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SEVENTY YEARS OF THE ARCHIVES OF MECHANICS

Free in-plane and out-of-plane vibrations of rotating thin ring based on the toroidal shell theory

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In this paper rigorous formulae for natural frequencies of in-plane and out-of-plane free vibrations of a rotating ring are derived. An in-plane vibration mode of the ring is characterised by coupled flexural and extensional deformations, whereas an out-of-plane mode is distinguished by coupled flexural and torsional deformations. The expressions for natural frequencies are derived from a generalised toroidal shell theory. For the in-plane vibrations, the ring is considered to be a short top segment of a toroidal shell. For the out-of-plane vibrations, the ring is considered to be a side segment of the shell. Natural vibrations are analysed by the energy approach. The expressions for the ring strain and kinetic energies are deduced from the corresponding expressions for the torus. It is shown that the ring rotation causes bifurcation of natural frequencies of the in-plane vibrations only. Bifurcation of natural frequencies of the out-of-plane vibrations does not occur. Otherwise, for non-rotating rings, the derived formulae for the natural frequencies of the in-plane and the out-of-plane flexural vibrations are very similar. The derived analytical results are validated by a comparison with FEM and FSM (Finite Strip Method) results, as well as with experimental results available in the literature.

Key words: rotating ring, in-plane vibration, out-of-plane vibration, toroidal shell, analytical solution, bifurcations of natural frequencies.

1. Introduction

Vibration analysis of either stationary or rotating rings is relatively simple due to their simple geometry. However, the dynamics of the ring can, to some extent, reflect global dynamics of more complex axisymmetric shell structures, such as cylindrical or toroidal shells. This is appearing because analytical modelling vibration of shells can lead to very involved mathematical formulations, [1, 2], in which the underlying physics may become blurred with mathematical complexity. It is thus instructive to study vibrations of simplified rotating structures, such as rings, in order to better understand vibrations of more complex rotating axisymmetric structures. For example, a number of authors have investigated vibrations of rotating rings as simplified models of rotating automotive tyres [3–5].

Vibration of rotating rings has been of interest for well over a century. The pioneering work on this subject was Bryan's investigation of a rotating wineglass from 1890, [6]. The thin ring vibration theory is presented in Love's book in 1927, [7]. Since those times, the knowledge of the rotating ring dynamic behaviour has been steadily improved by many authors. The governing differential equations were derived by Carrier and solutions of some special cases are shown in [8]. An expression for natural frequencies of a rotating ring is given by JOHNSON [9].

The effects of shear stiffness and rotary inertia of thick stationary and rotating rings, as well as the influence of the elastic foundation supporting the ring, have been systematically investigated by using different mathematical models, [10–17]. A special attention has been paid to the Coriolis coupling effects, [18, 19]. Forced vibrations of rotating rings due to both harmonic and periodic excitations have been investigated by modal expansion and Fourier series [20], and a closed form solution has been obtained. A stationary ring subjected to travelling loads has also been analysed in reference [20].

A very useful experimental investigation of vibrations of a thin rotating ring is presented in [15]. The natural frequencies of forward and backward flexural travelling waves were measured using strain gauges. Different mathematical models were evaluated by comparing numerical results with the measured ones. It was ascertained that no instability phenomenon exists, i.e. natural frequencies never become zero due to increasing rotation speed, in spite of some theoretical models anticipating such phenomena.

Non-linear partial differential equations for coupled in-plane and out-of-plane vibrations have been derived by the energy formulation, employing Hamilton's principle, [21]. Four mathematical models were established and linearized in [21]. The natural frequencies determined in the considered numerical examples were mutually compared in order to recommend an appropriate model to describe the non-linear dynamic behaviour more precisely [21].

In this paper a mathematical model for the in-plane and out-of-plane free vibrations of a rotating ring is formulated, which is based on the toroidal shell theory. This theory is rather complex due to the toroidal geometry involving double curvature. However, the theory is universal since a toroidal shell, Fig. 1, can take shape of the following basic shells: cylindrical shell ($a = \infty$, $\vartheta = \pi/2$, $a\mathrm{d}\vartheta = \mathrm{d}x$), circular membrane and plate ($a = \infty$, $\vartheta = 0$, $a\mathrm{d}\vartheta = \mathrm{d}r$), conical shell ($a = \infty$, $\vartheta = \vartheta_0$, $a\mathrm{d}\vartheta = \mathrm{d}x$), and spherical shell ($R$=0).

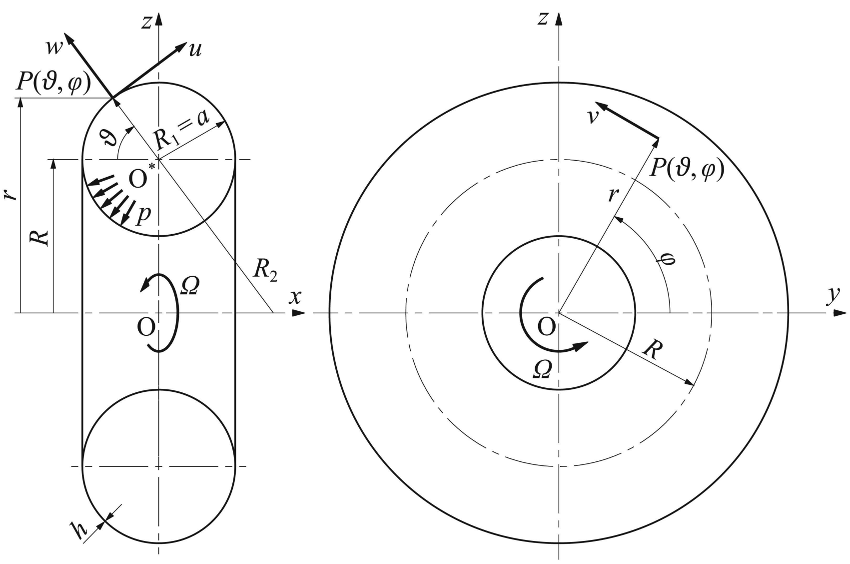

Fig. 1. Rotating toroidal shell, main dimensions and displacements.

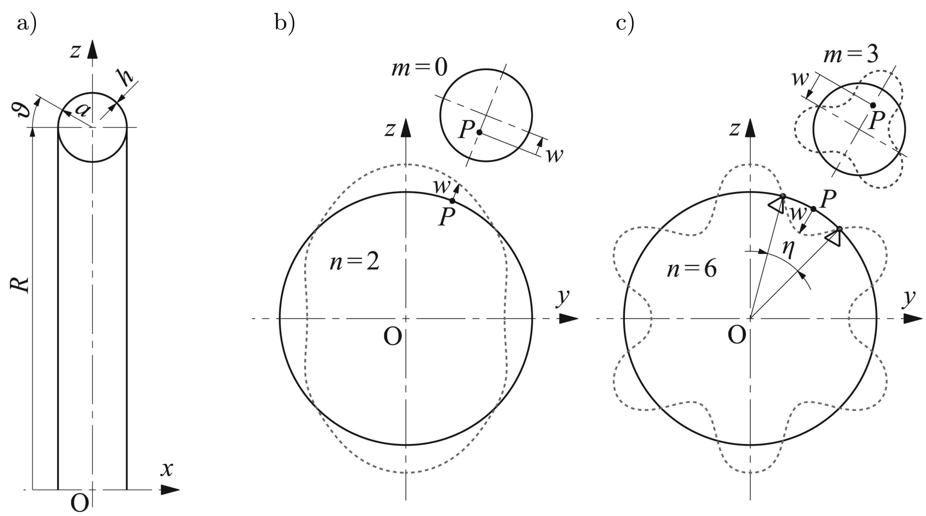

Fig. 2. Thin-walled toroidal ring.

Moreover, if the ratio of a toroidal shell radii $a/R \ll 1$, then it behaves like a thin-walled ring of a cross-section area $A = 2\pi ah$, and a moment of inertia $I = \pi a^3 h$, Fig. 2a. For small values of the circumferential wave number n vibrations are global, without deformations of the cross-section, Fig. 2b. However, at higher n-values, local deformations of the cross-section become noticeable, Fig. 2c, which are coupled with global deformations, shifting the natural frequencies downwards. If the central angle between two adjacent global vibration nodes, η, is small enough, such a segment of a thin-walled toroidal shell can be considered as a simply supported cylindrical shell, [22].

A ring can perform vibrations either in the φ-plane or out of the φ-plane, Fig. 1. In the former case the flexural vibrations are coupled with the extensional vibrations, and in the latter case they are coupled with torsional vibrations. These problems are analysed in the literature by solving differential equations of motion, usually derived by the energy approach and the application of Hamilton's principle, [21]. The intention of this paper is to demonstrate the universality of the toroidal shell theory in case of both in-plane and out-of-plane ring vibrations, and to present a rigorous analytical solution of the characteristic equation for the rotating ring in-plane vibrations.

The paper is structured in 6 sections. In Section 2 the basic theory for the forthcoming analysis is derived as a starting point. In Section 3 the in-plane vibrations of the rotating ring are discussed. In Section 4 the out-of-plane vibrations of the rotating ring are analysed. Finally, in Section 5, numerical examples are presented and the developed theories are validated by comparison with FEM, FSM, and experimental results.

2. Basic expressions for strain and kinetic energies of rotating toroidal shell

Vibration of a toroidal shell can be analysed by the Rayleigh–Ritz method, which is based on the minimisation of the strain and kinetic energies, [23]. The extensional displacement in the cross-sectional ϑ-plane, u, the circumferential displacement, v, and the radial deflection, w, Fig. 1, are assumed in the form

$$
\begin{aligned}
u(\vartheta,\varphi,t) &= U(\vartheta)\cos(n\varphi+\omega t),\\
\nu(\vartheta,\varphi,t) &= V(\vartheta)\sin(n\varphi+\omega t),\\
w(\vartheta,\varphi,t) &= W(\vartheta)\cos(n\varphi+\omega t),
\end{aligned}
\tag{2.1}
$$

where functions $U(\vartheta)$, $V(\vartheta)$ and $W(\vartheta)$ are the cross-sectional mode profiles, and ω is a natural frequency. The argument $n\varphi+\omega t$ in the trigonometric functions is used in order to be able to describe rotating modes which normally appear with rotating shells.

The modal strain energy, after integration in the circumferential direction, in the domain $0 \le \varphi \le 2\pi$, reads

$$(2.2)\qquad E_s = \int\limits_{\vartheta} \Big[\frac{1}{2}p_1(U')^2+\frac{1}{2}p_2U^2+p_3U'U+\frac{1}{2}p_4(V')^2+\frac{1}{2}p_5V^2+p_6V'V$$
$$+p_7U'V+p_8UV'+p_9UV$$
$$+\frac{1}{2}q_1(W'')^2+\frac{1}{2}q_2(W')^2+\frac{1}{2}q_3W^2+q_4W''W'+q_5W''W+q_6W'W$$
$$+q_7W''U'+q_8(W''U+W'U')+q_9W'U+q_{10}WU'+q_{11}WU$$
$$+q_{12}W''V+q_{13}W'V'+q_{14}W'V+q_{15}WV'+q_{16}WV\Big]\,\mathrm{d}\vartheta,$$

where $p_i(\vartheta)$, $i = 1, 2, \ldots, 9$ and $q_j(\vartheta)$, $j = 1, 2, \ldots, 16$ are variable coefficients which can be found in [23].

For the vibration analysis of a rotating toroidal shell the geometric strain energy due to the centrifugal and Coriolis forces is given by, [23],

$$(2.3)\qquad E_G = \int\limits_{\vartheta} \Big[\frac{1}{2}c_1(U')^2 + \frac{1}{2}c_2U^2 + \frac{1}{2}c_3(V')^2 + \frac{1}{2}c_4V^2$$
$$+ c_5V'V + \frac{1}{2}c_6(W')^2 + \frac{1}{2}c_7W^2 + c_8UV$$
$$+ c_9(U'W - UW') + c_{10}UW + c_{11}VW\Big]\,\mathrm{d}\vartheta,$$

where $c_i(\vartheta)$, $i = 1, 2, \ldots, 11$ are variable coefficients which can be found in [23].

The kinetic energy is presented in the form

$$(2.4)\qquad E_k = \frac{1}{2}\pi\rho ha \int\limits_{\vartheta} r[(\omega^2 + \Omega^2\cos^2\vartheta)U^2 + (\omega^2 + \Omega^2)V^2$$
$$+ (\omega^2 + \Omega^2\sin^2\vartheta)W^2 + 4\omega\Omega(\cos\vartheta UV + \sin\vartheta VW)$$
$$+ 2\Omega^2\sin\vartheta\cos\vartheta UW]\,\mathrm{d}\vartheta,$$

where ρ is the mass density, h is the shell thickness, a and r are cross-sectional and circumferential radii, respectively, and Ω is the rotational speed, Fig. 1.

The above equations are the starting point for the ring vibration analysis.

3. In-plane vibrations of rotating ring

For this purpose a toroidal shell segment in the vicinity of angle $\vartheta = \pi/2$ is considered, as shown in Fig. 3. For the in-plane vibrations the relevant displacements are the circumferential and the radial ones, V and W. The expressions for the strain energy, the geometric strain energy and the kinetic energy, Eqs. (2.2), (2.3) and (2.4), respectively, are after integration no longer functions of the angle ϑ. Therefore they are reduced to the following form for a unit length of the arch ($b = 1$):

$$
\begin{aligned}
E_s &= \frac{1}{2}p_5V^2 + \frac{1}{2}q_3W^2 + q_{16}VW,\\
E_G &= \frac{1}{2}c_4V^2 + \frac{1}{2}c_7W^2 + c_{11}VW,\\
E_K &= \frac{1}{2}\alpha[(\omega^2 + \Omega^2)V^2 + (\omega^2 + \Omega^2)W^2 + 4\omega\Omega VW],
\end{aligned}
\tag{3.1}
$$

where $\alpha = \pi\rho har$. The terms in E_K with Ω^2 and Ω represent the kinetic energy due to the centrifugal and Coriolis forces, respectively. Coefficients p_i, q_i and c_i in (3.1) are specified according to [23], taking into account that $\vartheta = \pi/2$ and $\nu = 0$ for the ring as a one-dimensional structural element

$$
\begin{aligned}
\frac{p_5}{\alpha} &= \frac{K}{\rho hr^2}n^2\left(1 + \frac{D}{Kr^2}\right),\\
\frac{q_3}{\alpha} &= \frac{K}{\rho hr^2}\left(1 + n^4\frac{D}{Kr^2}\right),\\
\frac{q_{16}}{\alpha} &= \frac{K}{\rho hr^2}n\left(1 + n^2\frac{D}{Kr^2}\right),\\
\frac{c_4}{\alpha} &= \frac{c_7}{\alpha} = (n^2 + 1)\Omega^2,\\
\frac{c_{11}}{\alpha} &= 2n\Omega^2,
\end{aligned}
\tag{3.2}
$$

where

$$
K = Eh, \qquad D = \frac{Eh^3}{12}.
\tag{3.3}
$$

The coefficients c_4, c_7 and c_{11} take into account the pre-stressing membrane force $N_\varphi = \rho hr^2\Omega^2$ due to the centrifugal load.

Minimizing the total energy $E = E_S + E_G - E_K$ by setting its derivatives per V and W equal to zero, yields a symmetric matrix equation

$$
\begin{bmatrix} a_{11} - \omega^2 & a_{12} - 2\Omega\omega \\ a_{21} - 2\Omega\omega & a_{22} - \omega^2 \end{bmatrix} \begin{Bmatrix} V \\ W \end{Bmatrix} = \{0\},
\tag{3.4}
$$

where

$$
\begin{aligned}
a_{11} &= \frac{p_5}{\alpha} + \frac{c_4}{\alpha} - \Omega^2 = \frac{p_5}{\alpha} + n^2\Omega^2,\\
a_{22} &= \frac{q_3}{\alpha} + \frac{c_7}{\alpha} - \Omega^2 = \frac{q_3}{\alpha} + n^2\Omega^2,\\
a_{12} = a_{21} &= \frac{q_{16}}{\alpha} + \frac{c_{11}}{\alpha} = \frac{q_{16}}{\alpha} + 2n\Omega^2.
\end{aligned}
\tag{3.5}
$$

Note that the coefficients related to the geometric strain energy, c_4 and c_7 in (3.2), are summed up in (3.5) with those related to the kinetic energy, Ω^2.

The non-trivial solution of Eq. (3.4) is obtained from the condition that the determinant of the matrix in Eq. (3.4) vanishes. Applying this condition results in the following characteristic equation in the form of a fourth order (quartic) polynomial:

$$\omega^4 - a_2\omega^2 + a_1\omega + a_0 = 0, \tag{3.6}$$

where

$$\begin{aligned} a_2 &= a_{11} + a_{22} + 4\Omega^2, \\ a_1 &= 4a_{12}\Omega, \\ a_0 &= a_{11}a_{22} - a_{12}^2. \end{aligned} \tag{3.7}$$

Substituting Eqs. (3.5) into (3.7) one obtains

$$\begin{aligned} a_2 &= \frac{K}{\rho h r^2}(n^2+1)\left(1 + n^2\frac{D}{Kr^2}\right) + 2(n^2+2)\Omega^2, \\ a_1 &= 4\Omega n\left[\frac{K}{\rho h r^2}\left(1 + n^2\frac{D}{Kr^2}\right) + 2\Omega^2\right], \\ a_0 &= \left(\frac{K}{\rho h r^2}\right)^2 n^2(n^2-1)^2\frac{D}{Kr^2} \\ &\quad + \frac{K}{\rho h r^2}n^2(n^2-3)\left(1 + n^2\frac{D}{Kr^2}\right)\Omega^2 + n^2(n^2-4)\Omega^4. \end{aligned} \tag{3.8}$$

Although it is possible to express the roots of the fourth order polynomial (3.6) using standard formulae, the expressions would be long, cumbersome, not useful for quick calculations, and not physically transparent. Therefore, equations such as Eq. (3.6) are ordinarily solved numerically. However, it is still possible to obtain approximate and reliable analytical solutions by combining some physical reasoning with the fact that the third order term is missing in Eq. (3.6).

As can be seen in Eqs. (3.8) the coefficients a_0 and a_2 include centrifugal force (terms with Ω^2), while coefficient a_1 includes both the Coriolis force (the term with Ω) and the centrifugal force. If the Coriolis force is ignored, $a_1 = 0$, then Eq. (3.6) can be reduced to a bi-quadratic polynomial with roots

$$\omega_{e,b} = \pm\sqrt{\frac{a_2}{2}}\sqrt{1 \pm \sqrt{1 - \frac{4a_0}{a_2^2}}}. \tag{3.9}$$

The second term in the inside square root, $4a_0/a_2^2$, is of order $D/(Kr^2) = (h/r)^2/12$, which is much smaller than 1. Therefore the inner square root can be expanded into power series. Taking into account only the first two terms of the expansion, $\sqrt{1-\varepsilon} \approx 1-\varepsilon/2$, the following expressions for the natural frequencies are obtained

$$\omega_e = \pm\sqrt{a_2}, \qquad \omega_b = \pm\sqrt{\frac{a_0}{a_2}}. \tag{3.10}$$

Using the same reasoning as for the second term in the inside square root of Eq. (3.9), it can be shown that the first root in Eq. (3.10) is much larger than the second one. In fact, the first root, ω_e, represents natural frequencies of a group of modes characterised solely by extensional deformations of the ring along the circumference. The second root, ω_b, represents natural frequencies of the in-plane bending modes.

The coefficient a_1 in Eq. (3.6) is related to the Coriolis force. It causes bifurcation of natural frequencies (without it the quartic polynomial becomes bi-quadratic). The value of a_1 is relatively small, starting from zero if $\varOmega = 0$, Eqs. (3.8). Therefore, the term $a_1\omega$ in Eq. (3.6) can be grouped with a_0, and the solution of the pseudo-bi-quadratic equation can be presented in the form

$$\omega = \pm\sqrt{\frac{a_2}{2}}\sqrt{1 \pm \sqrt{1 - \frac{4a_0 + a_1\omega}{a_2^2}}}. \tag{3.11}$$

Equation (3.11) can be solved iteratively. In the first step of iteration for the extensional vibrations $\omega = \omega_e = \pm\sqrt{a_2}$, Eqs. (3.10). Substituting ω_e into (3.11) and taking into account the fact that the natural frequencies of extensional vibrations are much higher than those corresponding to the flexural vibrations, yields

$$\tilde{\omega}_e = \pm\sqrt{\frac{a_2}{2}}\sqrt{1 + \sqrt{1 - 4\frac{a_0}{a_2^2} \mp 4\frac{a_1}{a_2^2}\sqrt{a_2}}}. \tag{3.12}$$

Furthermore, the second term in the inside square root in Eq. (3.12) is again negligible in comparison with unity, whereas the third term is very small in comparison with unity. Therefore, the inside square root can be expanded into power series. Taking into account only the first two terms of the expansion, one can write

$$\tilde{\omega}_e = \pm\sqrt{a_2} \mp \frac{a_1}{2a_2}. \tag{3.13}$$

Inserting Eqs. (3.8) for coefficients a_0, a_1 and a_2 into (3.13), and keeping only the dominant terms, one obtains an approximate formula for determining extensional natural frequencies of a rotating ring

$$\tilde{\omega}_e = \frac{2n}{n^2+1}\varOmega \pm \sqrt{(\omega_e^0)^2 + 2(n^2+2)\varOmega^2}, \tag{3.14}$$

where

$$\omega_e^0 = \sqrt{(n^2+1)\left(1+n^2\frac{D}{Kr^2}\right)}\sqrt{\frac{K}{\rho h r^2}} \tag{3.15}$$

are the extensional natural frequencies of the non-rotating ring.

Flexural bending frequency, ω_b, Eqs. (3.10), is the solution of the characteristic Eq. (3.6) without $a_1\omega$. If $\omega_b = \pm\sqrt{a_0/a_2}$ is inserted into (3.6), ω^4 may be ignored as a small quantity of higher order. The flexural natural frequency of the rotating ring, $\tilde{\omega}_b$, can also be derived under this assumption if rotation speed Ω is small. The solution of the corresponding quadratic equation

$$a_2\tilde{\omega}_b^2 - a_1\tilde{\omega}_b - a_0 = 0 \tag{3.16}$$

represents an approximate solution of the bi-quadratic Eq. (3.6)

$$\tilde{\omega}_b = \frac{a_1}{2a_2} \pm \sqrt{\left(\frac{a_1}{2a_2}\right)^2 + \frac{a_0}{a_2}}. \tag{3.17}$$

Substituting expressions (3.8) for coefficients a_0, a_1 and a_2 into (3.17) and taking into account only their dominant terms, one obtains, in case that n and Ω are relatively small,

$$\tilde{\omega}_b = \frac{2n}{n^2+1}\Omega \pm \sqrt{(\omega_b^0)^2 + \frac{n^2(n^2-1)^2}{(n^2+1)^2}\Omega^2}, \tag{3.18}$$

where

$$\omega_b^0 = \frac{n(n^2-1)}{\sqrt{(n^2+1)\left(1+n^2\frac{D}{Kr^2}\right)}}\sqrt{\frac{D}{\rho h r^2}} \tag{3.19}$$

is a flexural natural frequency of the non-rotating ring. If $n = 1$, $\omega_b^0 = 0$ and the ring performs rigid body motion in φ-plane.

Formula (3.18) is well-known in the relevant literature and has been derived by solving governing differential equations of motion under the assumption $w(\varphi,t) = -\partial v(\varphi,t)/\partial\varphi$, [1, 8, 9, 15]. Therefore, this intuitively introduced assumption is justified.

The structure of formulae (3.14) and (3.18) for extensional and flexural natural frequencies is similar. Positive and negative values are related to the forward and backward rotating modes. The values of natural frequencies of extensional vibrations are much larger than those of flexural vibrations since the stiffness ratio $D/(Kr^2) \ll 1$. However, the bifurcation of natural frequencies according to the first term in Eqs. (3.14) and (3.18) is the same in both cases.

Formulae (3.14) and (3.18) are applicable only for relatively small circumferential wave number n in the domain $0 < \Omega/\omega_e^0 < 1$ and $0 < \Omega/\omega_b^0 < 1$ respectively. Otherwise, more reliable formulae (3.12) and (3.17) are on disposal.

However, if the rigorous solution is desired, it is necessary to find the roots of a quartic polynomial, Eq. (3.6). For this purpose the characteristic equation (3.6) is solved exactly in a sophisticated way in Appendix to the paper.

In case of a thin-walled toroidal ring one can write for the ratios in Eqs. (3.15) and (3.19), $K/(hr^2) = E/R^2$, $D/(hr^2) = EI/(AR^2)$ and $D/(Kr^2) = I/(AR^2)$. Taking into account that $A = 2\pi ah$ and $I = \pi a^3 h$, yields $I/(AR^2) = (a/R)^2/2$. Inserting the above relations into Eqs. (3.15) and (3.19) one obtains

$$\omega_e^0 = \sqrt{(n^2+1)\left[1+2n^2\left(\frac{a}{R}\right)^2\right]}\sqrt{\frac{E}{\rho R^2}}, \tag{3.20}$$

$$\omega_b^0 = \frac{n(n^2-1)}{\sqrt{(n^2+1)\left[1+2n^2\left(\frac{a}{R}\right)^2\right]}}\left(\frac{a}{R}\right)\sqrt{\frac{E}{2\rho R^2}}. \tag{3.21}$$

Flexural natural frequencies linearly depend on ratio a/R.

4. Out-of-plane vibrations of rotating ring

This type of vibrations is analysed by considering the toroidal shell segment in the vicinity of angle $\vartheta = \pi$, with two degrees of freedom, i.e. deflection W and twist angle Ψ, Fig. 3. Since extensional displacements U and V are zero, the strain energy according to Eq. (2.2) is reduced to

$$E_S = \frac{1}{2}q_1(W'')^2 + \frac{1}{2}q_2(W')^2 + \frac{1}{2}q_3W^2 + q_4W''W' + q_5W''W + q_6W'W, \tag{4.1}$$

where referring to [23], and setting $K = 0$

$$\begin{aligned}
q_1 &= \pi\frac{D}{a^2}\,\frac{r}{a},\\
q_2 &= \pi\frac{D}{ar}[1+2(1-\nu)n^2],\\
q_3 &= \pi\frac{a}{r}\,\frac{D}{r^2}n^2[n^2+2(1-\nu)],\\
q_4 &= \pi\nu\frac{D}{a^2},\\
q_5 &= -\pi\nu\frac{D}{ar}n^2,\\
q_6 &= -\pi[1+2(1-\nu)]\frac{D}{r^2}n^2,
\end{aligned} \tag{4.2}$$

where K and D are defined with Eq. (3.3). Poisson's coefficient ν is not ignored in (4.2) (like in the case of in-plane vibrations), since it is introduced through the shear modulus $G = E/(2(1+\nu))$ at the very beginning of development of toroidal shell vibration theory.

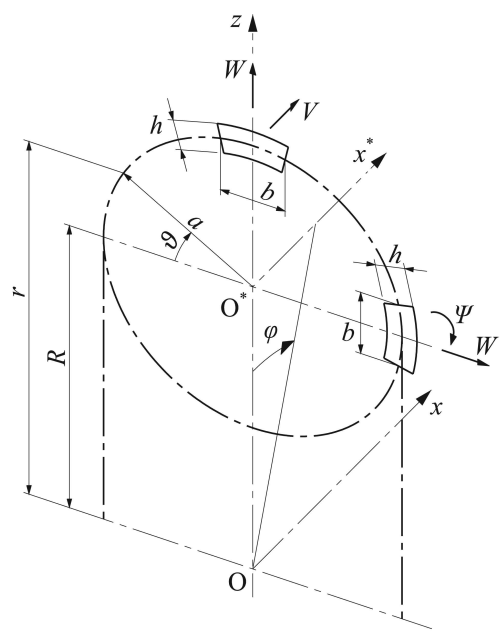

FIG. 3. Toroidal shell segments as rings.

The deflection derivative is actually the twist angle, Fig. 3, and a new variable is introduced for simplicity

$$W' = \frac{\mathrm{d}W}{\mathrm{d}\vartheta} = a\Psi = \frac{a}{r}X. \tag{4.3}$$

In a similar way one can write for the curvature

$$W'' = \frac{\mathrm{d}^2W}{\mathrm{d}\vartheta^2} = a^2\theta = \left(\frac{a}{r}\right)^2 Y. \tag{4.4}$$

Substituting expressions (4.3) and (4.4) into (4.1), yields

$$\begin{aligned} E_S = {} & \frac{1}{2}q_1\left(\frac{a}{r}\right)^4 Y^2 + \frac{1}{2}q_2\left(\frac{a}{r}\right)^2 X^2 + \frac{1}{2}q_3 W^2 \\ & + q_4\left(\frac{a}{r}\right)^3 XY + q_5\left(\frac{a}{r}\right)^2 YW + q_6\left(\frac{a}{r}\right) XW. \end{aligned} \tag{4.5}$$

Furthermore

$$\begin{aligned}\frac{\partial E_S}{\partial W} &= q_3 W + q_5\left(\frac{a}{r}\right)^2 Y + q_6\left(\frac{a}{r}\right)X,\\ \frac{\partial E_S}{\partial X} &= q_2\left(\frac{a}{r}\right)^2 X + q_4\left(\frac{a}{r}\right)^3 Y + q_6\left(\frac{a}{r}\right)W,\\ \frac{\partial E_S}{\partial Y} &= q_1\left(\frac{a}{r}\right)^4 Y + q_4\left(\frac{a}{r}\right)^3 X + q_5\left(\frac{a}{r}\right)^2 W = 0.\end{aligned} \tag{4.6}$$

Since the displacement Y is not accompanied with the inertia term and it is also not present in the geometric strain energy, the right hand side of the last equation of (4.6) is set to zero.

Hence, one obtains

$$Y = -\frac{q_4\left(\frac{a}{r}\right)^3}{q_1\left(\frac{a}{r}\right)^4}X - \frac{q_5\left(\frac{a}{r}\right)^2}{q_1\left(\frac{a}{r}\right)^4}W. \tag{4.7}$$

Substituting (4.7) into the first two equations of (4.6) the system of equations is reduced to

$$\begin{aligned}\frac{\partial E_S}{\partial W} &= a_{11}^* W + a_{12}^* X,\\ \frac{\partial E_S}{\partial X} &= a_{21}^* W + a_{22}^* X,\end{aligned} \tag{4.8}$$

where coefficients a_{ij}, taking into account Eq. (4.3), are given by

$$\begin{aligned}a_{11}^* &= \pi\frac{a}{r}\frac{D}{r^2}(1-\nu^2)n^2\left(n^2 + \frac{2}{1+\nu}\right),\\ a_{22}^* &= \pi\frac{a}{r}\frac{D}{r^2}(1-\nu^2)\left(1 + \frac{2}{1+\nu}n^2\right),\\ a_{12}^* = a_{21}^* &= -\pi\frac{a}{r}\frac{D}{r^2}(1-\nu^2)n^2\left(1 + \frac{2}{1+\nu}\right),\end{aligned} \tag{4.9}$$

The geometric strain energy, Eq. (2.3), has only one term, i.e.

$$E_G = \frac{1}{2}c_7 W^2, \tag{4.10}$$

where according to [23] and after applying the membrane force due to the centrifugal load $N_\varphi = \rho h r^2 \varOmega^2$

$$c_7 = \pi\frac{a}{r}n^2 N_\varphi = \pi\rho h a r \varOmega^2 n^2. \tag{4.11}$$

The kinetic energy, Eq. (2.4), has also only one term, $\rho h\omega^2 W$, which is related to the inertia force. Since a rotation of the ring cross-section Ψ is introduced, the rotary inertia must be taken into account, too. (If a thick toroidal shell is considered such a term is a priory included in E_K). Based on the analogy between inertia force and the moment of rotary inertia, as well as taking into account substitution $\Psi = \frac{1}{r}X$, Eq. (4.3), one can write

$$hW^2 : h\frac{h^2}{12}\Psi^2 = \frac{i_p}{r^2}X^2, \qquad i_p = \frac{h^3}{12}. \tag{4.12}$$

In this way the kinetic energy can be written as

$$E_K = \frac{1}{2}\pi\rho har\omega^2 W^2 + \frac{1}{2}\pi\rho\frac{a}{r}i_p\omega^2 X^2. \tag{4.13}$$

Now Eq. (4.8) is extended to the total energy $E = E_S + E_G - E_K$ and one can write

$$\begin{aligned} \frac{\partial E}{\partial W} &= a_{11}^* W + a_{12}^* X + \alpha\Omega^2 n^2 W - \alpha\omega^2 W = 0, \\ \frac{\partial E}{\partial X} &= a_{21}^* W + a_{22}^* X - \beta\omega^2 X = 0, \end{aligned} \tag{4.14}$$

where $\alpha = \pi\rho har$ and $\beta = \pi\rho(a/r)i_p$. If the first and the second equation of (4.14) are divided by α and β, respectively, one obtains an asymmetric matrix equation

$$\begin{bmatrix} a_{11} - \omega^2 & a_{12} \\ a_{21} & a_{22} - \omega^2 \end{bmatrix} \begin{Bmatrix} W \\ X \end{Bmatrix} = \{0\}, \tag{4.15}$$

where

$$\begin{aligned} a_{11} &= \frac{D}{\rho hr^2}(1-\nu^2)n^2\left(n^2 + \frac{2}{1+\nu}\right) + n^2\Omega^2, \\ a_{22} &= \frac{D}{\rho i_p r^2}(1-\nu^2)\left(1 + \frac{2}{1+\nu}n^2\right), \\ a_{12} &= -\frac{D}{\rho hr^2}(1-\nu^2)n^2\left(1 + \frac{2}{1+\nu}\right), \\ a_{21} &= -\frac{D}{\rho i_p r^2}(1-\nu^2)n^2\left(1 + \frac{2}{1+\nu}\right). \end{aligned} \tag{4.16}$$

The determinant of the matrix in Eq. (4.15) must vanish, i.e.

$$\omega^4 - a_2\omega^2 + a_0 = 0, \tag{4.17}$$

where

$$a_2 = a_{11} + a_{22}, \qquad a_0 = a_{11}a_{22} - a_{12}a_{21}. \tag{4.18}$$

Inserting (4.16) into (4.18), yields

$$\begin{aligned} a_2 &= (1-\nu^2)\left[\frac{D}{\rho h r^4} n^4 \left(n^2 + \frac{2}{1+\nu}\right) + \frac{D}{\rho i_p r^2}\left(1 + \frac{2}{1+\nu} n^2\right)\right] + n^2 \Omega^2, \\ a_0 &= \left[\frac{D}{\rho r^2}(1-\nu^2)\right]^2 \frac{1}{h i_p r^2} n^2 \\ &\quad \times \left[\left(n^2 + \frac{2}{1+\nu}\right)\left(1 + \frac{2}{1+\nu} n^2\right) - \left(1 + \frac{2}{1+\nu}\right)^2 n^2\right] \\ &\quad + \frac{D}{\rho i_p r^2}(1-\nu^2) n^2 \left(1 + \frac{2}{1+\nu} n^2\right)\Omega^2. \end{aligned} \tag{4.19}$$

Now, it is necessary to substitute all the shell parameters specified per unit length with the ring parameters of breadth b, Fig. 3,

$$\frac{D(1-\nu^2)}{i_p} = \frac{EI}{I_p}, \qquad \frac{D(1-\nu^2)}{h} = \frac{EI}{A}, \qquad \frac{i_p}{h} = \frac{I_p}{A}. \tag{4.20}$$

The moment of inertia of the shell cross-section $i_p = h^3/12$, related to the rotary inertia, is substituted by the equivalent ring polar moment of inertia $I_p = (h^2 + b^2)bh/12$.

Furthermore, formulae (4.19) are derived for a shell segment, and the strain energy includes the energy of both twist moments at the meridional and circumferential shell cross-sections, $M_{12} = M_{21}$. Their energy is represented in the formula for the total strain energy in [23] by the term $(1-\nu)\kappa_{12}^2/2$, where κ_{12} is the twist strain. This is shown in Eqs. (4.19) by the coefficient $2/(1+\nu)$. Therefore, only one half of this coefficient must be taken into account in Eqs. (4.19). In this way formulae (4.19) derived for the toroidal shell are modified in such a way to be valid for a ring

$$\begin{aligned} a_2 &= \frac{E}{\rho r^2}\left[\frac{I}{Ar^2} n^2 \left(n^2 + \frac{1}{1+\nu}\right) + \frac{I}{I_p}\left(1 + \frac{1}{1+\nu} n^2\right)\right] + n^2\Omega^2, \\ a_0 &= \left(\frac{E}{\rho r^2}\right)^2 \frac{I}{Ar^2}\,\frac{I}{I_p}\,\frac{1}{1+\nu} n^2 (n^2-1)^2 + \frac{E}{\rho r^2}\,\frac{I}{I_p} n^2 \left(1 + \frac{1}{1+\nu} n^2\right)\Omega^2. \end{aligned} \tag{4.21}$$

The solutions of the bi-quadratic equation (4.17) can be presented in the form

$$\omega_{t,b} = \sqrt{\frac{a_2}{2}\left[1 \pm \sqrt{1 - \frac{4a_0}{a_2^2}}\right]}. \tag{4.22}$$

The first solution represents natural frequencies of predominantly torsional vibrations, while the second one represents natural frequencies of predominantly flexural vibrations. This can be easily seen in the case of a non-rotating ring. By setting $\Omega = 0$ it becomes obvious that the second term under the square root of Eq. (4.22) is very small. By using the approximation $\sqrt{1-\varepsilon} \approx 1 - \varepsilon/2$ one obtains the following expressions for natural frequencies of torsional and flexural vibrations of a stationary ring:

$$\omega_t^0 = \sqrt{\Big(1 + \frac{n^2}{1+\nu}\Big) + \frac{I_p}{Ar^2} n^2 \Big(n^2 + \frac{1}{1+\nu}\Big)} \sqrt{\frac{E}{\rho r^2} \frac{I}{I_p}}, \tag{4.23}$$

$$\omega_b^0 = \frac{n(n^2-1)}{\sqrt{n^2 + 1 + \nu + \frac{I_p}{Ar^2} n^2[(1+\nu)n^2 + 1]}} \sqrt{\frac{E}{\rho r^2} \frac{I}{Ar^2}}. \tag{4.24}$$

For a toroidal ring $I/I_p = 1/2$ and $I_p/(AR^2) = (a/R)^2$, so that

$$\omega_t^0 = \sqrt{\Big(1 + \frac{n^2}{1+\nu}\Big) + \Big(\frac{a}{R}\Big)^2 n^2 \Big(n^2 + \frac{1}{1+\nu}\Big)} \sqrt{\frac{E}{2\rho R^2}}, \tag{4.25}$$

$$\omega_b^0 = \frac{n(n^2-1)}{\sqrt{n^2 + 1 + \nu + \big(\frac{a}{R}\big)^2 n^2[(1+\nu)n^2 + 1]}} \Big(\frac{a}{R}\Big) \sqrt{\frac{E}{2\rho R^2}}. \tag{4.26}$$

Formulae (4.25) and (4.26) for the out-of-plane vibrations are very similar to those for the in-plane vibrations, Eqs. (3.20) and (3.21), respectively. If the small terms $(a/R)^2$ in Eqs. (3.20), (3.21), (4.25) and (4.26)) are ignored, the only difference between these two sets of formulae is Poisson's coefficient ν in Eqs. (4.25) and (4.26). This fact explains why the FEM vibration analyses presented in the forthcoming Section 5 give pairs of the in-plane and out-of-plane bending modes with almost the same natural frequencies for the same n.

It is observed that both the centrifugal load and the Coriolis load induced by the ring rotation are involved in the ring in-plane vibrations (the terms with Ω^2 and Ω in Eqs. (3.8), respectively). With the out-of-plane vibrations only the centrifugal load participates (the terms with Ω^2 in Eqs. (4.19)). Hence, there is no bifurcation of natural frequencies in the latter case.

5. Numerical examples

5.1. Thick-walled toroidal ring

As explained in Introduction, a toroidal shell of the small radius ration a/R behaves like a ring, Fig. 2b. In this subsection a thick-walled toroidal ring with the following geometric and physical properties is considered: $R = 1$ m, $a = 0.05$ m, $h = 0.01$ m, $E = 2.1 \cdot 10^{11}$ N/m^2, $\nu = 0.3$, $\rho = 7850$ kg/m^3. Natural frequencies for the first four flexural modes of the non-rotating ring are determined by the corresponding formulae (3.21) and (4.26) and are listed in Table 1. Natural frequencies of the in-plane flexural vibrations are slightly higher than those of the out-of-plane vibrations due to the reasons explained in Section 5.

Table 1. Flexural natural frequencies of stationary thin-walled toroidal ring, ω (Hz), $R = 1$ m, $a = 0.05$ m, $h = 0.01$ m, $\Omega = 0$.

Mode no.	Mode type	Eq.	n	Ring	Shell, FEM 20×416	FSM (3,3) 200 FS
1	In-plane	(3.21)	2	77.32	75.02	84.85
2	Out-of-plane	(4.26)	2	75.41	72.98	80.94
3	In-plane	(3.21)	3	216.07	209.10	218.04
4	Out-of-plane	(4.26)	3	214.68	205.61	216.85

The same problem is also solved by considering the ring to be a thick toroidal shell. Software ABAQUS with 54R shell element is used [24]. The 3D FEM model includes $20 \times 416 = 8320$ finite elements. The first four natural modes are shown in Fig. 4. The natural frequencies determined by the FEM model agree very well with those obtained by the ring model and calculated by the simple formulae, Table 1.

Table 1 also includes natural frequencies determined by the finite strip method (FSM), [25]. The toroidal shell cross-section is modelled with 200 three nodes higher order strips. The natural frequencies calculated using the ring model are bounded by the FEM and FSM values.

Natural frequencies of the in-plane vibrations of the rotating ring, i.e. flexural and extensional, are determined analytically by employing the exact procedure, Eq. (A17), approximated formulae, Eqs. (3.17) and (3.12), and formulae for estimation, Eqs. (3.18) and (3.14), Table 2. Three values of the rotation speed

Table 2. Natural frequencies of rotating thin-walled toroidal ring in-plane vibrations, $\tilde{\omega}$ (Hz), $R = 1$ m, $a = 0.05$ m, $h = 0.01$ m, $n = 2$, $\omega_0 = 75.41$ Hz.

Ω/ω_0	Method	Flexural, $\tilde{\omega}_b$		Extensional, $\tilde{\omega}_e$	
		Forward	Backward	Forward	Backward
0	All	77.97	77.97	1843.6	1843.6
1	Rigorous, Eq. (A17)	58.54	184.09	1796.0	1921.6
	Approximated, Eqs. (3.17), (3.12)	58.53	182.73	1793.2	1919.9
	Approximated, Eqs. (3.18), (3.14)	59.36	184.00	1802.6	1927.2
2	Rigorous, Eq. (A17)	73.51	327.23	1776.8	2030.5
	Approximated, Eqs. (3.17), (3.12)	73.49	319.49	1762.2	2025.2
	Approximated, Eqs. (3.18), (3.14)	77.90	327.18	1797.9	2047.2

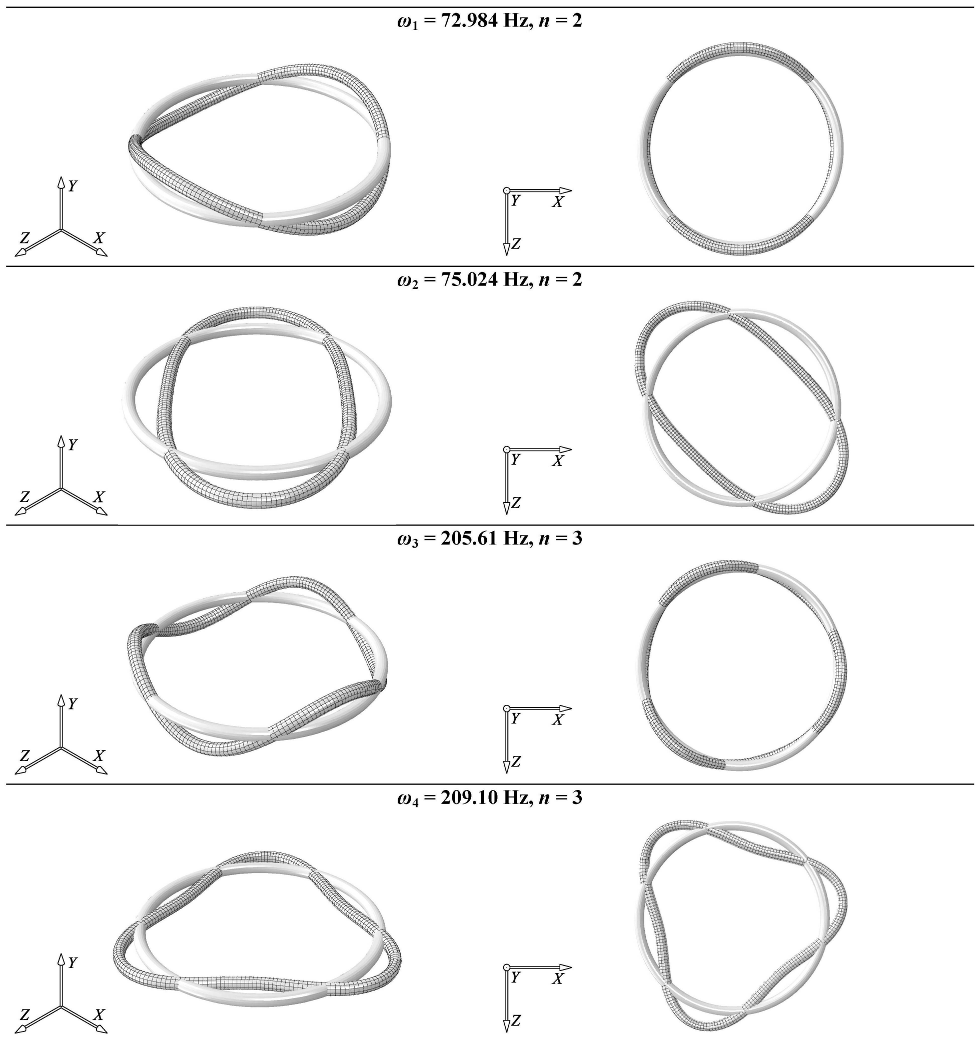

Fig. 4. Natural modes of thin-walled toroidal ring (Abaqus).

are selected, and $n = 2$ is assumed for illustration. The approximated formulae give values of natural frequencies very close to the exact ones. The accuracy of formulae for estimation of the natural frequencies are acceptable only for relatively small values of the rotational speed.

Natural frequencies of the rotating ring out-of-plane vibrations, i.e. flexural and torsional, determined by Eqs. (4.22), are shown in Table 3. In this case there is no bifurcation and the natural frequencies of both spectra are increased by an increasing rotation speed.

Table 3. Natural frequencies of rotating thin-walled toroidal ring out-of-plane vibrations, $\tilde{\omega}$ (Hz), $R = 1$ m, $a = 0.05$ m, $h = 0.01$ m, $\omega_0 = 75.41$ Hz.

n	Ω/ω_0	Flexural, $\tilde{\omega}_b$, Eq. (4.22)	Torsional, $\tilde{\omega}_t$, Eq. (4.22)
2	0	75.57	1179.73
	1	168.18	1179.80
	2	309.79	1180.04
3	0	216.53	1646.82
	1	312.30	1646.99
	2	499.32	1647.52
4	0	417.39	2135.85
	1	513.89	2136.12
	2	730.33	2136.98
5	0	676.69	2634.81
	1	773.38	2635.18
	2	1009.05	2636.41

Next, natural frequencies of the rotating ring modelled as a thick toroidal shell are determined by FEM in the fixed coordinate system for $n = 2$ and 3. The dimensionless rotational speed Ω/ω_0 is varied from 0 to 1. The obtained results for the forward and backward rotating modes are listed in Table 4. They are transformed into the rotating coordinate system by the following expres-

Table 4. Natural frequencies of rotating thin-walled toroidal ring, ω (Hz), $R = 1$ m, $a = 0.05$ m, $h = 0.01$ m, fixed coordinate system, FEM, 20×416 FE, $\omega_0 = 72.98$ Hz.

Ω/ω_0	Ω (Hz)	$n = 2$				$n = 3$			
		Out-of-plane		In-plane		Out-of-plane		In-plane	
		Forward	Backward	Forward	Backward	Forward	Backward	Forward	Backward
0	0	72.98	72.97	75.01	75.01	205.58	205.58	209.06	209.06
0.1	7.29	59.85	88.91	66.73	84.28	184.89	228.45	192.20	227.30
0.2	14.52	49.38	107.52	59.42	94.52	166.34	253.47	176.70	246.90
0.3	21.89	41.19	128.40	53.02	105.68	149.85	280.54	162.52	267.83
0.4	29.19	34.82	151.10	47.47	117.68	135.25	309.52	149.61	290.02
0.5	36.48	29.83	175.17	42.65	130.41	122.38	340.22	137.88	313.39
0.6	43.78	25.87	200.29	38.48	143.80	111.05	372.46	127.25	337.86
0.7	51.08	22.68	226.17	34.87	157.73	101.07	406.04	117.62	363.33
0.8	58.38	20.07	252.63	31.73	172.14	92.24	440.80	108.90	389.71
0.9	65.68	17.90	279.53	28.98	186.94	84.43	476.56	100.99	416.90
1.0	72.98	16.06	306.76	26.56	202.08	77.48	513.19	93.80	444.82

sions, [15], which take into account the Doppler effect

$$\tilde{\omega}_F = \omega_F + n\Omega, \qquad \tilde{\omega}_B = \omega_B - n\Omega. \tag{5.1}$$

and are presented in Table 5. In case of the out-of-plane vibrations there is no bifurcation of natural frequencies. The analytically determined natural frequencies for the rotating toroidal ring are compared with FEM values for the thick-walled toroidal shell and quite a good agreement can be noticed, Table 5. The corresponding diagrams of natural frequencies for the out-of-plane and in-plane vibrations are shown in Figs. 5 and 6, respectively.

Table 5. Natural frequencies of rotating thin-walled toroidal ring, ω (Hz), $R = 1$ m, $a = 0.05$ m, $h = 0.01$ m, rotating coordinate system, FEM, 20×416 FE, $\omega_0 = 72.98$ Hz; * – Eq. (4.22), ** – Eq. (A17).

Ω/ω_0	$n = 2$			$n = 3$		
	Out-of-plane	In-plane		Out-of-plane	In-plane	
		Forward	Backward		Forward	Backward
0	72.98 (75.57)*	75.01 (77.96)**	75.01 (77.96)**	205.58 (216.53)*	209.06 (220.43)**	209.06 (220.43)**
0.1	74.38 (76.95)	69.69 (72.60)	81.33 (84.32)	206.67 (217.62)	205.11 (216.68)	214.09 (225.56)
0.2	78.45 (80.97)	65.33 (68.18)	88.61 (91.62)	209.90 (220.87)	203.11 (214.28)	220.49 (232.06)
0.3	84.80 (87.25)	61.89 (64.66)	96.81 (99.83)	215.19 (226.18)	202.15 (213.21)	228.20 (239.88)
0.4	92.96 (95.36)	59.29 (61.98)	105.85 (108.87)	222.38 (233.40)	202.45 (213.39)	237.18 (248.97)
0.5	102.50 (104.86)	57.43 (60.02)	115.63 (118.66)	231.30 (242.37)	203.92 (214.76)	247.35 (259.26)
0.6	113.08 (115.42)	56.23 (58.71)	126.06 (129.10)	241.75 (252.91)	206.50 (217.21)	258.61 (270.66)
0.7	124.12 (126.77)	55.56 (57.34)	137.04 (140.10)	253.55 (264.82)	210.08 (220.66)	270.87 (283.08)
0.8	136.35 (138.71)	55.37 (57.63)	148.49 (151.57)	266.52 (277.92)	214.56 (225.01)	284.05 (296.42)
0.9	148.71 (151.11)	55.57 (57.71)	160.34 (163.46)	280.49 (292.07)	219.86 (230.15)	298.03 (310.60)
1.0	161.41 (163.86)	56.12 (58.11)	172.52 (175.68)	295.33 (307.10)	225.88 (236.01)	312.74 (325.53)

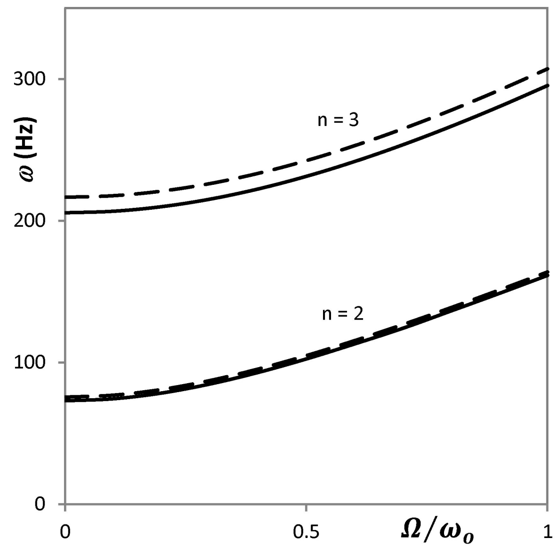

FIG. 5. Natural frequencies of rotating toroidal ring, out-of-plane vibrations, —— FEM, - - - analytical, (Table 5).

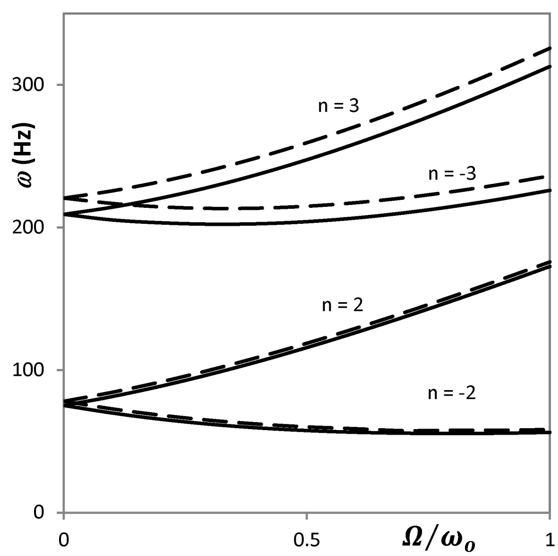

FIG. 6. Natural frequencies of rotating toroidal ring, in-plane vibrations, —— FEM, - - - analytical, (Table 5).

5.2. Thin ring

In this subsection a thin ring is considered, for which the experimentally determined natural frequencies are available, [15]. The main geometrical and material parameters of the ring considered are the following: mean radius

$R = 87.6$ mm, $b = 20$ mm, thickness $h = 0.88$ mm, Young's modulus $E = 2.1 \cdot 10^{11}$ N/m^2, and mass density $\rho = 7850$ kg/m^3.

The natural frequencies of the rotating ring for the mode number $n = 2$, determined rigorously and approximately by Eqs. (A17) and Eq. (3.18), respectively, are listed in Table 6. Discrete normalized rotational speeds are used, which correspond to the rotational speeds used in the experimental campaign reported in [15]. The approximated natural frequencies are almost the same as the rigorous ones in the considered case of a thin ring with $h/R = 0.01$.

Table 6. Natural frequencies of rotating thin ring, $\tilde{\omega}$ (rad/s), $R = 87.6$ mm, $b = 20$ mm, $h = 0.88$ mm, $n = 2$, $\omega_2^0 = 64.973$ rad/s.

Ω/ω_2^0	Rigorous, Eq. (A17)		Approximated, Eq. (3.18)	
	Forward	Backward	Forward	Backward
0	64.973	64.973	64.972	64.972
0.120	59.405	71.881	59.405	71.880
0.240	55.138	80.089	55.139	80.088
0.405	51.187	93.291	51.188	93.291
0.540	49.352	105.490	49.353	105.489
0.608	48.823	112.031	48.825	112.030
0.720	48.438	123.290	48.440	123.289
0.775	48.442	129.012	48.444	129.011
0.840	48.588	135.916	48.592	135.915
0.896	48.826	141.976	48.830	141.975
0.960	49.211	149.015	49.216	149.014
1.020	49.670	155.712	49.675	155.711
1.080	50.215	162.495	50.221	162.494

Table 7. Natural frequencies of rotating thin ring, $\tilde{\omega}$ (rad/s), $R = 87.6$ mm, $b = 20$ mm, $h = 0.88$ mm, $n = 3$, $\omega_3^0 = 183.771$ rad/s.

Ω/ω_3^0	Rigorous, Eq. (A17)		Approximated, Eq. (3.18)	
	Forward	Backward	Forward	Backward
0	183.771	183.771	183.771	183.771
0.070	178.628	194.066	178.62	194.061
0.090	178.085	197.934	178.08	197.929
0.160	179.210	214.498	179.20	214.493
0.210	182.633	228.949	182.63	228.944
0.245	186.167	240.202	186.16	240.197
0.284	191.078	253.716	191.08	253.711
0.344	200.369	276.241	200.37	276.237
0.385	207.747	292.664	207.75	292.659
0.426	215.831	309.792	215.84	309.788

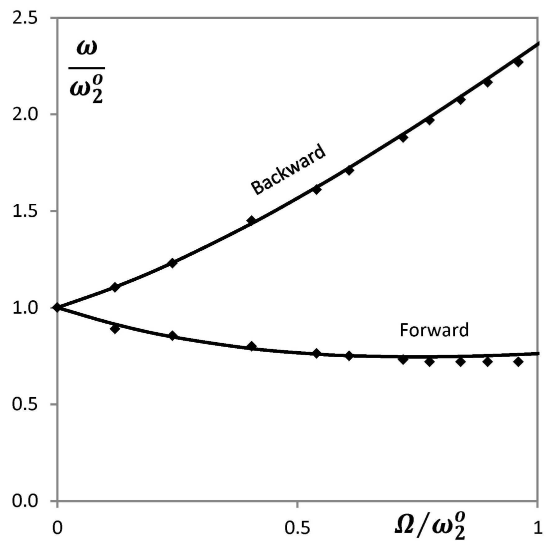

FIG. 7. Comparison of dimensionless natural frequencies of rotating thin ring, $n = 2$, —— theoretical, Eq. (A17), ◇ measured, [15].

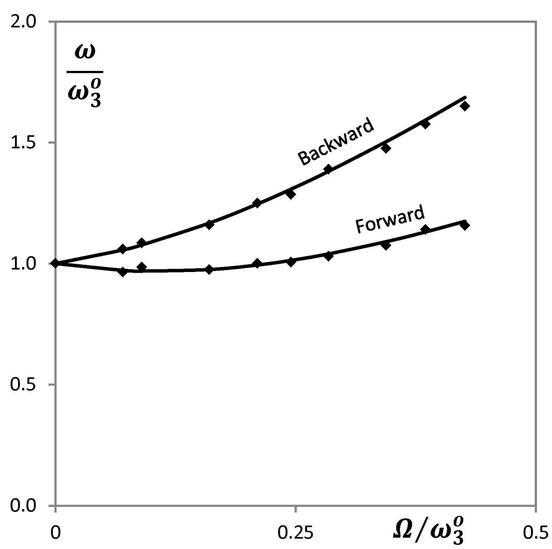

FIG. 8. Comparison of dimensionless natural frequencies of rotating thin ring, $n = 3$, —— theoretical, Eq. (A17), ◇ measured, [15].

The natural frequencies for the mode number $n = 3$ are shown in Table 7. Also in this case the approximated natural frequencies agree very well with the rigorous ones. Hence, one can conclude that the simple approximation formula (3.18) is quite reliable in the case of thin rings.

The normalized forward and backward natural frequencies for the mode number $n = 2$ and 3 are shown in Figs. 7 and 8, and compared with the measured values presented in [15]. A very good agreement of the calculated frequency values with the measured ones is observed in both cases.

6. Conclusions

In this paper investigations of the rotating ring in-plane and out-of-plane vibrations are carried out, which are based on the toroidal shell theory. The energy approach is used. The strain and the kinetic energies are formulated indirectly by deducing from the corresponding energies of a toroidal shell. In the relevant literature this problem is ordinarily analysed by solving differential equations of motion derived from balance of strain and kinetic energy via Hamilton's principle.

The in-plane vibration modes consist of combined flexural and extensional deformations, whereas the out-of-plane modes comprise combined flexural and torsional deformations. The problem is solved in an exact sophisticated way and in an approximate way that yields relatively simple formulae for practical use. The formulae for the natural frequencies of the in-plane and the out-of-plane flexural vibrations are very similar and give almost the same results assuming the same circumferential wave number. The simplified expression for the in-plane natural frequencies is identical to the well-known formula in the relevant literature.

The application of the developed ring vibration theory is illustrated by a number of numerical examples. The obtained results agree very well with those determined by the FEM analysis and the FSM analysis of a slender toroidal shell. The structure of the derived formulae for the in-plane vibrations indicates how centrifugal forces, induced by the ring rotation, increase the mean value of natural frequencies and the Coriolis forces cause their bifurcation.

The presented theory for the in-plane and out-of-plane free vibration of a rotating ring, based on the application of the toroidal shell theory, seems to be rather complicated. On the other side, it is very educative since it points out the universality of the toroidal shell theory and sheds more light to this still challenging problem.

7. Appendix: Rigorous solution of the characteristic equation for the rotating ring in-plane vibrations

Solving of a quartic equation has been a challenging subject of investigation from the 16-th century. Among scientists there are some well-known names: Lodovico Ferrari, Gerolamo Cardano, Descartes, Euler, [26, 27]. The problem is still relevant nowadays [28, 29].

The non-linear characteristic equation for the rotating ring in-plane vibrations, Eq. (3.6), is actually a depressed quartic equation, i.e. a quartic equation without the cubic term

$$\omega^4 - a_2\omega^2 + a_1\omega + a_0 = 0. \tag{A1}$$

It can be solved by following the mathematical procedure described in [30]. One of the possibilities to solve Eq. (A1) is to assume that it is reducible by factorization. Hence, the four roots of Eq. (A1) coincide with two pairs of roots of two quadratic equations

$$\omega^2 + \frac{1}{2}A\omega + \left(y - \frac{a_1}{A}\right) = 0, \tag{A2}$$

where

$$A = \pm\sqrt{8y + 4a_2} \tag{A3}$$

and y is a real root of the cubic resolvent of Eq. (A1)

$$8y^3 + 4a_2y^2 - 8a_0y - (4a_2a_0 + a_1^2) = 0. \tag{A4}$$

Equation (A4) can be condensed into a simpler form by shifting y. Substituting $y = x - a_2/6$ into (A4) yields

$$x^3 + 3px + 2q = 0, \tag{A5}$$

where

$$p = -\frac{1}{36}(12a_0 + a_2^2), \tag{A6}$$

$$q = \frac{1}{432}(2a_2^3 - 72a_2a_0 - 27a_1^2). \tag{A7}$$

The three real roots of Eq. (A5) are assumed in the form

$$x_1 = u + \nu, \qquad x_2 = \varepsilon_1 u + \varepsilon_2 \nu, \qquad x_3 = \varepsilon_2 u + \varepsilon_1 \nu, \tag{A8}$$

where ε_1 and ε_2 are the roots of equation $\varepsilon^2 + \varepsilon + 1 = 0$, i.e.

$$\varepsilon_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \tag{A9}$$

The first root x_1 in (A8) is determined by Cardano's Formula

$$u = \sqrt[3]{z_1}, \qquad \nu = \sqrt[3]{z_2}, \tag{A10}$$

where

(A11) $$z_{1,2} = -q \pm \sqrt{q^2 + p^3}$$

are roots of the equation $z^2 + 2qz - p^{=}0$. If the discriminant $D = q^2 + p^3 < 0$, one can write

(A12) $$z_{1,2} = -q + iw, \qquad w = \sqrt{|q^2 + p^3|}.$$

The complex quantity $z_{1,2}$ can be presented in the exponential form (De Moire's formula), i.e.

(A13) $$z_{1,2} = \rho \mathrm{e}^{\pm i\varphi}, \quad \rho = \sqrt{q^2 + w^2}, \quad \varphi = \mathrm{arctg}\left(\frac{w}{-q}\right),$$

where $-\pi \leq \varphi \leq \pi$. Substituting (A13) into (A10), yields

(A14) $$u, \nu = \sqrt[3]{z_{1,2}} = \sqrt[3]{\rho}\,\mathrm{e}^{\pm i\varphi/3} = \sqrt[3]{\rho}\,[\cos(\varphi/3) \pm i\sin(\varphi/3)].$$

Finally, one obtains for the first root of Eq. (A5), according to Eqs. (A8)

(A15) $$x_1 = 2\sqrt[3]{\rho}\,\cos(\varphi/3).$$

The values of x_1 are real since the imaginary parts of u and v cancel each other.

Furthermore, the solutions of Eq. (A2) read

(A16) $$\omega_{1,2} = -\frac{A}{4} \pm \sqrt{\left(\frac{A}{4}\right)^2 - \left(y - \frac{a_1}{A}\right)}.$$

Substituting (A3) and $y = x_1 - a_2/6$ into (A16) one obtains

(A17) $$\omega_{1,2,3,4} = \frac{1}{2\sqrt{3}}\left[-s\sqrt{2a_2 + 6x_1} \pm \sqrt{4a_2 - 6x_1 + \frac{6\sqrt{3}\,a_1}{s\sqrt{2a_2 + 6x_1}}}\right],$$

where $s = \mathrm{sign}\,(A)$, Eq. (A3).

The following example can be used as a benchmark for the application of the above procedure:

Data: $a_0 = 1$, $a_1 = \dfrac{32}{3}$, $a_2 = \dfrac{38}{3}$.

Eq. (A17): $\dfrac{1}{2\sqrt{3}}(-s \cdot 6.9282 \pm \sqrt{28 + s \cdot 16})$.

Solution: $\omega_1 = -3.91485$, $\omega_2 = -0.085146$, $\omega_3 = 1.0$, $\omega_4 = 3.0$.

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References

1. N. Alujević, N. Campillo-Davo, P. Kindt, W. Desmet, B. Pluymers, S. Vercammen, *Analytical solution for free vibrations of rotating cylindrical shells having free boundary conditions*, Engineering Structures, **132**, 152–171, 2017.
2. T. Balderes, A.E. Armenakas, *Free vibrations of ring-stiffened toroidal shells*, AIAA Journal, **11**, 12, 1637–1644, 1973.
3. Y.T. Wei, L. Nasdala, H. Rothert, *Analysis of tire rolling contact response by REF model*, Tire Science and Technology, **32**, 4, 214–235, 2004.
4. P. Kindt, P. Sas, W. Desmet, *Development and validation of a three-dimensional ring-based structural tyre model*, Journal of Sound and Vibration, **326**, 852–869, 2009.
5. M. Matsubara, D. Tajiri, T. Ise, S. Kawamura, *Vibrational response analysis of tires using a three-dimensional flexible ring-based model*, Journal of Sound and Vibration, **408**, 368–382, 2017.
6. G.H. Bryan, *On the beats in the vibrations of a revolving cylinder or bell*, Proceedings of the Cambridge Philosophical Society, 101–111, 1890.
7. A.E.H. Love, *A Treatise on the Mathematic Theory of Elasticity*, Dover Publications, 4th ed., Dover, 1927.
8. G.F. Carrier, *On the vibrations of the rotating ring*, Quarterly of Applied Mathematics, **3**, 235–245, 1945.
9. D.C. Johnson, *Free vibrations of a rotating elastic body*, Aircraft Engineering, **24**, 234–236, 1952.
10. D.A. Evensen, *Non-linear flexural vibrations of thin circular rings*, ASME Journal of Applied Mechanics, **33**, 553–560, 1966.
11. S.S. Rao, V. Sundarajan, *In-plane flexural vibrations of circular rings*, Journal of Applied Mechanics, **91**, 620–625, 1969.
12. J. Kirkhope, *Simple frequency expression for the in-plane vibration of thick circular rings*, The Journal of the Acoustical Society of America, **59**, 86–89, 1976.
13. D.L. Hawkings, *A generalized analysis of the vibration of circular rings*, Journal of Sound and Vibration, **54**, 67–74, 1977.
14. C.W. Bert, T.L.C. Chen, *On vibration of a thick flexible ring rotating at high speed*, Journal of Sound and Vibration, **61**, 517–530, 1978.

15. M. Endo, K. Hatamura, M. Sakata, O. Taniguchi, *Flexural vibration of a thin rotating ring*, Journal of Sound and Vibration, 82, 261–272, 1984.

16. W.B. Bickford, E.S. Reddy, *On the vibration of rotating rings*, Journal of Sound and Vibration, **101**, 13–22, 1985.

17. S.P. Maganty, W.B. Bickford, *Large amplitude oscillations of thin circular rings*, ASME Journal of Applied Mechanics, **54**, 315–322, 1987.

18. S.C. Huang, W. Soedel, *Effects of Coriolis acceleration on the free and forced in-plane vibrations of rotating rings on elastic foundation*, Journal of Sound and Vibration, **115**, 253–274, 1987.

19. R. Eley, C.H.J. Fox, S. Mcwilliam, *Coriolis coupling effects on the vibration of rotating rings*, Journal of Sound and Vibration, **238**, 459–480, 2000.

20. S.C. Huang, W. Soedel, *Response of rotating rings to harmonic and periodic loading and comparison with the inverted problem*, Journal of Sound and Vibration, **118**, 2, 253–270, 1987.

21. W. Kim, J. Chung, *Free non-linear vibrations of a rotating thin ring with the in-plane and out-of-plane motions*, Journal of Sound and Vibration, **258**, 1, 167–178, 2002.

22. W. Soedel, *Vibrations of Shells and Plates (3rd edition)*, Marcel Dekker, Inc. New York, 2004.

23. I. Senjanović, N. Alujević, I. Ćatipović, D. Čakmak, N. Vladimir, *Vibration analysis of rotating toroidal shell by Rayleigh–Ritz method and Fourier series*, Engineering Structures, **173**, 870–891, 2018.

24. Dassault Systèmes, ABAQUS 6.9 User's guide and theoretical manual, Hibbitt, Karlsson & Sorensen, Inc., California, 2009.

25. I. Senjanović, I. Ćatipović, N. Alujević, D. Čakmak, N. Vladimir, *A finite strip for vibration analysis of rotating toroidal shell under internal pressure*, ASME Journal of Vibration and Acoustics, doi: 10.1115/1.4041734.

26. University of Kentucky, College of Arts & Sciences, MA330 *History of Mathematics*, Lecture Notes (http://www.ms.uky.edu/~sohum/ma330/files/eqns_4.pdf).

27. UCD School of Mathematics and Statistics, MST3022 *History of Mathematics*, Lecture Notes (http://mathsa.ucd.ie/courses/mst3022/history4.pdf).

28. M.D. Yacoub, G. Fraidenraich, *A new simple solution of the general quartic equation*, The Mathematical Gazette, **95**, 2011.

29. S.L. Shmakov, *A universal method of solving quartic equation*, International Journal of Pure and Applied Mathematics, **71**, 251–253, 2011.

30. I.N. Bronstein, K.A. Semendjajew, G. Musiol, H. Mühlig, *Taschenbuch der Mathematik*, Thun und Frankfurt am Main, Verlag Harri Deutsch, Augsburg, 2001.

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