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Surface Green's functions in finite plane elastostatics of harmonic materials

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The closed-form representations of surface Green's functions corresponding to the action of a concentrated force applied at the boundary of a region occupied by a particular class of compressible hyperelastic materials of harmonic type, has been derived. In our analysis, we consider both a bounded region in the form of a circular disk and an unbounded region with either an elliptical hole or a parabolic boundary.

Key words: harmonic material, surface Green's function, concentrated force, complex variable formulation.

1. Introduction

Surface Green's functions which arise from problems concerning the action of a concentrated force on the boundary of a (bounded or unbounded) region occupied by (isotropic or anisotropic) linearly elastic materials have been the subject of intense research in the literature (see, for example, [1–4]). In contrast, the corresponding Green's functions arising from analogous problems in hyperelastic materials, appear rarely in the literature. We note, in particular, the contributions of [5–8] in which several Green's functions have been obtained for small deformations superimposed on finitely-strained nonlinear elastic materials. The model describing the plane strain deformations of a particular class of compressible hyperelastic materials (known as Harmonic Materials as first proposed by John [9]) affords a particular advantage in that its complex variable formulation (originally presented by Varley and Cumberbatch [10] and later developed by Ru [11]) is relatively straightforward allowing for the analysis of a range of problems in this class of hyperelastic materials (see, for example, [12–19]).

In this paper, we apply Ru's complex variable formulation [11] to derive the surface Green's functions corresponding to the plane strain deformations of a region occupied by compressible hyperelastic materials of harmonic type in which a concentrated force is applied to the boundary of the region. More precisely, we derive surface Green's functions in the case of: (i) a circular disk; (ii) an unbounded region with parabolic boundary; (iii) an unbounded region weakened by an elliptical hole. The corresponding Green's functions are found in a closed-form with the aid of analytic continuation and conformal mapping techniques.

2. Complex variable formulation

The model of compressible hyperelastic materials of harmonic type was first proposed by John [9]. In this section, we present the basic equations describing the complex variable formulation developed by Ru [11].

Let the complex variable $z = x_1 + \mathrm{i}x_2$ represent the initial coordinates of a material particle in the undeformed configuration and $w(z) = y_1(z) + \mathrm{i}y_2(z)$ the corresponding spatial coordinates in the deformed configuration. Define the deformation gradient tensor by the Cartesian components

$$F_{ij} = \frac{\partial y_i}{\partial x_j}. \tag{2.1}$$

We consider a particular class of harmonic materials, whose strain energy density W is given by

$$W = 2\mu[F(I) - J], \qquad F'(I) = \frac{1}{4\alpha}\left[I + \sqrt{I^2 - 16\alpha\beta}\right]. \tag{2.2}$$

where μ is the shear modulus and $1/2 \le \alpha < 1$, $\beta > 0$ are two material constants, and I and J are the scalar invariants of the tensor $\mathbf{F}\mathbf{F}^T$ given by

$$I = \lambda_1 + \lambda_2 = \sqrt{F_{ij}F_{ij} + 2J}, \qquad J = \lambda_1\lambda_2 = \det[F_{ij}]. \tag{2.3}$$

According to the formulation developed by Ru [11], the deformation w can be written in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ as

$$\mathrm{i}w(z) = \alpha\varphi(z) + \mathrm{i}\overline{\psi(z)} + \frac{\beta z}{\overline{\varphi'(z)}}, \tag{2.4}$$

and the complex Piola stress function χ is given by

$$\chi(z) = 2\mathrm{i}\mu\left[(\alpha - 1)\varphi(z) + \mathrm{i}\overline{\psi(z)} + \frac{\beta z}{\overline{\varphi'(z)}}\right]. \tag{2.5}$$

In addition, the Piola stress components can be written in terms of the Piola stress function χ as

$$-\sigma_{21} + \mathrm{i}\sigma_{11} = \chi_{,2}, \qquad \sigma_{22} - \mathrm{i}\sigma_{12} = \chi_{,1}. \tag{2.6}$$

3. Surface Green's functions

In this section, the complex variable formulation mentioned above is applied to the derivation of surface Green's functions in each of the cases of: (i) a circular disk; (ii) an unbounded domain with parabolic boundary; (iii) an unbounded domain with elliptical hole.

3.1. A circular disk

We first consider a circular disk of radius R with its center at the origin (Fig. 1). Two equal and opposite vertical forces P are applied at $z = z_0$ $(0 \leq \mathrm{Im}\{z_0\} \leq R)$ and $z = \bar{z}_0$ on the surface of the disk.

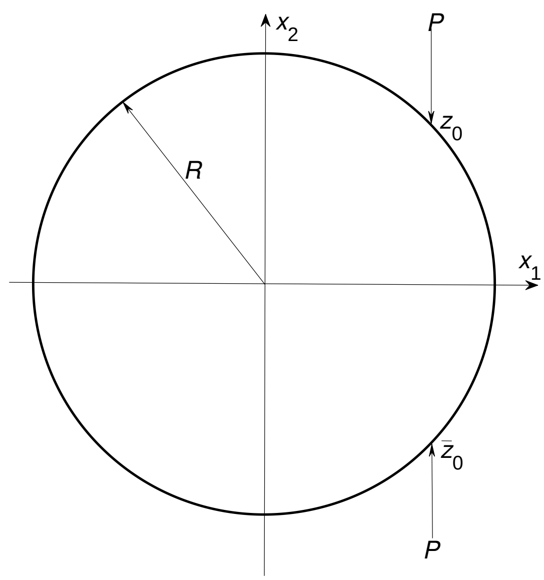

Fig. 1. A circular disk of radius R with two equal and opposite vertical forces P being applied at $z = z_0$ and $z = \bar{z}_0$ on its surface.

Using analytic continuation, the otherwise traction-free boundary condition on $|z| = R$ can be expressed as

$$(\alpha - 1)\varphi^{+}(z) + \mathrm{i}\bar{\psi}^{-}(R^2/z) + \frac{\beta z}{\bar{\varphi}'^{-}(R^2/z)} = 0, \qquad |z| = R. \tag{3.1}$$

The solution to the above is found using the generalized Liouville's theorem as

$$
\begin{aligned}
&\varphi(z) = \frac{P}{4\pi\mu(\alpha-1)} \ln\frac{z-z_0}{z-\bar{z}_0} - \frac{\beta z}{\overline{\varphi'(0)}(\alpha-1)},\\
&\psi(z) = -\frac{\mathrm{i}P}{4\pi\mu} \ln\frac{z-z_0}{z-\bar{z}_0}\\
&\quad + \frac{\beta R^2 P\,\mathrm{Im}\{z_0\}[z-2\,\mathrm{Re}\{z_0\}]}{\varphi'(0)\{-\mathrm{i}Pz\,\mathrm{Im}\{z_0\}[z-2\,\mathrm{Re}\{z_0\}]+2\pi\mu R^2(\alpha-1)\varphi'(0)(z-z_0)(z-\bar{z}_0)\}},
\end{aligned}
\tag{3.2}
$$

for $|z| \le R$.

The consistency condition for $\varphi'(0)$ will yield the following non-linear equation for $\varphi'(0)$

$$
(\alpha-1)\varphi'(0) + \frac{\beta}{\overline{\varphi'(0)}} = \frac{\mathrm{i}P\,\mathrm{Im}\{z_0\}}{2\pi\mu R^2},
\tag{3.3}
$$

the solution of which is given by

$$
\varphi'(0) = \frac{\mathrm{i}P\,\mathrm{Im}\{z_0\}}{4\pi\mu R^2(\alpha-1)} \pm \mathrm{i}\sqrt{\left(\frac{P\,\mathrm{Im}\{z_0\}}{4\pi\mu R^2(1-\alpha)}\right)^2 + \frac{\beta}{1-\alpha}}.
\tag{3.4}
$$

In order to ensure that $\varphi'(z) \neq 0$ for $|z| \le R$ [8], the following inequality should be satisfied

$$
\left|\mathrm{Re}\{z_0\} \pm \sqrt{(\mathrm{Re}\{z_0\})^2 - R^2 + \frac{\mathrm{i}P\overline{\varphi'(0)}\,\mathrm{Im}\{z_0\}}{2\pi\mu\beta}}\right| > R.
\tag{3.5}
$$

3.2. A parabolic boundary

Next, as shown in Fig. 2, we consider a harmonic material that occupies the region

$$
x_2 \le b x_1^2, \qquad b > 0,
\tag{3.6}
$$

the boundary of which is a parabola described by

$$
x_2 = b x_1^2.
\tag{3.7}
$$

A concentrated force is applied at $z = x_1^0 + \mathrm{i}x_2^0$ on the parabola. Let X and Y be the force components in the x_1 and x_2 directions, respectively. The parabola reduces to a plane boundary when $b = 0$ and to a semi-infinite crack when $b \to \infty$. We introduce the following conformal mapping function

$$
z = \omega(\xi) = \xi + \mathrm{i}b\xi^2, \quad \xi = \omega^{-1}(z) = \frac{\sqrt{1+4\mathrm{i}bz}-1}{2\mathrm{i}b}, \quad \mathrm{Im}\{\xi\} \le 0.
\tag{3.8}
$$

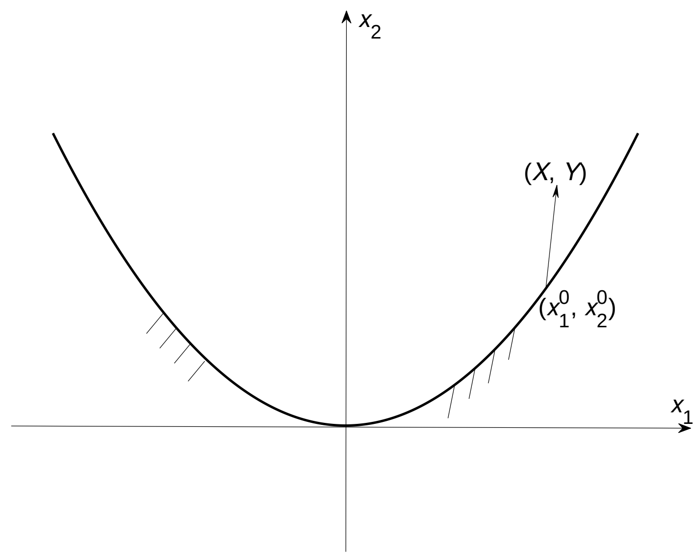

FIG. 2. A region with parabolic boundary.

Using this mapping function, the region occupied by the harmonic material in the z-plane is mapped onto the region $\mathrm{Im}\{\xi\} \leq 0$ in the ξ-plane and the parabola in the z-plane is mapped onto $\mathrm{Im}\{\xi\} = 0$ in the ξ-plane. Applying analytic continuation, the otherwise traction-free boundary condition on $\mathrm{Im}\{\xi\} = 0$ can be expressed as

$$(\alpha - 1)\varphi^{-}(\xi) + \mathrm{i}\bar{\psi}^{+}(\xi) + \frac{\beta\bar{\omega}'(\xi)\omega(\xi)}{\bar{\varphi}'^{+}(\xi)} = 0, \qquad \mathrm{Im}\{\xi\} = 0, \tag{3.9}$$

where $\varphi(\xi) = \varphi(\omega(\xi))$ and $\psi(\xi) = \psi(\omega(\xi))$.

The generalized Liouville's theorem now leads to the solution:

$$\begin{aligned} \varphi(\xi) &= \frac{Y - \mathrm{i}X}{4\pi\mu(1-\alpha)} \ln(\xi - x_1^0) + A(\xi + \mathrm{i}b\xi^2), \\ \psi(\xi) &= \frac{-X + \mathrm{i}Y}{4\pi\mu} \ln(\xi - x_1^0) \\ &\quad + \frac{\bar{A}(1-\alpha)(X + \mathrm{i}Y)(\xi - \mathrm{i}b\xi^2)}{Y - \mathrm{i}X + 4\pi\mu A(1-\alpha)(2\mathrm{i}b\xi + 1)(\xi - x_1^0)}, \quad \mathrm{Im}\{\xi\} \leq 0, \end{aligned} \tag{3.10}$$

where $|A| = \sqrt{\beta/(1-\alpha)}$ and the phase angle of A is arbitrary.

In order to ensure that $\varphi'(\xi) \neq 0$ for $Im\{\xi\} \leq 0$, the following inequality should be satisfied

$$1 \pm \mathrm{Re}\left\{\sqrt{(2\mathrm{i}bx_1^0 + 1)^2 - \frac{2b(X + \mathrm{i}Y)}{\pi\mu A(1-\alpha)}}\right\} > 0. \tag{3.11}$$

When $b = 0$, the solution in Eq. (3.10) reduces to that found in Wang et al. [14] for a concentrated force acting on the surface of a half-plane (the present authors note a sign error in Eq. (25) in [14]).

3.3. An elliptical hole

Finally, as shown in Fig. 3, we consider a harmonic material that occupies a region

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \geq 1, \qquad a \geq b \geq 0, \tag{3.12}$$

the boundary of which is an ellipse described by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1. \tag{3.13}$$

A concentrated force is applied at $z = z_0$ on the ellipse. Let X and Y be the force components in the x_1 and x_2 directions, respectively. In this case, we

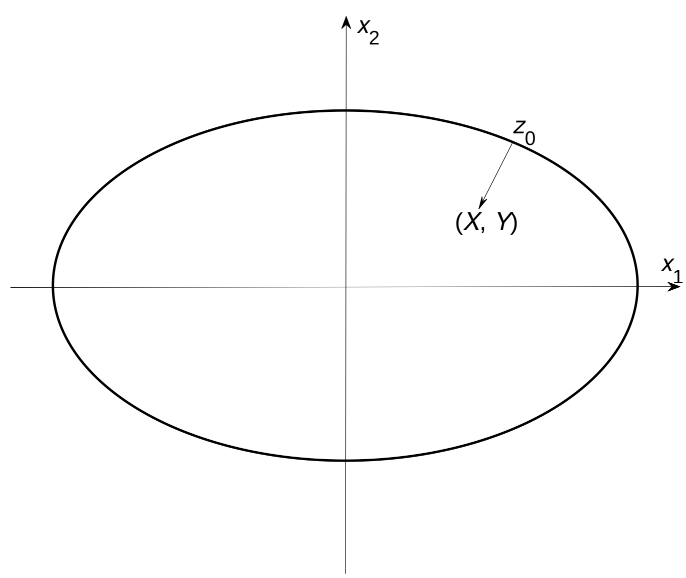

Fig. 3. An unbounded region weakened by an elliptical hole.

introduce the following conformal mapping function

$$z = \omega(\xi) = R(\xi + m\xi^{-1}), \tag{3.14}$$
$$\xi = \omega^{-1}(z) = \frac{z}{2R}\left[1 + \sqrt{1 - \frac{4mR^2}{z^2}}\right], \quad |\xi| \geq 1,$$

where

$$R = \frac{a+b}{2}, \qquad m = \frac{a-b}{a+b}. \tag{3.15}$$

Using this mapping function, the region occupied by the harmonic material in the z-plane is mapped onto $|\xi| \geq 1$ in the ξ-plane; the elliptical boundary in the z-plane is mapped onto $|\xi| = 1$, in the ξ-plane and the point $z = z_0$ is mapped to the point $\xi = \xi_0 = \omega^{-1}(z_0)$. Using analytic continuation, the otherwise traction-free boundary condition on $|\xi| = 1$ can be expressed as

$$(\alpha - 1)\varphi^-(\xi) + \mathrm{i}\bar{\psi}^+(1/\xi) + \frac{\beta\bar{\omega}'(1/\xi)\omega(\xi)}{\bar{\varphi}'^+(1/\xi)} = 0, \qquad |\xi| = 1, \tag{3.16}$$

where $\varphi(\xi) = \varphi(\omega(\xi))$ and $\psi(\xi) = \psi(\omega(\xi))$ as in Sec. 3.2.

Once again, the generalized Liouville's theorem is applied leading eventually to the solution

$$\varphi(\xi) = \frac{Y-\mathrm{i}X}{4\pi\mu(1-\alpha)}\ln(\xi-\xi_0) - \frac{\alpha(Y-\mathrm{i}X)}{4\pi\mu(1-\alpha)}\ln\xi + AR\xi + \frac{\beta mR}{\bar{A}(1-\alpha)}\xi^{-1},$$
$$\psi(\xi) = \frac{-X+\mathrm{i}Y}{4\pi\mu}\ln(\xi-\xi_0) + \frac{(\alpha-1)(-X+\mathrm{i}Y)}{4\pi\mu}\ln\xi - \mathrm{i}\bar{A}R(\alpha-1)\xi^{-1} \tag{3.17}$$
$$+ \frac{\frac{\beta mR(X+\mathrm{i}Y)}{4\pi\mu A(1-\alpha)}\frac{\xi_0}{\xi-\xi_0} + \frac{\beta mR(X+\mathrm{i}Y)}{4\pi\mu A} + \mathrm{i}\beta R^2(m\xi^{-3}-\xi^{-1})}{\frac{Y-\mathrm{i}X}{4\pi\mu(1-\alpha)}\frac{1}{\xi-\xi_0} - \frac{\alpha(Y-\mathrm{i}X)}{4\pi\mu(1-\alpha)}\xi^{-1} - \frac{\beta mR}{\bar{A}(1-\alpha)}\xi^{-2} + AR}, \quad |\xi| \geq 1,$$

where $|A| = \sqrt{\beta/(1-\alpha)}$ and the phase angle of A can be arbitrary.

From Eq. (3.17) we see that

$$\varphi(\xi) \cong \frac{Y - \mathrm{i}X}{4\pi\mu}\ln\xi + AR\xi + O(1), \tag{3.18}$$
$$\psi(\xi) \cong \frac{\alpha(-X + \mathrm{i}Y)}{4\pi\mu}\ln\xi + O(1), \quad \text{as } |\xi| \to \infty,$$

as expected. It is relatively straightforward to check that the displacement $w - z$ is single-valued for a contour C surrounding the elliptical hole and that

$$\int_C (\mathrm{d}\chi_1 + \mathrm{i}\mathrm{d}\chi_2) = Y - \mathrm{i}X, \tag{3.19}$$

with C taken in the counterclockwise direction.

In order to ensure that $\varphi'(\xi) \neq 0$ for $|\xi| \geq 1$, all three roots of the following cubic equation in ξ should lie within the unit circle

$$\xi^3 + \left(\frac{Y - \mathrm{i}X}{4\pi\mu AR} - \xi_0\right)\xi^2 - \left[m + \frac{\xi_0\alpha(Y - \mathrm{i}X)}{4\pi\mu AR(\alpha - 1)}\right]\xi + \xi_0 m = 0. \tag{3.20}$$

4. Conclusions

Using complex variable techniques, we derive closed-form representations of the surface Green's functions for the following regions occupied by a particular class of hyperelastic materials of harmonic type: (i) a circular disk; (ii) an unbounded region with a parabolic boundary; (iii) an unbounded region weakened by an elliptical hole. It is expected that our method can be applied successfully with changes only in detail, to the derivation of surface Green's functions in the case of an arbitrary-shaped hole in a region occupied by harmonic materials.

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