and the coefficient of the equilibrated inertia for l-th pore network, respectively; $\rho_l > 0\,(l = 1,2,3)$, η and $\mathbf{F}^{(1)} = (F_1, F_2, F_3)$ are the entropy and the body force per unit mass, respectively; $\mathbf{F}^{(2)} = (F_4, F_5, F_6)$ is the extrinsic equilibrated body force per unit mass associated to pore networks, F_7 is the heat supply per unit mass; $\lambda, \mu, b_j, d, \alpha_{lj}, \gamma_0, \gamma_l, a_{lj}, c, k$ $(l, j = 1,2,3)$ are constitutive coefficients, δ_{lj} is Kronecker's delta, ω is the oscillation frequency, $\omega > 0$, e_{lj} are the components of the strain tensor,

$$e_{lj} = \frac{1}{2}(u_{l,j} + u_{j,l}), \qquad l, j = 1,2,3, \tag{2.5}$$

the function $\xi^{(l)}$ is the intrinsic equilibrated body force for l-th pore network and defined by

$$\xi^{(l)} = -b_l e_{rr} - \alpha_{lj}\varphi_j + \gamma_l \theta. \tag{2.6}$$

Substituting Eqs. (2.1), (2.3), (2.5) and (2.6) into (2.2) and (2.4) we obtain the following system of equations of steady vibrations in the linear theory of thermoelasticity for materials with triple voids expressed in terms of the displacement vector $\mathbf{u}$, the changes of volume fractions vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and the variation of temperature θ:

$$\begin{aligned}
&(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda+\mu)\nabla \operatorname{div}\mathbf{u} + b_j \nabla\varphi_j - \gamma_0\,\nabla\theta = -\rho\mathbf{F}^{(1)},\\
&(a_{lj}\Delta + \beta_{lj})\varphi_j - b_l \operatorname{div}\mathbf{u} + \gamma_l\,\theta = -\rho\, F_{l+3},\\
&(k\Delta + c')\,\theta + \gamma_0' \operatorname{div}\mathbf{u} + \gamma_j'\,\varphi_j = -\rho\, F_7,
\end{aligned} \tag{2.7}$$

where Δ is the Laplacian operator, $\beta_{lj} = \rho_l\omega^2\delta_{lj} - \alpha_{lj}$ (no sum), $c' = i\omega c T_0$, $\gamma_m' = i\omega\gamma_m T_0$, $l, j = 1,2,3$, $m = 0,1,2,3$.

We introduce the matrix differential operator $\mathbf{A}(\mathbf{D_x}) = (A_{lj}(\mathbf{D_x}))_{7\times 7}$, where

$$\begin{aligned}
&A_{lj}(\mathbf{D_x}) = (\mu\,\Delta + \rho\omega^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j}, \qquad A_{l;j+3}(\mathbf{D_x}) = b_j\frac{\partial}{\partial x_l},\\
&A_{l7}(\mathbf{D_x}) = -\gamma_0\frac{\partial}{\partial x_l}, \qquad A_{l+3;j}(\mathbf{D_x}) = -b_l\frac{\partial}{\partial x_j}, \qquad A_{l+3;j+3}(\mathbf{D_x}) = a_{lj}\Delta + \beta_{lj},\\
&A_{l+3;7}(\mathbf{D_x}) = -\gamma_l, \qquad A_{7l}(\mathbf{D_x}) = \gamma_0'\frac{\partial}{\partial x_l}, \qquad A_{7;l+3}(\mathbf{D_x}) = \gamma_l',\\
&A_{77}(\mathbf{D_x}) = k\Delta + c', \qquad \mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad l, j = 1,2,3.
\end{aligned}$$

It is easily seen that the system (2.7) can be rewritten in the following matrix form

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{2.8}$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \theta)$ and $\mathbf{F} = (-\rho F_1, -\rho F_2, \dots, -\rho F_7)$ are seven-component vector functions.

3. Fundamental solution

In this section the fundamental solution of the system (2.7) is constructed explicitly by means of elementary functions and its basic properties are established.

Definition 1. The fundamental solution of system (2.7) is the matrix $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{7\times 7}$ satisfying the equation

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J} \tag{3.1}$$

in the class of generalized functions, where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{7\times 7}$ is the unit matrix, $\mathbf{x} \in \mathbb{R}^3$.

We introduce the notation:

1)

$$\Lambda_1(\Delta) = \frac{1}{k a_0 \mu_0} \det \mathbf{B}(\Delta),$$

where $a_0 = \det(a_{lj})_{3\times 3}$, $\mu_0 = \lambda + 2\mu$ and

$$\mathbf{B}(\Delta) = \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & -b_1\Delta & -b_2\Delta & -b_3\Delta & \gamma_0'\Delta \\ b_1 & a_{11}\Delta + \beta_{11} & a_{12}\Delta + \beta_{12} & a_{13}\Delta + \beta_{13} & \gamma_1' \\ b_2 & a_{21}\Delta + \beta_{21} & a_{22}\Delta + \beta_{22} & a_{23}\Delta + \beta_{23} & \gamma_2' \\ b_3 & a_{31}\Delta + \beta_{31} & a_{32}\Delta + \beta_{32} & a_{33}\Delta + \beta_{33} & \gamma_3' \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & -\gamma_3 & k\Delta + c' \end{pmatrix}_{5\times 5}.$$

We can consider $\Lambda_1(-\xi) = 0$ as an algebraic equation of the fifth degree, which admits five roots τ_1^2, τ_2^2, $\dots$, τ_5^2 (with respect to ξ). Then we have

$$\Lambda_1(\Delta) = \prod_{j=1}^{5}(\Delta + \tau_j^2).$$

We assume that the values $\tau_1^2, \tau_2^2, \dots, \tau_6^2$ are distinct and different from zero, where $\tau_6^2 = \rho\omega^2/\mu$.

2)

$$n_{j1}(\Delta)=-\frac{1}{ka_0\mu\mu_0}[(\lambda+\mu)B^*_{j1}(\Delta)-b_rB^*_{j;r+1}(\Delta)+\gamma_0'B^*_{j4}(\Delta)],$$
$$n_{jl}(\Delta)=\frac{1}{ka_0\mu_0}B^*_{jl}(\Delta),\qquad j=1,2,\dots,5,\ l=2,3,4,5,$$

where B^*_{lj} is the cofactor of the element B_{lj} of matrix $\mathbf{B}$.

3)

$$\mathbf{L}(\mathbf{D_x})=(L_{lj}(\mathbf{D_x}))_{7\times7},\qquad L_{lj}(\mathbf{D_x})=\frac{1}{\mu}\Lambda_1(\Delta)\,\delta_{lj}+n_{11}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},$$
$$L_{l;m+2}(\mathbf{D_x})=n_{1m}(\Delta)\frac{\partial}{\partial x_l},\qquad L_{m+2;l}(\mathbf{D_x})=n_{m1}(\Delta)\frac{\partial}{\partial x_l}, \tag{3.2}$$
$$L_{m+2;p+2}(\mathbf{D_x})=n_{mp}(\Delta),\qquad l,j=1,2,3,\qquad m,p=2,3,4,5.$$

4)

$$\mathbf{Y}(\mathbf{x})=(Y_{lm}(\mathbf{x}))_{7\times7},\qquad Y_{11}(\mathbf{x})=Y_{22}(\mathbf{x})=Y_{33}(\mathbf{x})=\sum_{j=1}^{6}\eta_{2j}\gamma^{(j)}(\mathbf{x}),$$
$$Y_{44}(\mathbf{x})=Y_{55}(\mathbf{x})=Y_{66}(\mathbf{x})=Y_{77}(\mathbf{x})=\sum_{j=1}^{5}\eta_{1j}\gamma^{(j)}(\mathbf{x}), \tag{3.3}$$
$$Y_{lm}(\mathbf{x})=0,\qquad l\neq m,\ l,m=1,2,\dots,7,$$

where

$$\gamma^{(j)}(\mathbf{x})=-\frac{e^{i\tau_j|\mathbf{x}|}}{4\pi|\mathbf{x}|} \tag{3.4}$$

and

$$\eta_{1m}=\prod_{l=1,\,l\neq m}^{5}(\tau_l^2-\tau_m^2)^{-1},\qquad \eta_{2j}=\prod_{l=1,\,l\neq j}^{6}(\tau_l^2-\tau_j^2)^{-1},$$
$$m=1,2,\dots,5,\ j=1,2,\dots,6.$$

We have the following

Theorem 1. *If*

$$ka_0\mu\mu_0\neq0, \tag{3.5}$$

then the matrix $\mathbf{\Gamma}(\mathbf{x})$ *defined by*

$$\mathbf{\Gamma}(\mathbf{x})=\mathbf{L}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) \tag{3.6}$$

is the fundamental solution of system (2.7)*, where the matrices* $\mathbf{L}(\mathbf{D_x})$ *and* $\mathbf{Y}(\mathbf{x})$ *are given by* (3.2) *and* (3.3)*, respectively.*

Proof. On the basis of (3.6) and identities

$$\mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta), \qquad \mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},$$

where

$$\begin{aligned}
&\mathbf{\Lambda}(\Delta) = (\Lambda_{lj}(\Delta))_{7\times 7},\\
&\Lambda_{11}(\Delta) = \Lambda_{22}(\Delta) = \Lambda_{33}(\Delta) = \Lambda_1(\Delta)(\Delta + \tau_6^2),\\
&\Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) = \Lambda_{66}(\Delta) = \Lambda_{77}(\Delta) = \Lambda_1(\Delta),\\
&\Lambda_{lj}(\Delta) = 0, \qquad l \neq j,\ l,j = 1,2,\ldots,7,
\end{aligned}$$

we have

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) = \mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Hence, the matrix $\mathbf{\Gamma}(\mathbf{x})$ is a solution of (3.1). □

Clearly, the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed by 6 metaharmonic functions $\gamma^{(j)}$ $(j = 1,\ldots,6)$ (see (3.4)).

Theorem 1 directly leads to the following basic properties of $\mathbf{\Gamma}(\mathbf{x})$.

Theorem 2. *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of the homogeneous equation*

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{3.7}$$

at every point $\mathbf{x} \in \mathbb{R}^3$ *except the origin.*

Theorem 3. *If condition* (3.5) *is satisfied, then the fundamental solution of the system*

$$\begin{aligned}
&\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\nabla\,\mathrm{div}\,\mathbf{u} = \mathbf{0},\\
&a_{lj}\Delta\varphi_j = 0, \qquad k\,\Delta\theta = 0, \qquad l = 1,2,3
\end{aligned}$$

is the matrix $\mathbf{\Psi}(\mathbf{x}) = (\Psi_{lj}(\mathbf{x}))_{7\times 7}$*, where*

$$\begin{aligned}
&\Psi_{lj}(\mathbf{x}) = \lambda'\,\frac{\delta_{lj}}{|\mathbf{x}|} + \mu'\,\frac{x_l x_j}{|\mathbf{x}|^3}, \qquad \Psi_{l+3;j+3}(\mathbf{x}) = \frac{a^*_{lj}}{a_0}\gamma^{(0)}(\mathbf{x}),\\
&\Psi_{77}(\mathbf{x}) = \frac{1}{k}\gamma^{(0)}(\mathbf{x}), \qquad \Psi_{lm}(\mathbf{x}) = \Psi_{ml}(\mathbf{x}) = \Psi_{l+3;7}(\mathbf{x}) = \Psi_{7;l+3}(\mathbf{x}) = 0,\\
&\gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \qquad \lambda' = -\frac{\lambda+3\mu}{8\pi\mu\mu_0}, \qquad \mu' = -\frac{\lambda+\mu}{8\pi\mu\mu_0},\\
&\qquad\qquad l,j = 1,2,3,\ m = 4,5,6,7
\end{aligned} \tag{3.8}$$

and a^*_{lj} *is the cofactor of the element* a_{lj} *of matrix* $\mathbf{a} = (a_{lj})_{3\times 3}$.

Obviously, (3.6) and (3.8) imply the following results.

COROLLARY 1. *The relations*

$$\Psi_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Psi_{l+3;j+3}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Psi_{77}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \tag{3.9}$$

hold in the neighborhood of the origin, where $l, j = 1, 2, 3$.

THEOREM 4. *The relations*

$$\begin{aligned} &\Gamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma_{l+3;j+3}(\mathbf{x}) = O(|\mathbf{x}|^{-1}),\\ &\Gamma_{77}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma_{lm}(\mathbf{x}) = O(1),\\ &\Gamma_{ml}(\mathbf{x}) = O(1), \qquad \Gamma_{l+3;7}(\mathbf{x}) = O(1), \qquad \Gamma_{7;l+3}(\mathbf{x}) = O(1) \end{aligned}$$

hold in the neighborhood of the origin, where $l, j = 1, 2, 3,\ m = 4, 5, 6, 7$.

On the basis of Theorem 4 and Corollary 1 we can prove the following

THEOREM 5. *The relations*

$$\Gamma_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|) \tag{3.10}$$

hold in the neighborhood of the origin, where $l, j = 1, \ldots, 7$.

Thus, in view of (3.9) and (3.10), the matrix $\mathbf{\Psi}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin.

4. Boundary value problems

In what follows we assume that $\operatorname{Im}\tau_j \geq 0$ $(j = 1, \ldots, 5)$, $\tau_6 > 0$ and the constitutive coefficients satisfy the conditions:

(i) $\mathbf{a} = (a_{lj})_{3\times 3}$ and $\boldsymbol{\alpha} = (\alpha_{lj})_{3\times 3}$ are positive definite matrices;

(ii)

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad c > 0, \qquad k > 0. \tag{4.1}$$

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{1,\nu}$, $0 < \nu \leq 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$; $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to S at $\mathbf{z}$. The scalar product of two vectors $\mathbf{U} = (u_1, \ldots, u_7)$ and $\mathbf{V} = (v_1, \ldots, v_7)$ is denoted by $\mathbf{U} \cdot \mathbf{V} = \sum_{j=1}^{7} u_j \bar{v}_j$, where $\bar{v}_j$ is the complex conjugate of v_j.

DEFINITION 2. A vector function $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \theta) = (U_1, \ldots, U_7)$ is called regular in Ω^- (or Ω^+) if

(i)

$$U_l \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \qquad (\text{or } U_l \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})),$$

(ii)

$$U_l = \sum_{j=1}^{6} U_l^{(j)}, \quad U_l^{(j)} \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-),$$

(iii) $(\Delta + \tau_j^2) U_l^{(j)}(\mathbf{x}) = 0$ and

$$\left(\frac{\partial}{\partial |\mathbf{x}|} - i\tau_j\right) U_l^{(j)}(\mathbf{x}) = e^{i\tau_j |\mathbf{x}|} o(|\mathbf{x}|^{-1}) \qquad \text{for } |\mathbf{x}| \gg 1, \tag{4.2}$$

where $U_m^{(6)} = 0$, $j = 1, \dots, 6$, $l = 1, \dots, 7$, $m = 4, 5, 6, 7$.

Obviously, the relation (4.2) implies (for details see Vekua [45])

$$U_l^{(j)}(\mathbf{x}) = e^{i\tau_j |\mathbf{x}|} O(|\mathbf{x}|^{-1}) \qquad \text{for } |\mathbf{x}| \gg 1, \tag{4.3}$$

where $j = 1, \dots, 6$, $l = 1, \cdots, 7$.

Relations (4.2) and (4.3) are the radiation conditions in the linear theory of thermoelasticity for materials with triple voids.

In the sequel, we use the matrix differential operator

$$\mathbf{P}(\mathbf{D_x}, \mathbf{n}) = (P_{lj}(\mathbf{D_x}, \mathbf{n}))_{7\times 7},$$

where

$$\begin{aligned} &P_{lj}(\mathbf{D_x}, \mathbf{n}) = \mu \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}, \qquad P_{l;j+3}(\mathbf{D_x}, \mathbf{n}) = b_j\, n_l, \\ &P_{l7}(\mathbf{D_x}, \mathbf{n}) = -\gamma_0\, n_l, \qquad P_{l+3;j+3}(\mathbf{D_x}, \mathbf{n}) = a_{lj} \frac{\partial}{\partial \mathbf{n}}, \\ &P_{77}(\mathbf{D_x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}}, \quad P_{l+3;j}(\mathbf{D_x}, \mathbf{n}) = P_{l+3;7}(\mathbf{D_x}, \mathbf{n}) = P_{7m}(\mathbf{D_x}, \mathbf{n}) = 0, \\ &\qquad\qquad l, j = 1, 2, 3, \ m = 1, \dots, 6 \end{aligned} \tag{4.4}$$

and $\partial/\partial\mathbf{n}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the linear theory of thermoelasticity for materials with triple voids are formulated as follows.

Find a regular (classical) solution to (2.8) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{4.5}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}(\mathbf{D_x}, \mathbf{n}(\mathbf{z})) \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z})) \mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{4.6}$$

in the internal *Problem (II)*$^{+}_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed seven-component vector functions.

Find a regular (classical) solution to (2.8) for $\mathbf{x} \in \Omega^{-}$ satisfying the boundary condition

$$\lim_{\Omega^{-} \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^{-} = \mathbf{f}(\mathbf{z})$$

in the external *Problem* $(I)^{-}_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^{-} \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^{-} = \mathbf{f}(\mathbf{z}) \tag{4.7}$$

in the external *Problem (II)*$^{-}_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed seven-component vector functions, $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^{-}.

5. Uniqueness theorems

In this section the uniqueness of regular solutions of the BVPs $(K)^{+}_{\mathbf{F},\mathbf{f}}$ and $(K)^{-}_{\mathbf{F},\mathbf{f}}$ is studied, where $K = I, II$. In the sequel we use the matrix differential operators:

1)

$$\begin{aligned}
&\mathbf{A}^{(0)}(\mathbf{D_x}) = (A^{(0)}_{lj}(\mathbf{D_x}))_{3\times3}, && A^{(0)}_{lj}(\mathbf{D_x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},\\
&\mathbf{A}^{(1)}(\mathbf{D_x}) = (A^{(1)}_{lr}(\mathbf{D_x}))_{3\times7}, && A^{(1)}_{lr}(\mathbf{D_x}) = A_{lr}(\mathbf{D_x}),\\
&\mathbf{A}^{(2)}(\mathbf{D_x}) = (A^{(2)}_{lr}(\mathbf{D_x}))_{3\times7}, && A^{(2)}_{lr}(\mathbf{D_x}) = A_{l+3;r}(\mathbf{D_x}),\\
&\mathbf{A}^{(3)}(\mathbf{D_x}) = (A^{(3)}_{1r}(\mathbf{D_x}))_{1\times7}, && A^{(3)}_{1r}(\mathbf{D_x}) = A_{7r}(\mathbf{D_x});
\end{aligned}$$

2)

$$\begin{aligned}
&\mathbf{P}^{(0)}(\mathbf{D_x}, \mathbf{n}) = (P^{(0)}_{lj}(\mathbf{D_x}, \mathbf{n}))_{3\times3}, && P^{(0)}_{lj}(\mathbf{D_x}, \mathbf{n}) = P_{lj}(\mathbf{D_x}, \mathbf{n}),\\
&\mathbf{P}^{(1)}(\mathbf{D_x}, \mathbf{n}) = (P^{(1)}_{lr}(\mathbf{D_x}, \mathbf{n}))_{3\times7}, && P^{(1)}_{lr}(\mathbf{D_x}, \mathbf{n}) = P_{lr}(\mathbf{D_x}, \mathbf{n}),
\end{aligned}$$

where $l, j = 1, 2, 3$, $m = 2, 3$ and $r = 1, \dots, 7$.

We introduce the notation

$$\begin{aligned}
W^{(0)}(\mathbf{u}) &= \frac{1}{3}(3\lambda + 2\mu)\,|\operatorname{div}\mathbf{u}|^2\\
&\quad + \frac{\mu}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right|^2 + \frac{\mu}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2,\\
W^{(1)}(\mathbf{U}) &= W^{(0)}(\mathbf{u}) - \rho\omega^2|\mathbf{u}|^2 + (b_j\varphi_j - \gamma_0\theta)\operatorname{div}\overline{\mathbf{u}},\\
W^{(2)}(\mathbf{U}) &= a_{lj}\nabla\varphi_j \cdot \nabla\varphi_l - \beta_{lj}\varphi_j\overline{\varphi_l} + b_l\overline{\varphi_l}\operatorname{div}\mathbf{u} - \gamma_l\theta\,\overline{\varphi_l},\\
W^{(3)}(\mathbf{U}) &= k|\nabla\theta|^2 - c'|\theta|^2 - (\gamma'_0\operatorname{div}\mathbf{u} + \gamma'_l\varphi_l)\,\overline{\theta}.
\end{aligned} \tag{5.1}$$

We have the following

LEMMA 1. *If* $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\theta)$ *is a regular vector in* Ω^+, *then*

$$\begin{aligned}&\int\limits_{\Omega^+}[\mathbf{A}^{(1)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+W^{(1)}(\mathbf{U})]d\mathbf{x}\\ &\qquad\qquad=\int\limits_S\mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{u}(\mathbf{z})\,d_{\mathbf{z}}S,\\ &\int\limits_{\Omega^+}[\mathbf{A}^{(2)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x})+W^{(2)}(\mathbf{U})]d\mathbf{x}=\int\limits_S\mathbf{a}\frac{\partial\boldsymbol{\varphi}(\mathbf{z})}{\partial\mathbf{n}}\cdot\boldsymbol{\varphi}(\mathbf{z})\,d_{\mathbf{z}}S,\\ &\int\limits_{\Omega^+}[\mathbf{A}^{(3)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\overline{\theta(\mathbf{x})}+W^{(3)}(\mathbf{U})]d\mathbf{x}=\int\limits_S k\frac{\partial\theta(\mathbf{z})}{\partial\mathbf{n}}\overline{\theta(\mathbf{z})}\,d_{\mathbf{z}}S.\end{aligned}\tag{5.2}$$

Proof. It is well known (see e.g. [44]) that on the basis of the divergence theorem the following identities hold

$$\begin{aligned}&\int\limits_{\Omega^+}[\mathbf{A}^{(0)}(\mathbf{D_x})\,\mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+W^{(0)}(\mathbf{u})]d\mathbf{x}\\ &\qquad\qquad=\int\limits_S\mathbf{P}^{(0)}(\mathbf{D_z},\mathbf{n})\mathbf{u}(\mathbf{z})\cdot\mathbf{u}(\mathbf{z})\,d_{\mathbf{z}}S,\\ &\int\limits_{\Omega^+}[\Delta\varphi_j(\mathbf{x})\,\overline{\varphi_l(\mathbf{x})}+\nabla\varphi_j(\mathbf{x})\cdot\nabla\varphi_l(\mathbf{x})]d\mathbf{x}=\int\limits_S\frac{\partial\varphi_j(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\overline{\varphi_l(\mathbf{z})}\,d_{\mathbf{z}}S,\\ &\int\limits_{\Omega^+}\Big[\nabla\varphi_j(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+\varphi_j(\mathbf{x})\,\mathrm{div}\,\overline{\mathbf{u}(\mathbf{x})}\Big]\,d\mathbf{x}=\int\limits_S\varphi_j(\mathbf{z})\mathbf{n}(\mathbf{z})\cdot\mathbf{u}(\mathbf{z})\,d_{\mathbf{z}}S.\end{aligned}\tag{5.3}$$

Keeping in mind (5.1), from (5.3) we obtain the identities (5.2). □

We are now in a position to study the uniqueness of regular solutions of the BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K=I,II$. We have the following results.

THEOREM 6. *Two regular solutions of the internal BVP* $(I)^+_{\mathbf{F},\mathbf{f}}$, *may differ only for an additive vector* $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\theta)$, *where*

$$\theta(\mathbf{x})=0\qquad \textit{for }\ \mathbf{x}\in\Omega^+\tag{5.4}$$

and the six-component vector $\mathbf{v}=(\mathbf{u},\boldsymbol{\varphi})$ *is a regular solution of the following system*

$$\begin{aligned}&(\mu\Delta+\rho\omega^2)\mathbf{u}+(\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u}+b_j\nabla\varphi_j=\mathbf{0},\\ &(a_{lj}\Delta+\beta_{lj})\varphi_j-b_l\,\mathrm{div}\,\mathbf{u}=0,\\ &\gamma_0\,\mathrm{div}\,\mathbf{u}+\gamma_j\,\varphi_j=0,\qquad l=1,2,3,\end{aligned}\tag{5.5}$$

satisfying the boundary condition

$$\{\mathbf{v}(\mathbf{z})\}^+ = 0 \qquad \textit{for } \mathbf{z} \in S. \tag{5.6}$$

In addition, problems $(I)^+_{\mathbf{0},\mathbf{0}}$ *and* (5.5), (5.6) *have the same eigenfrequencies.*

Proof. Suppose that there are two regular solutions of problem $(I)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous system of Eqs. (3.7) in Ω^+ satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = 0 \qquad \text{for } \mathbf{z} \in S. \tag{5.7}$$

On the basis of (3.7) and (5.7), from (5.2) we obtain

$$\int\limits_{\Omega^+} W^{(j)}(\mathbf{U})d\mathbf{x} = 0, \qquad j = 1, 2, 3. \tag{5.8}$$

Clearly, from (5.1) we have

$$\begin{aligned}
\operatorname{Im} W^{(1)}(\mathbf{U}) &= \operatorname{Im}[b_j \varphi_j \operatorname{div} \overline{\mathbf{u}}] - \gamma_0 \operatorname{Im}[\theta \operatorname{div} \overline{\mathbf{u}}],\\
\operatorname{Im} W^{(2)}(\mathbf{U}) &= -\operatorname{Im}[b_j \varphi_j \operatorname{div} \overline{\mathbf{u}}] - \operatorname{Im}[\gamma_l \theta \overline{\varphi_l}],\\
\operatorname{Re} W^{(3)}(\mathbf{U}) &= k|\nabla\theta|^2 - \omega T_0 \gamma_0 \operatorname{Im}[\theta \operatorname{div} \overline{\mathbf{u}}] - \omega T_0 \operatorname{Im}[\gamma_l \theta \overline{\varphi_l}].
\end{aligned}$$

Consequently, we get

$$\operatorname{Re} W^{(3)}(\mathbf{U}) - \omega T_0 \operatorname{Im}[W^{(1)}(\mathbf{U}) + W^{(2)}(\mathbf{U})] = k|\nabla\theta|^2$$

and from (5.8) it follows that

$$\int\limits_{\Omega^+} |\nabla\theta(\mathbf{x})|^2 d\mathbf{x} = 0.$$

Hence, $\nabla\theta(\mathbf{x}) \equiv 0$ in Ω^+ and we may derive

$$\theta(\mathbf{x}) = \text{const} \qquad \text{for } \mathbf{x} \in \Omega^+. \tag{5.9}$$

On the basis of homogeneous boundary condition (5.7) from (5.9) we obtain the relation (5.4). By virtue of (5.4) from (3.7) we get the system (5.5). Obviously, in view of conditions (5.4) and (5.7) the six-component vector $\mathbf{v} = (\mathbf{u}, \boldsymbol{\varphi})$ satisfies the boundary condition (5.6).

Finally, it is easy to see that the homogeneous boundary value problems $(I)^+_{\mathbf{0},\mathbf{0}}$ and (5.5), (5.6) have the same eigenfrequencies. □

Let $\mathbf{R}(\mathbf{D_z},\mathbf{n})$ be the following matrix differential operator

$$\mathbf{R}(\mathbf{D_z},\mathbf{n}) = (R_{lj}(\mathbf{D_z},\mathbf{n}))_{6\times 6}, \qquad R_{lj} = P_{lj}, \qquad l,j = 1,\dots,6,$$

where P_{lj} is given by (4.4).

THEOREM 7. *Two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$, *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u},\boldsymbol{\varphi},\theta)$, *where* θ *satisfies the condition* (5.4), *the vector* $\mathbf{v} = (\mathbf{u},\boldsymbol{\varphi})$ *is a regular solution of the system* (5.5) *satisfying the boundary condition*

$$\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{v}(\mathbf{z})\}^+ = 0 \qquad \text{for } \mathbf{z}\in S. \tag{5.10}$$

In addition, problems $(II)^+_{\mathbf{0},\mathbf{0}}$ *and* (5.5), (5.10) *have the same eigenfrequencies.*

Proof. Suppose that there are two regular solutions of the problem $(II)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous system of Eqs. (3.7) in Ω^+ satisfying the homogeneous boundary condition

$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = 0 \qquad \text{for } \mathbf{z}\in S. \tag{5.11}$$

Quite similarly as in Theorem 6, we obtain the relation (5.9). On the other hand, from (3.7) it follows that

$$\Lambda_1(\Delta)\theta(\mathbf{x}) = 0. \tag{5.12}$$

By virtue of (5.9) and the relation $\tau_j \neq 0$ $(j = 1,\dots,5)$ from (5.12) we have (5.4) and consequently, the system (3.7) implies (5.5). Obviously, in view of conditions (5.4) and (5.11) the vector $\mathbf{v}$ satisfies the boundary condition (5.10).

Finally, it is easy to see that the homogeneous boundary value problems $(\mathrm{II})^+_{\mathbf{0},\mathbf{0}}$ and (5.5), (5.10) have the same eigenfrequencies. □

THEOREM 8. *The external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Theorem 8 can be proved similarly to Theorems 6 and 7 using the radiation conditions (4.2) and (4.3).

6. Existence theorems

In the sequel we use the matrix differential operator

$$\tilde{\mathbf{P}}(\mathbf{D_x},\mathbf{n}) = (\tilde{P}_{lj}(\mathbf{D_x},\mathbf{n}))_{7\times 7},$$

where

$$\tilde{P}_{lm}(\mathbf{D_x},\mathbf{n}) = P_{lm}(\mathbf{D_x},\mathbf{n}), \quad \tilde{P}_{l7}(\mathbf{D_x},\mathbf{n}) = -\gamma_0' n_l, \quad \tilde{P}_{rj}(\mathbf{D_x},\mathbf{n}) = P_{rj}(\mathbf{D_x},\mathbf{n}),$$
$$l = 1,2,3,\ m = 1,\dots,6,\ j = 1,\dots,7,\ r = 4,5,6,7.$$

It is easy to verify that the operator $\tilde{\mathbf{P}}(\mathbf{D_x}, \mathbf{n})$ may be obtained from the operator $\mathbf{P}(\mathbf{D_x}, \mathbf{n})$ by replacing γ_0 by γ_0' and vice versa.

We introduce the following notation:

1) $\mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{g}) = \int\limits_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the single-layer potential,

2) $\mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{g}) = \int\limits_S [\tilde{\mathbf{P}}(\mathbf{D_y}, \mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top \mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the double-layer potential, and

3) $\mathbf{Q}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^\pm) = \int\limits_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y})d\mathbf{y}$ is the volume potential,

where $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$ and defined by (3.6), $\mathbf{g}$ and $\boldsymbol{\phi}$ are seven-component vector functions, $\mathbf{\Gamma}^\top$ is the transpose of the matrix $\mathbf{\Gamma}$.

On the basis of properties of the matrix $\mathbf{\Gamma}(\mathbf{x})$ we have the following results.

THEOREM 9. *If $S \in C^{m+1,\nu}$, $\mathbf{g} \in C^{m,\nu'}(S)$, $0 < \nu' < \nu \le 1$, and m is a non-negative integer, then:*

(a)
$$\mathbf{Q}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\nu'}(\mathbb{R}^3) \cap C^{m+1,\nu'}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm),$$

(b)
$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0},$$

(c)
$$\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z}, \mathbf{g})\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z}, \mathbf{g}), \tag{6.1}$$

(d)
$$\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z}, \mathbf{g})$$

is a singular integral, where $\mathbf{z} \in S$, $\mathbf{x} \in \Omega^\pm$.

THEOREM 10. *If $S \in C^{m+1,\nu}$, $\mathbf{g} \in C^{m,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then:*

(a)
$$\mathbf{Q}^{(2)}(\cdot, \mathbf{g}) \in C^{m,\nu'}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm),$$

(b)
$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0},$$

(c)
$$\{\mathbf{Q}^{(2)}(\mathbf{z}, \mathbf{g})\}^\pm = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z}, \mathbf{g}) \tag{6.2}$$

for the non-negative integer m,

(d) $\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral, where* $\mathbf{z}\in S$,

(e)

$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{+}=\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{-},$$

for the natural number m, *where* $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^{\pm}$.

THEOREM 11. *If* $S\in C^{1,\nu}$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^+)$, $0<\nu'<\nu\le 1$, *then:*

(a)

$$\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^+)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^+)\cap C^{2,\nu'}(\overline{\Omega_0^+}),$$

(b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^+)=\boldsymbol{\phi}(\mathbf{x}),$$

where $\mathbf{x}\in\Omega^+$, Ω_0^+ *is a domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^+}\subset\Omega^+$.

THEOREM 12. *If* $S\in C^{1,\nu}$, $\mathrm{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^-)$, $0<\nu'<\nu\le 1$, *then:*

(a)

$$\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^-)\cap C^{2,\nu'}(\overline{\Omega_0^-}),$$

(b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-)=\boldsymbol{\phi}(\mathbf{x}),$$

where $\mathbf{x}\in\Omega^-$, Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^-}\subset\Omega^-$.

We introduce the notation

$$\begin{aligned}
\mathcal{K}^{(1)}\,\mathbf{g}(\mathbf{z})&\equiv\tfrac12\,\mathbf{g}(\mathbf{z})+\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(2)}\,\mathbf{g}(\mathbf{z})&\equiv-\tfrac12\,\mathbf{g}(\mathbf{z})+\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z})&\equiv-\tfrac12\,\mathbf{g}(\mathbf{z})+\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(4)}\mathbf{g}(\mathbf{z})&\equiv\tfrac12\,\mathbf{g}(\mathbf{z})+\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}_\chi\mathbf{g}(\mathbf{z})&\equiv\tfrac12\,\mathbf{g}(\mathbf{z})+\chi\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})
\end{aligned}\tag{6.3}$$

for $\mathbf{z}\in S$, where χ is a complex number. Obviously, on the basis of Theorems 9 and 10, $\mathcal{K}_j$ and $\mathcal{K}_\chi$ are the singular integral operators $(j=1,2,3,4)$.

Let $\boldsymbol{\sigma}^{(j)}=(\sigma^{(j)}_{lm})_{7\times 7}$ be the symbol of the singular integral operator $\mathcal{K}^{(j)}$ $(j=1,2,3,4)$ (see [44]). Taking into account (6.3) we find

$$\begin{aligned}
\det\boldsymbol{\sigma}^{(1)}&=-\det\boldsymbol{\sigma}^{(2)}=-\det\boldsymbol{\sigma}^{(3)}=\det\boldsymbol{\sigma}^{(4)}\\
&=-\frac{1}{128}\left[1-\frac{\mu^2}{(\lambda+2\mu)^2}\right]=-\frac{(\lambda+\mu)(\lambda+3\mu)}{128(\lambda+2\mu)^2}<0.
\end{aligned}\tag{6.4}$$

Hence, the operator $\mathcal{K}^{(j)}$ is of the normal type, where $j=1,2,3,4$.

Let $\boldsymbol{\sigma}_\chi$ and $\operatorname{ind}\mathcal{K}_\chi$ be the symbol and the index of the operator $\mathcal{K}_\chi$, respectively. It may be easily shown that

$$\det \boldsymbol{\sigma}_\chi = -\frac{(\lambda+2\mu)^2-\mu^2\chi^2}{128(\lambda+2\mu)^2}$$

and $\det \boldsymbol{\sigma}_\chi$ vanishes only at two points χ_1 and χ_2 of the complex plane. By virtue of (6.4) and $\det \boldsymbol{\sigma}_1 = \det \boldsymbol{\sigma}^{(1)}$ we get $\chi_j \neq 1$ $(j=1,2)$ and

$$\operatorname{ind}\mathcal{K}_1 = \operatorname{ind}\mathcal{K}^{(1)} = \operatorname{ind}\mathcal{K}_0 = 0.$$

Quite similarly we obtain $\operatorname{ind}\mathcal{K}^{(2)} = -\operatorname{ind}\mathcal{K}^{(3)} = 0$ and $\operatorname{ind}\mathcal{K}^{(4)} = -\operatorname{ind}\mathcal{K}^{(1)} = 0$.

Thus, the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1,2,3,4)$ is of the normal type with an index equal to zero. Consequently, Fredholm's theorems are valid for $\mathcal{K}^{(j)}$.

The definitions of a normal type singular integral operator, the symbol and the index of operator, and Fredholm's theorems for the singular integral equations are given in [44] and [46].

By theorems 11 and 12 the volume potential $\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{F},\Omega^\pm)$ is a regular solution of (2.8), where $\mathbf{F} \in C^{0,\nu'}(\Omega^\pm)$, $0<\nu'\le 1$; $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-. Therefore, further we consider problems $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$, and we prove the existence theorems of a regular (classical) solution of these BVPs, where $K = I, II$.

Problem $(I)^+_{\mathbf{0},\mathbf{f}}$. Let us assume that ω is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. We seek a regular solution to this problem in the form of the double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) \qquad \text{for } \mathbf{x}\in\Omega^+, \tag{6.5}$$

where $\mathbf{g}$ is the required seven-component vector function.

Obviously, by Theorem 10 the vector function $\mathbf{U}$ is a solution of (3.7) for $\mathbf{x}\in\Omega^+$. Keeping in mind the boundary condition (4.5) and using (6.2), from (6.5) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(1)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z}\in S. \tag{6.6}$$

We prove that Eq. (6.6) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the associate homogeneous equation

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \qquad \text{for } \mathbf{z}\in S, \tag{6.7}$$

where $\mathbf{h}$ is the required seven-component vector function. Now, we prove that (6.7) has only the trivial solution.

Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (6.7). On the basis of Theorem 9 and Eq. (6.1) the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of the external homogeneous BVP $(II)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 8, problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^-. \tag{6.8}$$

On the other hand, by Theorem 9 and (6.8) we get

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \qquad \text{for } \mathbf{z} \in S,$$

i.e., on the basis of Theorem 9 the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 6 and the assumption that ω is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$, the problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^+. \tag{6.9}$$

By virtue of (6.8), (6.9) and identity (6.1) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for } \mathbf{z} \in S.$$

Thus, the homogeneous equation (6.7) has only the trivial solution and therefore on the basis of Fredholm's theorem the integral equation (6.6) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved

Theorem 13. *If* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *and* ω *is not an eigenfrequency of the BVP* $(I)^+_{\mathbf{0},\mathbf{0}}$, *then a regular solution of the internal BVP* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by the double-layer potential* (6.5), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (6.6) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$. Let us assume that ω is not an eigenfrequency of the BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. We seek a regular solution to this problem in the form of the single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{g}) \qquad \text{for } \mathbf{x} \in \Omega^+, \tag{6.10}$$

where $\mathbf{g}$ is the required seven-component vector function.

Obviously, by Theorem 9 the vector function $\mathbf{U}$ is a solution of (3.7) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition (4.6) and using (6.1), from (6.10) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(2)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S. \tag{6.11}$$

We prove that Eq. (6.11) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the homogeneous equation

$$(6.12)\qquad -\frac{1}{2}\,\mathbf{g}_0(\mathbf{z})+\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}_0)=\mathbf{0}\qquad \text{for } \mathbf{z}\in S,$$

where $\mathbf{g}_0$ is the required seven-component vector function. Now, we prove that (6.12) has only the trivial solution. On the basis of Theorem 9 and Eq. (6.12) the vector function $\mathbf{V}(\mathbf{x})=\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}_0)$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 7 and the assumption that ω is not an eigenfrequency of the problem $(II)^+_{\mathbf{0},\mathbf{0}}$, this problem has only the trivial solution, that is

$$(6.13)\qquad \mathbf{V}(\mathbf{x})\equiv\mathbf{0}\qquad \text{for } \mathbf{x}\in\Omega^+.$$

On the other hand, by Theorem 9 and (6.13) we get

$$\{\mathbf{V}(\mathbf{z})\}^-=\{\mathbf{V}(\mathbf{z})\}^+=\mathbf{0}\qquad \text{for } \mathbf{z}\in S,$$

i.e., on the basis of Theorem 9 the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 8 the problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$(6.14)\qquad \mathbf{V}(\mathbf{x})\equiv\mathbf{0}\qquad \text{for } \mathbf{x}\in\Omega^-.$$

By virtue of (6.13), (6.14) and identity (6.1) we obtain

$$\mathbf{g}_0(\mathbf{z})=\{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^+=\mathbf{0}\qquad \text{for } \mathbf{z}\in S.$$

Thus, the homogeneous equation (6.12) has only the trivial solution and therefore on the basis of Fredholm's theorem the integral equation (6.11) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

THEOREM 14. *If* $S\in C^{2,\nu}$, $\mathbf{f}\in C^{0,\nu'}(S)$, $0<\nu'<\nu\le 1$, *and* ω *is not an eigenfrequency of the BVP* $(II)^+_{\mathbf{0},\mathbf{0}}$, *then a regular solution of the internal BVP* $(II)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by the single-layer potential* (6.10), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (6.11) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$. Quite similarly the following theorem is proved.

THEOREM 15. *If* $S\in C^{2,\nu}$, $\mathbf{f}\in C^{1,\nu'}(S)$, $0<\nu'<\nu\le 1$, *then a regular solution* $\mathbf{U}$ *of the external BVP* $(I)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by a sum of double-layer and single-layer potentials*

$$\mathbf{U}(\mathbf{x})=\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g})+(1-i)\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g})\qquad \textit{for } \mathbf{x}\in\Omega^-,$$

where $\mathbf{g}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(3)}\,\mathbf{g}(\mathbf{z})+(1-i)\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})=\mathbf{f}(\mathbf{z})\qquad \textit{for } \mathbf{z}\in S,$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Problem $(II)^{-}_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to this problem in the form

$$\mathbf{U}(\mathbf{x})=\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h})+\mathbf{U}^{*}(\mathbf{x})\qquad \text{for } \mathbf{x}\in\Omega^{-}, \tag{6.15}$$

where $\mathbf{h}$ is the required seven-component vector function and the six-component vector function $\mathbf{U}^{*}$ is a regular solution of the equation

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\mathbf{U}^{*}(\mathbf{x})=\mathbf{0}\qquad \text{for } \mathbf{x}\in\Omega^{-}. \tag{6.16}$$

Keeping in mind the boundary condition (4.7) and using (6.1), from (6.15) we obtain the following singular integral equation for determining the unknown vector $\mathbf{h}$

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z})=\mathbf{f}^{*}(\mathbf{z})\qquad \text{for } \mathbf{z}\in S, \tag{6.17}$$

where

$$\mathbf{f}^{*}(\mathbf{z})=\mathbf{f}(\mathbf{z})-\{\mathbf{P}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{U}^{*}(\mathbf{z})\}^{-}. \tag{6.18}$$

Now, we prove that Eq. (6.17) is always solvable for an arbitrary vector $\mathbf{f}$. We assume that the homogeneous equation

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z})=\mathbf{0} \tag{6.19}$$

has m linearly independent solutions $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^{m}$ that are assumed to the orthonormal. By Fredholm's theorem the solvability condition of Eq. (6.17) can be written as

$$\int_S \{\mathbf{P}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{U}^{*}(\mathbf{z})\}^{-}\cdot\boldsymbol{\psi}^{(l)}(\mathbf{z})d_{\mathbf{z}}S=N_l, \tag{6.20}$$

where

$$N_l=\int_S \mathbf{f}(\mathbf{z})\cdot\boldsymbol{\psi}^{(l)}(\mathbf{z})d_{\mathbf{z}}S$$

and $\{\boldsymbol{\psi}^{(l)}(\mathbf{z})\}_{l=1}^{m}$ is a complete system of solutions of the homogeneous associated equation of (6.19), i.e.

$$\mathcal{K}^{(1)}\,\boldsymbol{\psi}^{(l)}=\mathbf{0},\qquad l=1,\dots,m.$$

It is easy to see that the condition (6.20) takes the form (for details see [44])

$$\int_S \mathbf{h}^{(l)}(\mathbf{z}) \cdot \{\mathbf{U}^*(\mathbf{z})\}^- d_{\mathbf{z}}S = -N_l, \qquad l = 1, \dots, m. \tag{6.21}$$

Let the vector $\mathbf{U}^*$ be a solution of (6.16) and satisfies the boundary condition

$$\{\mathbf{U}^*(\mathbf{z})\}^- = \hat{\mathbf{f}}(\mathbf{z}), \tag{6.22}$$

where

$$\hat{\mathbf{f}}(\mathbf{z}) = \sum_{l=1}^{m} N_l \mathbf{h}^{(l)}(\mathbf{z}). \tag{6.23}$$

By virtue of Theorem 15 the BVP (6.16), (6.22) is always solvable. Because of the orthonormalization of $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^m$, the condition (6.21) is fulfilled automatically and the solvability of (6.17) is proved. Consequently, the existence of regular solution of the problem $(II)^-_{\mathbf{0},\mathbf{f}}$ is proved too. Thus, the following theorem has been proved.

Theorem 16. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then a regular solution $\mathbf{U}$ of the external BVP $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the sum* (6.15)*, where $\mathbf{h}$ is a solution of the singular integral equation* (6.17) *which is always solvable, $\mathbf{U}^*$ is the solution of BVP* (6.16),(6.22) *which is always solvable; and the vector functions $\mathbf{f}^*$ and $\hat{\mathbf{f}}$ are determined by* (6.18) *and* (6.23)*, respectively.*

7. Concluding remarks

1. In the present paper the linear theory of thermoelasticity for materials with triple voids is considered and the following results are obtained:

(i) the fundamental solution of the system of equations of steady vibrations is constructed explicitly by means of elementary functions and its basic properties are established;

(ii) the radiation conditions are established and the uniqueness theorems of the basic internal and external BVPs of steady vibrations are proved;

(iii) the basic properties of the surface (single-layer and double-layer) and volume potentials are established;

(iv) the existence theorems for regular (classical) solutions of the above mentioned BVPs are proved by using the potential method and the theory of singular integral equations.

2. On the basis of results of this paper it is possible to investigate the nonclassical BVPs in the linear theories of elasticity and thermoelasticity for materials with multiple voids by using the potential method and the theory of singular integral equations.

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