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SEVENTY YEARS OF THE ARCHIVES OF MECHANICS

Generalized theory of thermoelastic diffusion with double porosity

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The present paper focuses on the derivation of the constitutive relations and field equations for anisotropic thermoelastic medium with mass diffusion and double porosity. The variational principle, uniqueness and reciprocity theorems are also derived.

Key words: thermoelastic diffusion, pores, fissures.

1. Introduction

The classical theory of thermoelasticity assumes that when an elastic solid is subjected to a thermal disturbance, the effect is felt at a location far from the source, instantaneously. It means that the thermal waves propagate with infinite speed which is impossible from the physical point of view. In [1], Lord and Shulman introduced a new theory of thermoelasticity, known as generalized thermoelasticity, by incorporating a flux-rate term into Fourier's law of heat conduction which gives a hyperbolic heat transport equation admitting finite speed for thermal signals. This theory was further extended by Dhaliwal and Sherief [2] to include the anisotropic case.

Diffusion can be defined as the transfer of mass of a substance from the high concentration regions to low concentration regions. There is now a great deal of interest in the study of this phenomenon due to its widespread applications in the domains of geophysics, microelectronics, environmental mechanics, biomedical engineering, and so on. Nowacki [3–6] proposed the classical diffusion-thermoelasticity to describe the coupled mechanical behaviour among temperature, concentration, and strain fields in elastic solids. Sherief *et al.* [7], Aouadi [8] and Kansal and Kumar [9] established the different theories of generalized diffusion-thermoelasticity to eliminate the shortcomings of classical diffusion-thermoelasticity.

The study of the interaction of elastic waves with fluid-loaded solids has been recognized as a viable means for non-destructive evaluation of solid structures. The theory of linear elastic materials with voids is one of the most important generalizations of the classical theory of elasticity. This theory has a practical use for investigating various types of geological and biological materials for which the elastic theory is inadequate. This theory is concerned with elastic materials consisting of a distribution of small pores (voids), in which the voids volume is included among the kinematics variables and in the limiting case of volume tending to zero, the theory reduces to the classical theory of elasticity. Goodman and Cowin [10] established a continuum theory for granular materials, whose matrix material (or skeletal) is elastic and interstices are voids. They formulated this theory from the formal arguments of continuum mechanics and introduced the concept of distributed body, which represents a continuum model for granular materials (sand, grain, powder, etc.) as well as porous materials (rock, soil, sponge, pressed powder, cork etc.). The basic concept underlying this theory is that the bulk density of the material is written as the product of two fields, the density field of the matrix material and the volume fraction field (the ratio of the volume occupied by grains to the bulk volume at a point of the material). This representation was employed by Nunziato and Cowin [11] to develop a nonlinear theory of elastic material with voids. Cowin and Nunziato [12] presented a linear theory of elastic material with voids for the mathematical study of the mechanical behaviour of porous solids. They considered several applications of the linear theory by investigating the response of the materials to homogeneous deformations, pure bending of beams and small amplitudes of acoustic waves. Iesan [13] proved the uniqueness, reciprocity and variational theorems for the basic governing equations of elastic materials with voids and also studied the propagation of acceleration waves in such materials. Iesan [14] extended the linear theory of elastic materials with voids to include the thermal effect. Aouadi [15] developed a theory of thermoelastic diffusion materials with voids and derived the uniqueness, reciprocity, continuous dependence and existence theorems.

Biot [16] presented the first model for single porosity deformable solid by using the classical Darcy's law. Barenblatt *et al.* [17] and Warren and Root [18] extended this law to describe fluid flow through undeformable double porosity materials. The double porosity model represents a double porous structure, one is macro porosity which is connected to pores and other is micro porosity which is connected to fissures. Wilson and Aifantis [19] developed the theory for deformable materials with double porosity. This theory unifies the earlier proposed models of Barenblatt *et al.* [17] for porous media with double porosity and Biot [16] for porous media with single porosity. Bai *et al.* [20], Moutsopoulos *et al.* [21] and Straughan [22] presented various mathematical models of elasticity and thermoelasticity with multiple porosity by using the extended Darcy's law. In these models, the dependent variables are the displacement vector, the pressures in the pore networks and the variation of temperature. Iesan and Quintanilla [23] derived a non-linear theory of thermoelastic solids with double porosity structure based upon Nunziato–Cowin theory of materials with voids. This theory was not based upon Darcy's law. Various authors [24–34] discussed different types of problems on elastic solids, viscoelastic solids and thermoelastic solids with double porosity.

In the present article, the constitutive relations, field equations, variational principle, uniqueness and reciprocity theorems for anisotropic generalized thermoelasticity with mass diffusion and double porosity based upon the Lord–Shulman model [1] are derived.

2. Basic equations

The law of conservation of energy for an arbitrary material volume V bounded by a surface A at time t can be written as

$$\int_V \rho[\dot{u}_i\ddot{u}_i + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}]dV = \int_V \rho[F_i\dot{u}_i + g\dot{\nu}_1 + l\dot{\nu}_2]dV + \int_A [f_i\dot{u}_i + \Omega_i n_i\dot{\nu}_1 + \chi_i n_i\dot{\nu}_2 - q_i n_i]dA, \tag{2.1}$$

where U is the internal energy per unit mass, ρ is the density, q_i are the components of heat flux vector $\boldsymbol{q}$, F_i are the components of the external forces per unit mass, u_i are the components of the displacement vector $\boldsymbol{u}$, f_i are the components of the surface traction vector $\boldsymbol{f}$ occurring on the surface A, ν_1 and ν_2 are the volume fraction fields corresponding to pores and fissures respectively, k_1 and k_2 are coefficients of equilibrated inertia, g and l are, respectively, extrinsic equilibrated body forces per unit mass associated to macro pores and fissures, Ω_i, χ_i are the components of equilibrated stress vectors corresponding to ν_1, ν_2 measured per unit area of the surface A respectively, n_i are the components of outward unit normal vector $\boldsymbol{n}$ to the surface A.

The components f_i are connected to the stress vector by the relation

$$f_i = \sigma_{ji}n_j, \tag{2.2}$$

where $\sigma_{ji}(=\sigma_{ij})$ are the components of the stress tensor.

Using Eq. (2.2) in Eq. (2.1) and applying the divergence theorem, we obtain

$$\text{(2.3)}\qquad \int_V \rho[\dot{u}_i\ddot{u}_i + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}]dV = \int_V \rho[F_i\dot{u}_i + g\dot{\nu}_1 + l\dot{\nu}_2]dV + \int_V [\sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \Omega_{i,i}\dot{\nu}_1 + \Omega_i\dot{\nu}_{1,i} + \chi_{i,i}\dot{\nu}_2 + \chi_i\dot{\nu}_{2,i} - q_{i,i}]dV.$$

Equation (2.3) is valid for every part of the body. Therefore, we obtain the local form of conservation of energy

$$\text{(2.4)}\qquad \rho[\dot{u}_i\ddot{u}_i + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}] = \rho[F_i\dot{u}_i + g\dot{\nu}_1 + l\dot{\nu}_2] + \sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \Omega_{i,i}\dot{\nu}_1 + \Omega_i\dot{\nu}_{1,i} + \chi_{i,i}\dot{\nu}_2 + \chi_i\dot{\nu}_{2,i} - q_{i,i}.$$

Let us consider a second motion which differs from the given motion only by a constant superposed rigid body translational velocity. Let us assume that k_1, k_2, U, g, l, ρ, Ω_i, χ_i, q_i, F_i, σ_{ji} are not changed by such superposed rigid body velocity. The above equation is also true when $\dot{u}_i$ is replaced by $\dot{u}_i + s_i$, where s_i are arbitrary constants, all other terms being unchanged. Therefore, from Eq. (2.4), we have

$$\text{(2.5)}\qquad \rho[(\dot{u}_i + s_i)\ddot{u}_i + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}] = \rho[F_i(\dot{u}_i + s_i) + g\dot{\nu}_1 + l\dot{\nu}_2] + \sigma_{ji,j}(\dot{u}_i + s_i) + \sigma_{ji}\dot{u}_{i,j} + \Omega_{i,i}\dot{\nu}_1 + \Omega_i\dot{\nu}_{1,i} + \chi_{i,i}\dot{\nu}_2 + \chi_i\dot{\nu}_{2,i} - q_{i,i}.$$

Subtracting Eq. (2.4) from Eq. (2.5), we get

$$\text{(2.6)}\qquad s_i[\sigma_{ji,j} + \rho F_i - \rho\ddot{u}_i] = 0.$$

Since the quantities in the square brackets are independent of s_i, therefore from the above equation, we obtain

$$\text{(2.7)}\qquad \sigma_{ji,j} + \rho F_i = \rho\ddot{u}_i.$$

Taking into account Eq. (2.7), we get a simplified law of energy balance from Eq. (2.4), namely

$$\text{(2.8)}\qquad \rho\dot{U} = \sigma_{ji}\dot{u}_{i,j} + \Omega_i\dot{\nu}_{1,i} + \chi_i\dot{\nu}_{2,i} - q_{i,i} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2,$$

where ξ and ζ satisfy the relations

$$\Omega_{i,i} + \xi + \rho g = k_1 \ddot{\nu}_1, \quad \chi_{i,i} + \zeta + \rho l = k_2 \ddot{\nu}_2. \tag{2.9}$$

The balance of entropy [35] can be written as

$$\int_V \rho \dot{S} dV + \int_A \left(\frac{q_i}{T}\right) n_i dA - \int_A \left(\frac{P\eta_i}{T}\right) n_i dA = -\int_V \frac{q_i}{T^2} T_{,i} dV - \int_V \frac{P_{,i}}{T} \eta_i dV + \int_V \frac{P}{T^2} \eta_i T_{,i} dV, \tag{2.10}$$

where S, P, are entropy and chemical potential per unit mass respectively, η_i is the mass diffusion flux vector $\boldsymbol{\eta}$, T is the absolute temperature.

Equation (2.10) can be represented in the form

$$\rho \dot{S} + \left(\frac{q_i}{T}\right)_{,i} - \left(\frac{P\eta_i}{T}\right)_{,i} = -\frac{q_i}{T^2} T_{,i} - \frac{P_{,i}}{T} \eta_i + \frac{P}{T^2} \eta_i T_{,i}. \tag{2.11}$$

The right hand side of the above equation is the entropy source

$$\Re = -\frac{q_i}{T^2} T_{,i} - \frac{P_{,i}}{T} \eta_i + \frac{P}{T^2} \eta_i T_{,i} \ \geq \ 0.$$

In view of the above equation, Eq. (2.11) can be written in the form of an inequality called Clausius–Duhem inequality

$$\rho \dot{S} + \frac{q_{i,i}}{T} - \frac{q_i}{T^2} T_{,i} - \frac{P}{T} \eta_{i,i} - \frac{P_{,i}}{T} \eta_i + \frac{P}{T^2} \eta_i T_{,i} \ \geq \ 0. \tag{2.12}$$

The equation of conservation of mass is

$$\eta_{j,j} = -\dot{C}, \tag{2.13}$$

where C is the concentration of the diffusion material in the elastic body.

Equation (2.12) with the help of Eq. (2.8) and (2.13) becomes

$$\rho T \dot{S} - \rho \dot{U} + \sigma_{ij} \dot{e}_{ij} + \Omega_i \dot{\nu}_{1,i} + \chi_i \dot{\nu}_{2,i} - \xi \dot{\nu}_1 - \zeta \dot{\nu}_2 - \frac{q_i}{T} T_{,i} + P\dot{C} - P_{,i} \eta_i + \frac{P}{T} \eta_i T_{,i} \geq 0, \tag{2.14}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are components of strain tensor.

Helmholtz free energy function Γ is defined as

$$\Gamma = U - TS. \tag{2.15}$$

Using Eq. (2.15) in Eq. (2.14), we get

$$-\rho[\dot{\Gamma} + \dot{T}S] + \sigma_{ij}\dot{e}_{ij} + \Omega_i\dot{\nu}_{1,i} + \chi_i\dot{\nu}_{2,i} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2 - \frac{q_i}{T}T_{,i} + P\dot{C} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0. \tag{2.16}$$

The function Γ can be expressed in terms of independent variables $e_{ij}, \nu_1, \nu_{1,i}, \nu_2, \nu_{2,i}, T, T_{,i}, C$ and $C_{,i}$. Therefore, we have

$$\dot{\Gamma} = \frac{\partial\Gamma}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\Gamma}{\partial\nu_1}\dot{\nu}_1 + \frac{\partial\Gamma}{\partial\nu_{1,i}}\dot{\nu}_{1,i} + \frac{\partial\Gamma}{\partial\nu_2}\dot{\nu}_2 + \frac{\partial\Gamma}{\partial\nu_{2,i}}\dot{\nu}_{2,i} + \frac{\partial\Gamma}{\partial T}\dot{T} + \frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial\Gamma}{\partial C}\dot{C} + \frac{\partial\Gamma}{\partial C_{,i}}\dot{C}_{,i}. \tag{2.17}$$

Introducing Eq. (2.17) into Eq. (2.16), we get

$$\left[\sigma_{ij} - \rho\frac{\partial\Gamma}{\partial e_{ij}}\right]\dot{e}_{ij} + \left[\Omega_i - \rho\frac{\partial\Gamma}{\partial\nu_{1,i}}\right]\dot{\nu}_{1,i} + \left[\chi_i - \rho\frac{\partial\Gamma}{\partial\nu_{2,i}}\right]\dot{\nu}_{2,i} - \left[\xi + \rho\frac{\partial\Gamma}{\partial\nu_1}\right]\dot{\nu}_1 - \left[\zeta + \rho\frac{\partial\Gamma}{\partial\nu_2}\right]\dot{\nu}_2 - \rho\left[S + \frac{\partial\Gamma}{\partial T}\right]\dot{T} + \left[P - \rho\frac{\partial\Gamma}{\partial C}\right]\dot{C} - \rho\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} - \rho\frac{\partial\Gamma}{\partial C_{,i}}\dot{C}_{,i} - \frac{q_i}{T}T_{,i} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0. \tag{2.18}$$

The inequality should be satisfied for all rates $\dot{e}_{ij}$, $\dot{\nu}_1$, $\dot{\nu}_{1,i}$, $\dot{\nu}_2$, $\dot{\nu}_{2,i}$, $\dot{T}$, $\dot{T}_{,i}$, $\dot{C}$ and $\dot{C}_{,i}$. Hence the coefficients of above variables must vanish, that is

$$\sigma_{ij} = \rho\frac{\partial\Gamma}{\partial e_{ij}}, \tag{2.19}$$

$$\Omega_i = \rho\frac{\partial\Gamma}{\partial\nu_{1,i}}, \tag{2.20}$$

$$\chi_i = \rho\frac{\partial\Gamma}{\partial\nu_{2,i}}, \tag{2.21}$$

$$\xi = -\rho\frac{\partial\Gamma}{\partial\nu_1}, \tag{2.22}$$

$$\zeta = -\rho\frac{\partial\Gamma}{\partial\nu_2}, \tag{2.23}$$

$$S = -\frac{\partial\Gamma}{\partial T}, \tag{2.24}$$

$$P = \rho\frac{\partial\Gamma}{\partial C}, \tag{2.25}$$

$$\frac{\partial\Gamma}{\partial T_{,i}} = \frac{\partial\Gamma}{\partial C_{,i}} = 0, \tag{2.26}$$

$$-\frac{q_i}{T}T_{,i} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geq 0. \tag{2.27}$$

Let us introduce the notations

$$\phi = \nu_1 - (\nu_1)_0, \qquad \psi = \nu_2 - (\nu_2)_0, \qquad \theta = T - T_0, \tag{2.28}$$

where T_0 is the reference temperature of the body chosen such that $|\frac{\theta}{T_0}| \ll 1$, $(\nu_1)_0$ and $(\nu_2)_0$ are the volume fraction fields in the reference configuration.

The independent variables in the linear theory are e_{ij}, ϕ, $\phi_{,i}$, ψ, $\psi_{,i}$, θ and C. We assume that the undeformed body is free from stresses and has zero intrinsic equilibrated body forces and entropy. If the body has a centre of symmetry, then we have

$$\begin{aligned}2\rho\Gamma = {} & c_{ijkl}e_{ij}e_{kl} + 2p_{ij}e_{ij}\phi + 2\gamma_{ij}e_{ij}\psi - 2a_{ij}e_{ij}\theta - 2b_{ij}e_{ij}C \\ & + q_{ij}\phi_{,i}\phi_{,j} + 2\alpha_{ij}\phi_{,i}\psi_{,j} + f_{ij}\psi_{,i}\psi_{,j} + d^*\phi^2 + f\psi^2 + 2\alpha_1\phi\psi \\ & - 2\gamma_1\phi\theta - 2v\phi C - 2\gamma_2\psi\theta - 2m\psi C - \frac{\rho C_e\theta^2}{T_0} - 2a\theta C + bC^2.\end{aligned} \tag{2.29}$$

Using the above equation in Eqs. (2.19)–(2.25), we obtain the following constitutive equations

$$\sigma_{ij} = c_{ijkl}e_{kl} + p_{ij}\phi + \gamma_{ij}\psi - a_{ij}\theta - b_{ij}C, \tag{2.30}$$

$$\Omega_i = q_{ij}\phi_{,j} + \alpha_{ij}\psi_{,j}, \tag{2.31}$$

$$\chi_i = \alpha_{ij}\phi_{,j} + f_{ij}\psi_{,j}, \tag{2.32}$$

$$\xi = -p_{ij}e_{ij} - d^*\phi - \alpha_1\psi + \gamma_1\theta + vC, \tag{2.33}$$

$$\zeta = -\gamma_{ij}e_{ij} - \alpha_1\phi - f\psi + \gamma_2\theta + mC, \tag{2.34}$$

$$\rho S = a_{ij}e_{ij} + \gamma_1\phi + \gamma_2\psi + \frac{\rho C_e\theta}{T_0} + aC, \tag{2.35}$$

$$P = -b_{ij}e_{ij} - v\phi - m\psi - a\theta + bC. \tag{2.36}$$

Equations (2.7) and (2.9) with the aid of Eqs. (2.30)–(2.34) become

$$c_{ijkl}e_{kl,j} + p_{ij}\phi_{,j} + \gamma_{ij}\psi_{,j} - a_{ij}\theta_{,j} - b_{ij}C_{,j} + \rho F_i = \rho\ddot{u}_i, \tag{2.37}$$

$$-p_{ij}e_{ij} + q_{ij}\phi_{,ij} - d^*\phi + \alpha_{ij}\psi_{,ij} - \alpha_1\psi + \gamma_1\theta + vC + \rho g = k_1\ddot{\phi}, \tag{2.38}$$

$$-\gamma_{ij}e_{ij} + \alpha_{ij}\phi_{,ij} - \alpha_1\phi + f_{ij}\psi_{,ij} - f\psi + \gamma_2\theta + mC + \rho l = k_2\ddot{\psi}. \tag{2.39}$$

The linearized form of Eq. (2.11) is

$$\rho T_0\dot{S} = -q_{i,i}. \tag{2.40}$$

Using Eq. (2.35) in Eq. (2.40), we get

$$a_{ij}T_0\dot{e}_{ij} + \gamma_1 T_0\dot{\phi} + \gamma_2 T_0\dot{\psi} + \rho C_e\dot{\theta} + aT_0\dot{C} = -q_{i,i}. \tag{2.41}$$

The generalized Fourier's law of a heat conduction equation is

$$q_i + \tau_0\dot{q}_i = -K_{ij}\theta_{,j}, \tag{2.42}$$

where K_{ij} are coefficients of thermal conductivity tensor, τ_0 is the thermal relaxation time which ensures that the heat conduction equation predicts finite speeds of heat propagation.

The above equation with the help of Eq. (2.41) becomes

$$a_{ij}T_0(\dot{e}_{ij} + \tau_0\ddot{e}_{ij}) + \gamma_1 T_0(\dot{\phi} + \tau_0\ddot{\phi}) + \gamma_2 T_0(\dot{\psi} + \tau_0\ddot{\psi}) + \rho C_e(\dot{\theta} + \tau_0\ddot{\theta}) + aT_0(\dot{C} + \tau_0\ddot{C}) = K_{ij}\theta_{,ij}. \tag{2.43}$$

Similar to Eq. (2.42), the generalized Fick's law of mass diffusion is

$$\eta_i + \tau^0\dot{\eta}_i = -d_{ij}P_{,j}, \tag{2.44}$$

where d_{ij} are coefficients of diffusion tensor, τ^0 is the diffusion relaxation time which ensures that the equation satisfied by the concentration will also predict finite speeds of propagation of matter from one medium to the other.

Using Eqs. (2.13) and (2.36) in Eq. (2.44), we get

$$-d_{ij}[b_{kl}e_{kl,ij} + v\phi_{,ij} + m\psi_{,ij} + a\theta_{,ij} - bC_{,ij}] = \dot{C} + \tau^0\ddot{C}. \tag{2.45}$$

In the upcoming sections, the chemical potential is used as a state variable instead of the concentration.

Using Eq. (2.36) in Eqs. (2.30), (2.33)–(2.35), (2.37)–(2.39), (2.43) and (2.44), we get

$$\sigma_{ij} = d_{ijkl}e_{kl} + g_{ij}\phi + h_{ij}\psi - s_{ij}\theta - l_{ij}P, \tag{2.46}$$

$$\xi = -g_{ij}e_{ij} - d_1\phi - \beta_1\psi + \kappa_1\theta + wP, \tag{2.47}$$

$$\zeta = -h_{ij}e_{ij} - \beta_1\phi - f_1\psi + \kappa_2\theta + \nu P, \tag{2.48}$$

$$\rho S = s_{ij}e_{ij} + \kappa_1\phi + \kappa_2\psi + z\theta + sP, \tag{2.49}$$

$$d_{ijkl}e_{kl,j} + g_{ij}\phi_{,j} + h_{ij}\psi_{,j} - s_{ij}\theta_{,j} - l_{ij}P_{,j} + \rho F_i = \rho\ddot{u}_i, \tag{2.50}$$

$$-g_{ij}e_{ij} + q_{ij}\phi_{,ij} - d_1\phi + \alpha_{ij}\psi_{,ij} - \beta_1\psi + \kappa_1\theta + wP + \rho g = k_1\ddot{\phi}, \tag{2.51}$$

$$-h_{ij}e_{ij} + \alpha_{ij}\phi_{,ij} - \beta_1\phi + f_{ij}\psi_{,ij} - f_1\psi + \kappa_2\theta + \nu P + \rho l = k_2\ddot{\psi}. \tag{2.52}$$

$$\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)T_0[s_{ij}e_{ij} + \kappa_1\phi + \kappa_2\psi + z\theta + sP] = K_{ij}\theta_{,ij}, \tag{2.53}$$

$$\left(\frac{\partial}{\partial t} + \tau^0\frac{\partial^2}{\partial t^2}\right)[l_{ij}e_{ij} + w\phi + \nu\psi + s\theta + nP] = d_{ij}P_{,ij}, \tag{2.54}$$

where

$$n = \frac{1}{b},\ l_{ij} = nb_{ij},\ g_{ij} = p_{ij} - vl_{ij},\ s_{ij} = a_{ij} + al_{ij},\ h_{ij} = \gamma_{ij} - ml_{ij},$$

$$d_{ijkl} = c_{ijkl} - l_{ij}b_{kl},\ s = an,\ w = vn,\ \nu = mn,\ d_1 = d^* - vw,\ \beta_1 = \alpha_1 - v\nu,$$

$$\kappa_1 = \gamma_1 + vs,\ f_1 = f - m\nu,\ \kappa_2 = \gamma_2 + ms,\ z = \frac{\rho C_e}{T_0} + as. \tag{2.55}$$

3. Variational principle

The principle of virtual work with variation of displacements for the elastic deformable body with double porosity is written as

$$\int\limits_V [\rho(F_i - \ddot{u}_i)\delta u_i + (\rho g + \xi - k_1\ddot{\phi})\delta\phi + (\rho l + \zeta - k_2\ddot{\psi})\delta\psi]dV + \int\limits_A [f_i\delta u_i + \Omega\delta\phi + \chi\delta\psi]dA = \int\limits_V [\sigma_{ji}\delta u_{i,j} + \Omega_i\delta\phi_{,i} + \chi_i\delta\psi_{,i}]dV. \tag{3.1}$$

On the left hand side, we have the virtual work of body forces F_i, inertial forces $\rho\ddot{u}_i, k_1\ddot{\phi}, k_2\ddot{\psi}$, surface forces $f_i = \sigma_{ji}n_j$, $\Omega = \Omega_i n_i, \chi = \chi_i n_i$, whereas on the right hand side, we have the virtual work of internal forces.

Using the symmetry of the stress tensor and the definition of the strain tensor, the Eq. (3.1) can be rewritten as

$$\int\limits_V [\rho(F_i - \ddot{u}_i)\delta u_i + (\rho g + \xi - k_1\ddot{\phi})\delta\phi + (\rho l + \zeta - k_2\ddot{\psi})\delta\psi]dV + \int\limits_A [f_i\delta u_i + \Omega\delta\phi + \chi\delta\psi]dA = \int\limits_V [\sigma_{ij}\delta e_{ij} + \Omega_i\delta\phi_{,i} + \chi_i\delta\psi_{,i}]dV. \tag{3.2}$$

Using Eqs. (2.31)–(2.32) and (2.46) in the above equation, we get

$$\int\limits_V [\rho(F_i - \ddot{u}_i)\delta u_i + (\rho g + \xi - k_1\ddot{\phi})\delta\phi + (\rho l + \zeta - k_2\ddot{\psi})\delta\psi]dV + \int\limits_A [f_i\delta u_i + \Omega\delta\phi + \chi\delta\psi]dA$$
$$= \delta(W + R + X + Y) + \int\limits_V g_{ij}\phi\delta e_{ij}dV + \int\limits_V h_{ij}\psi\delta e_{ij}dV - \int\limits_V s_{ij}\theta\delta e_{ij}dV - \int\limits_V l_{ij}P\delta e_{ij}dV. \tag{3.3}$$

where

$$W = \frac{1}{2}\int\limits_V d_{ijkl}e_{ij}e_{kl}dV, \qquad R = \frac{1}{2}\int\limits_V q_{ij}\phi_{,i}\phi_{,j}dV,$$

$$X = \frac{1}{2}\int\limits_V f_{ij}\psi_{,i}\psi_{,j}dV, \qquad Y = \int\limits_V \alpha_{ij}\phi_{,i}\psi_{,j}dV.$$

Since we are taking the coupling of the deformation field with the temperature, chemical potential, pores and fissures, therefore two additional relations are necessary which characterize the phenomena of the thermal conductivity and mass diffusion.

We define a vector **J** [36] connected with the entropy through the relation

$$\rho S = -J_{i,i}. \tag{3.4}$$

Combining Eqs. (2.40), (2.42), (2.49) and (3.4), we obtain

$$T_0K^*_{ij}\left(\frac{d}{dt} + \tau_0\frac{d^2}{dt^2}\right)J_i + \theta_{,j} = 0, \tag{3.5}$$

$$-J_{i,i} = s_{ij}e_{ij} + \kappa_1\phi + \kappa_2\psi + z\theta + sP, \tag{3.6}$$

where K^*_{ij}, the resistivity matrix, is the inverse of the thermal conductivity K_{ij}.

Multiplying both sides of Eq. (3.5) by δJ_j and integrating over the region of the body, we get

$$\int\limits_V \left[\theta_{,j} + T_0K^*_{ij}\left(\frac{dJ_i}{dt} + \tau_0\frac{d^2J_i}{dt^2}\right)\right]\delta J_j dV = 0. \tag{3.7}$$

Now

$$\int\limits_V \theta_{,j}\delta J_j dV = \int\limits_V (\theta\delta J_j)_{,j}dV - \int\limits_V \theta\delta J_{j,j}dV. \tag{3.8}$$

Applying the divergence theorem defined by,

$$\int\limits_V (\theta\delta J_j)_{,j}dV = \int\limits_A (\theta\delta J_j)n_j dA, \tag{3.9}$$

in Eq. (3.8), we obtain

$$\int\limits_V \theta_{,j}\delta J_j dV = \int\limits_A (\theta\delta J_j)n_j dA - \int\limits_V \theta\delta J_{j,j}dV. \tag{3.10}$$

Substituting Eq. (3.10) in Eq. (3.7), we obtain

$$\int_A (\theta\delta J_j) n_j dA - \int_V \theta\delta J_{j,j} dV + T_0 \int_V K^*_{ij}\left(\frac{dJ_i}{dt} + \tau_0\frac{d^2J_i}{dt^2}\right)\delta J_j dV = 0. \tag{3.11}$$

Making use of Eq. (3.6) in Eq. (3.11), we obtain the second variational equation

$$\int_A (\theta\delta J_j) n_j dA + \int_V s_{ij}\theta\delta e_{ij} dV + \kappa_1 \int_V \theta\delta\phi dV \\ + \kappa_2 \int_V \theta\delta\psi dV + s\int_V \theta\delta P dV + \delta(E+H) = 0, \tag{3.12}$$

where the function of thermal potential E is defined by

$$E = \frac{z}{2}\int_V \theta^2 dV, \qquad \delta E = z\int_V \theta\delta\theta dV, \tag{3.13}$$

and the function of thermal dissipation H is defined by

$$H = \frac{T_0}{2}\int_V K^*_{ij}\left(\frac{dJ_i}{dt} + \tau_0\frac{d^2J_i}{dt^2}\right)J_j dV, \\ \delta H = T_0\int_V K^*_{ij}\left(\frac{dJ_i}{dt} + \tau_0\frac{d^2J_i}{dt^2}\right)\delta J_j dV. \tag{3.14}$$

In order to obtain the last of the variational equations, we now introduce the vector function $\mathbf{N}$ defined as follows

$$C = -N_{i,i}. \tag{3.15}$$

Combining Eqs. (2.13), (2.36), (2.44) and (3.15), we obtain

$$d^*_{ij}\left(\frac{d}{dt} + \tau^0\frac{d^2}{dt^2}\right)N_i + P_{,j} = 0, \tag{3.16}$$

$$-N_{i,i} = l_{ij}e_{ij} + w\phi + \nu\psi + s\theta + nP, \tag{3.17}$$

where d^*_{ij} is the inverse of the diffusion tensor d_{ij}.

Multiplying Eq. (3.16) by δN_j and integrating over the region of the body, we obtain

$$\int_V \left[d^*_{ij}\left(\frac{dN_i}{dt} + \tau^0\frac{d^2N_i}{dt^2}\right) + P_{,j}\right]\delta N_j dV = 0, \tag{3.18}$$

Consider

$$\int_V P_{,j}\delta N_j dV = \int_V (P\delta N_j)_{,j} dV - \int_V P\delta N_{j,j} dV. \tag{3.19}$$

We know that

$$\int_V (P\delta N_j)_{,j} dV = \int_A (P\delta N_j) n_j dA. \tag{3.20}$$

Thus, Eq. (3.19) becomes

$$\int_V P_{,j}\delta N_j dV = \int_A (P\delta N_j) n_j dA - \int_V P\delta N_{j,j} dV. \tag{3.21}$$

Making use of Eq. (3.21) in Eq. (3.18) yields

$$\int_A (P\delta N_j) n_j dA - \int_V P\delta N_{j,j} dV + \int_V d^*_{ij}\left(\frac{dN_i}{dt} + \tau^0 \frac{d^2 N_i}{dt^2}\right)\delta N_j dV = 0. \tag{3.22}$$

Substituting the value of $N_{i,i}$ from Eq. (3.17) in the above equation, we obtain the third variational equation

$$\begin{aligned} \int_A (P\delta N_j) n_j dA + \int_V l_{ij} P\delta e_{ij} dV + w \int_V P\delta\phi dV \\ + \nu \int_V P\delta\psi dV + s \int_V P\delta\theta dV + \delta(G+F) = 0, \end{aligned} \tag{3.23}$$

where, the function of diffusion potential G is defined by

$$G = \frac{n}{2}\int_V P^2 dV, \qquad \delta G = n\int_V P\delta P dV, \tag{3.24}$$

and the function of diffusion dissipation F is defined by

$$\begin{aligned} F &= \frac{1}{2}\int_V d^*_{ij}\left(\frac{dN_i}{dt} + \tau^0 \frac{d^2 N_i}{dt^2}\right) N_j dV, \\ \delta F &= \int_V d^*_{ij}\left(\frac{dN_i}{dt} + \tau^0 \frac{d^2 N_i}{dt^2}\right)\delta N_j dV. \end{aligned} \tag{3.25}$$

Eliminating integrals $\int_V s_{ij}\theta\delta e_{ij}dV$ and $\int_V l_{ij}P\delta e_{ij}dV$ from Eqs. (3.3), (3.12), (3.23) and using Eqs. (2.47) and (2.48), we obtain the variational principle in the following form

$$\delta\Big[W+R+X+Y+E+H+G+F+L+K+M+Z \\ +s\int_V P\theta dV+\beta_1\int_V \phi\psi dV\Big] \\ =\int_V[\rho(F_i-\ddot{u}_i)\delta u_i+(\rho g-k_1\ddot{\phi})\delta\phi+(\rho l-k_2\ddot{\psi})\delta\psi]dV \\ +\int_A[f_i\delta u_i+\Omega\delta\phi+\chi\delta\psi]dA-\int_A(\theta\delta J_i)n_i dA-\int_A(P\delta N_i)n_i dA. \tag{3.26}$$

where

$$L=\int_V g_{ij}\phi e_{ij}dV,\qquad K=\int_V h_{ij}\psi e_{ij}dV,$$
$$M=\frac{d_1}{2}\int_V\phi^2 dV,\qquad Z=\frac{f_1}{2}\int_V\psi^2 dV.$$

On the right-hand side of the above equation, we find all the causes, the body forces, inertial forces, the surface forces, the heating and the chemical potential on the surface A bounding the body.

4. Uniqueness theorem

We assume that the virtual displacements δu_i, the virtual increment of the temperature $\delta\theta$, etc. correspond to the increments occurring in the body. Then

$$\delta u_i=\frac{\partial u_i}{\partial t}dt=\dot{u}_i dt,\qquad \delta\theta=\frac{\partial\theta}{\partial t}dt=\dot{\theta}dt,\qquad \text{etc.} \tag{4.1}$$

and Eq. (3.26) reduces to the following relation

$$\frac{d}{dt}\Big[W+R+X+Y+E+H+G+F+L+K+M+Z \\ +s\int_V P\theta dV+\beta_1\int_V\phi\psi dV\Big] \\ =\int_V[\rho(F_i-\ddot{u}_i)\dot{u}_i+(\rho g-k_1\ddot{\phi})\dot{\phi}+(\rho l-k_2\ddot{\psi})\dot{\psi}]dV \\ +\int_A[f_i\dot{u}_i+\Omega\dot{\phi}+\chi\dot{\psi}]dA-\int_A(\theta\dot{J}_i)n_i dA-\int_A(P\dot{N}_i)n_i dA. \tag{4.2}$$

Now

$$\int_V \rho \ddot{u}_i \dot{u}_i dV = \frac{\partial \aleph}{\partial t}, \tag{4.3}$$

where $\aleph = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}_i dV$, is the kinetic energy of the body enclosed by the volume V. We also have

$$E + G + s\int_V P\theta dV = \frac{1}{2}\int_V (z\theta^2 + nP^2 + 2sP\theta)dV. \tag{4.4}$$

Using Eqs. (4.3) and (4.4) in Eq. (4.2), we obtain

$$\begin{aligned}\frac{d}{dt}\Big[W + R + X + Y + H + \aleph + F + L + K + M + Z \\ + \frac{1}{2}\int_V (z\theta^2 + nP^2 + 2sP\theta)dV + \beta_1 \int_V \phi\psi dV\Big] \\ = \int_V [\rho F_i \dot{u}_i + (\rho g - k_1\ddot{\phi})\dot{\phi} + (\rho l - k_2\ddot{\psi})\dot{\psi}]dV \\ + \int_A [f_i \dot{u}_i + \Omega\dot{\phi} + \chi\dot{\psi}]dA - \int_A (\theta \dot{J}_i)n_i dA - \int_A (P\dot{N}_i)n_i dA.\end{aligned} \tag{4.5}$$

Let

$$G^* = \frac{k_1}{2}\int_V \dot{\phi}^2 dV, \qquad H^* = \frac{k_2}{2}\int_V \dot{\psi}^2 dV, \tag{4.6}$$

so that

$$\frac{dG^*}{dt} = k_1\int_V \dot{\phi}\ddot{\phi} dV, \qquad \frac{dH^*}{dt} = k_2\int_V \dot{\psi}\ddot{\psi} dV.$$

Also, we have

$$M + Z + \beta_1 \int_V \phi\psi dV = \frac{1}{2}\int_V [d_1\phi^2 + f_1\psi^2 + 2\beta_1\phi\psi]dV. \tag{4.7}$$

With the help of above two equations, Eq. (4.5) becomes

$$\begin{aligned}\frac{d}{dt}\Big[W + R + X + Y + H + \aleph + F + L + K + G^* + H^* \\ + \frac{1}{2}\int_V (z\theta^2 + nP^2 + 2sP\theta)dV + \frac{1}{2}\int_V (d_1\phi^2 + f_1\psi^2 + 2\beta_1\phi\psi)dV\Big]\end{aligned} \tag{4.8}$$

$$= \int_V \rho[F_i \dot{u}_i + g\dot{\phi} + l\dot{\psi}]dV + \int_A [f_i \dot{u}_i + \Omega\dot{\phi} + \chi\dot{\psi}]dA - \int_A (\theta \dot{J}_i) n_i dA - \int_A (P\dot{N}_i) n_i dA.$$

To prove the uniqueness theorem, following results need to be proved:

Theorem 1. *If K^*_{ij} and d^*_{ij} satisfy the symmetry relations*

$$K^*_{ij} = K^*_{ji},\ d^*_{ij} = d^*_{ji}, \tag{4.9}$$

then

$$\frac{dH}{dt} = T_0 \int_V K^*_{ij} \dot{J}_i \dot{J}_j dV + \frac{d}{dt}\left[\frac{T_0 \tau_0}{2} \int_V K^*_{ij} \dot{J}_i \dot{J}_j dV\right], \tag{4.10}$$

and

$$\frac{dF}{dt} = \int_V d^*_{ij} \dot{N}_i \dot{N}_j dV + \frac{d}{dt}\left[\frac{\tau^0}{2} \int_V d^*_{ij} \dot{N}_i \dot{N}_j dV\right]. \tag{4.11}$$

Proof. From Eqs. (3.14) and (4.1), we get

$$\frac{dH}{dt} = T_0 \int_V K^*_{ij} \left(\frac{dJ_i}{dt} + \tau_0 \frac{d^2 J_i}{dt^2}\right) \frac{dJ_j}{dt} dV. \tag{4.12}$$

Now using Eq. $(4.9)_1$,

$$\frac{d}{dt}(K^*_{ij} \dot{J}_i \dot{J}_j) = 2K^*_{ij} \ddot{J}_i \dot{J}_j. \tag{4.13}$$

Substituting last equation in Eq. (4.12), we arrive at Eq. (4.10). Similarly using Eq. $(4.9)_2$, Eq. (4.11) can be proved.

Theorem 2. *If z, s, n and d_1, f_1, β_1 are constants satisfying the inequalities*

$$0 < s^2 < zn, \tag{4.14}$$

$$0 < \beta_1^2 < d_1 f_1 \tag{4.15}$$

respectively, then

$$z\theta^2 + nP^2 + 2sP\theta > 0, \tag{4.16}$$

and

$$d_1 \phi^2 + f_1 \psi^2 + 2\beta_1 \phi\psi > 0. \tag{4.17}$$

Proof. Proofs of this theorem are obvious because proofs of positive definiteness of second degree polynomials in θ and ϕ are obvious.

Using theorems 1 and 2, now we prove the uniqueness theorem

THEOREM. *There is only one solution of the problem of generalized thermoelastic diffusion with double porosity, subject to the boundary conditions on the surface A*

$$f_i = \sigma_{ij} n_j = f_{i1}, \quad \phi = \phi_1, \quad \psi = \psi_1, \quad \theta = \theta_1, \quad P = P_1,$$

and the initial conditions at $t = 0$

$u_i = u_i^0$, $\dot{u}_i = \dot{u}_i^0$, $\phi = \phi^0$, $\dot{\phi} = \dot{\phi}^0$, $\dot{\psi} = \dot{\psi}^0$, $\theta = \theta^0$, $\dot{\theta} = \dot{\theta}^0$, $P = P^0$ *and* $\dot{P} = \dot{P}^0$,

where f_{i1}, θ_1, P_1, u_i^0, $\dot{u}_i^0$, ϕ^0, $\dot{\phi}^0$, ψ^0, $\dot{\psi}^0$, θ^0, $\dot{\theta}^0$, P^0 *and* $\dot{P}^0$ *are known functions.*

We assume that the material parameters satisfy the inequalities

$$T_0 > 0, \quad \tau_0 > 0, \quad C_e > 0, \quad \rho > 0, \quad \tau^0 > 0, \quad k_1 > 0, \quad k_2 > 0, \tag{4.18}$$

the constitutive coefficients satisfy the symmetry relations

$$\begin{gathered} d_{ijkl} = d_{klij}, \quad s_{ij} = s_{ji}, \quad l_{ij} = l_{ji}, \quad g_{ij} = g_{ji}, \\ h_{ij} = h_{ji}, \quad q_{ij} = q_{ji}, \quad f_{ij} = f_{ji}, \quad \alpha_{ij} = \alpha_{ji}, \quad K^*_{ij} = K^*_{ji}, \quad d^*_{ij} = d^*_{ji}, \end{gathered} \tag{4.19}$$

d_{ijkl}, K^*_{ij}, d^*_{ij}, q_{ij}, f_{ij}, g_{ij}, h_{ij} *and* α_{ij} *are positive definite and* $0 < s^2 < zn$, $0 < \beta_1^2 < d_1 f_1$.

Proof. Let $u_i^{(1)}$, ϕ^1, ψ^1, $\theta^{(1)}$, $P^{(1)}$, ... and $u_i^{(2)}$, ϕ^2, ψ^2, $\theta^{(2)}$, $P^{(2)}$, ... be two solutions sets of Eqs. (2.7),(2.46) and (2.51)–(2.54). Let us take

$$\begin{gathered} u_i = u_i^{(1)} - u_i^{(2)}, \quad \phi = \phi^{(1)} - \phi^{(2)}, \quad \psi = \psi^{(1)} - \psi^{(2)}, \\ \theta = \theta^{(1)} - \theta^{(2)}, \text{ and } P = P^{(1)} - P^{(2)}. \end{gathered} \tag{4.20}$$

The functions u_i, ϕ, ψ, θ, and P satisfy the governing equations with zero body forces, homogeneous initial and boundary conditions. Thus, these functions satisfy an equation similar to the equation (4.8) with zero right hand side, that is,

$$\begin{aligned} \frac{d}{dt}\Bigg[& W + R + X + Y + H + \aleph + F + L + K + G^* + H^* \\ & + \frac{1}{2}\int_V (z\theta^2 + nP^2 + 2sP\theta)dV + \frac{1}{2}\int_V (d_1\phi^2 + f_1\psi^2 + 2\beta_1\phi\psi)dV \Bigg] = 0. \end{aligned} \tag{4.21}$$

Substituting Eqs. (4.10) and (4.11) in Eq. (4.21), we obtain

$$\frac{d}{dt}\Big[W+R+X+Y+\aleph+L+K+G^*+H^*+\frac{T_0\tau_0}{2}\int_V K^*_{ij}\dot{J}_i\dot{J}_j dV + \frac{\tau^0}{2}\int_V d^*_{ij}\dot{N}_i\dot{N}_j dV+\frac{1}{2}\int_V(z\theta^2+nP^2+2sP\theta)dV + \frac{1}{2}\int_V(d_1\phi^2+f_1\psi^2+2\beta_1\phi\psi)dV\Big]+T_0\int_V K^*_{ij}\dot{J}_i\dot{J}_j dV+\int_V d^*_{ij}\dot{N}_i\dot{N}_j dV=0. \tag{4.22}$$

Using the inequalities (4.18) and (4.19) in Eq. (4.22), we obtain

$$\frac{d}{dt}\Big[W+R+X+Y+\aleph+L+K+G^*+H^*+\frac{T_0\tau_0}{2}\int_V K^*_{ij}\dot{J}_i\dot{J}_j dV + \frac{\tau^0}{2}\int_V d^*_{ij}\dot{N}_i\dot{N}_j dV+\frac{1}{2}\int_V(z\theta^2+nP^2+2sP\theta)dV+\frac{1}{2}\int_V(d_1\phi^2+f_1\psi^2+2\beta_1\phi\psi)dV\Big]\le 0. \tag{4.23}$$

Thus

$$W+R+X+Y+\aleph+L+K+G^*+H^*+\frac{T_0\tau_0}{2}\int_V K^*_{ij}\dot{J}_i\dot{J}_j dV + \frac{\tau^0}{2}\int_V d^*_{ij}\dot{N}_i\dot{N}_j dV+\frac{1}{2}\int_V(z\theta^2+nP^2+2sP\theta)dV+\frac{1}{2}\int_V(d_1\phi^2+f_1\psi^2+2\beta_1\phi\psi)dV \tag{4.24}$$

is a decreasing function of time.

Since

$$0<s^2<zn, \qquad 0<\beta_1^2<d_1f_1,$$

therefore

$$\int_V[z\theta^2+nP^2+2sP\theta]dV>0, \qquad \int_V[d_1\phi^2+f_1\psi^2+2\beta_1\phi\psi]dV>0.$$

Thus, the expression (4.24) vanishes for $t=0$, due to the homogeneous initial conditions, and it must be always non-positive for $t>0$.

Using Eq. (4.19), it follows immediately that the expression (4.24) must be identically zero for $t>0$. We thus have

$$u_i=\phi=\psi=\theta=P=e_{ij}=\sigma_{ij}=0.$$

This proves the uniqueness of the solution to the complete system of field equations subjected to the displacement-temperature-chemical potential-pores-fissures initial and boundary conditions.

5. Reciprocity theorem

Let us consider a homogeneous anisotropic generalized thermoelastic diffusion body with double porosity occupying the region V and bounded by the surface A. We assume that the stresses σ_{ij} and the strains e_{ij} are continuous together with their first derivatives whereas the displacements u_i, temperature θ, concentration C, chemical potential P and volume fraction fields ϕ, ψ are continuous and have continuous derivatives up to the second order, for $\mathbf{x} \,\epsilon\, V + A$, $t > 0$. We denote

$$\begin{gathered} q = K_{ij}\theta_{,j}n_i, \qquad p = d_{ij}P_{,j}n_i, \qquad \wp = q_{ij}\phi_{,j}n_i, \\ \hbar = f_{ij}\psi_{,j}n_i, \qquad y = \alpha_{ij}\psi_{,j}n_i, \qquad x = \alpha_{ij}\phi_{,j}n_i. \end{gathered} \tag{5.1}$$

To the system of field equations, we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions:

$$\begin{gathered} u_i(\mathbf{x},t) = U_i(\mathbf{x},t), \quad \phi(\mathbf{x},t) = \varPhi(\mathbf{x},t), \quad \psi(\mathbf{x},t) = \varPsi(\mathbf{x},t), \\ \theta(\mathbf{x},t) = \varpi(\mathbf{x},t), \quad P(\mathbf{x},t) = \varsigma(\mathbf{x},t), \end{gathered} \tag{5.2}$$

for all $\mathbf{x} \,\epsilon\, A$, $t > 0$; and the homogeneous initial conditions

$$\begin{gathered} u_i(\mathbf{x},0) = \dot{u}_i(\mathbf{x},0) = 0, \qquad \phi(\mathbf{x},0) = \dot{\phi}(\mathbf{x},0) = \psi(\mathbf{x},0) = \dot{\psi}(\mathbf{x},0) = 0, \\ \theta(\mathbf{x},0) = \dot{\theta}(\mathbf{x},0) = 0, \qquad P(\mathbf{x},0) = \dot{P}(\mathbf{x},0) = 0, \end{gathered} \tag{5.3}$$

for all $\mathbf{x} \,\epsilon\, V$, $t = 0$.

We derive the dynamic reciprocity relationship for a generalized thermoelastic diffusion bounded body V with double porosity, which satisfies Eqs. (2.7),(2.46) and (2.51)-(2.54), the boundary conditions (5.2) and the homogeneous initial conditions (5.3).

We define the Laplace transform as

$$\bar{f}(x,r) = \pounds(f(\mathbf{x},t)) = \int_0^\infty f(\mathbf{x},t)e^{-rt}dt. \tag{5.4}$$

Applying the Laplace transform defined by Eq. (5.4) on Eqs. (2.7),(2.46) and (2.51)–(2.54) and omitting the bars for simplicity, we obtain

$$\sigma_{ij,j} + \rho F_i = \rho r^2 u_i, \tag{5.5}$$

$$\sigma_{ij} = d_{ijkl}e_{kl} + g_{ij}\phi + h_{ij}\psi - s_{ij}\theta - l_{ij}P, \tag{5.6}$$

$$-g_{ij}e_{ij} + q_{ij}\phi_{,ij} - d_1\phi + \alpha_{ij}\psi_{,ij} - \beta_1\psi + \kappa_1\theta + wP + \rho g = k_1 r^2\phi, \tag{5.7}$$

$$-h_{ij}e_{ij} + \alpha_{ij}\phi_{,ij} - \beta_1\phi + f_{ij}\psi_{,ij} - f_1\psi + \kappa_2\theta + \nu P + \rho l = k_2 r^2\psi, \tag{5.8}$$

$$(r + \tau_0 r^2)T_0[s_{ij}e_{ij} + \kappa_1\phi + \kappa_2\psi + z\theta + sP] = K_{ij}\theta_{,ij}, \tag{5.9}$$

$$(r + \tau^0 r^2)[l_{ij}e_{ij} + w\phi + \nu\psi + s\theta + nP] = d_{ij}P_{,ij}. \tag{5.10}$$

We now consider two problems where applied body forces, chemical potential, the surface temperature and volume fraction fields are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have $u_i^{(1)}$, $e_{ij}^{(1)}$, $\sigma_{ij}^{(1)}$, $\phi^{(1)}$, $\psi^{(1)}$, $\theta^{(1)}$, $P^{(1)}$, ... for the first problem and $u_i^{(2)}$, $e_{ij}^{(2)}$, $\sigma_{ij}^{(2)}$, $\phi^{(2)}$,$\psi^{(2)}$, $\theta^{(2)}$, $P^{(2)}$, ... for the second problem. Each set of variables satisfies Eqs. (2.7), (2.46) and (2.51)–(2.54).

Using the assumption $\sigma_{ij} = \sigma_{ji}$, we obtain

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \int_V \sigma_{ij}^{(1)} u_{i,j}^{(2)} dV = \int_V (\sigma_{ij}^{(1)} u_i^{(2)})_{,j} dV - \int_V \sigma_{ij,j}^{(1)} u_i^{(2)} dV. \tag{5.11}$$

Using the divergence theorem in the first term of the right hand side of eq. (5.11) yields

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \int_A (\sigma_{ij}^{(1)} u_i^{(2)}) n_j dA - \int_V \sigma_{ij,j}^{(1)} u_i^{(2)} dV. \tag{5.12}$$

Equation (5.12) with the use of Eqs. (2.2) and (5.5) gives

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \int_A f_i^{(1)} u_i^{(2)} dA - \rho \int_V r^2 u_i^{(1)} u_i^{(2)} dV + \rho \int_V F_i^{(1)} u_i^{(2)} dV. \tag{5.13}$$

A similar expression is obtained for the integral $\int_V \sigma_{ij}^{(2)} e_{ij}^{(1)} dV$, from which together with Eq. (5.13), it follows that

$$\int_V [\sigma_{ij}^{(1)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)}] dV = \int_A [f_i^{(1)} u_i^{(2)} - f_i^{(2)} u_i^{(1)}] dA + \rho \int_V [F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}] dV. \tag{5.14}$$

Now multiplying Eq. (5.6) by $e_{ij}^{(2)}$ and $e_{ij}^{(1)}$ for the first and second problems respectively, subtracting and integrating over the region V, we obtain

$$\int_V [\sigma_{ij}^{(1)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)}] dV = \int_V d_{ijkl} (e_{kl}^{(1)} e_{ij}^{(2)} - e_{kl}^{(2)} e_{ij}^{(1)}) dV + \int_V g_{ij} (\phi^{(1)} e_{ij}^{(2)} - \phi^{(2)} e_{ij}^{(1)}) dV + \int_V h_{ij} (\psi^{(1)} e_{ij}^{(2)} - \psi^{(2)} e_{ij}^{(1)}) dV - \int_V s_{ij} (\theta^{(1)} e_{ij}^{(2)} - \theta^{(2)} e_{ij}^{(1)}) dV - \int_V l_{ij} (P^{(1)} e_{ij}^{(2)} - P^{(2)} e_{ij}^{(1)}) dV.$$

Using the symmetry properties of d_{ijkl}, we obtain

$$\int_V [\sigma_{ij}^{(1)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)}] dV = \int_V g_{ij}(\phi^{(1)} e_{ij}^{(2)} - \phi^{(2)} e_{ij}^{(1)}) dV + \int_V h_{ij}(\psi^{(1)} e_{ij}^{(2)} - \psi^{(2)} e_{ij}^{(1)}) dV - \int_V s_{ij}(\theta^{(1)} e_{ij}^{(2)} - \theta^{(2)} e_{ij}^{(1)}) dV - \int_V l_{ij}(P^{(1)} e_{ij}^{(2)} - P^{(2)} e_{ij}^{(1)}) dV. \tag{5.15}$$

Equating Eqs. (5.14) and (5.15), we get the first part of the reciprocity theorem

$$\int_A [f_i^{(1)} u_i^{(2)} - f_i^{(2)} u_i^{(1)}] dA + \rho \int_V [F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}] dV = \int_V g_{ij}(\phi^{(1)} e_{ij}^{(2)} - \phi^{(2)} e_{ij}^{(1)}) dV + \int_V h_{ij}(\psi^{(1)} e_{ij}^{(2)} - \psi^{(2)} e_{ij}^{(1)}) dV - \int_V s_{ij}(\theta^{(1)} e_{ij}^{(2)} - \theta^{(2)} e_{ij}^{(1)}) dV - \int_V l_{ij}(P^{(1)} e_{ij}^{(2)} - P^{(2)} e_{ij}^{(1)}) dV. \tag{5.16}$$

To derive the second part, multiply Eq. (5.7) by $\phi^{(2)}$ and $\phi^{(1)}$ for the first and second problems respectively, subtracting and integrating over V, we get

$$\int_V q_{ij}[\phi_{,ij}^{(1)} \phi^{(2)} - \phi_{,ij}^{(2)} \phi^{(1)}] dV + \int_V \alpha_{ij}[\psi_{,ij}^{(1)} \phi^{(2)} - \psi_{,ij}^{(2)} \phi^{(1)}] dV - \int_V g_{ij}[e_{ij}^{(1)} \phi^{(2)} - e_{ij}^{(2)} \phi^{(1)}] dV - \beta_1 \int_V [\psi^{(1)} \phi^{(2)} - \psi^{(2)} \phi^{(1)}] dV + \kappa_1 \int_V [\theta^{(1)} \phi^{(2)} - \theta^{(2)} \phi^{(1)}] dV + w \int_V [P^{(1)} \phi^{(2)} - P^{(2)} \phi^{(1)}] dV + \rho g \int_V [\phi^{(2)} - \phi^{(1)}] dV = 0. \tag{5.17}$$

Now

$$\phi_{,ij}^{(1)} \phi^{(2)} = (\phi_{,j}^{(1)} \phi^{(2)})_{,i} - \phi_{,j}^{(1)} \phi_{,i}^{(2)}, \qquad \phi_{,ij}^{(2)} \phi^{(1)} = (\phi_{,j}^{(2)} \phi^{(1)})_{,i} - \phi_{,j}^{(2)} \phi_{,i}^{(1)}, \tag{5.18}$$

and

$$\psi_{,ij}^{(1)} \phi^{(2)} = (\psi_{,j}^{(1)} \phi^{(2)})_{,i} - \psi_{,j}^{(1)} \phi_{,i}^{(2)}, \qquad \psi_{,ij}^{(2)} \phi^{(1)} = (\psi_{,j}^{(2)} \phi^{(1)})_{,i} - \psi_{,j}^{(2)} \phi_{,i}^{(1)}. \tag{5.19}$$

Equation (5.17) with the help of equations (5.1), (5.2), (5.18), (5.19) and divergence theorem gives

$$\int_A [\wp^{(1)}\Phi^{(2)} - \wp^{(2)}\Phi^{(1)}]dA + \int_A [y^{(1)}\Phi^{(2)} - y^{(2)}\Phi^{(1)}]dA - \int_V g_{ij}[e_{ij}^{(1)}\phi^{(2)} - e_{ij}^{(2)}\phi^{(1)}]dV - \beta_1 \int_V [\psi^{(1)}\phi^{(2)} - \psi^{(2)}\phi^{(1)}]dV + \kappa_1 \int_V [\theta^{(1)}\phi^{(2)} - \theta^{(2)}\phi^{(1)}]dV + w\int_V [P^{(1)}\phi^{(2)} - P^{(2)}\phi^{(1)}]dV + \rho g \int_V [\phi^{(2)} - \phi^{(1)}]dV = 0. \tag{5.20}$$

For the derivation of the third part, multiply Eq. (5.8) by $\psi^{(2)}$ and $\psi^{(1)}$ for the first and second problems respectively, subtracting and integrating over V, we get

$$\int_V \alpha_{ij}[\phi_{,ij}^{(1)}\psi^{(2)} - \phi_{,ij}^{(2)}\psi^{(1)}]dV + \int_V f_{ij}[\psi_{,ij}^{(1)}\psi^{(2)} - \psi_{,ij}^{(2)}\psi^{(1)}]dV - \int_V h_{ij}[e_{ij}^{(1)}\psi^{(2)} - e_{ij}^{(2)}\psi^{(1)}]dV - \beta_1 \int_V [\phi^{(1)}\psi^{(2)} - \phi^{(2)}\psi^{(1)}]dV + \kappa_2 \int_V [\theta^{(1)}\psi^{(2)} - \theta^{(2)}\psi^{(1)}]dV + \nu\int_V [P^{(1)}\psi^{(2)} - P^{(2)}\psi^{(1)}]dV + \rho l \int_V [\psi^{(2)} - \psi^{(1)}]dV = 0. \tag{5.21}$$

Now

$$\psi_{,ij}^{(1)}\psi^{(2)} = (\psi_{,j}^{(1)}\psi^{(2)})_{,i} - \psi_{,j}^{(1)}\psi_{,i}^{(2)}, \quad \psi_{,ij}^{(2)}\psi^{(1)} = (\psi_{,j}^{(2)}\psi^{(1)})_{,i} - \psi_{,j}^{(2)}\psi_{,i}^{(1)}, \tag{5.22}$$

and

$$\phi_{,ij}^{(1)}\psi^{(2)} = (\phi_{,j}^{(1)}\psi^{(2)})_{,i} - \phi_{,j}^{(1)}\psi_{,i}^{(2)}, \quad \phi_{,ij}^{(2)}\psi^{(1)} = (\phi_{,j}^{(2)}\psi^{(1)})_{,i} - \phi_{,j}^{(2)}\psi_{,i}^{(1)}. \tag{5.23}$$

Equation (5.21) with the aid of Eqs. (5.1), (5.2), (5.22), (5.23) and divergence theorem gives

$$\int_A [\hbar^{(1)}\Psi^{(2)} - \hbar^{(2)}\Psi^{(1)}]dA + \int_A [x^{(1)}\Psi^{(2)} - x^{(2)}\Psi^{(1)}]dA - \int_V h_{ij}[e_{ij}^{(1)}\psi^{(2)} - e_{ij}^{(2)}\psi^{(1)}]dV - \beta_1 \int_V [\phi^{(1)}\psi^{(2)} - \phi^{(2)}\psi^{(1)}]dV + \kappa_2 \int_V [\theta^{(1)}\psi^{(2)} - \theta^{(2)}\psi^{(1)}]dV + \nu\int_V [P^{(1)}\psi^{(2)} - P^{(2)}\psi^{(1)}]dV + \rho l \int_V [\psi^{(2)} - \psi^{(1)}]dV = 0, \tag{5.24}$$

To derive the fourth part, multiplying Eq. (5.9) by $\theta^{(2)}$ and $\theta^{(1)}$ for the first and second problems respectively, subtracting and integrating over V, we get

$$\int_V K_{ij}[\theta^{(1)}_{,ij}\theta^{(2)} - \theta^{(2)}_{,ij}\theta^{(1)}]dV$$
$$= (r+\tau_0 r^2)T_0[\kappa_1 \int_V [\phi^{(1)}\theta^{(2)} - \phi^{(2)}\theta^{(1)}]dV + \kappa_2 \int_V [\psi^{(1)}\theta^{(2)} - \psi^{(2)}\theta^{(1)}]$$
$$+ \int_V s_{ij}[e^{(1)}_{ij}\theta^{(2)} - e^{(2)}_{ij}\theta^{(1)}]dV + s\int_V [P^{(1)}\theta^{(2)} - P^{(2)}\theta^{(1)}]dV], \tag{5.25}$$

Now

$$\theta^{(1)}_{,ij}\theta^{(2)} = (\theta^{(1)}_{,j}\theta^{(2)})_{,i} - \theta^{(1)}_{,j}\theta^{(2)}_{,i}, \qquad \text{and}$$
$$\theta^{(2)}_{,ij}\theta^{(1)} = (\theta^{(2)}_{,j}\theta^{(1)})_{,i} - \theta^{(2)}_{,j}\theta^{(1)}_{,i}. \tag{5.26}$$

Equation (5.25) with the help of Eqs. (5.1), (5.2), (5.26) and the divergence theorem can be written as

$$\int_A [q^{(1)}\varpi^{(2)} - q^{(2)}\varpi^{(1)}]dA$$
$$= (r+\tau_0 r^2)T_0[\kappa_1 \int_V [\phi^{(1)}\theta^{(2)} - \phi^{(2)}\theta^{(1)}]dV + \kappa_2 \int_V [\psi^{(1)}\theta^{(2)} - \psi^{(2)}\theta^{(1)}]$$
$$+ \int_V s_{ij}[e^{(1)}_{ij}\theta^{(2)} - e^{(2)}_{ij}\theta^{(1)}]dV + s\int_V [P^{(1)}\theta^{(2)} - P^{(2)}\theta^{(1)}]dV]. \tag{5.27}$$

To derive the last part, multiplying Eq. (5.10) by $P^{(2)}$ and $P^{(1)}$ for the first and second problems respectively, subtracting and integrating over V, we obtain

$$\int_V d_{ij}[P^{(1)}_{,ij}P^{(2)} - P^{(2)}_{,ij}P^{(1)}]dV$$
$$= (r+\tau^0 r^2)[\int_V l_{ij}[e^{(1)}_{ij}P^{(2)} - e^{(2)}_{ij}P^{(1)}]dV + w\int_V [\phi^{(1)}P^{(2)} - \phi^{(2)}P^{(1)}]dV$$
$$+ \nu\int_V [\psi^{(1)}P^{(2)} - \psi^{(2)}P^{(1)}]dV + s\int_V [\theta^{(1)}P^{(2)} - \theta^{(2)}P^{(1)}]dV]. \tag{5.28}$$

Consider

$$P^{(1)}_{,ij}P^{(2)} = (P^{(1)}_{,j}P^{(2)})_{,i} - P^{(1)}_{,j}P^{(2)}_{,i}, \qquad \text{and}$$
$$P^{(2)}_{,ij}P^{(1)} = (P^{(2)}_{,j}P^{(1)})_{,i} - P^{(2)}_{,j}P^{(1)}_{,i}. \tag{5.29}$$

Equation (5.28) with the aid of Eqs. (5.1), (5.2), (5.29) and the divergence theorem yields

$$(5.30)\qquad \int_A (p^{(1)}\varsigma^{(2)} - p^{(2)}\varsigma^{(1)})dA$$
$$= (r+\tau^0 r^2)[\int_V l_{ij}[e_{ij}^{(1)}P^{(2)} - e_{ij}^{(2)}P^{(1)}]dV + w\int_V [\phi^{(1)}P^{(2)} - \phi^{(2)}P^{(1)}]dV$$
$$+ \nu \int_V [\psi^{(1)}P^{(2)} - \psi^{(2)}P^{(1)}]dV + s\int_V [\theta^{(1)}P^{(2)} - \theta^{(2)}P^{(1)}]dV].$$

Eliminating the integrals $\int_V g_{ij}[e_{ij}^{(2)}\phi^{(1)} - e_{ij}^{(1)}\phi^{(2)}]dV$, $\int_V h_{ij}[e_{ij}^{(2)}\psi^{(1)} - e_{ij}^{(1)}\psi^{(2)}]dV$, $\int_V s_{ij}[e_{ij}^{(2)}\theta^{(1)} - e_{ij}^{(1)}\theta^{(2)}]dV$, $\int_V l_{ij}[e_{ij}^{(2)}P^{(1)} - e_{ij}^{(1)}P^{(2)}]dV$, $s\int_V [P^{(2)}\theta^{(1)} - P^{(1)}\theta^{(2)}]dV$, $\kappa_1 \int_V [\phi^{(2)}\theta^{(1)} - \phi^{(1)}\theta^{(2)}]dV$ and $\kappa_2 \int_V [\psi^{(2)}\theta^{(1)} - \psi^{(1)}\theta^{(2)}]dV$ from Eqs. (5.16), (5.20), (5.24), (5.27) and (5.30), we obtain

$$(5.31)\qquad r(1+\tau_0 r)(1+\tau^0 r)T_0\Big[\int_A [f_i^{(1)}u_i^{(2)} - f_i^{(2)}u_i^{(1)}]dA$$
$$+ \rho\int_V [F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)}]dV + \rho\int_V [g(\phi^{(2)} - \phi^{(1)}) + l(\psi^{(2)} - \psi^{(1)})]dV\Big]$$
$$- (1+\tau^0 r)\int_A (q^{(1)}\varpi^{(2)} - q^{(2)}\varpi^{(1)})dA - (1+\tau_0 r)T_0\int_A [p^{(1)}\varsigma^{(2)} - p^{(2)}\varsigma^{(1)}]dA$$
$$+ r(1+\tau_0 r)(1+\tau^0 r)T_0[\int_A [(\wp^{(1)} + y^{(1)})\phi^{(2)} - (\wp^{(2)} + y^{(2)})\phi^{(1)}$$
$$+ (\hbar^{(1)} + x^{(1)})\psi^{(2)} - (\hbar^{2)} + x^{(2)})\psi^{(1)}]dA] = 0.$$

This is the general reciprocity theorem in the Laplace transform domain.

For applying the inverse Laplace transform on Eqs. (5.16), (5.20), (5.24), (5.27), (5.30) and (5.31), we shall use the convolution theorem

$$(5.32)\qquad \pounds^{-1}\{F(r)G(r)\} = \int_0^t \tilde{f}(t-\upsilon)\tilde{g}(\upsilon)d\upsilon = \int_0^t \tilde{g}(t-\upsilon)\tilde{f}(\upsilon)d\upsilon,$$

and the symbolic notations

$$(5.33)\qquad \hat{F}_1(\tilde{f}) = 1 + \tau_0\frac{\partial \tilde{f}(\mathbf{x},\upsilon)}{\partial \upsilon}, \qquad \hat{F}_2 = 1 + \tau^0\frac{\partial \tilde{f}(\mathbf{x},\upsilon)}{\partial \upsilon},$$

$$(5.34)\qquad \hat{F}_3 = 1 + (\tau_0 + \tau^0)\frac{\partial \tilde{f}(\mathbf{x},\upsilon)}{\partial \upsilon} + \tau_0\tau^0\frac{\partial^2 \tilde{f}(\mathbf{x},\upsilon)}{\partial \upsilon^2}.$$

Equations (5.16), (5.20), (5.24), (5.27) and (5.30) with the aid of Eq. (5.32) yield the first, second, third, fourth and fifth parts of the reciprocity theorem in the final form

$$\begin{aligned}\int_A\int_0^t f_i^{(1)}(\mathbf{x},t-\upsilon)u_i^{(2)}(\mathbf{x},\upsilon)d\upsilon dA+\rho\int_V\int_0^t F_i^{(1)}(\mathbf{x},t-\upsilon)u_i^{(2)}(\mathbf{x},\upsilon)d\upsilon dV\\ -\int_V\int_0^t g_{ij}\phi^{(1)}(\mathbf{x},t-\upsilon)e_{ij}^{(2)}(\mathbf{x},\upsilon)d\upsilon dV-\int_V\int_0^t h_{ij}\psi^{(1)}(\mathbf{x},t-\upsilon)e_{ij}^{(2)}(\mathbf{x},\upsilon)d\upsilon dV\\ +\int_V\int_0^t s_{ij}\theta^{(1)}(\mathbf{x},t-\upsilon)e_{ij}^{(2)}(\mathbf{x},\upsilon)d\upsilon dV+\int_V\int_0^t l_{ij}P^{(1)}(\mathbf{x},t-\upsilon)e_{ij}^{(2)}(\mathbf{x},\upsilon)d\upsilon dV=S_{21}^{12},\end{aligned} \tag{5.35}$$

$$\begin{aligned}\int_A\int_0^t[\wp^{(1)}(\mathbf{x},t-\upsilon)+y^{(1)}(\mathbf{x},t-\upsilon)]\Phi^{(2)}(\mathbf{x},\upsilon)d\upsilon dA\\ +\int_V\int_0^t g_{ij}\phi^{(1)}(\mathbf{x},t-\upsilon)e_{ij}^{(2)}(\mathbf{x},\upsilon)d\upsilon dV+\beta_1\int_V\int_0^t\phi^{(1)}(\mathbf{x},t-\upsilon)\psi^{(2)}(\mathbf{x},\upsilon)d\upsilon dV\\ -\kappa_1\int_V\int_0^t\phi^{(1)}(\mathbf{x},t-\upsilon)\theta^{(2)}(\mathbf{x},\upsilon)d\upsilon dV-w\int_V\int_0^t\phi^{(1)}(\mathbf{x},t-\upsilon)P^{(2)}(\mathbf{x},\upsilon)d\upsilon dV\\ -\rho g\int_V\phi^{(1)}(\mathbf{x},t)dV=S_{21}^{12},\end{aligned} \tag{5.36}$$

$$\begin{aligned}\int_A\int_0^t[\hbar^{(1)}(\mathbf{x},t-\upsilon)+x^{(1)}(\mathbf{x},t-\upsilon)]\Psi^{(2)}(\mathbf{x},\upsilon)d\upsilon dA\\ +\int_V\int_0^t h_{ij}\psi^{(1)}(\mathbf{x},t-\upsilon)e_{ij}^{(2)}(\mathbf{x},\upsilon)d\upsilon dV+\beta_1\int_V\int_0^t\psi^{(1)}(\mathbf{x},t-\upsilon)\phi^{(2)}(\mathbf{x},\upsilon)d\upsilon dV\\ -\kappa_2\int_V\int_0^t\psi^{(1)}(\mathbf{x},t-\upsilon)\theta^{(2)}(\mathbf{x},\upsilon)d\upsilon dV-\nu\int_V\int_0^t\psi^{(1)}(\mathbf{x},t-\upsilon)P^{(2)}(\mathbf{x},\upsilon)d\upsilon dV\\ -\rho l\int_V\psi^{(1)}(\mathbf{x},t)dV=S_{21}^{12},\end{aligned} \tag{5.37}$$

$$\begin{aligned}(5.38)\qquad &\int_A \int_0^t q^{(1)}(\mathbf{x},t-\upsilon)\varpi^{(2)}(\mathbf{x},\upsilon)d\upsilon dA\\
&+\kappa_1 T_0 \int_V \int_0^t \theta^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_1(\phi^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\\
&+\kappa_2 T_0 \int_V \int_0^t \theta^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_1(\psi^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\\
&+T_0 \int_V \int_0^t s_{ij}\theta^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_1(e_{ij}^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\\
&+sT_0 \int_V \int_0^t \theta^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_1(P^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV = S_{21}^{12},\end{aligned}$$

$$\begin{aligned}(5.39)\qquad &\int_A \int_0^t p^{(1)}(\mathbf{x},t-\upsilon)\varsigma^{(2)}(\mathbf{x},\upsilon)d\upsilon dA\\
&+w \int_V \int_0^t P^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_2(\phi^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\\
&+\nu \int_V \int_0^t P^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_2(\psi^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\\
&+\int_V \int_0^t l_{ij}P^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_2(e_{ij}^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\\
&+s \int_V \int_0^t P^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_2(\theta^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV = S_{21}^{12},\end{aligned}$$

Here S_{21}^{12} indicates the same expression as on the left-hand side except that the superscripts (1) and (2) are interchanged.

Finally, Eq. (5.31) with the aid of Eq. (5.32) gives the general reciprocity theorem in the final form

$$\begin{aligned}(5.40)\qquad &\int_A \int_0^t f_i^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_3(u_i^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dA\\
&+\rho \int_V \int_0^t F_i^{(1)}(\mathbf{x},t-\upsilon)\frac{\partial \hat{F}_3(u_i^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dV\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{T_0}\int\limits_A \int\limits_0^t q^{(1)}(\mathbf{x}, t-\upsilon)\hat{F}_2(\varpi^{(2)})(\mathbf{x},\upsilon)d\upsilon dA \\
&- \int\limits_A \int\limits_0^t p^{(1)}(\mathbf{x}, t-\upsilon)\hat{F}_1(\varsigma^{(2)})(\mathbf{x},\upsilon)d\upsilon dA \\
&+ \int\limits_A \int\limits_0^t [\wp^{(1)}(\mathbf{x}, t-\upsilon) + y^{(1)}(\mathbf{x}, t-\upsilon)]\frac{\partial \hat{F}_3(\varPhi^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dA \\
&+ \int\limits_A \int\limits_0^t [\hbar^{(1)}(\mathbf{x}, t-\upsilon) + x^{(1)}(\mathbf{x}, t-\upsilon)]\frac{\partial \hat{F}_3(\varPsi^{(2)})}{\partial \upsilon}(\mathbf{x},\upsilon)d\upsilon dA \\
&- \rho g \int\limits_V \left(1 + \tau^0 \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\phi^{(1)}(\mathbf{x},t)dV \\
&- \rho l \int\limits_V \left(1 + \tau^0 \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\psi^{(1)}(\mathbf{x},t)dV = S_{21}^{12}.
\end{aligned}
$$

6. Conclusions

The following results are obtained in the current paper:

1) The linear theory of thermoelastic diffusion with double porosity has derived without using Darcy's law. This theory can be useful for finding fundamental solutions, studying the wave phenomenon etc.

2) The variational principle and uniqueness theorems have been proved on the basis of Biot's principle [36].

3) The reciprocity theorem has been derived with the help of Laplace and inverse Laplace transforms.

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