

## Analyses of elastic limit heat loads in thick walled tubes subjected to periodic surface temperatures: analytical treatment

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ANALYTICAL SOLUTIONS ARE DERIVED TO ANALYZE elastic limit heat loads in tubes subjected to periodic surface temperatures. The tube is initially at zero temperature and for the times greater than zero one of the surfaces of the cylinder is subject to a periodic boundary condition while the other surface is insulated. For the transient temperature distribution, the heat conduction equation is solved by using Duhamel's theorem. The uncoupled theory of thermoelasticity is used as the cylinder is heated or cooled slowly. Tresca's yield criterion is used to monitor the yielding of the tube. The generalized plane strain condition is assumed. It is observed that yielding always occurs at the surface subject to a periodic boundary condition. It is also observed that, depending on the material properties of the tube and the amplitude of the boundary condition, yielding commences with different stress states.

**Key words:** thermoelasticity, transient heat conduction, periodic boundary condition, Duhamel's theorem, elastic tubes.

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### 1. Introduction

AXIALLY SYMMETRIC MACHINE PARTS like the cylinders, tubes and disks are widely used in engineering applications [1–3]. Some of the applications like heat exchangers, heat pipes, manufacturing processes, nuclear engineering structures, nozzle section of rockets include transient thermal conditions such as periodic boundary conditions with various forms [4–6]. Because the problem of heat conduction with transient boundary conditions cannot be solved directly by using the method of separation of variables, researchers solved the problem by using dif-

ferent methods like Laplace transforms, Hankel transforms, Fourier transforms, and Duhamel's theorem. These methods can be found in the book by OZISIK [7].

Some of the works related to transient temperature fields and related thermal stresses in tubes subjected to transient boundary conditions like in periodic, cyclic or exponential forms are summarized subsequently. In the studies of KANDIL *et al.* [4] and KANDIL [5], the thermoelastic stresses for thick-walled tubes subjected to several boundary conditions including cyclic function at the inner surface [4] and cyclic temperature and cyclic pressure at the inner surface [5] have been obtained by using a finite difference method. SEGALL [8, 9] has investigated the transient response of thick-walled tubes subjected to boundary conditions in exponential form [8] and a polynomial form [9] at the inner surface with external convection to the surrounding environment by using Duhamel's theorem. ATEFI *et al.* [10] have also used Duhamel's theorem to solve the heat conduction problem of a hollow cylinder subjected to a periodic boundary condition decomposed by the Fourier series at the outer surface while the inner surface is insulated. LU *et al.* [11] have investigated the transient temperature distribution for n-layer composite cylinder subjected to time-dependent boundary conditions assumed to be changed with the Fourier series at both the inner and outer surfaces. MAHMOUDI and ATEFI [12] have obtained the thermoelastic stresses in a finite length tube subjected to periodic thermal loading on the inner circular surface while the outer circular surface was insulated when both lateral surfaces were kept at constant temperature. They have obtained the transient temperature distribution by using the Fourier series for the periodic boundary condition function. In this study, they have also derived the temperature distribution in axial direction and showed that according to the same boundary conditions at ends of the tube, temperature distribution and induced thermal stresses do not vary on changing the axial coordinate. RADU *et al.* [13] have developed a set of analytical solutions for the transient temperature field by using the finite Hankel transform and related thermoelastic stress distributions for a hollow cylinder subjected to sinusoidal transient thermal loading only at the inner surface while the outer surface is kept at zero temperature. LEE and HUANG [14] have developed an analytical solution method, without integral transformation, to find the exact solutions for the transient heat conduction in functionally graded circular hollow cylinders with time-dependent boundary conditions. They have considered different boundary conditions with the forms of exponential and sinusoidal. KAYA and ERASLAN [15] have used a temperature cycle to obtain an analytical solution toward the prediction of the thermoelastic response of a long tube heated by a temperature cycle at one surface while the other was insulated. TU and LEE [16] have used the shifting function method to obtain an analytical solution for the heat transfer problem in hollow cylinders with the time-dependent boundary condition and time-dependent heat transfer coefficient simultaneously.

They have assumed that the surface is subject to a time-dependent temperature field at the inner surface with the combination of exponential and cosine functions, whereas the heat is dissipated by time-dependent convection from the outer surface into a surrounding environment at zero temperature. In their recent work related to the transient thermal stress behavior of a cylinder subjected to a periodic boundary condition, ERASLAN and APATAY [17] have investigated the transient thermoelastoplastic behavior of a solid cylinder.

Some of the recent studies solve transient problems in tubes and cylinders assuming functionally graded materials. For example, TAKABI [18] has investigated the thermomechanical behavior of the thick hollow FGM cylinder subjected to a pressure and a thermal load in the transient condition. MANTHENA *et al.* [19] have investigated the thermoelastic behavior of a hollow FGM cylinder subjected to internal transient heat generation. They have obtained the solution of the two dimensional heat conduction equation in the transient state in terms of Bessel and trigonometric functions. AYOUBI and ALIBEIGLOO [20] have obtained the three dimensional elasticity solution for transient thermoelasticity analysis of FGM cylindrical shell subjected to both thermal and mechanical loading analytically for the case of a simply supported boundary condition using the Fourier series expansions and semianalytically for the edge boundary conditions. MANTHENA and KEDAR [21] have investigated two dimensional temperature distribution and associated thermal stresses of a thick hollow FGM cylinder subjected to a varying point heat source. The solutions are obtained in the transient state in the form of Bessel functions. NAJIBI and TALEBITOOTI [22] have investigated the transient thermo-elastic analysis of a thick hollow finite length FGM cylinder. They have solved the transient heat conduction and the thermo-elastic equations for the cylinder subjected to thermal loading utilizing the finite element method.

In the present work, analytical models are developed to investigate the stress states at elastic limits of sufficiently long tubes subjected to periodic heating on one face while the other one isolated. It is observed that yielding commences on the face under periodic heating. Moreover, different stress states emerge at the yielding surface depending on the amplitude of heating and the material properties. Parametric analyses are performed to draw charts in order to identify when and according to which stress state yielding occurs. These charts can be used in investigations which go beyond elastic states of stress. In the following sections we outline our model and present its results.

## 2. Temperature distributions

A long tube of inner radius  $a$  and outer radius  $b$  is taken into account. The tube is initially at zero temperature but for the times  $t > 0$  either the outer or

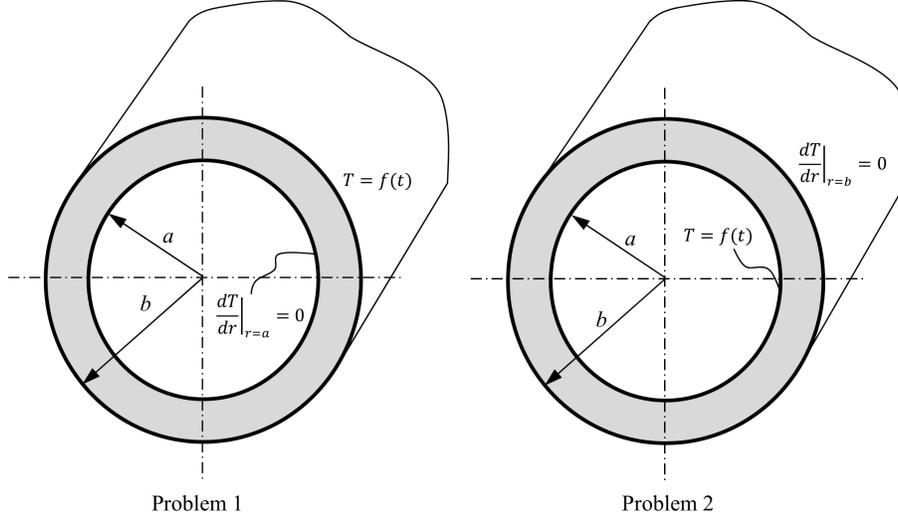


FIG. 1. Geometry and boundary conditions for problems 1 and 2.

the inner surface is subjected to a periodic boundary condition, while the other surface is isolated. These boundary conditions are explained in Fig. 1.

### 2.1. Problem 1 – Periodic boundary condition at the outer surface

This heat transfer problem is described by a transient one-dimensional heat conduction equation

$$(2.1) \quad \frac{1}{\alpha_T} \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}; \quad a < r < b, \quad t > 0,$$

which is accompanied by the boundary and initial conditions

$$(2.2) \quad \begin{aligned} \frac{\partial T}{\partial r} \Big|_{r=a} &= 0, & T(b, t) &= f(t), \\ T(r, 0) &= 0. \end{aligned}$$

In these equations  $\alpha_T$  is the thermal diffusivity,  $T(r, t)$  the temperature at the radial position  $r$  at time  $t$ , and  $f(t)$  a function defining the time dependent boundary condition. Introducing the dimensionless variables: temperature  $\bar{T} = T/T_0$ , time  $\tau = \alpha_T t/b^2$  and radial coordinate  $\bar{r} = r/b$ , where  $T_0$  being a reference temperature, the dimensionless form of the conduction equation and the accom-

panied conditions take the forms

$$(2.3) \quad \frac{\partial \bar{T}}{\partial \tau} = \frac{1}{\bar{r}} \frac{\partial \bar{T}}{\partial \bar{r}} + \frac{\partial^2 \bar{T}}{\partial \bar{r}^2}; \quad \bar{a} < \bar{r} < 1, \tau > 0,$$

$$(2.4) \quad \left. \frac{\partial \bar{T}}{\partial \bar{r}} \right|_{\bar{r}=\bar{a}} = 0, \quad \bar{T}(1, \tau) = F(\tau), \\ \bar{T}(\bar{r}, 0) = 0.$$

The equations given below are written in terms of these variables, but to simplify the notation overbars are not used. Because of the nonhomogeneous and time-dependent boundary condition the solution is obtained by using the Duhamel's theorem as (OZISIK [7])

$$(2.5) \quad T(r, \tau) = \int_0^\tau F(\beta) \frac{\partial}{\partial \tau} \Phi(r, \tau - \beta) d\beta,$$

where  $\Phi(r, \tau)$  is the solution of the auxiliary problem given by

$$(2.6) \quad \frac{\partial \Phi}{\partial \tau} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial r^2}; \quad a < r < 1, \tau > 0,$$

with the conditions

$$(2.7) \quad \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0, \quad \Phi(1, \tau) = 1, \\ \Phi(r, 0) = 0.$$

The nonhomogeneous boundary condition  $\Phi(1, \tau) = 1$  is handled by proposing a solution of the form

$$(2.8) \quad \Phi(r, \tau) = Y(r, \tau) + Z(r).$$

Substituting  $\Phi(r, \tau)$  into the auxiliary problem, Eq. (2.6), the following differential equation is obtained

$$(2.9) \quad \frac{\partial Y}{\partial \tau} = \frac{1}{r} \frac{\partial Y}{\partial r} + \frac{\partial^2 Y}{\partial r^2} + \frac{1}{r} \frac{dZ}{dr} + \frac{d^2 Z}{dr^2}; \quad a < r < 1, \tau > 0,$$

Letting

$$(2.10) \quad \frac{1}{r} \frac{dZ}{dr} + \frac{d^2 Z}{dr^2} = 0,$$

Eq. (2.9) is divided into two parts as

$$(2.11) \quad \frac{1}{r} \frac{dZ}{dr} + \frac{d^2 Z}{dr^2} = 0; \quad \left. \frac{dZ}{dr} \right|_{r=a} = 0, \quad Z(1) = 1,$$

and

$$(2.12) \quad \frac{\partial Y}{\partial \tau} = \frac{1}{r} \frac{\partial Y}{\partial r} + \frac{\partial^2 Y}{\partial r^2}; \quad a < r < 1, \tau > 0,$$

subject to

$$(2.13) \quad \left. \frac{\partial Y}{\partial r} \right|_{r=a} = 0, \quad Y(1, \tau) = 0, \\ Y(r, 0) = -Z(r).$$

The solutions are obtained as

$$(2.14) \quad Z(r) = 1,$$

and

$$(2.15) \quad Y(r, \tau) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 \tau} [J_0(\lambda_n r) Y_1(\lambda_n a) - J_1(\lambda_n a) Y_0(\lambda_n r)],$$

where  $\lambda_n$  for  $n = 1, 2, \dots$  are the positive roots of

$$(2.16) \quad J_0(\lambda) Y_1(\lambda a) - J_1(\lambda a) Y_0(\lambda) = 0,$$

in which  $J_0(\lambda)$  and  $J_1(\lambda)$  represent the Bessel functions of the first kind of order zero and one, and  $Y_0(\lambda)$  and  $Y_1(\lambda)$  represent the Bessel functions of the second kind of order zero and one, respectively. Applying the boundary condition of  $Y(r, 0) = -1$  and using the relationship (SPIEGEL [23])

$$(2.17) \quad Y_1(\lambda_n a) J_0(\lambda_n a) - Y_0(\lambda_n a) J_1(\lambda_n a) = -\frac{2}{a\pi\lambda_n},$$

the constant  $B_n$  is obtained as

$$(2.18) \quad B_n = \frac{2\pi^2 \lambda_n [Y_1(\lambda_n) J_1(\lambda_n a) - Y_1(\lambda_n a) J_1(\lambda_n)]}{-4 + \lambda_n^2 \pi^2 [Y_1(\lambda_n) J_1(\lambda_n a) - Y_1(\lambda_n a) J_1(\lambda_n)]^2}.$$

Hence, the solution of the auxiliary problem defined by Eqs. (2.6) and (2.7) turns out

$$(2.19) \quad \Phi(r, \tau) = 1 + \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 \tau} [J_0(\lambda_n r) Y_1(\lambda_n a) - J_1(\lambda_n a) Y_0(\lambda_n r)].$$

According to Eq.(2.5) the temperature distribution in the tube is obtained as

$$(2.20) \quad T(r, \tau) = - \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 \tau} \lambda_n^2 [J_0(\lambda_n r) Y_1(\lambda_n a) - J_1(\lambda_n a) Y_0(\lambda_n r)] \\ \times \int_0^{\tau} e^{\lambda_n^2 \beta} F(\beta) d\beta,$$

and integrating by parts

$$(2.21) \quad \int_0^\tau e^{\lambda_n^2 \beta} F(\beta) d\beta = \frac{1}{\lambda_n^2} \left[ e^{\lambda_n^2 \tau} F(\tau) - F(0) - \int_0^\tau e^{\lambda_n^2 \beta} \frac{dF(\beta)}{d\beta} d\beta \right],$$

it becomes

$$(2.22) \quad T(r, \tau) = F(\tau) + \sum_{n=1}^{\infty} B_n [J_0(\lambda_n r) Y_1(\lambda_n a) - J_1(\lambda_n a) Y_0(\lambda_n r)] \int_0^\tau e^{-\lambda_n^2 (\tau - \beta)} \frac{dF(\beta)}{d\beta} d\beta.$$

Here, the initial condition of  $F(0) = 0$  is utilized. Note that at  $r = 1$ , i.e., at the outer surface, based on Eq. (2.16) the temperature becomes  $T(1, \tau) = F(\tau)$  as required. In addition, the temperature gradient is derived as

$$(2.23) \quad \frac{\partial T(r, \tau)}{\partial r} = - \sum_{n=1}^{\infty} B_n \lambda_n [J_1(\lambda_n r) Y_1(\lambda_n a) - J_1(\lambda_n a) Y_1(\lambda_n r)] \times \int_0^\tau e^{-\lambda_n^2 (\tau - \beta)} \frac{dF(\beta)}{d\beta} d\beta$$

## 2.2. Problem 2 – Periodic boundary condition at the inner surface

The conditions that accompany Eq. (2.3) becomes

$$(2.24) \quad \begin{aligned} T(a, \tau) &= F(\tau), & \left. \frac{\partial T}{\partial r} \right|_{r=1} &= 0, \\ T(r, 0) &= 0. \end{aligned}$$

Thus, the conditions of the auxiliary problem turn into

$$\begin{aligned} \Phi(a, \tau) &= 1, & \left. \frac{\partial \Phi}{\partial r} \right|_{r=1} &= 0, \\ \Phi(r, 0) &= 0 \end{aligned}$$

Using the procedure described in the Problem 1, the solution of the auxiliary problem is obtained as

$$(2.25) \quad \Phi(r, \tau) = 1 + \sum_{n=1}^{\infty} D_n e^{-\lambda_n^2 \tau} [J_0(\lambda_n r) Y_0(\lambda_n a) - J_0(\lambda_n a) Y_0(\lambda_n r)],$$

in which the eigenvalues  $\lambda_n$  for  $n = 1, 2, \dots$  are the roots of

$$(2.26) \quad J_0(\lambda a) Y_1(\lambda) - J_1(\lambda) Y_0(\lambda a) = 0,$$

and

$$(2.27) \quad D_n = \frac{4\pi}{-4 + \lambda_n^2 \pi^2 [Y_0(\lambda_n) J_0(\lambda_n a) - Y_0(\lambda_n a) J_0(\lambda_n)]^2}.$$

By Duhamel's theorem, Eq. (2.5), the temperature distribution and its gradient in the tube for this problem are obtained, respectively, as

$$(2.28) \quad T(r, \tau) = F(\tau) + \sum_{n=1}^{\infty} D_n [J_0(\lambda_n r) Y_0(\lambda_n a) - J_0(\lambda_n a) Y_0(\lambda_n r)] \\ \times \int_0^{\tau} e^{-\lambda_n^2 (\tau - \beta)} \frac{dF(\beta)}{d\beta} d\beta,$$

and

$$(2.29) \quad \frac{\partial T(r, \tau)}{\partial r} = - \sum_{n=1}^{\infty} D_n \lambda_n [J_1(\lambda_n r) Y_0(\lambda_n a) - J_0(\lambda_n a) Y_1(\lambda_n r)] \\ \times \int_0^{\tau} e^{-\lambda_n^2 (\tau - \beta)} \frac{dF(\beta)}{d\beta} d\beta.$$

Note again that, according to Eq. (2.26),  $T(a, \tau) = F(\tau)$  as required.

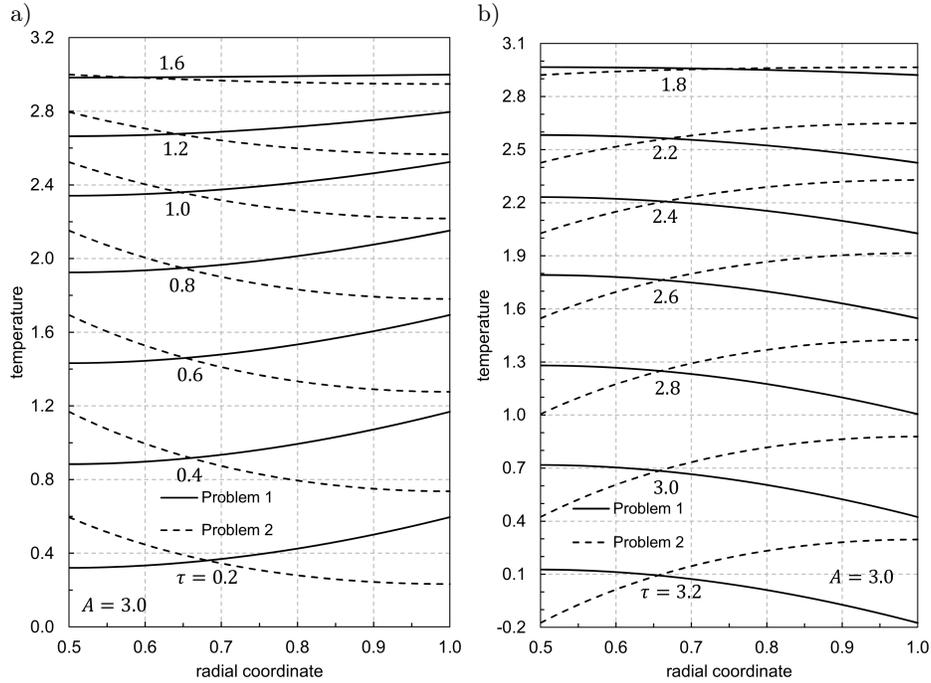


FIG. 2. Temperature distributions in time intervals: a)  $0 < \tau \leq 1.6$ , b)  $1.8 \leq \tau \leq 3.2$  corresponding to  $A = 3.0$ .

For both of the problems the periodic function  $F(\tau) = A \sin \tau$ , with  $A$  being the thermal load, is used at the periodic boundary. As a result of which

$$(2.30) \quad \int_0^\tau e^{-\lambda_n^2(\tau-\beta)} \frac{dF(\beta)}{d\beta} d\beta = \frac{A[\lambda_n^2 \cos \tau + \sin \tau - \lambda_n^2 e^{-\lambda_n^2 \tau}]}{1 + \lambda_n^4}.$$

Assigning  $A = 3.0$ , the results of the temperature distributions for both problems are plotted in Figs. 2a and 2b within  $0 < \tau < 3.2$ . The increase in temperature within a tube of  $a = 0.5$  with increasing surface temperature may be visualized in Fig. 2a, while its decrease with decreasing surface temperature can be seen in Fig. 2b.

### 3. Thermoelastic solution

Because of the temperature distribution is time-dependent, all stresses, strains and the radial displacement are functions of time. The mechanical properties of the material are assumed to be constant. Dimensionless forms of the elastic equations are used. The equilibrium equation (TIMOSHENKO and GOODIER [2])

$$(3.1) \quad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0,$$

the equations of the generalized Hooke's law

$$(3.2) \quad \epsilon_r = \sigma_r - \nu(\sigma_\theta + \sigma_z) + \alpha T,$$

$$(3.3) \quad \epsilon_\theta = \sigma_\theta - \nu(\sigma_r + \sigma_z) + \alpha T,$$

$$(3.4) \quad \epsilon_z = \sigma_z - \nu(\sigma_r + \sigma_\theta) + \alpha T,$$

form a basis. In these equations  $\sigma_j$  is the dimensionless stress (stress/ $\sigma_Y$ ),  $\epsilon_j$  the normalized strain (strain  $\times E/\sigma_Y$ ),  $\nu$  the Poisson's ratio and  $\alpha$  the dimensionless coefficient of thermal expansion (the coefficient of thermal expansion  $\times ET_0/\sigma_Y$ ). Furthermore,  $E$  is the modulus of elasticity and  $\sigma_Y$  the yield stress. In a state of generalized plain strain  $\epsilon_z = \epsilon_0$  is constant and Eq. (3.4) can be solved for the axial stress to give

$$(3.5) \quad \sigma_z = \epsilon_0 + \nu(\sigma_r + \sigma_\theta) - \alpha T,$$

The axial stress is eliminated from the radial and circumferential strain expressions and substituted in the strain-displacement relations:  $\epsilon_\theta = u/r$  and  $\epsilon_r = du/dr$  to obtain the stress-displacement relations

$$(3.6) \quad \sigma_r = \frac{1}{(1+\nu)(1-2\nu)} \left[ \nu\epsilon_0 + \frac{\nu u}{r} + (1-\nu)u' \right] - \frac{\alpha T}{1-2\nu},$$

$$(3.7) \quad \sigma_\theta = \frac{1}{(1+\nu)(1-2\nu)} \left[ \nu\epsilon_0 + \frac{(1-\nu)u}{\bar{r}} + \nu u' \right] - \frac{\alpha T}{1-2\nu},$$

where  $u$  is the dimensionless radial displacement (radial displacement  $\times E/\sigma_Y b$ ). A prime in above equations denotes differentiation with respect to the radial coordinate  $r$ . Substituting the stress expressions Eq. (3.6) and Eq. (3.7) into Eq. (3.1) yields

$$(3.8) \quad r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - u = \frac{1 + \nu}{1 - \nu} \alpha r^2 \frac{\partial T}{\partial r},$$

which is the governing differential equation in terms of the radial displacement. The general solution is

$$(3.9) \quad u(r) = C_1 r + \frac{C_2}{r} + \left( \frac{1 + \nu}{1 - \nu} \right) \left( \frac{\alpha}{r} \right) \int_a^r \eta T(\eta, \tau) d\eta$$

where  $C_1$  and  $C_2$  are the integration constants. Having the analytical expression for  $u(r)$ , the stresses are obtained by the use of Eqs. (3.6), (3.7) and (3.5). As these expressions contain the unknowns  $C_1$ ,  $C_2$  and  $\epsilon_0$ , the following boundary conditions and the free end condition are then used

$$(3.10) \quad 1. \sigma_r(a) = 0, \quad 2. \sigma_r(1) = 0, \quad 3. \int_a^1 r \sigma_z dr = 0,$$

to determine

$$(3.11) \quad C_1 = \frac{\alpha(1 - 3\nu)}{(a^2 - 1)(\nu - 1)} \int_a^1 r T(r, \tau) dr,$$

$$(3.12) \quad C_2 = \frac{a^2 \alpha(1 + \nu)}{(a^2 - 1)(\nu - 1)} \int_a^1 r T(r, \tau) dr,$$

$$(3.13) \quad \epsilon_0 = -\frac{2\alpha}{a^2 - 1} \int_a^1 r T(r, \tau) dr.$$

As a result, the thermoelastic solution of the tube is completed with the following stress and displacement expressions

$$(3.14) \quad \sigma_r = \frac{\alpha}{1 - \nu} \left[ \frac{r^2 - a^2}{r^2(1 - a^2)} \int_a^1 r T(r, \tau) dr - \frac{1}{r^2} \int_a^r \eta T(\eta, \tau) d\eta \right],$$

$$(3.15) \quad \sigma_\theta = \frac{\alpha}{1 - \nu} \left[ \frac{r^2 + a^2}{r^2(1 - a^2)} \int_a^1 r T(r, \tau) dr + \frac{1}{r^2} \int_a^r \eta T(\eta, \tau) d\eta - T(r, \tau) \right],$$

$$(3.16) \quad \sigma_z = \frac{\alpha}{1-\nu} \left[ \frac{2}{1-a^2} \int_a^1 rT(r, \tau) dr - T(r, \tau) \right],$$

$$(3.17) \quad u = \frac{\alpha}{1-\nu} \left[ \frac{r^2(1-3\nu) + a^2(1+\nu)}{r(1-a^2)} \int_a^1 rT(r, \tau) dr + \frac{1+\nu}{r} \int_a^r \eta T(\eta, \tau) d\eta \right].$$

Note that, analytical evaluation of the integrals

$$(3.18) \quad \int_a^r \eta T(\eta, \tau) d\eta \quad \text{and} \quad \int_a^1 rT(r, \tau) dr,$$

can be performed by the use of formulae

$$(3.19) \quad \int_a^r \eta J_0(\lambda_n \eta) d\eta = \frac{1}{\lambda_n} [rJ_1(\lambda_n r) - aJ_1(\lambda_n a)],$$

and

$$(3.20) \quad \int_a^r \eta Y_0(\lambda_n \eta) d\eta = \frac{1}{\lambda_n} [rY_1(\lambda_n r) - aY_1(\lambda_n a)].$$

#### 4. Results and discussion

In the following calculations the Poisson's ratio is  $\nu = 0.3$  and the inner radius is  $a = 0.5$ . Furthermore, Tresca's yield criterion is used to examine elastic limits. In this regard the equivalent stress is defined as

$$(4.1) \quad \sigma_{EQ} = \sigma_{\max} - \sigma_{\min}.$$

Firstly, for both problems an attempt is made to show how the stress states change in tubes with time. For this purpose the stress distributions are drawn corresponding to the parameters  $A = 3.0$  and  $\alpha = 1.75$  at two different time instants. At an early time  $\tau = 0.2$ , the unknowns of the problem 1 are determined as  $C_1 = 0.0539$ ,  $C_2 = 0.1750$ ,  $\epsilon_0 = 0.7540$  and of the problem 2 as  $C_1 = 0.0405$ ,  $C_2 = 0.1310$ ,  $\epsilon_0 = 0.5663$ . At time  $\tau = 2.8$  these are  $C_1 = 0.1462$ ,  $C_2 = 0.4751$ ,  $\epsilon_0 = 2.046$  for the problem 1 and  $C_1 = 0.1647$ ,  $C_2 = 0.5354$ ,  $\epsilon_0 = 2.306$  for the problem 2. As it is seen in Figs. 3a and 3b, for the problem 1  $\sigma_{EQ}$  is always at the outer surface of the tube, while at  $\tau = 0.2$  the stress state is  $\sigma_r > \sigma_\theta (= \sigma_z)$ ,

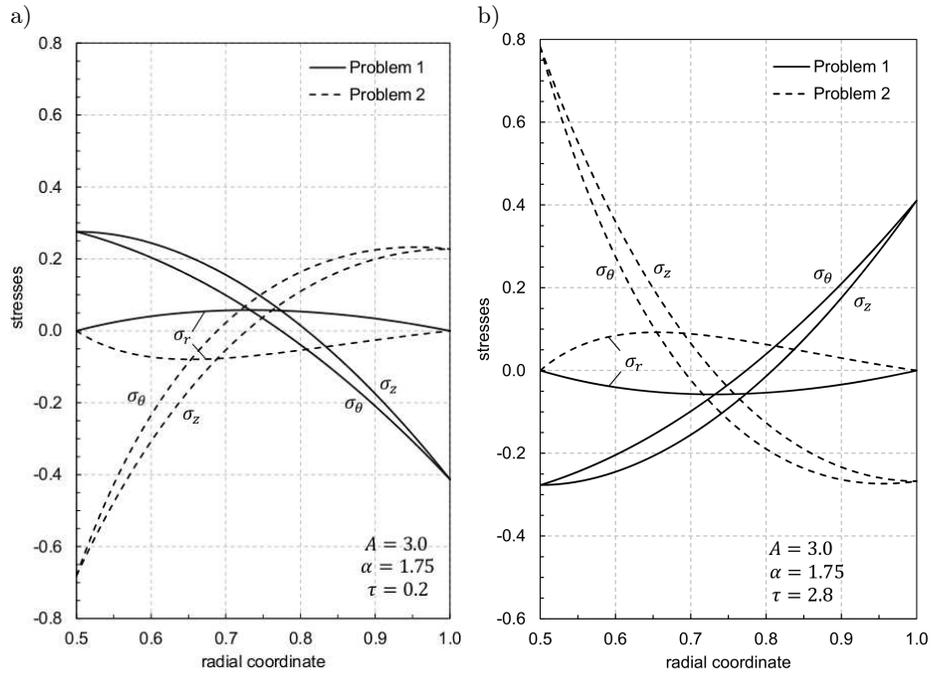


FIG. 3. Stress distributions for  $A = 3.0$ ,  $\alpha = 1.75$  at: a)  $\tau = 0.2$ , b)  $\tau = 2.8$ .

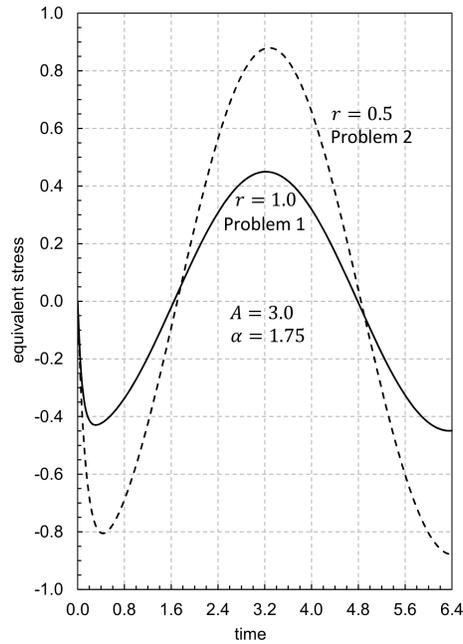


FIG. 4. Change of equivalent stress at the boundaries with time for  $A = 3.0$ ,  $\alpha = 1.75$ .

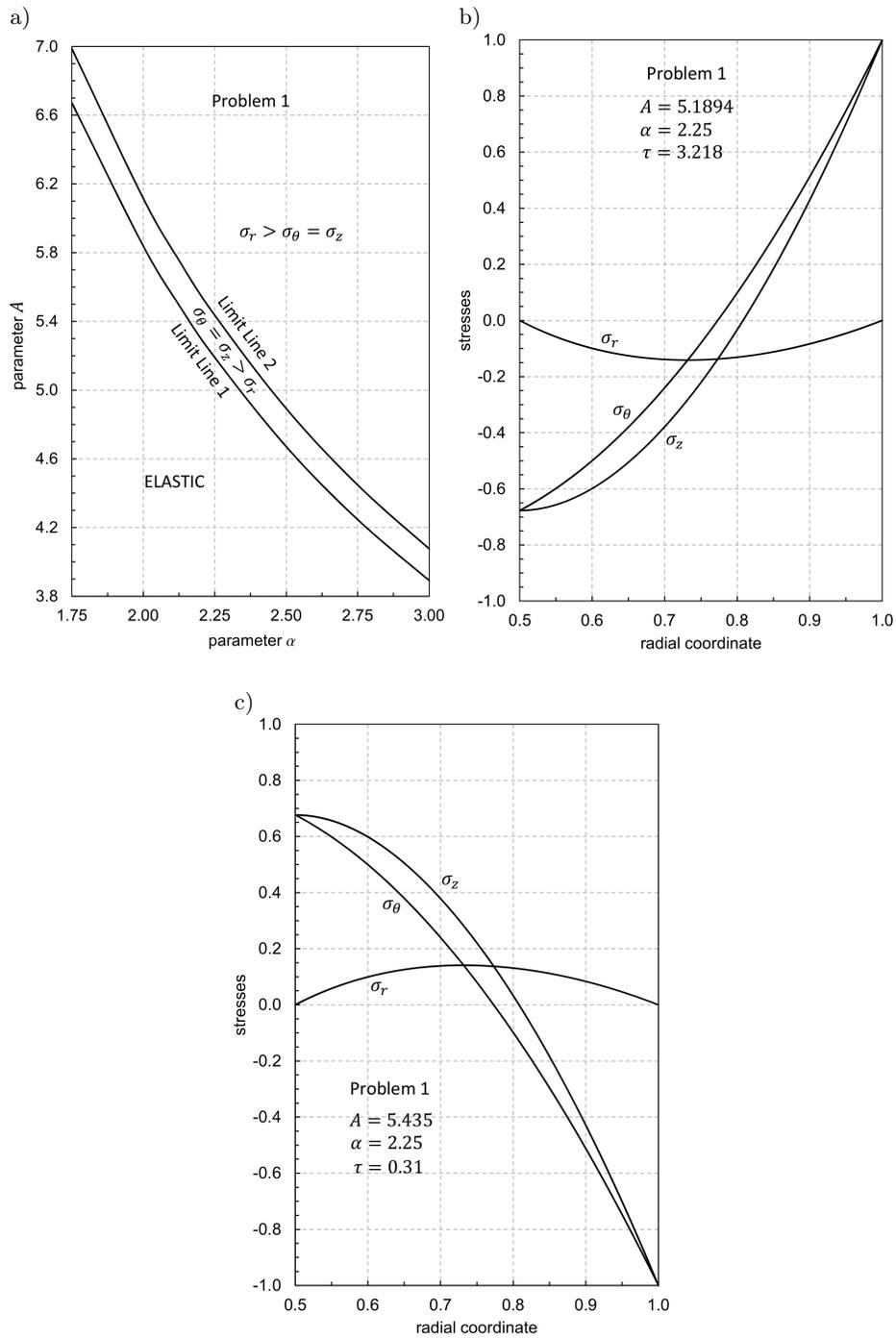


FIG. 5. a) Deformation chart based on the numerical values of the set  $\{A, \bar{\alpha}\}$  for Problem 1; b) stress distributions on Limit Line 1, c) stress distributions on Limit Line 2.

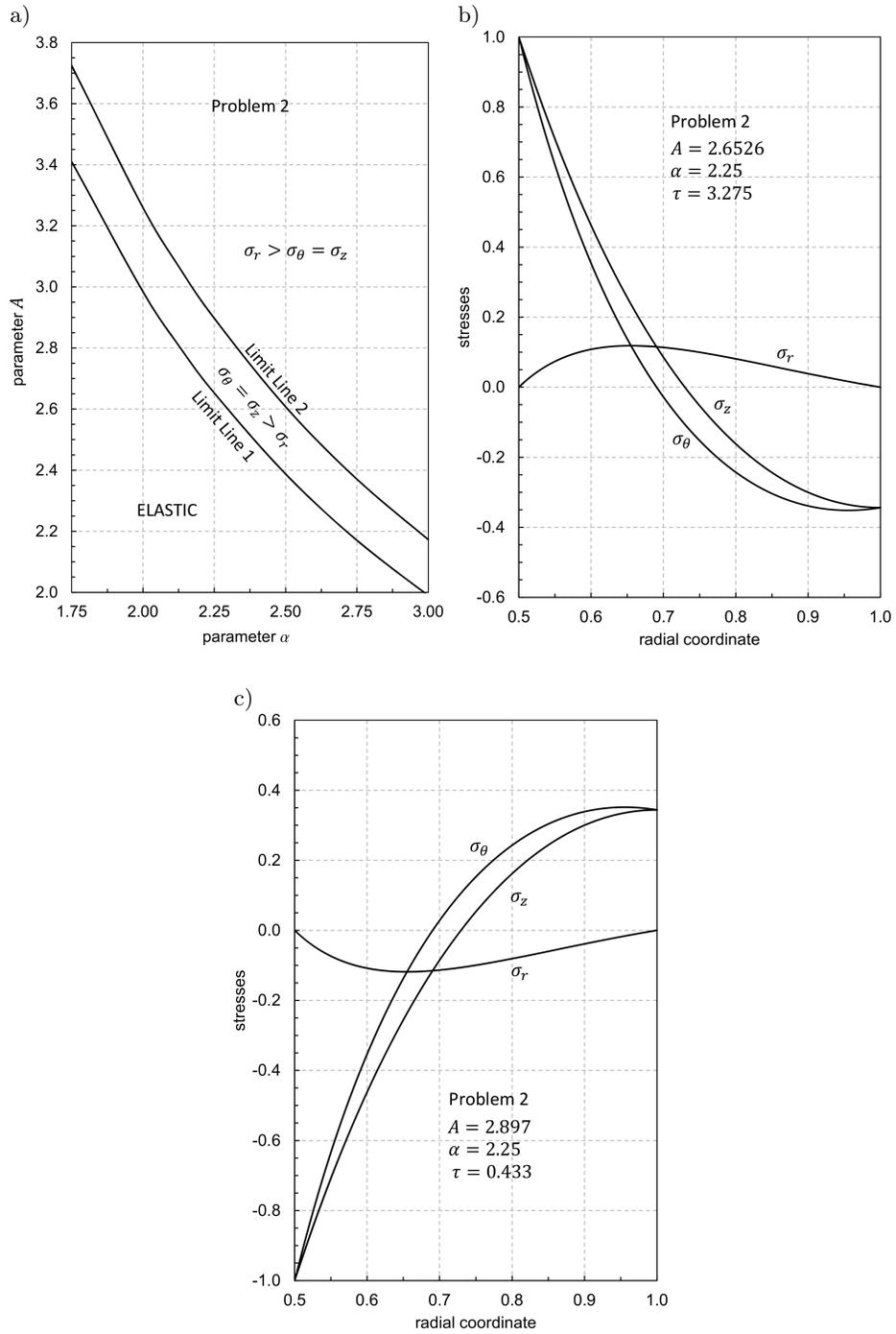


FIG. 6. a) Deformation chart based on the numerical values of the set  $\{A, \bar{\alpha}\}$  for Problem 2; b) stress distributions on Limit Line 1, c) stress distributions on Limit Line 2.

it turns into  $\sigma_\theta (= \sigma_z) > \sigma_r$  at  $\tau = 2.8$ . A similar situation is observed for the problem 2 as well. As it is seen in Figs. 3a and 3b, for this problem  $\sigma_{EQ}$  is always at the inner surface of the tube with the stress state  $\sigma_r > \sigma_\theta (= \sigma_z)$  at time  $\tau = 0.2$  and with  $\sigma_\theta (= \sigma_z) > \sigma_r$  at time  $\tau = 2.8$ . These two similar states indicate that  $\sigma_{EQ}$  is always located at the periodically heated surface.

As mentioned before,  $\sigma_{EQ}$  takes its maximum value at the periodically heated boundaries. This is the outer surface of the tube for the problem 1 and inner surface for the problem 2. The time rate of change of  $\sigma_{EQ}$  at these locations are calculated and plotted in Fig. 4. As it is seen in this figure, there are two peaks for both problems. For the problem 1, these peaks correspond to times  $\tau = 0.31$  and  $\tau = 3.218$  and for the problem 2 they correspond to  $\tau = 0.433$  and  $\tau = 3.275$ . For both problems, the tubes may first yield either from the first peak or the second depending on the values of the set  $\{A, \alpha\}$ .

Parametric analyses are performed to identify when and according to which stress state yielding occurs in the tubes. Figure 5a shows the results of these calculations for the tube of the problem 1. The tube always behaves in elastic way if the parameter set  $\{A, \alpha\}$  is below Limit Line 1. If the set is on Limit Line 1, yielding commences on the second peak with Tresca's criterion  $\sigma_\theta (= \sigma_z) > \sigma_r$ . On the other hand, if the set is on Limit Line 2, yielding commences on the first peak according to  $\sigma_r > \sigma_\theta (= \sigma_z)$ . Finally, if the parameter set happens to be located just above these lines, yielding may take place a little earlier than  $\tau = 0.31$  or  $\tau = 3.218$  with the criteria indicated above. Keeping  $\alpha$  constant at  $\alpha = 2.25$  and changing  $A$  values, two example figures are drawn and presented in Figs. 5b and 5c. In Fig. 5b  $A = 5.1894$  which is exactly located on Limit Line 1. As a result yielding commences at the outer surface when  $\tau = 3.218$  with the condition  $\sigma_\theta (= \sigma_z) > \sigma_r$ . In the second example,  $A$  is selected on the Limit Line 2 as  $A = 5.435$ . The corresponding stress state is shown in Fig. 5c. Yielding begins at the outer surface when  $\tau = 0.31$  according to the criterion  $\sigma_r > \sigma_\theta (= \sigma_z)$ .

For problem 2, the results of similar analyses can be followed in Figs. 6a–6c.

## 5. Concluding remarks

Thermoelastic behavior of tubes near and at elastic limits subjected to periodic heating have been investigated. Two possible cases have been considered; heating from the inner surface and from the outer. The function  $F(\tau) = A \sin \tau$  is used to define the periodic surface temperature. Assuming that the other face of the tube has been isolated, the corresponding transient heat conduction problem has been solved by the use of Duhamel's theorem. An uncoupled, thermoelastic solution has been carried out under the state of generalized plane strain. Tresca's yield criterion has been utilized to monitor yield points.

It is observed that yielding commences at the surface that is subject to the periodic heating. Depending on the parameters  $A$  and  $\alpha$ , there might be two different stress states when yielding begins. One possible stress state is  $\sigma_r > \sigma_\theta = \sigma_z$  and the other one is  $\sigma_\theta = \sigma_z > \sigma_r$ . The numerical values of the pair  $A$ - $\alpha$  that leads to different stress states at the yield point are shown in Figs. 5a and 6a. These figures are especially useful in doing research to understand loading and unloading behavior of such tubes.

## References

1. D.W.A. REES, *The Mechanics of Solids and Structures*, McGraw-Hill, London, New York, 1990.
2. S. TIMOSHENKO, J.N. GOODIER, *Theory of Elasticity*, McGraw-Hill, New York, 1970.
3. R.B. HETNARSKI, M.R. ESLAMI, *Thermal Stresses – Advanced Theory and Applications*, Springer, Netherlands, 2009.
4. A. KANDIL, A.A. EL-KADY, A. EL-KAFRAWY, *Transient thermal stress analysis of thick-walled cylinders*, International Journal of Mechanical Sciences, **37**, 721–732, 1995.
5. A. KANDIL, *Analysis of thick-walled cylindrical pressure vessels under the effect of cyclic internal pressure and cyclic temperature*, International Journal of Mechanical Sciences, **38**, 1319–1332, 1996.
6. Y. SUN, X. ZHANG, *Transient heat transfer of a hollow cylinder subjected to periodic boundary conditions*, Journal of Pressure Vessel Technology, **137**, 5, 2348–2351, 2015.
7. M.N. OZISIK, *Heat Conduction*, John Wiley, New York, 1980.
8. A.E. SEGALL, *Thermoelastic analysis of thick-walled vessels subjected to transient thermal loading*, Journal of Pressure Vessel Technology, **123**, 1, 146–149, 2001.
9. A.E. SEGALL, *Transient analysis of thick-walled piping under polynomial thermal loading*, Nuclear Engineering and Design, **226**, 183–191, 2003.
10. G. ATEFI, M.A. ABDOUS, A. GANJEHKAVIRI, *Analytical solution of temperature field in hollow cylinder under time dependent boundary condition using Fourier series*, American Journal of Engineering and Applied Sciences, **1**, 2, 141–148, 2008.
11. X. LU, P. TERVOLA, M. VILJANEN, *Transient analytical solution to heat conduction in composite circular cylinder*, International Journal of Heat and Mass Transfer, **49**, 1, 2, 341–348, 2006.
12. H. MAHMOUDI, G. ATEFI, *Analytical solution for thermal stresses in a hollow cylinder under periodic thermal loading*, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, **226** 7, 1705–1725, 2012.
13. V. RADU, N. TAYLOR, E. PAFFUMI, *Development of new analytical solutions for elastic thermal stress components in a hollow cylinder under sinusoidal transient thermal loading*, International Journal of Pressure Vessels and Piping, **85**, 885–893, 2008.
14. S.-Y. LEE, C.-C. HUANG, *Analytic solutions for heat conduction in functionally graded circular hollow cylinders with time-dependent boundary conditions*, Mathematical Problems in Engineering, Hindawi Publishing Corporation, Article ID 816385.

15. Y. KAYA, A.N. ERASLAN, *Thermoelastic response of a long tube subjected to periodic heating*, Mathematical Sciences and Applications E-Notes, **2**, 1, 14–27, 2014.
16. T.-W. TU, S.-Y. LEE, *Analytical solution of heat conduction for hollow cylinders with time-dependent boundary condition and time-dependent heat transfer coefficient*, Journal of Applied Mathematics, Hindawi Publishing Corporation, Article ID 203404, 2015.
17. A.N. ERASLAN, T. APATAY, *Analytical solution to thermal loading and unloading of a cylinder subjected to periodic surface heating*, Journal of Thermal Stresses, **39**, 8, 928–941, 2016.
18. B. TAKABI, *Thermomechanical transient analysis of a thick-hollow FGM cylinder*, Engineering Solid Mechanics, **4**, 25–32, 2016.
19. V.R. MANTHENA, N.K. LAMBA, G.D. KEDAR, *Thermoelastic analysis of a nonhomogeneous hollow cylinder with internal heat generation*, Applications and Applied Mathematics: An International Journal (AAM), **12**, 2, 946–967, 2017.
20. P. AYOUBI, A. ALIBEIGLOO, *Three-dimensional transient analysis of FGM cylindrical shell subjected to thermal and mechanical loading*, Journal of Thermal Stresses, **40**, 9, 1166–1183, 2017.
21. V.R. MANTHENA, G.D. KEDAR, *Transient thermal stress analysis of a functionally graded thick hollow cylinder with temperature-dependent material properties*, Journal of Thermal Stresses, <https://doi.org/10.1080/01495739.2017.1402669>, 2017.
22. A. NAJIBI, R. TALEBITOOTI, *Nonlinear transient thermo-elastic analysis of a 2D-FGM thick hollow finite length cylinder*, Composites Part B, **111**, 211–217, 2017.
23. M.R. SPIEGEL, *Mathematical Handbook*, Mc Graw-Hill, London, New York, 1990.

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