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Horizontally polarized shear waves in stratified anisotropic (monoclinic) media

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The dispersion relations are derived for the horizontally polarized shear (SH) waves in stratified anisotropic plates with arbitrary elastic anisotropy. Analytical expressions for the vectorial group and ray velocities of SH waves propagating in anisotropic layers with monoclinic symmetry are obtained. Closed form relations between velocities and specific kinetic and strain energy for SH waves are derived and analyzed.

Key words: SH-wave, surface wave, anisotropy, dispersion, specific energy.

1. Introduction

Horizontally polarized shear surface waves (SH-waves) propagating in multilayered plates resemble Love waves [1] in polarization, but differ in absence of a contacting half-space (substrate), and, hence excluding necessity of Sommerfeld's emission condition

$$\mathbf{u}(\mathbf{x},t) = O(|x'|^{-1}), \qquad |x'| \to \infty, \tag{1.1}$$

where $\mathbf{u}$ is the displacement field in the substrate; $x' \equiv \boldsymbol{\nu} \cdot \mathbf{x}$ is the coordinate along depth of the substrate, and $\boldsymbol{\nu}$ is the unit normal to the plane boundary of the substrate.

As it is shown later, the absence of the condition (1.1) results in different behavior of SH-waves comparing to Love waves. Love's inequality [1, 2] for a genuine Love wave propagating in an isotropic traction-free layer contacting with the isotropic substrate

$$(c^T_{\mathbf{nm}})_{\text{layer}} > (c^T_{\mathbf{nm}})_{\text{substrate}} \tag{1.2}$$

is necessary and sufficient for existence of the Love wave. In (1.2) $c^T_{\mathbf{nm}}$ denotes the speed of the corresponding shear bulk wave propagating in $\mathbf{n}$, and polarized in $\mathbf{m}$ direction. However, SH waves in laminated plates with isotropic homogeneous layers propagate at the violating inequality (1.2). It should also be noted that if the inequality (1.2) does not hold, then no Love wave can propagate in a layer and a halfspace.

Concerning energy of surface acoustic waves (SAW), the first works [3–7] on derivation of expressions for the kinetic and elastic specific energy revealed that in contrast to bulk waves, for which the kinetic and elastic specific energies coincide, in case of SAW these energies differ. Other theoretical studies of SAW energy in elastic and piezoelectric media are contained in recent works [8–11]. The analysis presented below shows that the difference between these specific energies is associated with the transverse non-uniform distribution of displacement magnitudes in SAW.

The principle method used for constructing analytical solutions for velocities and specific energy of SH waves, is based on a combination of the three dimensional complex formalism [12] and the modified transfer matrix (MTM) method [13, 14]. The latter being rather fast and numerically stable allows us to construct analytical solutions for inhomogeneous plates containing several anisotropic layers.

The main results presented below concern along with specific energy analyses, relations between phase, group, and ray velocities for SH waves propagating in anisotropic layers with monoclinic symmetry. The developed method is also suitable for analyzing dispersion of these waves propagating in layered composites having at least one common plane of elastic symmetry. Such a common plane of elastic symmetry is actually necessary for the existence of Love and SH waves.

2. Basic notations

All the regarded layers of a plate are assumed homogeneous, anisotropic and linearly hyperelastic. Equations of motion for a homogeneous anisotropic elastic medium can be written in the form:

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \operatorname{div}_x \mathbf{C}\cdot\cdot\nabla_x\mathbf{u} - \rho\ddot{\mathbf{u}} = 0, \tag{2.1}$$

where ρ is the material density, and $\mathbf{C}$ is the elasticity tensor assumed to be *positive definite*

$$\underset{\mathbf{A}\in sym(R^3\otimes R^3),\,\mathbf{A}\neq 0}{\forall \mathbf{A}} (\mathbf{A}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{A}) \equiv \sum_{i,j,m,n} A_{ij}C^{ijmn}A_{mn} > 0. \tag{2.2}$$

Remark 2.1. a) The other assumption concerns symmetry of the elasticity tensor. It is assumed that all the regarded materials possess planes of elastic symmetry coinciding with the sagittal plane $\mathbf{m} \cdot \mathbf{x} = 0$, where the vector $\mathbf{m}$ is the polarization vector of the SH-wave. This is achieved by the elasticity tensor belonging to the *monoclinic* system, and the latter is equivalent to vanishing all of the decomposable components of the tensor $\mathbf{C}$ having an odd number of entries of the vector $\mathbf{m}$ in the orthogonal basis in R^3 generated by the vector $\mathbf{m}$ and any two orthogonal vectors belonging to the sagittal plane.

b) It is shown later that assuming monoclinic symmetry provides a sufficient condition for the surface tractions acting on any plane $\boldsymbol{\nu}\cdot\mathbf{x} = \text{const}$ to be collinear with the vector $\mathbf{m}$.

Following [15, 16], a horizontally polarized harmonic shear wave in a layer can be represented in a form

$$\mathbf{u}(\mathbf{x}) = \mathbf{m}\, f(irx')\, e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}, \tag{2.3}$$

where coordinate $x' = \boldsymbol{\nu}\cdot\mathbf{x}$ is as defined in (1.1); f is the unknown scalar complex-valued function; the exponential multiplier $ir(\mathbf{n}\cdot\boldsymbol{\nu} - ct)$ in (2.3) corresponds to propagation of the plane wave front along the direction $\mathbf{n}$ with the phase speed c; r is the wave number,

$$r = \frac{\omega}{c}. \tag{2.4}$$

Remark 2.2. A displacement field defined by (2.3) is generally complex. In reality, either real or imaginary part of the right-hand side of (2.3) represents the physical displacement field. However, retaining complex expressions for the displacement field, allows us to describe situations with the phase shift in a more convenient manner.

Substituting representation (2.3) into Eq. (2.1) and taking into account Remark 2.1.a, yields the following differential equation:

$$\left(\begin{array}{r} (\mathbf{m}\otimes\boldsymbol{\nu}\cdot\cdot\,\mathbf{C}\cdot\cdot\,\boldsymbol{\nu}\otimes\mathbf{m})\, f''_{x'} + 2(\mathbf{m}\cdot sym(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n})\cdot\mathbf{m})\, f'_{x'} \\ + (\mathbf{m}\otimes\mathbf{n}\cdot\cdot\,\mathbf{C}\cdot\cdot\,\mathbf{n}\otimes\mathbf{m} - \rho c^2) f \end{array} \right) = 0. \tag{2.5}$$

The characteristic equation for the differential Eq. (2.5), known also as the Christoffel equation, has the form:

$$\begin{aligned} (\mathbf{m}\otimes\boldsymbol{\nu}\cdot\cdot\,\mathbf{C}\cdot\cdot\,\boldsymbol{\nu}\otimes\mathbf{m})\, \gamma^2 + 2(\mathbf{m}\cdot sym(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n})\cdot\mathbf{m})\, \gamma& \\ + (\mathbf{m}\otimes\mathbf{n}\cdot\cdot\,\mathbf{C}\cdot\cdot\,\mathbf{n}\otimes\mathbf{m} - \rho c^2) &= 0. \end{aligned} \tag{2.6}$$

The left-hand side of Eq. (2.6) is a polynomial of degree 2 with respect to the Christoffel parameter γ. Thus, for a layer with monoclinic symmetry only two partial waves form SH wave.

The following lemma flows out from solving the Cauchy problem for Eq. (2.5):

Lemma 2.1. *A necessary and sufficient condition for the real-analytic solution of Eq.* (2.5) *to be non-trivial, is a simultaneous non-vanishing f and its first derivative at some x'.*

Remark 2.3. For the orthotropic medium and SH wave propagating in the direction of principle elasticity, Eq. (2.6) can be simplified

$$(\mathbf{m} \otimes \boldsymbol{\nu} \cdot \cdot \mathbf{C} \cdot \cdot \boldsymbol{\nu} \otimes \mathbf{m})\, \gamma^2 + (\mathbf{m} \otimes \mathbf{n} \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{n} \otimes \mathbf{m} - \rho c^2) = 0. \tag{2.7}$$

Its immediate solution becomes

$$\gamma_{1,2} = \pm \sqrt{\frac{\rho c^2 - \mathbf{m} \otimes \mathbf{n} \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{n} \otimes \mathbf{m}}{\mathbf{m} \otimes \boldsymbol{\nu} \cdot \cdot \mathbf{C} \cdot \cdot \boldsymbol{\nu} \otimes \mathbf{m}}}. \tag{2.8}$$

For the considered case, the general solution of Eq. (2.5) can be represented in the form:

$$f(irx') = C_1 \sin(r\gamma x') + C_2 \cos(r\gamma x'), \tag{2.9}$$

where γ is generally the complex root with a positive sign in (2.8).

3. Energy of SH-waves

3.1. Specific kinetic and elastic (potential) energy

Herein, we derive expressions for specific kinetic and elastic (potential) energies of the SH-waves. Taking into account the representation (2.3) and assuming $|\mathbf{m}| = 1$, the specific kinetic energy can be defined by

$$E_{\text{kin}} \equiv \frac{1}{2} \rho\, \dot{\mathbf{u}} \cdot \bar{\dot{\mathbf{u}}} = \frac{1}{2} \rho \omega^2 |\mathbf{m}|^2 |f|^2, \tag{3.1}$$

where the following relation between the phase speed and frequency is used

$$\omega = rc. \tag{3.2}$$

Equations (2.5), (3.1), and (3.2) yield the following representation for the specific kinetic energy

$$E_{\text{kin}} \equiv \frac{1}{2} r^2 \overline{f} \left[\begin{aligned} (\mathbf{m} \otimes \boldsymbol{\nu} \cdot \cdot \mathbf{C} \cdot \cdot \boldsymbol{\nu} \otimes \mathbf{m}) f'' + 2(\mathbf{m} \cdot sym(\boldsymbol{\nu} \cdot \mathbf{C} \cdot \mathbf{n}) \cdot \mathbf{m}) f' \\ + (\mathbf{m} \otimes \mathbf{n} \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{n} \otimes \mathbf{m}) f \end{aligned} \right]. \tag{3.3}$$

Another useful expression flows out from (3.1) and (3.2):

$$\omega^2 = \frac{2E_{\text{kin}}}{\rho |f|^2}. \tag{3.4}$$

Now, the specific elastic energy can be defined by

$$E_{\text{elast}} \equiv \frac{1}{2}\nabla\mathbf{u}\cdot\cdot\mathbf{C}\cdot\cdot\nabla\bar{\mathbf{u}} = \frac{1}{2}r^2\mathbf{m}\otimes(f'\boldsymbol{\nu}+f\mathbf{n})\cdot\cdot\mathbf{C}\cdot\cdot(\overline{f\mathbf{n}+f'\boldsymbol{\nu}})\otimes\bar{\mathbf{m}}. \tag{3.5}$$

REMARK 3.1. a) In view of Remark 2.2, expressions (3.1) and (3.5) coincide with the corresponding expressions for kinetic and elastic specific energies, obtained without using complex displacement fields.

b) Analysis of expressions (3.1)–(3.5) reveals that for the considered SH waves $E_{\text{kin}} \neq E_{\text{elast}}$, due to the presence of the generally non-constant function f. At the same time, for bulk waves $f = \text{const}$, and hence from (2.1) and (2.3) we arrive at $E_{\text{kin}} = E_{\text{elast}}$; see also [2, 7] for discussions.

PROPOSITION 3.1. a) *At vanishing circular frequency ω both specific kinetic and elastic energies vanish.*

b) *The specific kinetic energy vanishes at any ω, on longitudinal planes $x' \equiv \boldsymbol{\nu}\cdot\mathbf{x} = \text{const}$, if function f vanishes at the corresponding x'.*

c) *The specific elastic energy does not vanish at any non-vanishing frequency ω.*

Proofs of conditions a) and b) are obvious. Proof of the condition c) follows from the positive definite condition for the elasticity tensor, Lemma 2.1, expression (3.5) and orthogonality of vectors $\boldsymbol{\nu}$ and $\mathbf{n}$.

3.2. Group velocity

Herein, the vector-valued group velocity $\mathbf{v}_{\text{group}}$ is defined by [2]:

$$\mathbf{v}_{\text{group}} = \nabla_{(r\mathbf{n})}\omega, \tag{3.6}$$

where $\nabla_{(r\mathbf{n})}$ denotes the gradient with respect to the independent spatial variable $(r\mathbf{n})$. For the subsequent analysis the scalar group speed c_{group} is also needed:

$$c_{\text{group}} \equiv |\mathbf{v}_{\text{group}}| = \sqrt{\nabla_{(r\mathbf{n})}\omega\cdot\overline{\nabla_{(r\mathbf{n})}\omega}}. \tag{3.7}$$

Now, combining (3.3), (3.4), and (3.7) yields

$$c_{\text{group}} = \frac{\sqrt{(f'\boldsymbol{\nu}+f\mathbf{n})\cdot(\mathbf{m}\cdot\mathbf{C}\cdot\mathbf{m})^2\cdot(\overline{f\mathbf{n}+f'\boldsymbol{\nu}})}}{c\rho|f|}, \tag{3.8}$$

where c stands for the phase speed.

PROPOSITION 3.2. a) *At any physically admissible properties of a medium and any SH-wave propagating with the finite phase speed $c \neq 0$, the corresponding group speed c_{group} is delimited from zero.*

b) *If $f \to 0$ at $x' \to x'_0$, where x'_0 takes some finite value, then $c_{\text{group}} \to \infty$.*

Proof a) flows out from observation that the radicand in (3.8) is strictly positive due to (2.2) and Lemma 2.1. Proof b) is obvious.

REMARK 3.2. According to the definition (3.7) the group velocity c_{group} cannot be negative, since according to (3.7) c_{group} is defined as the length of the (possibly complex) vector (3.6). However, there are other definitions for the group velocity, that allows negative values for c_{group}; see [17–19], where the following definition is adopted

$$c_{\text{group}} = \frac{\partial \omega}{\partial r}. \tag{3.9}$$

A more detailed analysis [7] of expressions (3.6) and (3.9) reveals that the latter expression yields projection of the vector valued velocity (3.6) onto the wave normal $\mathbf{n}$. This provides the explanation of possible appearing negative values of the group speed.

3.3. Ray speed

The vector-valued ray speed can be defined by (see [7]):

$$\mathbf{v}_{\text{ray}} = \frac{\mathbf{J}_{\text{elast}}}{E_{\text{kin}} + E_{\text{elast}}}, \tag{3.10}$$

where $\mathbf{J}_{\text{elast}}$ is the flux of elastic energy:

$$\mathbf{J}_{\text{elast}} \equiv \dot{\mathbf{u}} \cdot \mathbf{C} \cdot \cdot \nabla \mathbf{u}. \tag{3.11}$$

The corresponding scalar ray speed is:

$$c_{\text{ray}} \equiv |\mathbf{v}_{\text{ray}}| = \frac{\sqrt{\mathbf{J}_{\text{elast}} \cdot \overline{\mathbf{J}_{\text{elast}}}}}{E_{\text{kin}} + E_{\text{elast}}}. \tag{3.12}$$

Substituting (2.3) into (3.12) and exploiting (3.3), (3.5), yields:

$$c_{\text{ray}} = \frac{2c|f|\sqrt{(f'\boldsymbol{\nu} + f\mathbf{n}) \cdot (\mathbf{m} \cdot \mathbf{C} \cdot \mathbf{m})^2 \cdot (\overline{f\mathbf{n} + f'\boldsymbol{\nu}})}}{\rho c^2 |f|^2 + (f'\boldsymbol{\nu} + f\mathbf{n}) \cdot (\mathbf{m} \cdot \mathbf{C} \cdot \mathbf{m}) \cdot (\overline{f\mathbf{n} + f'\boldsymbol{\nu}})}. \tag{3.13}$$

PROPOSITION 3.3. a) *At any physically admissible properties of a medium and any SH-wave propagating with the finite phase speed $c \neq 0$, the corresponding ray speed c_{ray} is delimited from zero.*

b) *If $f \to 0$ at $x' \to x'_0$, where x'_0 is finite, then $c_{\text{ray}} \to \infty$.*

c) *A necessary and sufficient condition for $c_{\text{group}} = c_{\text{ray}}$, is as follows:*

$$E_{\text{kin}} = E_{\text{elast}}. \tag{3.14}$$

Proofs a) and b) are analogous to the proof of Proposition 3.2. Proof c) follows directly from (3.8), (3.13), with account of (3.1), (3.5).

4. Single-layered orthotropic plate

Hence it is assumed that vectors $\boldsymbol{\nu}, \mathbf{m}$, and $\mathbf{n}$ coincide with the axes of elastic symmetry of an orthotropic medium.

REMARK 4.1. It can be shown (see [13, 14], where similar arguments are applied to analysis of Love waves) that regardless of boundary conditions and at imaginary roots of Eq. (2.7), no SH-wave can propagate in directions of elastic symmetry of an orthotropic single-layered plate. Thus, the following inequality

$$c > \sqrt{\frac{\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m}}{\rho}}, \tag{4.1}$$

naturally arising from (2.8), delivers a necessary condition for existing surface SH-wave. Thus, for the regarded plate all surface SH-waves are necessary supersonic, since the radicand in the right-hand side of (4.1) defines speed of the corresponding shear bulk wave $c^T_{\mathbf{nm}}$. In this section we assume that the condition (4.1) holds.

4.1. Traction-free plate

Herein we consider a single-layered plate with the traction-free boundary conditions:

$$\begin{cases} \mathbf{t}_{\boldsymbol{\nu}}(h/2) = 0, \\ \mathbf{t}_{\boldsymbol{\nu}}(-h/2) = 0, \end{cases} \tag{4.2}$$

where h is the thickness of the plate (we choose origin of coordinates at the median plane).

For such a plate, finding function f from (2.7), (4.2), yields:

$$f(irx') = \begin{cases} \cos(r\gamma x') & \text{at } r = \dfrac{2n\pi}{\gamma h}, \\ \sin(r\gamma x') & \text{at } r = \dfrac{(2n-1)\pi}{\gamma h}, \end{cases} \qquad n = 1, 2, \ldots, \tag{4.3}$$

where γ is defined by (2.8).

PROPOSITION 4.1. a) *On planes* $x' = \text{const}$, *where*

$$x' = \begin{cases} \dfrac{\frac{1}{2}+k}{2n}h & \textit{at } r = \dfrac{2n\pi}{\gamma h},\ -n \le k < n, \\ \dfrac{k}{2n-1}h & \textit{at } r = \frac{(2n-1)\pi}{\gamma h},\ -n \le k < n, \end{cases} \qquad n, k \in Z, \tag{4.4}$$

the displacement field and specific kinetic energy vanish. That is equivalent to existence of the internal immovable layers under propagating SH-wave on a traction-free plate.

b) *At any finite phase speed satisfying the inequality* (4.1), *there are no waves propagating at vanishing frequency* (*both phase speed and frequency are delimited from zero*).

Proof a) follows from considering zeroes of the function, defined by (4.3). Proof b) follows from analyzing expressions (4.3), (3.2). It reveals that no non-trivial solutions exist at $\omega = 0$.

4.2. Clamped plate

For a single-layered plate with clamped outer surfaces, boundary conditions are

$$\begin{cases} \mathbf{u}(h/2) = 0, \\ \mathbf{u}(-h/2) = 0. \end{cases} \tag{4.5}$$

Finding the function f from Eq. (2.5) and satisfying boundary conditions (4.5), yields:

$$f(irx') = \begin{cases} \sin(r\gamma x') & \text{at } r = \dfrac{2n\pi}{\gamma h}, \\ \cos(r\gamma x') & \text{at } r = \dfrac{(2n-1)\pi}{\gamma h}, \end{cases} \qquad n = 1, 2, \ldots. \tag{4.6}$$

Similarly to Proposition 4.1, we have

Proposition 4.2. a) *On planes* $x' = \text{const}$, *where*

$$x' = \begin{cases} \dfrac{k}{2n} h & \text{at } r = \dfrac{2n\pi}{\gamma h}, \ -n \le k \le n, \\ \dfrac{\frac{1}{2} + k}{2n-1} h & \text{at } r = \dfrac{(2n-1)\pi}{\gamma h}, \ -n-1 \le k \le n-1, \end{cases} \qquad n, k \in Z, \tag{4.7}$$

both the displacement field and specific kinetic energy vanish. That is equivalent to existence of the internal immovable layers under propagating surface SH-wave on a clamped plate.

b) *At any finite phase speed satisfying the inequality* (4.1), *there are no waves propagating at vanishing frequency* (*both phase speed and frequency are delimited from zero*).

4.3. Plate with mixed boundary conditions

Herein we consider a plate with traction-free upper and clamped bottom surface:

$$\begin{cases} \mathbf{t}_{\boldsymbol{\nu}}(h/2) = 0, \\ \mathbf{u}(-h/2) = 0. \end{cases} \tag{4.8}$$

Direct analysis reveals that the function f satisfying homogeneous boundary conditions (4.8) takes the form:

$$f(irx') = \begin{cases} \sin\left(r\gamma x' - \dfrac{\pi}{4}\right) & \text{at } r = \dfrac{2(n-\frac{1}{4})\pi}{\gamma h},\ n = 1, 2, \ldots, \\ \sin\left(r\gamma x' + \frac{\pi}{4}\right) & \text{at } r = \dfrac{2(n+\frac{1}{4})\pi}{\gamma h},\ n = 0, 1, \ldots. \end{cases} \tag{4.9}$$

PROPOSITION 4.3. a) *On planes* $x' = \text{const}$*, where*

$$x' = \begin{cases} \dfrac{(k+\frac{1}{4})}{2(n-\frac{1}{4})}h & \textit{at } r = \dfrac{2(n-\frac{1}{4})\pi}{\gamma h},\ -n \le k < n, \\ \dfrac{(k-\frac{1}{4})}{2(n+\frac{1}{4})}h & \textit{at } r = \dfrac{2(n+\frac{1}{4})\pi}{\gamma h},\ -n \le k \le n, \end{cases} \qquad n, k \in Z, \tag{4.10}$$

both the displacement field and specific kinetic energy vanish. That is equivalent to existence of immovable layers under propagating surface SH-wave on a clamped plate.

b) *At any finite phase speed satisfying the inequality* (4.1)*, there are no waves propagating at vanishing frequency* (*both phase speed and frequency are delimited from zero*).

5. Concluding remarks

Considering specific energy, it was proved that kinetic and elastic energies of SH waves generally differ; they coincide if only if the displacement distribution is uniform at the cross section of a plate.

For SH waves the explicit expressions for the group and ray speeds were derived; it was shown that both group and ray speeds defined by Eqs. (3.7) and (3.12) are positive and delimited from zero.

For monoclinic and homogeneous plates and all the considered boundary conditions:

(i) the admissible speed interval is transonic:

$$c \in (c^T_{\mathbf{mn}}; \infty); \tag{5.1}$$

(ii) at any phase speed satisfying (5.1) there are immovable longitudinal layers, and (iii) there are no limiting SH-waves corresponding to the vanishing frequency.

It should also be noted that energy considerations associated with propagation of the surface acoustic waves not restricted to SH and Love waves, were analyzed in [20–22], and propagation of the nonlinear SH waves was analyzed in recent papers [23–25].

In a case of horizontally polarized waves in piezoelectric homogeneous (monoclinic) medium the governing equation of motion does not reduce to a single scalar equation; actually there will be a system of two coupled equations. Hence, for stratified piezoelectric monoclinic media, the transfer matrix dispersion solution will involve a composition of transfer matrices, similarly to the matrices appearing in Lamb wave analyses [26].

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