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A circular inclusion with inhomogeneous sliding imperfect interface in harmonic materials

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In the following study we rigorously analyze the problem of a circular inclusion with inhomogeneous imperfect sliding interface in finite deformation of harmonic materials. The work begins by defining the inhomogeneous sliding boundary conditions characterized by two interface parameters corresponding to the normal and tangential coordinate directions (with respect to the interface boundary curve), respectively. Then, through the process of analytic continuation the problem is eventually reduced to the determination of a single analytic function given by an ordinary differential equation with variable coefficients. A specific example is selected to illustrate the method. The effects of the circumferential variation of the interface parameter on the mean stress at the interface and the average mean stress in the inclusion are discussed.

Key words: inclusion, harmonic material, imperfect interface.

1. Introduction

The treatment of inclusion problems in the context of linear elasticity has seen a great deal of development over the past decades. Research conducted in this area ranges from the fundamental works of Eshelby [1] and Muskhelishvili [2], amongst others, to the application of arbitrary inclusion geometries (see, for example [3, 4, 5]), imperfect bonding models (see, for example [6, 7, 8]) and complex interphase models (see, for example [9]). However in finite elasticity theory, specifically in the area of harmonic materials, the study of inclusion problems has not witnessed the same level of awareness. The works of Fritz [10], Ogden and Isherwood [11], Varley and Cumberbatch [12], Knowles and Sternberg [13] laid the foundation for the finite deformation of harmonic materials. However, it was not until Ru [14] who developed a more convenient form of the complex variable formulation for harmonic materials that research into inclusion problems experienced a rapid onset of development. Building on this work, finite elasticity problems of elliptical inclusions with uniform internal stress fields [15], designing an inclusion with uniform interior stress [16], partially debonded circular inclusions [17] and a circular inclusion with homogeneous imperfect interface [18] have been studied just to name a few. However, in many real world problems a homogeneous imperfect interface is not a realistic assumption. What is of particular interest is a model that captures the variability of interface damage (such as the presence of microcracks, voids and impurities) which is referred to as an inhomogeneous imperfect interface. This assertion is supported, in part, by previous works (see [19, 20]) corresponding to an inhomogeneous spring-type interface and an inhomogeneous non-slipping interface, respectively.

Recently, the concept of a sliding boundary has been receiving increasing interest in the literature since the study of sliding boundaries is necessary for modelling critical features of material behaviour. In 1993 Mijailovich *et al.* [21] hypothesized that dissipative stresses arise in the interaction among fibers in the connective lung tissue matrix. They established a mechanistic model by reducing the complicated three dimensional fiber network to the interaction of two ideal fibers that dissipate energy along their common slipping interface surface. The resulting model illustrates that a slipping interface is critical in understanding the mechanisms behind connective tissue elasticity. In the area of material science, it has been demonstrated through atomistic simulations (see, for example, Van Swygenhoven [22]) that for nanocrystalline materials macroscopic imposed deformations are accommodated by grain boundary slipping and separation. Following this fact Wei *et al.* [23] considered the effects of grain boundaries on polycrystalline materials. By incorporating crystal plasticity for the grain interior together with an interface constitutive model that takes into account grain boundary related deformation at the interface the authors illustrate that grain boundary slip-separation deformation has a significant effect on material response. Barton *et al.* [24] developed a multi-material numerical scheme for non-linear elastic solids that examines interfacial boundary conditions with particular emphasis placed on a sliding interface. Several examples are provided to illustrate the scheme.

In this work we consider the inhomogeneous sliding imperfect interface where the interphase layer is modelled as a two-dimensional curve of vanishing thickness and the material properties of the interphase layer are given in terms of two spring-type interface parameters. Such an interface condition allows for a relative tangential displacement but maintains continuity of radial displacements across the interphase layer. Use of the inhomogeneous sliding imperfect interface for the case of finite deformation is suitable for applications where type 1 harmonic materials are considered. For example, interface design of cord-reinforced rubber composites such as those found in tires, belts and various attenuation constructions is important. The optimal design of interfacial properties is of paramount importance for improving product quality especially when interface damage surrounding the cord is variable. The work begins with Section 2 where the fundamental equations of type 1 harmonic materials are presented. Following Section 2, Section 3 discusses the formulation of the problem and the sliding boundary conditions where eventually a first order linear ordinary differential equation with variable coefficients is developed for the inclusion function. Section 4 illustrates the analysis for a specific class of imperfect interface and in Section 5 an example is given to illustrate the method. In Section 6 the average mean stress in the inclusion and the mean stress at a point on the interface is evaluated and compared to the corresponding homogeneous imperfect interface. Finally a summary of the results are presented in Section 7.

2. Mathematical preliminaries

Consider a single simply connected domain of radius R bounded by a continuous circular curve ∂D_1, embedded in an infinite matrix in $\mathbb{R}^2$ (Fig. 1). Let us assume that any deformation relative to the reference configuration is confined to the x_1x_2 plane. Let $z = x_1 + ix_2$ be the Lagrangian coordinates of a particle in the reference configuration and let $w(z) = y_1(z) + iy_2(z)$ be the Eulerian coordinates of a particle in the current configuration. The inclusion is denoted by D_1 and endowed with material properties μ_1, α_1, β_1. The matrix is denoted by domain D_2 with material properties μ_2, α_2, β_2 where $\frac{1}{2} \leq \alpha_k < 1$, $\beta_k > 0$, $k = 1, 2$. In both cases, μ represents the material shear modulus, and α, β are derived from the ratios of the principal stretches of a harmonic material under

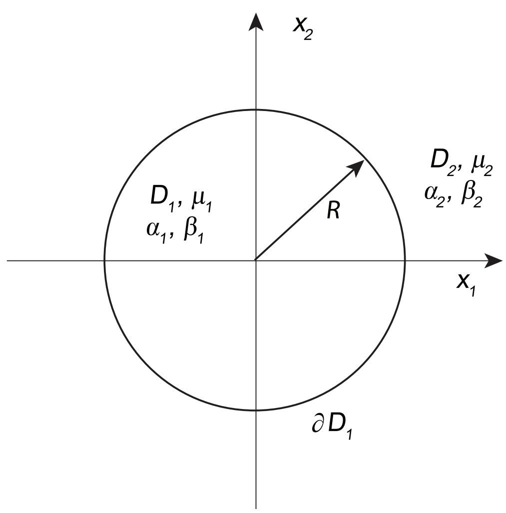

FIG. 1. Elastic circular inclusion (D_1) bounded by curve ∂D_1 embedded in a infinite matrix (D_2).

uni-axial tension. The matrix and inclusion are assumed to be type 1 harmonic material with a strain energy function $W(I, J)$ defined as follows

$$W(I,J) = 2\mu[H(I) - J], \quad F_{ij} = \frac{\partial y_i}{\partial x_j}, \quad H'(I) = \frac{1}{4\alpha}[I + \sqrt{I^2 - 16\alpha\beta}], \tag{2.1}$$

where I and J are the scalar invariants of the right Cauchy Green tensor $\mathbf{F}^T\mathbf{F}$ corresponding to the two dimensional deformation as noted above and are given by

$$I = \lambda_1^2 + \lambda_2^2 = \mathrm{tr}[\mathbf{C}], \qquad J = \lambda_1\lambda_2 = \sqrt{\det[\mathbf{C}]} = \det[\mathbf{F}], \tag{2.2}$$

($'$) denotes differentiation with respect to I and λ_1, λ_2 are the principal stretches. According to Ru [14], the deformation map and the Piola stress function can be given in terms of two complex potential functions $\phi_k(z)$ and $\psi_k(z)$ as follows

$$\begin{aligned} iw_k(z,\overline{z}) &= \alpha_k\phi_k(z) + i\overline{\psi_k(z)} + \frac{\beta_k z}{\overline{\phi_k'(z)}}, \\ \chi_k(z,\overline{z}) &= 2i\mu_k\left[(\alpha_k - 1)\phi_k(z) + i\overline{\psi_k(z)} + \frac{\beta_k z}{\overline{\phi_k'(z)}}\right] \qquad \text{for } k = 1,2. \end{aligned} \tag{2.3}$$

Equation (2.3) gives rise to the following Cartesian expressions for the displacement and stress fields

$$\begin{aligned} w_k(z,\overline{z}) - z &= (u_1 + iu_2)_k, \\ \chi_k(z,\overline{z})_{,1} = (P_{22} - iP_{12})_k, \quad \chi_k(z,\overline{z})_{,2} &= (-P_{21} + iP_{11})_k, \quad k = 1,2. \end{aligned} \tag{2.4}$$

where the subscript k refers to the either the inclusion $k = 1$ or the matrix $k = 2$ and P_{ij} are the components of the Piola stress tensor.

In order to illustrate the significance of the inhomogeneous sliding imperfect interface we consider the scenario where the rotations are neglected. Then Eq. (2.4) may be transformed into polar coordinates as shown:

$$\frac{R}{z}w_k(z,\overline{z}) - R = (u_r + iu_\theta)_k, \quad \chi_k'(z,\overline{z}) = (P_{rr} + iP_{\theta r})_k, \quad k = 1,2. \tag{2.5}$$

where a prime ($'$) denotes differentiation with respect to z.

3. Formulation

Although there are many interface models reported in the literature, in this particular work interface damage is considered to be distributed circumferentially around the inclusion. Thus, in order to capture the circumferential variation in interface damage let us consider the inclusion to be imperfectly bonded to the matrix along ∂D_1 (ie. the tractions are continuous across the material interface and the jump in the displacements across the material interface are proportional to their respective traction components) via the general spring model for an imperfect interface condition given by

$$\|P_{rr}+iP_{\theta r}\|=0, \quad P_{rr}=m(\theta)\|u_r\|, \quad P_{\theta r}=n(\theta)\|u_\theta\|, \qquad z\in\partial D_1, \tag{3.1}$$

where $m(\theta)$ and $n(\theta)$ are two non-negative imperfect interface parameters describing the variability in interface damage and $\|\cdot\|=(\cdot)_2-(\cdot)_1$ is the quantitative jump across ∂D_1. It is assumed that the potential functions $\phi_2(z)$ and $\psi_2(z)$ exhibit the following asymptotic behavior as $|z|\to\infty$

$$\phi_2(z)=Az+O(1), \qquad \psi_2(z)=Bz+O(1), \qquad |z|\to\infty, \tag{3.2}$$

where A and B are complex constants that reflect the far-field loading and are given by [19]

$$A=i\left[\frac{\frac{P_{22}^{\infty}+P_{11}^{\infty}}{4\mu_2}\pm\sqrt{(\frac{P_{22}^{\infty}+P_{11}^{\infty}}{4\mu_2})^2+4(1-\alpha_2)\beta_2}}{2(1-\alpha_2)}\right], \tag{3.3}$$

$$B=\frac{P_{11}^{\infty}-P_{22}^{\infty}-2iP_{12}^{\infty}}{4\mu_2}, \tag{3.4}$$

and the $O(1)$ are some first order constant terms. Furthermore, since ϕ_k,ψ_k are potential functions, we only consider the case where they are analytic and hence the potentials $\phi_k(z)$ and $\psi_k(z)$, $k=1,2$ admit the following series expansions

$$\begin{aligned}\phi_1(z)&=X_0+\sum_{k=1}^{\infty}X_kz^k, \qquad \psi_1(z)=Y_0+\sum_{k=1}^{\infty}Y_kz^k, \quad z\in D_1,\\ \phi_2(z)&=Az+\sum_{k=0}^{\infty}A_kz^{-k}, \quad\ \psi_2(z)=Bz+\sum_{k=0}^{\infty}B_kz^{-k}, \quad z\in D_2.\end{aligned} \tag{3.5}$$

Remark 1. From (3.5) we require that $X_1\neq 0$ for $|z|\leq R$ and $A\neq 0$ for $|z|\geq R$. This guarantees that $H'(I)=|\phi_k'(z)|\neq 0\ \forall z\in\mathbb{C}$.

In the present work we do not consider a rigid displacement of the inclusion, hence, without loss of generality, it is admissible to set both $X_0,Y_0=0$ and the continuity of traction condition from (3.1) gives

$$\begin{aligned}&\mu_1\left[(\alpha_1-1)\phi_1(z)+i\overline{\psi_1}(R^2/z)+\frac{\beta_1 z}{\overline{\phi_1}'(R^2/z)}\right]\\ &\qquad=\mu_2\left[(\alpha_2-1)\phi_2(z)+i\overline{\psi_2}(R^2/z)+\frac{\beta_2 z}{\overline{\phi_2}'(R^2/z)}\right], \quad z\in\partial D_1.\end{aligned} \tag{3.6}$$

Substituting $\Gamma = \frac{\mu_1}{\mu_2}$ into the above yields

$$\begin{aligned}(3.7)\qquad &\Gamma(\alpha_1-1)\phi_1(z) - i\overline{\psi_2}(R^2/z) - \frac{\beta_2 z}{\overline{\phi_2'}(R^2/z)} \\ &= (\alpha_2-1)\phi_2(z) - \Gamma i\overline{\psi_1}(R^2/z) - \frac{\Gamma\beta_1 z}{\overline{\phi_1'}(R^2/z)}, \quad z\in\partial D_1.\end{aligned}$$

The LHS of (3.7) is analytic for $z \in D_1$ and the RHS is analytic for $z \in D_2$. Utilizing the principle of analytic continuation on (3.7) we arrive at the following

$$\begin{aligned}(3.8)\qquad &i\overline{\psi_2}(R^2/z) + \frac{\beta_2 z}{\overline{\phi_2'}(R^2/z)} \\ &= \Gamma(\alpha_1-1)\phi_1(z) - (\alpha_2-1)Az + \frac{\Gamma\beta_1 z}{\overline{X_1}} + \frac{i\overline{B}R^2}{z}, \quad z\in\partial D_1,\end{aligned}$$

and

$$\begin{aligned}(3.9)\qquad &i\overline{\psi_1}(R^2/z) + \frac{\beta_1 z}{\overline{\phi_1'}(R^2/z)} \\ &= \frac{(\alpha_2-1)}{\Gamma}\phi_2(z) - \frac{(\alpha_2-1)}{\Gamma}Az + \frac{\beta_1 z}{\overline{X_1}} + \frac{i\overline{B}R^2}{\Gamma z}, \quad z\in\partial D_1.\end{aligned}$$

Thus, the problem is now reduced to determining two unknown analytic functions $\phi_1(z)$ and $\phi_2(z)$ complying with the interface condition and the asymptotic condition for $\phi_2(z)$.

3.1. Inhomogeneous imperfect sliding interface

We shall now consider a circular inclusion for which the inhomogeneous imperfect interface is characterized by $m(\theta) \to \infty$, $n(\theta) =$ finite. For this so-called sliding interface the displacement jump boundary conditions,(3.1), take the form

$$(3.10)\qquad \frac{(P_{r\theta})_2}{n(\theta)} = \|u_\theta\|, \qquad \|u_r\| = 0, \quad z\in\partial D_1,$$

where $n(\theta)$ is non-negative and periodic along ∂D_1. The displacement continuity condition is evaluated as follows

$$(3.11)\quad \|u_r\| = \frac{R}{z}(iw_2(z) - iw_1(z)) + \frac{z}{R}(\overline{iw_1}(R^2/z) - \overline{iw_2}(R^2/z)), \qquad z\in\partial D_1.$$

Inserting (2.3) in combination with (3.8, 3.9) into (3.11) gives

$$\begin{aligned}(3.12)\qquad &[\Gamma(\alpha_1-1)-\alpha_1]\frac{R}{z}\phi_1(z) + \left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z) \\ &+ \frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}AR + \frac{iBz^2}{R}\frac{\Gamma-1}{\Gamma} + \frac{\beta_1}{X_1}(1-\Gamma)\end{aligned}$$

$$= \left[\Gamma(\alpha_1 - 1) - \alpha_1\right] \frac{z}{R}\overline{\phi_1}(R^2/z) + \left[\frac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}\right]\frac{R}{z}\phi_2(z) + \frac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}\overline{A}R + \frac{i\overline{B}R^3}{z^2}\frac{1-\Gamma}{\Gamma} + \frac{\beta_1}{\overline{X_1}}(1-\Gamma), \quad z \in \partial D_1.$$

In (3.12) the left hand side is analytic in D_1 and the right hand side is analytic in D_2 except for possibly at the point $|z| = 0$ and as $|z| \to \infty$, respectively. Employing the technique of analytic continuation, we first analyze the behavior of the left hand side of (3.12) as $|z| \to 0$ as follows

$$(\Gamma(\alpha_1 - 1) - \alpha_1)X_1 R + \frac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}\overline{A}R + \frac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}AR + \frac{\beta_1}{X_1}(1-\Gamma), \quad |z| \to 0. \tag{3.13}$$

Since (3.13) does not contain any strictly singular terms, we may conclude that the left hand side of (3.12) is analytic in D_1 as $|z| \to 0$. Moving on to the right hand side of (3.12), we observe the following as $|z| \to \infty$

$$(\Gamma(\alpha_1 - 1) - \alpha_1)\overline{X_1}R + \frac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}AR + \frac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}\overline{A}R + \frac{\beta_1}{\overline{X_1}}(1-\Gamma), \quad |z| \to \infty. \tag{3.14}$$

Equation (3.14) represents the asymptotic behavior of the right hand side of (3.12) and subtracting (3.14) from both sides of (3.12), we may form the following function

$$D(z) = \begin{cases} [\Gamma(\alpha_1 - 1) - \alpha_1]\dfrac{R}{z}\phi_1(z) + \left[\dfrac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}\right]\dfrac{z}{R}\overline{\phi_2}(R^2/z) \\ \quad + \dfrac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}AR + \dfrac{iBz^2}{R}\dfrac{\Gamma - 1}{\Gamma} + \dfrac{\beta_1}{X_1}(1-\Gamma) \\ \quad -(\Gamma(\alpha_1 - 1) - \alpha_1)\overline{X_1}R - \dfrac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}AR \\ \quad - \dfrac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}\overline{A}R - \dfrac{\beta_1}{\overline{X_1}}(1-\Gamma), \quad z \in D_1, \\ [\Gamma(\alpha_1 - 1) - \alpha_1]\dfrac{z}{R}\overline{\phi_1}(R^2/z) + \left[\dfrac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}\right]\dfrac{R}{z}\phi_2(z) \\ \quad + \dfrac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}\overline{A}R + \dfrac{i\overline{B}R^3}{z^2}\dfrac{1-\Gamma}{\Gamma} + \dfrac{\beta_1}{\overline{X_1}}(1-\Gamma) \\ \quad - \dfrac{\alpha_2 - 1 - \Gamma\alpha_2}{\Gamma}AR - \dfrac{\alpha_2 - 1 + \Gamma(1-\alpha_2)}{\Gamma}\overline{A}R \\ \quad - (\Gamma(\alpha_1 - 1) - \alpha_1)\overline{X_1}R - \dfrac{\beta_1}{\overline{X_1}}(1-\Gamma), \quad z \in D_2. \end{cases} \tag{3.15}$$

Now, since $D(z)$ is well defined and analytic in the entire plane including as $|z| \to \infty$, Liouville's theorem states that every bounded entire function must be constant. Hence, it must be that $D(z) = \textit{constant}$ and through the subtraction of (3.14) on the left hand and right hand side of (3.12), it is concluded that $D(z) = 0$ and hence we arrive at the following two equations

$$\begin{aligned}(3.16)\qquad &[\Gamma(\alpha_1-1)-\alpha_1]\frac{R}{z}\phi_1(z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{z}{R}\overline{\phi_2}(R^2/z)\\ &+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}AR+\frac{iBz^2}{R}\frac{\Gamma-1}{\Gamma}+\frac{\beta_1}{X_1}(1-\Gamma)-(\Gamma(\alpha_1-1)-\alpha_1)\overline{X_1}R\\ &-\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}AR-\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R-\frac{\beta_1}{\overline{X_1}}(1-\Gamma)=0,\quad z\in D_1,\end{aligned}$$

$$\begin{aligned}(3.17)\qquad &[\Gamma(\alpha_1-1)-\alpha_1]\frac{z}{R}\overline{\phi_1}(R^2/z)+\left[\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}\right]\frac{R}{z}\phi_2(z)\\ &+\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R+\frac{i\overline{B}R^3}{z^2}\frac{1-\Gamma}{\Gamma}+\frac{\beta_1}{\overline{X_1}}(1-\Gamma)-(\Gamma(\alpha_1-1)-\alpha_1)\overline{X_1}R\\ &-\frac{\alpha_2-1-\Gamma\alpha_2}{\Gamma}AR-\frac{\alpha_2-1+\Gamma(1-\alpha_2)}{\Gamma}\overline{A}R-\frac{\beta_1}{\overline{X_1}}(1-\Gamma)=0,\quad z\in D_2.\end{aligned}$$

The compatability requirement between Eqs. (3.16) and (3.17) is given by

$$(3.18)\qquad (\Gamma(\alpha_1-1)-\alpha_1)R(X_1-\overline{X_1})+\beta_1(1-\Gamma)\left(\frac{1}{X_1}-\frac{1}{\overline{X_1}}\right)=-2AR.$$

We now consider the tangential stress-displacement interface condition which may be written as

$$(3.19)\qquad \frac{(P_{r\theta})_2}{n(\theta)}=-i\|u_r+iu_\theta\|,\qquad z\in\partial D_1,$$

which, in terms of (2.3), (2.4) and (3.8), (3.9, (3.17), (3.19) becomes

$$\begin{aligned}(3.20)\qquad &\left[\frac{\Gamma(1-\alpha_2+\Gamma(1-\alpha_1))}{\alpha_2-1-\Gamma\alpha_2}\right]\phi_1'(z)+\left[\frac{\Gamma(1-\alpha_2+\Gamma(1-\alpha_1))}{\alpha_2-1-\Gamma\alpha_2}\right]\overline{\phi_1}'(R^2/z)\\ &+\frac{2\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\alpha_2-1-\Gamma\alpha_2}\frac{\phi_1(z)}{z}\\ &+\frac{2\Gamma(\alpha_2-1)(\alpha_1-\Gamma(\alpha_1-1))}{\alpha_2-1-\Gamma\alpha_2}\overline{\phi_1'}(R^2/z)\frac{z}{R^2}\\ &+\frac{i\overline{B}R^2}{z^2}\frac{\Gamma}{\alpha_2-1-\Gamma\alpha_2}-\frac{iBz^2}{R^2}\frac{\Gamma}{\alpha_2-1-\Gamma\alpha_2}\\ &+\frac{\Gamma(\alpha_2-1)(\Gamma(\alpha_1-1)-\alpha_1)}{\alpha_2-1-\Gamma\alpha_2}\overline{X_1}\\ &+\frac{\Gamma(\alpha_2-1)(\Gamma(\alpha_1-1)-\alpha_1)}{\alpha_2-1-\Gamma\alpha_2}X_1+\frac{\Gamma\beta_1}{\overline{X_1}}+\frac{\Gamma\beta_1}{X_1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{n(\theta)R}{\mu_2}\bigg[(\alpha_1 - \varGamma(\alpha_1 - 1)\frac{\phi_1(z)}{z} + (\alpha_1 - \varGamma(\alpha_1 - 1)\frac{z}{R^2}\overline{\phi_1}(R^2/z) \\
&\quad + (\varGamma(\alpha_1 - 1) - \alpha_1)\frac{\overline{X_1}}{2} + \frac{\beta_1}{2\overline{X_1}}(1 - \varGamma) \\
&\quad + (\varGamma(\alpha_1 - 1) - \alpha_1)\frac{X_1}{2} + \frac{\beta_1}{2X_1}(1 - \varGamma)\bigg], \qquad z \in \partial D_1.
\end{aligned}
$$

The problem is now reduced to determining the single analytic function $\phi_1(z)$. It should be noted that direct substitution of the power series expansion of $\phi_1(z)$ into (3.20) will result in an unsolveable system of equations for the coefficients of $\phi_1(z)$. As such, we shall seek to use analytic continuation to further reduce (3.20) into an ordinary differential equation with variable coefficients for $\phi_1(z)$. To aid in this process we define a new imperfect interface parameter to replace $n(\theta)$ in (3.20) as follows

$$
\delta(\theta) = \frac{n(\theta)R}{\mu_2}, \qquad \delta(\theta) > 0, \tag{3.21}
$$

and since $1/\delta(\theta)$ is a non-negative and periodic function on ∂D_1, we may write

$$
\frac{\delta_0}{\delta(\theta)} = 1 + f(\theta), \quad \delta_0 > 0, \quad f(\theta) > -1, \tag{3.22}
$$

where δ_0 is real, $f(\theta)$ is 2π periodic on ∂D_1, and as $f(\theta) \to -1$, $n(\theta) \to \infty$, which is the case of a perfectly bonded interface. Given $f(\theta)$ is 2π periodic on ∂D_1, we are afforded a Fourier series expansion for $(1+f(z))$ which we may then rewrite as a function of the complex variable z as follows

$$
f(z) = \frac{1}{2}\sum_{k=1}^{s}(b_k + ia_k)\frac{R^k}{z^k} + (b_k - ia_k)\frac{z^k}{R^k}, \quad \forall z \in \partial D_1,\ f(\theta) = f(z). \tag{3.23}
$$

3.2. The differential equation for $\phi_1(z)$

Before returning to (3.20) we introduce the following two material parameters

$$
\begin{aligned}
\varOmega &= \frac{(1-\alpha_2)(\alpha_1 - \varGamma(\alpha_1 - 1))}{1 - \alpha_2 + \varGamma(1-\alpha_1)} > 0, \\
\omega &= \frac{\varGamma\alpha_2 - \alpha_2 + 1}{1 - \alpha_2 + \varGamma(1 - \alpha_1)} > 0.
\end{aligned} \tag{3.24}
$$

Using (3.22), (3.23), (3.24) we may rewrite (3.20) as

$$(1+f(z))\phi_1'(z)+\left[\frac{\omega(\alpha_1-\Gamma(\alpha_1-1)}{\Gamma}\frac{\delta_0}{z}-\frac{2\Omega(1+f(z))}{z}\right]\phi_1(z) \tag{3.25}$$
$$+(1+f(z))\left[\Omega X_1-\frac{iBz^2}{R^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}-\frac{\omega\beta_1}{X_1}\right]$$
$$-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}X_1+\frac{\beta_1}{2X_1}\frac{(1-\Gamma)\omega\delta_0}{\Gamma}$$
$$=-(1+f(z))\overline{\phi_1'}(R^2/z)+\left[\frac{2\Omega(1+f(z))z}{R^2}-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0 z}{R^2}\right]\overline{\phi_1}(R^2/z)$$
$$-(1+f(z))\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}+\frac{i\overline{B}R^2}{z^2}\frac{\omega}{\Gamma\alpha_2-\alpha_2+1}\right]$$
$$+\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}-\frac{\beta_1}{2\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{\Gamma},\quad z\in\partial D_1.$$

The left hand side of (3.25) is analytic in D_1 and the right hand side is analytic in D_2. Using the technique of analytic continuation we may construct an entire function by studying the behaviour of the left hand and right hand sides of (3.25) as $|z|=0$, and $|z|\to\infty$, respectively. Beginning with the left hand side we allow $|z|\to 0$ and recover the following singular terms

$$\sum_{j=1}^{k} jX_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-2\Omega\sum_{j=1}^{k}X_j z^{j-1}\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k} \tag{3.26}$$
$$+\left[\Omega X_1-\frac{\omega\beta_1}{X_1}\right]\sum_{k=1}^{s}\frac{b_k+ia_k}{2}\frac{R^k}{z^k}-\frac{iB\omega}{\Gamma\alpha_2-\alpha_2+1}\sum_{k=3}^{s}\frac{b_k+ia_k}{2}\frac{R^{k-2}}{z^{k-2}},\quad |z|\to 0.$$

Proceeding to the right hand side of (3.25), we find the following asymptotic and singular behavior as $|z|\to\infty$

$$\overline{X_1}(\Omega-1)-\sum_{k=1}^{k}j\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k} \tag{3.27}$$
$$-\sum_{k=1}^{s}(k+1)\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k$$
$$+2\Omega\left[\sum_{j=1}^{k}\overline{X_j}\left(\frac{R^2}{z}\right)^{j-1}\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}+\sum_{k=1}^{s}\overline{X_{k+1}}\frac{b_k-ia_k}{2}R^k\right]$$
$$-\frac{\omega(\alpha_1-\Gamma(\alpha_1-1))}{\Gamma}\frac{\delta_0}{2}\overline{X_1}+\frac{\omega\beta_1}{\overline{X_1}}-\left[\Omega\overline{X_1}-\frac{\omega\beta_1}{\overline{X_1}}\right]\sum_{k=1}^{s}\frac{b_k-ia_k}{2}\frac{z^k}{R^k}$$
$$-\frac{i\overline{B}\omega}{\Gamma\alpha_2-\alpha_2+1}\sum_{k=2}^{s}\frac{b_k-ia_k}{2}\frac{z^{k-2}}{R^{k-2}}-\frac{\beta_1}{\overline{X_1}}\frac{(1-\Gamma)\delta_0\omega}{2\Gamma},\quad |z|\to\infty.$$

The sum of (3.26) and (3.27) is defined by $L(z)$

$$L(z) = \sum_{j=1}^{k} jX_j z^{j-1} \sum_{k=1}^{s} \frac{b_k + ia_k}{2} \frac{R^k}{z^k} - 2\Omega \sum_{j=1}^{k} X_j z^{j-1} \sum_{k=1}^{s} \frac{b_k + ia_k}{2} \frac{R^k}{z^k}$$
$$+ \left[\Omega X_1 - \frac{\omega\beta_1}{X_1}\right] \sum_{k=1}^{s} \frac{b_k + ia_k}{2} \frac{R^k}{z^k} - \frac{iB\omega}{\Gamma\alpha_2 - \alpha_2 + 1} \sum_{k=3}^{s} \frac{b_k + ia_k}{2} \frac{R^{k-2}}{z^{k-2}}$$
$$+ \overline{X_1}(\Omega - 1) - \sum_{k=1}^{k} j\overline{X_j} \left(\frac{R^2}{z}\right)^{j-1} \sum_{k=1}^{s} \frac{b_k - ia_k}{2} \frac{z^k}{R^k}$$
$$- \sum_{k=1}^{s} (k+1)\overline{X_{k+1}} \frac{b_k - ia_k}{2} R^k$$
$$+ 2\Omega \left[\sum_{j=1}^{k} \overline{X_j} \left(\frac{R^2}{z}\right)^{j-1} \sum_{k=1}^{s} \frac{b_k - ia_k}{2} \frac{z^k}{R^k} + \sum_{k=1}^{s} \overline{X_{k+1}} \frac{b_k - ia_k}{2} R^k \right]$$
$$- \frac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} \frac{\delta_0}{2} \overline{X_1} + \frac{\omega\beta_1}{\overline{X_1}} - \left[\Omega\overline{X_1} - \frac{\omega\beta_1}{\overline{X_1}}\right] \sum_{k=1}^{s} \frac{b_k - ia_k}{2} \frac{z^k}{R^k}$$
$$- \frac{i\overline{B}\omega}{\Gamma\alpha_2 - \alpha_2 + 1} \sum_{k=2}^{s} \frac{b_k - ia_k}{2} \frac{z^{k-2}}{R^{k-2}} - \frac{\beta_1}{\overline{X_1}} \frac{(1-\Gamma)\delta_0\omega}{2\Gamma}, \tag{3.28}$$

such that by subtracting $L(z)$ from both the left hand side and right hand side of (3.25) we obtain the following entire function

$$E(z) = \begin{cases} (1+f(z))\phi_1'(z) + \left[\dfrac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1)}{\Gamma} \dfrac{\delta_0}{z} - \dfrac{2\Omega(1+f(z))}{z}\right]\phi_1(z) \\ \quad +(1+f(z))\left[\Omega X_1 - \dfrac{iBz^2}{R^2} \dfrac{\omega}{\Gamma\alpha_2 - \alpha_2 + 1} - \dfrac{\omega\beta_1}{X_1}\right] \\ \quad -\dfrac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} \dfrac{\delta_0}{2} X_1 + \dfrac{\beta_1}{2X_1} \dfrac{(1-\Gamma)\omega\delta_0}{\Gamma} - L(z), \quad z \in D_1, \\ -(1+f(z))\overline{\phi_1'}(R^2/z) \\ \quad +\left[\dfrac{2\Omega(1+f(z))z}{R^2} - \dfrac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} \dfrac{\delta_0 z}{R^2}\right]\overline{\phi_1}(R^2/z) \\ \quad -(1+f(z))\left[\Omega\overline{X_1} - \dfrac{\omega\beta_1}{\overline{X_1}} + \dfrac{i\overline{B}R^2}{z^2} \dfrac{\omega}{\Gamma\alpha_2 - \alpha_2 + 1}\right] \\ \quad + \dfrac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} \dfrac{\delta_0}{2} \overline{X_1} - \dfrac{\beta_1}{2\overline{X_1}} \dfrac{(1-\Gamma)\delta_0\omega}{\Gamma} - L(z), \quad z \in D_2. \end{cases} \tag{3.29}$$

Once again we seek to take advantage of Liouville's theorem whereby it is realized that $E(z) = \mathit{constant}$ in (3.29). Owing to the subtraction of $L(z)$, $E(z) = 0$ and we generate the following two equations

$$(1+f(z))\phi_1'(z) + \left[\frac{\omega(\alpha_1 - \mathit{\Gamma}(\alpha_1-1)}{\mathit{\Gamma}}\frac{\delta_0}{z} - \frac{2\mathit{\Omega}(1+f(z))}{z}\right]\phi_1(z)$$
$$+ (1+f(z))\left[\mathit{\Omega}X_1 - \frac{iBz^2}{R^2}\frac{\omega}{\mathit{\Gamma}\alpha_2-\alpha_2+1} - \frac{\omega\beta_1}{X_1}\right]$$
$$- \frac{\omega(\alpha_1-\mathit{\Gamma}(\alpha_1-1))}{\mathit{\Gamma}}\frac{\delta_0}{2}X_1 + \frac{\beta_1}{2X_1}\frac{(1-\mathit{\Gamma})\omega\delta_0}{\mathit{\Gamma}} - L(z) = 0, \quad z \in D_1, \tag{3.30}$$

$$- (1+f(z))\overline{\phi_1'}(R^2/z)$$
$$+ \left[\frac{2\mathit{\Omega}(1+f(z))z}{R^2} - \frac{\omega(\alpha_1-\mathit{\Gamma}(\alpha_1-1))}{\mathit{\Gamma}}\frac{\delta_0 z}{R^2}\right]\overline{\phi_1}(R^2/z)$$
$$- (1+f(z))\left[\mathit{\Omega}\overline{X_1} - \frac{\omega\beta_1}{\overline{X_1}} + \frac{i\overline{B}R^2}{z^2}\frac{\omega}{\mathit{\Gamma}\alpha_2-\alpha_2+1}\right]$$
$$+ \frac{\omega(\alpha_1-\mathit{\Gamma}(\alpha_1-1))}{\mathit{\Gamma}}\frac{\delta_0}{2}\overline{X_1} - \frac{\beta_1}{2\overline{X_1}}\frac{(1-\mathit{\Gamma})\delta_0\omega}{\mathit{\Gamma}} - L(z) = 0, \quad z \in D_2. \tag{3.31}$$

The compatibility requirement between (3.30) and (3.31) is given by allowing $|z| \to 0$ in (3.31)

$$L_0 = -\overline{L_0}, \tag{3.32}$$

where

$$L_0 = X_1(1-\mathit{\Omega}) - \frac{\omega\beta_1}{X_1} + \sum_{k=1}^{s}(k+1-2\mathit{\Omega})X_{k+1}\frac{b_k+ia_k}{2}R^k$$
$$+ \frac{\omega(\alpha_1-\mathit{\Gamma}(\alpha_1-1))}{\mathit{\Gamma}}\frac{\delta_0}{2}X_1 + \frac{\beta_1(1-\mathit{\Gamma})\omega\delta_0}{2\mathit{\Gamma}X_1} - \frac{iB\omega}{\mathit{\Gamma}\alpha_2-\alpha_2+1}\frac{b_2+ia_2}{2}. \tag{3.33}$$

Given Eq. (3.32), Eqs. (3.30) and (3.31) are equivalent and hence we may use (3.30) to define a simplified differential equation for $\phi_1(z)$ as follows

$$\phi_1'(z) + \left[\frac{\omega(\alpha_1-\mathit{\Gamma}(\alpha_1-1)}{\mathit{\Gamma}}\frac{\delta_0}{z(1+f(z))} - \frac{2\mathit{\Omega}}{z}\right]\phi_1(z) = P(z), \quad z \in D_1, \tag{3.34}$$

where

$$P(z) = \frac{\omega\beta_1}{X_1} - \mathit{\Omega}X_1 + \frac{iBz^2}{R^2}\frac{\omega}{\mathit{\Gamma}\alpha_2-\alpha_2+1}$$
$$- \frac{\delta_0/2}{1+f(z)}\left[\frac{\beta_1(1-\mathit{\Gamma})\omega}{\mathit{\Gamma}X_1} - \frac{\omega(\alpha_1-\mathit{\Gamma}(\alpha_1-1))}{\mathit{\Gamma}}X_1\right] + \frac{L(z)}{1+f(z)}. \tag{3.35}$$

Equation (3.34) is a first order ordinary differential equation with variable coefficients which has the following general solution

$$\phi_1(z) = e^{-T(z)} \int_{z_1}^{z} e^{T(z)} P(z) dz + C_0 e^{-T(z)}, \quad z \in D_1, \tag{3.36}$$

where

$$T(z) = \int \left(\frac{\omega(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} \frac{\delta_0}{z(1 + f(z))} - \frac{2\Omega}{z} \right) dz, \tag{3.37}$$

and z_1 is any point in D_1 and C_0 is an arbitrary constant of integration. In light of the fact that $P(z)$ in (3.36) contains the X_{s+1} coefficients of the power series expansion of $\phi_1(z)$, any solution of (3.36) must satisfy the consistency condition given by

$$X_k = \frac{\phi_1^k(0)}{k!}, \qquad k = 1, 2, \ldots, s, s+1, \tag{3.38}$$

We may derive Eq. (3.38) by first recalling that since $\phi_1(z)$ is analytic it has a Taylor series expansion in D_1 given by

$$\phi_1(z) = \sum_{k=0}^{\infty} Q_k z^k, \qquad Q_k = \frac{\phi_1^k(0)}{k!}. \tag{3.39}$$

Then, by substituting (3.39) into (3.30) and comparing coefficients of negative powers of z as we arrive at the following

$$\begin{aligned} \sum_{j=1}^{k} (j - 2\Omega) Q_j z^{j-1} \sum_{k=1}^{s} \frac{(b_k + i a_k)}{2} \frac{R^k}{z^k} \\ = \sum_{j=1}^{k} (j - 2\Omega) X_j z^{j-1} \sum_{k=1}^{s} \frac{(b_k + i a_k)}{2} \frac{R^k}{z^k}. \end{aligned} \tag{3.40}$$

Careful inspection of (3.40) reveals that when $\Omega \neq 1/2$ (3.38) is true for all s. However, for the case of $\Omega = 1/2$ we see that the first statement of (3.40) will be an identity, which provides no information on the form of the coefficient X_1 and implies (3.38) is not automatically satisfied for $k = 1$. Hence we must impose the additional requirement that

$$X_1 = \phi_1'(0). \tag{3.41}$$

In general, the solution for $\phi_1(z)$ in (3.36) is not holomorphic in the uncut domain D_1 due to the presence of multivalued logarithmic functions from under the integral and from isolated singular points stemming from the zeros of the interface function $(1 + f(z))$. To ensure the holomorphicity of $\phi_1(z)$ the domain must be cut appropriately such that $\phi_1(z)$ both single valued and bounded at all isolated singular points.

4. A specific class of inhomogeneous interface

In order to illustrate the method, let us consider an example where we select a specific form of the interface function $\delta(\theta)$ as follows

$$\delta(\theta) = \frac{\delta_0}{1 + b_s cos(s\theta)}, \qquad \delta_0 > 0, \quad -1 < b_s < 1. \tag{4.1}$$

Upon converting (4.1) into a complex variable form it is seen that there are singularities in the interface function originating from the roots of the following polynomial of degree $2s$

$$\frac{2}{b_s}\left(\frac{z}{R}\right)^s + \left(\frac{z}{R}\right)^{2s} + 1 = 0. \tag{4.2}$$

Of the $2s$ roots of (4.2), s will lie inside D_1 and the remaining s will lie in D_2. Let the s roots inside D_1 be denoted by

$$\rho_1, \rho_2, \rho_3, \ldots, \rho_s, \tag{4.3}$$

where $\rho_{(1,2,\ldots,s)} = \rho^*$ and ρ^* is real and given by

$$\rho^* = \begin{cases} \sqrt{\dfrac{1}{b_s^2} - 1} - \dfrac{1}{b_s} < 0, & b_s > 0, \\ -\sqrt{\dfrac{1}{b_s^2} - 1} - \dfrac{1}{b_s} > 0, & b_s < 0, \end{cases} \tag{4.4}$$

such that $-1 < \rho^* < 1$, and the remaining s roots in D_2 are given by $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \ldots, \frac{1}{\rho_s}$. As a consequence of the above interface definitions we make note of the following

$$\begin{aligned} -\frac{2}{b_s} &= \frac{1 + \rho^{*2}}{\rho^*}, \\ \frac{R\delta_0}{z(1 + f(z))} &= -\frac{\lambda(\frac{z}{R})^{s-1}}{(\frac{z}{R})^s - \rho^*} + \frac{\lambda(\frac{z}{R})^{s-1}}{(\frac{z}{R})^s - \frac{1}{\rho^*}}, \\ \lambda &= -\delta_0\left(\frac{1 + \rho^{*2}}{1 - \rho^{*2}}\right) < 0, \\ \frac{1}{1 + f(z)} &= \frac{\frac{2}{b_s}(\frac{z}{R})^s}{\left[(\frac{z}{R})^s - \rho^*\right]\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]}. \end{aligned} \tag{4.5}$$

Utilizing (4.5) we may express (3.36) as follows

$$\begin{aligned}\phi_1(z) = {} & \left(\frac{z}{R}\right)^{2\Omega}\left[\left(\frac{z}{R}\right)^s - \rho^*\right]^{\frac{\lambda\Omega\eta}{s}}\left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}} \\ & \times \int_{z_1}^{z}\left(\frac{t}{R}\right)^{-2\Omega+1}\left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\lambda\Omega\eta}{s}} \\ & \times \left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}\frac{P(t)}{\frac{t}{R}}dt, \quad z \in D_1, \\ \eta = {} & \frac{\Gamma\alpha_2 - \alpha_2 + 1}{\Gamma(1-\alpha_2)} > 0,\end{aligned} \tag{4.6}$$

where the integration path is taken along the edge of any branch cuts originating from each of the s branch points. In addition, to ensure boundedness of $\phi_1(z)$ at $z = R\rho_k$ we set $C_0 = 0$ and we require that

$$\int_{R\rho_1}^{R\rho_k}\left(\frac{t}{R}\right)^{-2\Omega+1}\left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\lambda\Omega\eta}{s}}\left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}\frac{P(t)}{\frac{t}{R}}dz = 0, \qquad k = 2, 3, \ldots, s, \tag{4.7}$$

in order to maintain boundedness of $\phi_1(z)$ at any of the potential isolated singular points $R\rho_k, k = 2, 3, \ldots, s$ in D_1. Additionally, by taking the difference

$$\phi_1(z^+) - \phi_1(z^-) = 0, \tag{4.8}$$

we may prove that (4.6) is continuous across any of the s branch cuts by noting that, due to the sign change of the exponents in and outside of the integral, any increments in the multivalued logarithmic terms that will arise from inside the integral will be nullified from which (4.8) is easily confirmed. The remaining irregular point to be considered is when $z = 0$. Closer inspection of (4.6) reveals that there are three cases to be considered as $z \to 0$.

4.1. Case one: $\Omega > \frac{1}{2}$

When $\Omega > \frac{1}{2}$ we see from (4.6) that $\phi_1(z) \to 0$ as $z \to 0$. However, in order to ensure the holomorphicity of $\phi_1(z)$ we must ensure that $\phi_1(z)$ is continuous across the branch cut formed from $z = R\rho^*$ along the real axis inside D_1. Closer inspection of (4.6) reveals the presence of an unintegrable singularity at $z = 0$. Hence we must define a new path of integration, L^*, to skirt around a neighborhood of $z = 0$ and set $z = z^*$,where z^* is any particular point on the branch cut from $z = 0$, to compensate for this change. In this way the continuity condition becomes

$$\int\limits_{L^*} \left(\frac{z^*}{t}\right)^{-2\Omega} \left[\frac{(\frac{z^*}{R})^s - \rho^*}{(\frac{t}{R})^s - \rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}} \left[\frac{(\frac{t}{R})^s - \frac{1}{\rho^*}}{(\frac{z^*}{R})^s - \frac{1}{\rho^*}}\right]^{\frac{\lambda\Omega\eta}{s}} P(t)dt = 0. \tag{4.9}$$

We may then solve for the X_{s+1} unknown coefficients using (3.18), (3.32), (4.10) and in cases of $s > 1$, (4.7).

4.2. Case two: $\Omega < \frac{1}{2}$

For this case we shall rewrite (4.6) in the form

$$\begin{aligned}\frac{\phi_1(z)}{\frac{z}{R}} = \left(\frac{z}{R}\right)^{2\Omega-1} \left[\left(\frac{z}{R}\right)^s - \rho^*\right]^{\frac{\lambda\Omega\eta}{s}} \left[\left(\frac{z}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{-\lambda\Omega\eta}{s}} \int\limits_{R\rho_1}^{z} \left(\frac{t}{R}\right)^{-2\Omega} \\ \times \left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\lambda\Omega\eta}{s}} \left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}} P(t)dt, \quad z \in D_1.\end{aligned} \tag{4.10}$$

Given that $X_0 = 0$, the LHS of (4.10) is analytic within D_1. As a consequence, $\frac{\phi_1(z)}{\frac{z}{R}}$ must be bounded at $z = 0$ and since $\Omega < \frac{1}{2}$ this implies that

$$\int\limits_{R\rho_1}^{0} \left(\frac{t}{R}\right)^{-2\Omega} \left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\lambda\Omega\eta}{s}} \left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}} P(t)dt = 0, \quad \Omega < \frac{1}{2}. \tag{4.11}$$

Note that in (4.11) there is a singularity in the integrand owing to the term $(\frac{t}{R})^{-2\Omega}$ for $\Omega < \frac{1}{2}$. Due to the fact that the path of integration in (4.11) lies on the real axis we may treat

$$K(\rho^*, t) = \left(\frac{t}{R}\right)^{-2\Omega} \left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\lambda\Omega\eta}{s}} \left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\Omega\eta}{s}}, \tag{4.12}$$

as a proper singular kernel function on such that (4.11) belongs to a class of Hölder continuous functions of ρ^* and is thusly integrable along such a domain [26]. We may then solve for the X_{s+1} unknown coefficients using (3.18), (3.32), (4.11) and in cases of $s > 1$, (4.7).

4.3. Case three: $\Omega = \frac{1}{2}$

In this case from (4.6) we see that $z = 0$ is not a singular point of $\phi_1(z)$ and hence $\phi_1(0) = 0$. We may then proceed to solve for the X_{s+1} unknown coefficients by recalling relation (3.32) and by evaluating (3.38) as

$$
RX_1 = [-\rho^*]^{\frac{\lambda\eta}{2s}} \left[-\frac{1}{\rho^*}\right]^{-\frac{\lambda\eta}{2s}} \int\limits_{R\rho_1}^{0} \left[\left(\frac{t}{R}\right)^s - \rho^*\right]^{\frac{-\lambda\eta}{2s}} \times \left[\left(\frac{t}{R}\right)^s - \frac{1}{\rho^*}\right]^{\frac{\lambda\eta}{2s}} \frac{P(t)}{\frac{t}{R}} dt = 0. \tag{4.13}
$$

The $s+1$ unknown coefficients are then determined from (3.18), (3.32), (4.13)and in cases of $s > 1$, (4.7).

5. Example

For ease of analysis in illustrating the method we shall assume that $\Omega = \frac{1}{2}$, $\lambda = -1$, $\eta = 2$ and we shall confine ourselves to the case $s = 1$. From these preliminaries we may evaluate (4.6) as

$$
\begin{aligned}
\phi_1(z) = \frac{z}{R}\left(\frac{z/R - 1/\rho^*}{z/R - \rho^*}\right)\Bigg[& I_1(z)\left(\omega\beta_1\left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right) - \frac{1}{2}\left(X_1 + \overline{X_1}\right)\right) \\
& + X_2 R I_2(z) + \delta_0(2/b_1)\frac{\omega\beta_1(\alpha_1 - \Gamma(\alpha_1 - 1))}{\Gamma} X_1 I_2(z) \\
& + \frac{iB\omega}{\Gamma\alpha_2 - \alpha_2 + 1} I_3(z)\Bigg], \quad z \in D_1,
\end{aligned} \tag{5.1}
$$

where

$$
\begin{aligned}
I_1(z) &= \int\limits_{R\rho^*}^{z} \frac{t/R}{(t/R - 1/\rho^*)^2} dt, \\
I_2(z) &= \int\limits_{R\rho^*}^{z} \frac{1}{(t/R - 1/\rho^*)^2} dt, \\
I_3(z) &= \int\limits_{R\rho^*}^{z} \frac{t/R(t/R - \rho^*)}{(t/R - 1/\rho^*)^2} dt, \quad z \in D_1.
\end{aligned} \tag{5.2}
$$

The unknown coefficients $X_1, \overline{X_1}, X_2, \overline{X_2}$ are then evaluated from (3.32), (4.13) as follows

$$
\begin{aligned}
\frac{1}{2}(X_1 + \overline{X_1}) - \omega\beta_1\left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right) + \frac{b_1}{2}R(X_2 + \overline{X_2}) \\
= \frac{\delta_0}{2}\left[\frac{\omega(\Gamma(\alpha_1 - 1) - \alpha_1)}{\Gamma}(X_1 + \overline{X_1}) + \frac{\beta_1(\Gamma - 1)\omega}{\Gamma}\left(\frac{1}{X_1} + \frac{1}{\overline{X_1}}\right)\right],
\end{aligned} \tag{5.3}
$$

$$\left[\frac{R\rho^{*2}}{\rho^{*2}-1}+Rln\left(\frac{1}{1-\rho^{*2}}\right)\right]\left(\omega\beta_1\left(\frac{1}{X_1}+\frac{1}{\overline{X_1}}\right)-\frac{1}{2}(X_1+\overline{X_1})\right)$$
$$+X_2R\left(\frac{R\rho^{*3}}{\rho^{*2}-1}\right)+\frac{iB\omega}{\Gamma\alpha_2-\alpha_2+1}\left[\frac{R(1-\rho^{*2})}{\rho^{*2}}ln\left(\frac{1}{1-\rho^{*2}}\right)+\frac{R\rho^{*2}}{2}-R\right]=0,$$
$$-1<\rho_1<1. \tag{5.4}$$

Equations (5.3) and (5.4) represent the exact solution to the inhomogeneous sliding interface. Noting that $\frac{1}{2}=1-\Omega$ it can be shown that when $\rho^*\to 0$, (5.3) and (5.4) reduce to the corresponding homogeneous imperfect sliding interface given in [18].

6. Results

Having verified the formulation we may now proceed to compare the homogeneous imperfect interface to the inhomogeneous one. For the purpose of this example we compare the inhomogeneous interface of the form

$$\frac{n(\theta)R}{\mu_2}=\frac{\delta_0}{1+b_1cos(\theta)},\quad \delta_0=\frac{1-\rho_1^2}{1+\rho_1^2},\quad -1<b_1<1, \tag{6.1}$$

to the homogeneous imperfect interface given by

$$\frac{nR}{\mu_2}=\delta_0. \tag{6.2}$$

Close inspection of the expression given by (3.4) reveals that in the cases of either a uniaxial or biaxial remote loading, B is in fact purely real. Hence we may prove from (5.3), (5.4) that X_1, X_2 must both be purely imaginary and we may solve for them using (3.32), (5.3), (5.4). Computing the average mean stress on the boundary defined by

$$(P_{11}+P_{22})_{2,Avg}=\frac{1}{C_{\partial D_1}}\int\limits_{\partial D_1}4\mu_2Im\left[\Gamma(1-\alpha_1)X_1+\frac{\Gamma\beta_1}{X_1}\right]dS, \tag{6.3}$$

the ratio of the inhomogeneous to homogeneous interfaces will be one to one since X_1 is identical in both interface conditions. In an attempt to explore further the results, we compute the ratio of the mean stress at $z=R$ given by the relations

$$(P_{11}+P_{22})_{2,Homogeneous}=4\mu_2Im\left[\Gamma(1-\alpha_1)(X_1)+\frac{\Gamma\beta_1}{X_1}\right], \tag{6.4}$$

$$(P_{11}+P_{22})_{2,Inhomogeneous}=4\mu_2Im\left[\Gamma(1-\alpha_1)(X_1+2X_2z)+\frac{\Gamma\beta_1}{X_1+2X_2z}\right], \tag{6.5}$$

from which the following trend is observed as per Fig. 2.

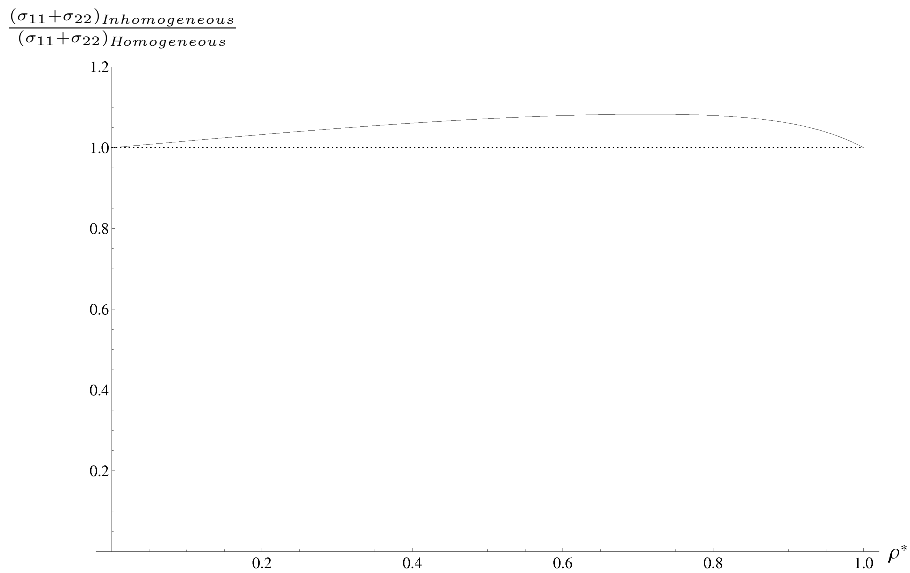

FIG. 2. Ratio of inhomogeneous to homogeneous mean stress at $z = R$ for the remote loading $P_{11}^{\infty} = 0$, $P_{22}^{\infty} = 1$, $P_{12}^{\infty} = 0$; — $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1 = 0.5$, $\Gamma = 1$; $\cdots\cdots$ reference value of 1.0.

From Fig. 2 we conclude that the inhomogeneous interface parameter ρ^* does have an influence on the mean stress at the point $z = R$ on the boundary ∂D_1, which at its peak reaches an error of 13 percent. In contrast, RU [25] observed a relative error in the mean stress of up to 80 percent in the case of an inhomogeneous sliding interface in linear elasticity. While the present work does not reach relative errors of a similar magnitude we cannot simply ignore the effects of the circumferential variation of the interface in the finite deformation setting. Therefore, replacing the circumferential variation of the interface by its homogenous counterpart will contribute to a modest relative error.

7. Conclusions

A precise mathematical description of material interface behaviour has been a central focus of the mechanics of composite materials. For example, in cord-reinforced rubber composites interfacial properties are of great importance for improving and enhancing the product quality. While there are many interface models reported in the literature the majority of them are not able to successfully capture the influence of interface damage such as damage arising from imperfect adhesion, mircocracks and voids. Interface imperfections in composite materials are almost always inhomogeneous along the material interface. Hence a successful interface model, that takes into account interfacial properties, will have a tremendous impact on the design, analysis and overall life cycle performance of composite materials. In this work, a general solution for the case of an inhomogeneous imperfect sliding interface characterized by the imperfect interface parameters $m(\theta) \to \infty, n(\theta) = finite$ is presented. The formulation has been validated analytically and subsequent results were presented for the mean stress at a specific point along the inclusion matrix boundary curve under remote loads. From these results it was observed that there was a maximum error of 13 percent when comparing the mean stress at a point on the interface between the inhomogeneous and homogeneous models. While the error is modest, it does follow the behaviour demonstrated in the linear theory for the sliding interface. Therefore, the error associated with the inhomogeneous sliding imperfect interface in finite elasticity cannot be simply dismissed from the analysis. Thus, replacing the circumferential variation of the sliding interface by its homogenous counterpart will contribute to a relative error in the overall prediction of material behaviour.

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