

## A study of critical point instability of micro and nano beams under a distributed variable-pressure force in the framework of the inhomogeneous non-linear nonlocal theory

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FRACTIONAL DERIVATIVE MODELS (FDMs) result from introduction of fractional derivatives (FDs) into the governing equations of the differential operator type of linear solid materials. FDMs are more general than those of integer derivative models (IDMs) so they are more fixable to describe physical phenomena. In this paper the inhomogeneous nonlocal theory has been introduced based on conformable fractional derivatives (CFD) to study the critical point instability of micro/nano beams under a distributed variable-pressure force. The phase of distributed variable-pressure force is used for electrostatic force, electromagnetic force and so on. This model has two free parameters: i) parameter to control the order of inhomogeneity in constitutive relation that gives a general form to the model, and ii) a nonlocal parameter to consider size dependence effects in micron and sub-micron scales. As a case study the theory has been used to model micro cantilever (C-F) and doubly-clamped (C-C) silicon beams under a distributed uniform electrostatic force in the presence of von-Karman non-linearity and their static critical point (static pull-in instability), moreover, effects of different inhomogeneity have been shown on the pull-in instability.

**Key words:** inhomogeneous nonlocal theory, conformable fractional derivative, pull-in voltage, electrostatic force.

## 1. Introduction

MICRO/NANO STRUCTURES ARE WIDELY USED in the micro/nano electro-mechanical systems (MEMS & NEMS), for example in micro-pumps, micro/nano mirrors, accelerometers, micro and nano switches. A critical point of such micro structures under a distributed variable-pressure force (electrostatic, electromagnetic and so on) is one of the most important phenomena that should be considered in the design, analysis and simulation of the MEMS/NEMS systems.

Let one assume a micro/nano beam in the electrostatic field that contains two electrodes, upper movable electrode which can be modelled as an elastic beam deflects towards the fixed electrode due to electrostatic attraction as an example. As the voltage increases, the deflection of the micro beam also increases. At a certain voltage, the movable electrode becomes unstable and collapses to the fixed electrode, the excitation voltage corresponding to the instability is called the static pull-in instability, and in this case, the critical point is the pull-in voltage. Many works have been done on the dynamic and static instability [1–9] but none of them considered the effects of FDs.

At the small-scale, the size of micro and nano structures often becomes prominent. Both experimental [10] and atomistic simulation [11] results have shown a significant ‘size-effect’ in the mechanical properties when the dimensions of these structures become small. From this point of view, scale-free continuum models may not be directly applicable in the nano and micro world, thus size-dependent continuum models are necessary [12]. The vast majority of structural theories are derived using the constitutive assumptions that the stress at a point depends only on the strain at the point, so-called local formulations. On the other hand, the nonlocal constitutive behaviour advanced by ERINGEN [13, 14] and ERINGEN and EDELEN [15] is based on the hypothesis that the stress at a point is a function of strains at all points in the continuum.

In Eringen nonlocal theory (ENT) only the integer gradient of stress (usually the second gradient of stress) has been used in constitutive equations but more recently CHALLAMEL *et al.* [16] by using a fractional gradient of the stress generalized Eringen nonlocal theory (FNT) showed that the optimized fractional derivative model gives perfect matching with the dispersive wave properties of the Born–Karman model of lattice dynamics and it is better than ENT (cf. also SUMELKA [17]).

FDMs have more advantages in comparison with classical integer-order models [18]. In the last few decades many authors pointed out that derivatives and integrals of non-integer are very suitable for description of properties of various real materials. It has been shown that fractional-order models are more adequate than previously used integer-order models [19] therefore many efforts concentrate on this field for instance: DEMIR *et al.* [20] studied application of fractional calculus in the dynamic of beam and concluded that the order and the coefficient of

the fractional derivative have a significant effect on the natural frequency and the amplitude of vibrations. LAZOPOULOS [21] introduced fractional derivative strain in the strain energy formulation. FDs have been also recently used to model non-local elastic media for wave propagation applications, which leads to some new spatial nonlocality kernels [22–24]. ATANACKOVIC *et al.* [23] generalized Hooke’s law by replacing the displacement gradient with the symmetrized Caputo spatial fractional derivative. MICHELITSCH [25, 26] performed the generalization of the classical theory on the level of the potential energy, which then led to a fractional derivative of the Marchaud type. Sumelka by using fractional calculus generalized the Kirchhoff–Love plate’s theory [27] and also studied the new concept of nonlocal continuum body definition by utilizing the fractional calculus [18]. More recently, RAHIMI *et al.* [28] have presented a generalized form of nonlocal elasticity theory using conformable fractional derivatives definition (CFDD) and showed its application in non-linear and linear free vibration of nano beams.

In present work like CHALLAMEL *et al.* [16] by introduction of FD a generalization form of ENT [28] has been presented, but here the CFD definition has been used instead of Caputo definition, which leads to a very different form of governing equations that could be solved easily by numerical solution just like an equation with integer derivatives. The resulting model can be classified as a nonlocal theory with inhomogeneous length scale effects based on Eringen’s stress gradient model. The non-linear motion equation of a beam under a distributed variable-pressure force in presence of von-Karman non-linearity has been obtained and as a case study the pull-in voltage of C-C and C-F silicon micro beams have been studied. Herein, it should be pointed out, that there is a debate in the literature about the meaning of the CFD concept [29].

## 2. Mathematical modelling

### 2.1. FDs definitions

Different types of definitions for FDs exist [30–32]. Two of the most popular ones are as bellow:

I: Riemann–Liouville definition. For  $\alpha \in [n - 1, n)$ , the  $\alpha$  derivative of  $f$  is [30–32]

$$(2.1) \quad D_a^\alpha(f)(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(\tau)}{(x - \tau)^{\alpha-n+1}} d\tau.$$

II: Caputo definition. For  $\alpha \in [n - 1, n)$ , the  $\alpha$  derivative of  $f$  is [30–32]

$$(2.2) \quad D_a^\alpha(f)(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{f^n(\tau)}{(x - \tau)^{\alpha-n+1}} d\tau,$$

where  $n - 1 \leq \alpha < n$ , and  $\Gamma$  represents the Euler-gamma function as below:

$$(2.3) \quad \Gamma = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

These definitions have some defects (see KHALIL *et al.* [33]) as a rule solution of governing fractional differential equations is difficult [36–37]. In the present work the new definition of FDs has been used that presented by KHALIL *et al.* [33]. The form of the definition shows that it is the most natural definition, and the very fruitful one [23–37]. Based on this definition, the FDs change to integer forms and they do not comprise integrals, so they can be solved easily by numerical methods (cf. also [29] for debate on CFD controversies).

The CFDD of multi-variables function for  $f : [0, \infty) \rightarrow \mathbb{R}$ , is defined as bellow (Appendix A):

$$(2.4) \quad \begin{aligned} D_x^n(f)(x, y) &= \frac{d^\alpha f(x, y)}{dx^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{[\alpha]-1}(x + \varepsilon x^{[\alpha]-\alpha}, y) - f^{[\alpha]-1}(x, y)}{\varepsilon}, \\ D_y^n(f)(x, y) &= \frac{d^\alpha f(x, y)}{dy^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{[\alpha]-1}(x, y + \varepsilon y^{[\alpha]-\alpha}) - f^{[\alpha]-1}(x, y)}{\varepsilon}, \end{aligned}$$

where  $n - 1 < \alpha \leq n$  and  $[\alpha]$  is the smallest integer number bigger or equal to  $\alpha$ . In the case of  $\alpha = n$  it reduces to:

$$(2.5) \quad \begin{aligned} D_x^n(f)(x, y) &= x^{(n-n)} \frac{d^n f(x, y)}{dx^n} \rightarrow \frac{d^n f(x, y)}{dx^n} = \frac{d^n f(x, y)}{dx^n}, \\ D_y^n(f)(x, y) &= y^{(n-n)} \frac{d^n f(x, y)}{dy^n} \rightarrow \frac{d^n f(x, y)}{dy^n} = \frac{d^n f(x, y)}{dy^n}. \end{aligned}$$

## 2.2. Formulation

In this section, we obtain the equation of motion of the Euler–Bernoulli beam in the presence of von-Karman non-linearity under a distributed variable-pressure force.

Assume the displacement field of the Euler–Bernoulli beam as bellow:

$$(2.6) \quad u_1(x, z) = -z \frac{dw}{dx}, \quad u_2(x, z) = 0, \quad u_3(x, z) = w,$$

where  $w$  is the transverse displacement of the point  $(x, 0)$  on the mid-plane (i.e.,  $z = 0$ ) of the beam. The only nonzero von-Karman non-linear strain in the plane strain condition is:

$$(2.7) \quad \varepsilon_{xx} = -z \frac{d^2 w}{dx^2} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 = \varepsilon + zk, \quad \varepsilon = \frac{1}{2} \left( \frac{dw}{dx} \right)^2, \quad k = -z \frac{d^2 w}{dx^2},$$

where  $\varepsilon$  is the non-linear extensional strain and  $k$  is the bending strain.

Next, the principle of a virtual displacement has the form

$$(2.8) \quad \int (-N_0 \delta \varepsilon - M \delta \kappa + q(x) \delta w) dx = 0,$$

where  $q(x)$  is the transverse distributed force (measured per unit unreformed length) and  $N$  and  $M$  are the stress resultants as bellow:

$$(2.9) \quad N_0 = \int \sigma dA, \quad M = \int \sigma z dA.$$

Finally, one obtains the following Euler–Lagrange equations in  $0 < x < L$

$$(2.10) \quad \frac{dN_0}{dx} = 0 \rightarrow N_0 = cte,$$

$$(2.11) \quad \frac{d^2 M}{dx^2} - \frac{d}{dx} (N_0 \frac{dw}{dx}) + q = 0 \rightarrow \frac{d^2 M}{dx^2} - N_0 \frac{d^2 w}{dx^2} + q(x) = 0.$$

### 2.3. The inhomogeneous nonlocal theory

The general form of the Eringen theory or FNT is as bellow [16]:

$$(2.12) \quad \sigma - \mu^\alpha \frac{d^\alpha \sigma}{dx^\alpha} = E \varepsilon,$$

where  $\sigma$  and  $\varepsilon$  are the uniaxial stress and strain, respectively,  $E$  is the Young modulus,  $\mu = e_0 a$ ,  $e_0$  is a material constant to be determined experimentally,  $a$  is the internal (e.g. lattice parameter, granular size) characteristic length and  $\alpha$  is integer or non-integer number  $n - 1 < \alpha \leq n$ , herein  $n = 2$ . The ENT is obtained when  $\alpha = 2$  and the local form of strain-stress is achieved when  $\mu = 0$ .

As opposed to the ENT, the FNT results in fractional differential relations involving the stress resultants and the strain. In the following, we present these relations for homogeneous isotropic beams under the assumption that the nonlocal behaviour is negligible in the thickness direction. Here we assume  $1 < \alpha \leq 2$  and govern the equations of the motion of the Euler–Bernoulli beam based on a conformable fractional derivative [33].

By using Eq. (2.4) when  $1 < \alpha \leq 2$  we have:

$$(2.13) \quad \frac{d^\alpha \sigma}{dx^\alpha} = x^{2-\alpha} \frac{d^2 \sigma}{dx^2}.$$

Next, by substituting Eq. (2.13) into the Eq. (2.12), the fractional derivative in Eq. (2.12) changes to the integer derivative as follow:

$$(2.14) \quad \sigma - \mu^\alpha x^{2-\alpha} \frac{d^2 \sigma}{dx^2} = E \varepsilon.$$

Herein it should be noticed that application of CFD, results in Eringen's stress gradient model with an inhomogeneous length scale effect – parameter  $\alpha$  which defines the order of CFD is in this sense an inhomogeneity parameter). Integrating Eq. (2.14) over the beam's cross-section area leads to the axial force-strain relation as:

$$(2.15) \quad N_0 - \mu^\alpha x^{2-\alpha} \frac{d^2 N_0}{dx^2} = EA\varepsilon.$$

Next, according to Eq. (2.10)

$$(2.16) \quad N_0 = EA\varepsilon = EA \left( \frac{dw}{dx} \right)^2,$$

where  $A$  is a cross section of the beam. For convenience, the non-linear term of the strain tensor can be averaged along the beam length and consequently a mean value of the generated axial force can be given as [5, 38]:

$$(2.17) \quad N = \frac{EA}{2L} \int_0^L \left( \frac{dw}{dx} \right)^2 dx,$$

where  $L$  is the length of a beam. Integrating over the cross-section area, we obtain the moment curvature relation as bellow:

$$(2.18) \quad M - \mu^\alpha x^{2-\alpha} \frac{d^2 M}{dx^2} = EI\kappa.$$

Finally, taking the second derivative of Eq. (2.18) and substituting the second derivative of  $M$  from Eq. (2.11) into Eq. (2.18), the inhomogeneous nonlocal equation of the Euler–Bernoulli beam is obtained as bellow:

$$(2.19) \quad -EI \frac{d^4 w}{dx^4} + \mu^\alpha N \left[ x^{2-\alpha} \frac{d^4 w}{dx^4} + x^{1-\alpha} 2(2-\alpha) \frac{d^3 w}{dx^3} + x^{-\alpha} (2-\alpha)(1-\alpha) \frac{d^2 w}{dx^2} \right] \\ - N \frac{d^2 w}{dx^2} - \mu^\alpha \left[ x^{2-\alpha} \frac{d^2 q}{dx^2} + x^{1-\alpha} 2(2-\alpha) \frac{dq}{dx} + x^{-\alpha} (2-\alpha)(1-\alpha) q \right] + q = 0,$$

where the external lateral distributed force per unit length  $q(x, t)$  is written as:

$$(2.20) \quad q = \frac{\beta}{(g_0 - w)^n},$$

where  $\beta$  is the parameter which depends on the kind of the force and  $g_0$  is an initial gap. (For instance in the electrostatic force  $\beta$  is  $\gamma V$  that  $\gamma$  is a constant and  $V$  is voltage). As an example  $n = 2$  and  $n = 3$  are for electrostatic and electromagnetic forces and whereas the van der Waals and Casimir force  $n = 4$ .

### 3. Case study

In this part, as a case study, the critical point (static pull-in instability) of a micro/nano beam under electrostatic force has been presented and the effects of inhomogeneity and nonlocal parameters on the pull-in voltages have been shown. Finally, to show the functionality of FDs model versus integer derivatives model, the calculated voltages from FNT, CT and ENT have been compared with experimental data for micro cantilever silicon beams.

One obtains Eq. (2.19) for a micro/nano beam under electrostatic force when the electrostatic force states as bellow [39]:

$$(3.1) \quad q = \frac{\varepsilon_0 b V^2}{2(g_0 - w)^2},$$

where  $\varepsilon_0$  and  $b$  are the dielectric constant of the gap medium and width of the beam, respectively. Figure 1 shows a schematic view of the micro beam in an electrostatic field.

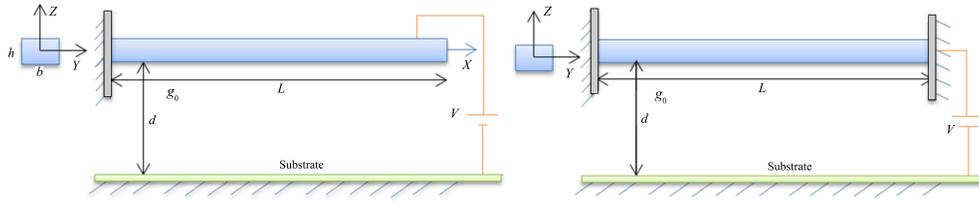


FIG. 1. Cantilever and Fixed-Fixed micro beam in the electrostatic field.

For convenience, the non-dimensional form of Eq. (3.1) is obtained by utilizing:  $w = w/g_0$ ,  $x = \frac{x}{L}$

$$(3.2) \quad -\frac{d^4 w}{dx^4} + [A_1 x^{2-\alpha} \frac{d^4 w}{dx^4} + x^{1-\alpha} A_8 \frac{d^3 w}{dx^3} + x^{-\alpha} A_2 \frac{d^2 w}{dx^2}] - A_3 \frac{d^2 w}{dx^2} \\ - \left[ x^{2-\alpha} \frac{A_4 V^2}{(1-w)^3} \frac{d^2 w}{dx^2} + \frac{A_5 V^2}{(1-w)^4} \left( \frac{dw}{dx} \right)^2 \right. \\ \left. + x^{1-\alpha} A_9 \frac{V^2}{(1-w)^3} \frac{dw}{dx} + x^{-\alpha} A_7 \frac{V^2}{(1-w)^2} \right] + A_6 \frac{V^2}{(1-w)^2} = 0,$$

where

$$A_1 = \frac{ANg_0^2 \mu^\alpha L^{2-\alpha}}{L^2 I}, \quad A_2 = \frac{ANg_0^2 L^{-\alpha} \mu^\alpha (2-\alpha)(1-\alpha)}{I}, \\ A_3 = \frac{Ng_0^2 A}{I}, \quad A_4 = \frac{\varepsilon_0 b \mu^\alpha L^2 L^{2-\alpha}}{EIg_0^3},$$

$$\begin{aligned}
A_5 &= \frac{3\varepsilon_0 b \mu^\alpha L^2 L^{2-\alpha}}{EI g_0^3}, & A_6 &= \frac{\varepsilon_0 b L^4}{2EI g_0^3}, \\
A_7 &= \frac{(2-\alpha)(1-\alpha)\varepsilon_0 b \mu^\alpha L^{4-\alpha}}{2EI g_0^3}, & A_8 &= \frac{2AN g_0^2 \mu^\alpha (2-\alpha)L^{1-\alpha}}{L^2 I}, \\
A_9 &= \frac{2\mu^\alpha \varepsilon_0 b (2-\alpha)L^{1-\alpha} L^3}{g_0^3 EI}, & N &= \int_0^1 \frac{1}{2} \left( \frac{dw}{dx} \right)^2 dx.
\end{aligned}$$

### 3.1. Numerical solution

The CFD definition gives the ability to solve the governing equation by using the numerical methods which are used for the integer differential equations. Therefore, we use Galerkin and step-by-step linearization (SSLM) methods [39–41]. Because of non-linearity of the governing equation, a SSLM method is used to make it linear. Afterwards, the obtained linear differential equation is solved using the Galerkin weighted residual method. Using SSLM, the voltage applied to the micro beam and substrate are increased from zero to its final value gradually;  $w_s^k$  is the displacement of the micro beam due to the applied voltage  $V^k$ . In the next step by increasing voltage, the displacement at the step of  $(k+1)$  can be obtained as:

$$(3.3) \quad \hat{w}_s^{k+1} = w_s^k + \delta w = w_s^k + \psi(x),$$

when

$$(3.4) \quad V^{k+1} = V^k + \delta V.$$

So, the equation of the static deflection of the micro beam (Eq. (3.2)) by using the Taylor series expansion about  $w_s$  at step of  $k+1$  can be rewritten as follows:

$$\begin{aligned}
(3.5) \quad & - \left( \frac{d^4 w^i}{dx^4} + \frac{d^4 \psi}{dx^4} \right) + A_1 x^{2-\alpha} \left( \frac{d^4 w^i}{dx^4} + \frac{d^4 \psi}{dx^4} \right) + \{A_2 x^{-\alpha} - A_3\} \left( \frac{d^2 w^i}{dx^2} + \frac{d^2 \psi}{dx^2} \right) \\
& - \left\{ x^{2-\alpha} \left\{ \left( \frac{A_4 V_{i+1}^2}{(1-w_i)^3} + \frac{3A_4 V_{i+1}^2}{(1-w_i)^4} \psi \right) \left( \frac{d^2 w^i}{dx^2} + \frac{d^2 \psi}{dx^2} \right) \right. \right. \\
& \left. \left. - \left( \frac{A_5 V_{i+1}^2}{(1-w_i)^4} + \frac{4A_5 V_{i+1}^2}{(1-w_i)^5} \psi \right) \left( \left( \frac{dw_i}{dx} \right)^2 + \left( \frac{dw_i}{dx} \right) \left( \frac{d\psi}{dx} \right) \right) \right\} \right\} \\
& + A_8 x^{1-\alpha} \left( \frac{d^3 w^i}{dx^3} + \frac{d^3 \psi}{dx^3} \right) - A_9 x^{1-\alpha} \left( \frac{V_{i+1}^2}{(1-w)^3} + \frac{3V_{i+1}^2}{(1-w)^4} \psi \right) \left( \frac{dw^i}{dx} + \frac{d\psi}{dx} \right) \\
& + \{A_6 - A_7 x^{-\alpha}\} \left( \frac{V_{i+1}^2}{(1-w_i)^2} + \frac{2V_{i+1}^2}{(1-w_i)^3} \psi \right) = 0.
\end{aligned}$$

It is expected that  $\psi(x)$  would be small enough by considering a small value of  $\delta V$ , hence it is possible to obtain the desired accuracy. The linear equation to calculate  $w(x)$  can be expressed as:

$$\begin{aligned}
 (3.6) \quad & -\frac{d^4\psi}{dx^4} + A_1x^{2-\alpha}\frac{d^4\psi}{dx^4} + [A_2x^{-\alpha} - A_3]\frac{d^2\psi}{dx^2} - x^{2-\alpha}\frac{A_4V_{i+1}^2}{(1-w_i)^3}\frac{d^2\psi}{dx^2} \\
 & -3x^{2-\alpha}\frac{A_4V_{i+1}^2}{(1-w_i)^4}\psi\frac{d^2w^i}{dx^2} - 2x^{2-\alpha}\frac{A_5V_{i+1}^2}{(1-w_i)^4}\left(\frac{dw^i}{dx}\right)\left(\frac{d\psi}{dx}\right) \\
 & -4x^{2-\alpha}\frac{A_5V_{i+1}^2}{(1-w_i)^5}\psi\left(\frac{dw^i}{dx}\right)^2 + A_8x^{1-\alpha}\left(\frac{d^3\psi}{dx^3}\right) \\
 & -A_9x^{1-\alpha}\left(\frac{3V_{i+1}^2}{(1-w)^4}\psi\frac{dw^i}{dx}\right) + [2A_6 - 2A_7x^{-\alpha}]\frac{V_{i+1}^2}{(1-w_i)^3}\psi \\
 = & x^{2-\alpha}\frac{A_5(V_{i+1}^2 - V_i^2)}{(1-w^i)^4}\left(\frac{dw^i}{dx}\right)^2 + x^{2-\alpha}\frac{A_4(V_{i+1}^2 - V_i^2)}{(1-w^i)^3} \\
 & + A_9x^{1-\alpha}\frac{(V_{i+1}^2 - V_i^2)}{(1-w^i)^3}\frac{dw^i}{dx} - [A_6 - A_7x^{-\alpha}]\frac{(V_{i+1}^2 - V_i^2)}{(1-w^i)^2}.
 \end{aligned}$$

In the obtained linear differential equation that is solved by the Galerkin method,  $\psi(x)$  can be expressed based on function spaces as:

$$(3.7) \quad \psi(x) = \sum_{j=1}^{\infty} a_j \phi_j(x) \approx \sum_{j=1}^N a_j \phi_j(x).$$

In this paper  $\phi_j(x)$  is selected as  $j$ -th un-damped mode shape of the straight micro beam. Substituting Eq. (3.7) into Eq. (3.6), and multiplying by  $\phi_i(x)$  as a weight function in the Galerkin method and then integrating the outcome from  $\hat{x} = 0$  to 1 leads to a set of linear algebraic equations as:

$$(3.8) \quad F_i = \sum_{j=1}^n k_{ij} a_j, \quad i = 1, \dots, n,$$

where

$$\begin{aligned}
 K_{ij} &= K_{ij}^m + K_{ij}^{f1} + K_{ij}^{f2} + K_{ij}^{f3} + K_{ij}^{f4} + K_{ij}^{f5} + K_{ij}^{f6} + K_{ij}^{f7}, \\
 K_{ij}^m &= -\int_0^1 \phi_j'''' \phi_i dx, \quad K_{ij}^{f1} = \int_0^1 A_1 x^{2-\alpha} \phi_j'''' \phi_i dx, \\
 K_{ij}^{f2} &= \int_0^1 (A_2 x^{-\alpha} - A_3) \phi_j'' \phi_i dx,
 \end{aligned}$$

$$\begin{aligned}
K_{ij}^{f3} &= \int_0^1 \frac{-A_4 x^{2-\alpha} V_{i+1}^2}{(1-w_i^3)} \phi_j'' \phi_i \, dx, \\
K_{ij}^{f4} &= \int_0^1 \frac{-3A_4 x^{2-\alpha} V_{i+1}^2 \frac{d^2 w}{dx^2}}{(1-w_i^4)} \phi_j \phi_i \, dx, \\
K_{ij}^{f5} &= \int_0^1 \frac{-A_5 2x^{2-\alpha} V_{i+1}^2 \frac{dw}{dx}}{(1-w_i^4)} \phi_j' \phi_i \, dx, \\
K_{ij}^{f6} &= \int_0^1 \frac{-4A_5 x^{2-\alpha} V_{i+1}^2 \left(\frac{dw}{dx}\right)^2}{(1-w_i^5)} \phi_j \phi_i \, dx, \\
K_{ij}^{f7} &= \int_0^1 \frac{2(A_6 - A_7 x^{-\alpha}) V_{i+1}^2}{(1-w_i^3)} \phi_j \phi_i \, dx, \\
K_{ij}^{f8} &= \int_0^1 (A_8) x^{1-\alpha} \phi_j''' \phi_i \, dx, \\
K_{ij}^{f9} &= \int_0^1 \left( \frac{-3A_9 x^{1-\alpha} V_{i+1}^2}{(1-w_i)^4} \right) \frac{dw}{dx} \phi_j \phi_i \, dx, \\
K_{ij}^{f10} &= \int_0^1 \left( \frac{-A_9 x^{1-\alpha} V_{i+1}^2}{(1-w_i)^3} \right) \phi_j'' \phi_i \, dx, \\
F_i &= F_1 + F_2 + F_3 + F_4, \\
F_1 &= \int_0^1 x^{2-\alpha} \frac{A_5 (V_{i+1}^2 - V_i^2)}{(1-w_i)^4} \left( \frac{dw^i}{dx} \right)^2 \phi_i \, dx, \\
F_2 &= \int_0^1 x^{2-\alpha} \frac{A_4 (V_{i+1}^2 - V_i^2)}{(1-w_i)^3} \frac{d^2 w^i}{dx^2} \phi_i \, dx, \\
F_3 &= \int_0^1 x^{1-\alpha} \frac{A_9 (V_{i+1}^2 - V_i^2)}{(1-w_i)^3} \frac{dw^i}{dx} \phi_i \, dx, \\
F_4 &= \int_0^1 \frac{-(A_6 - A_7 x^{-\alpha})(V_{i+1}^2 - V_i^2)}{(1-w_i)^2} \phi_i \, dx.
\end{aligned}$$

**3.2. Numerical results**

By applying a voltage between the beam and the substrate and slowly increasing the voltage, the system shifts from a stable to an unstable equilibrium. The instantaneous collapse (pull-in instability) of the beam was observed as the voltage increased enough and that voltage was named pull-in voltage.

The effects of the inhomogeneity parameter ( $\alpha$ ) (which includes both integer and non-integer numbers) and the dimensionless nonlocal parameter ( $\mu/L$ ) on the pull-in voltage of micro C-F and C-C beams have been shown. Note that ENT is a subset of the FNT but here to compare the theories ENT has been separated from the FNT and it has been told the FNT, does not comprise ENT.

In Table 1 the geometrical and material properties of the micro beams are shown. In Table 2 to validate the results the calculated pull-in for clamped-clamped (C-C) and clamped-free (C-F) beams have been compared with those that exist in the literature.

**Table 1. Geometrical and material properties.**

	Cantilever beam	Fixed-Fixed beam
Variables	Value	Value
Length	250 $\mu\text{m}$	350 $\mu\text{m}$
Wide	50 $\mu\text{m}$	50 $\mu\text{m}$
Height	2.94 $\mu\text{m}$	3 $\mu\text{m}$
Initial gap	1.05 $\mu\text{m}$	1 $\mu\text{m}$
Dielectric constant	$8.8541 \times 10^{-12}$	$8.8541 \times 10^{-12}$
Young modulus	169 Gpa	169 Gpa

**Table 2. Comparison of calculated pull-in voltage from CT.**

	BAGHANI [1]	ROKNI <i>et al.</i> [44]	Present
C-F	6.31 V	–	6.57 V
C-C	–	20.10 V	20.06 V

The FNT enables us to see the effect of different length scale inhomogeneity on the static instability of the beams. As mentioned above the inhomogeneity parameter control it therefore in Fig. 2 and Fig. 3 pull-in voltages of micro C-C and C-F beams are obtained based on the FNT for three values of the fractional parameter  $\alpha = 1.84, 1.92, 2$  (when  $\alpha = 2$  the FNT reduces to ENT) and also CT ( $C_g$  is non-dimensional center deflection and end deflection of the C-C and C-F micro beams  $C_g = 1 - w$ ). In Fig. 2 and Fig. 3 from the comparison of results of CT and ENT models, it can be seen that for the micro C-C beam the calculated pull-in voltages by ENT (when  $\alpha = 2$ ) is smaller than a classical

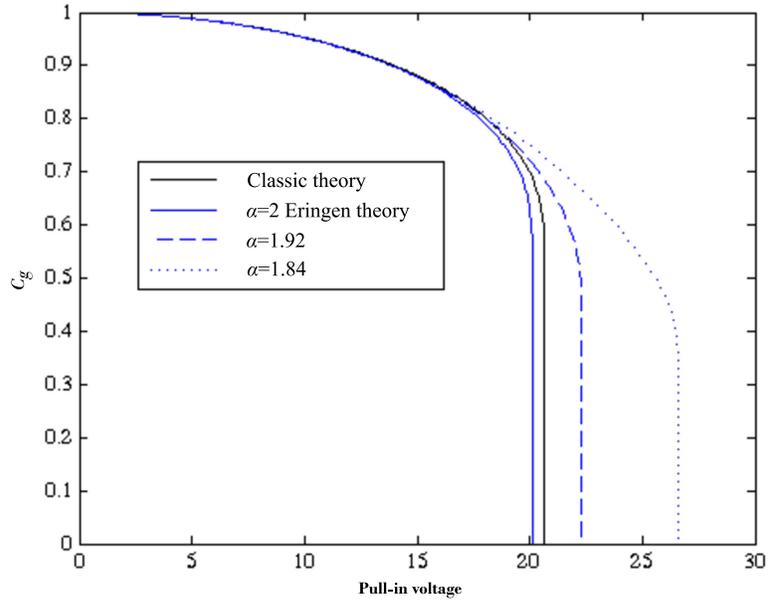


FIG. 2. Comparison of pull-in voltages for micro C-C beam based on different theories (the FNT ( $\alpha = 1.92, 1.84$ ), ENT ( $\alpha = 2$ ) and CT) ( $\mu/L = 0.1$ ).

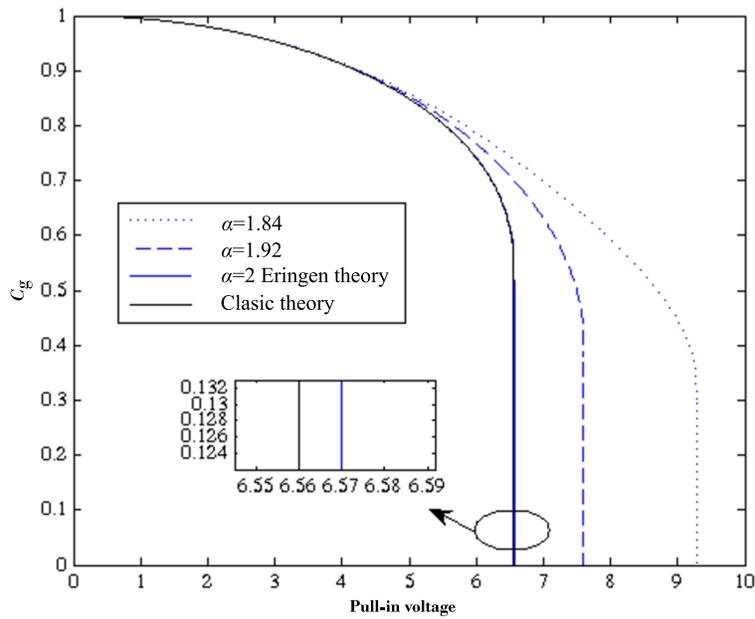


FIG. 3. Comparison of pull-in voltages for C-F micro beam based on different theories (the FNT ( $\alpha = 1.92, 1.84$ ), ENT ( $\alpha = 2$ ) and CT) ( $\mu/L = 0.1$ ).

value and for the micro C-F beam it is larger than the classical calculated value. These results for both C-C and C-F beams are in agreement with those from literature [42, 43]. On the other hand, from Fig. 2 and Fig. 3 it is visible that decreasing of  $\alpha$  leads to increase of pull-in voltage and the pull-in voltages based on the FNT are greater than calculated pull-in voltage based on ENT and CT results.

In Figs. 4a and 4b the calculated voltages based on different values of the parameter for five values of the nonlocal parameter ( $\mu/L = 0.01, 0.02, 0.03, 0.04, 0.05$ ) have been shown. In these figures variation of pull-in voltages with the inhomogeneity parameter is more visible. As it can be seen for both of the C-C and C-F micro beam decrease of  $\alpha$  parameter from its classical value ( $\alpha = 2$ ) causes increase in pull-in voltage. This effect is more pronounced, as the values of  $\mu/L$  increase.

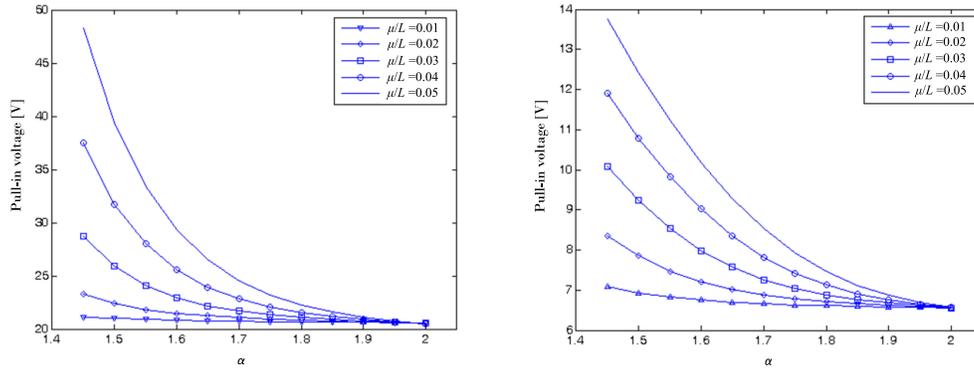


FIG. 4. Comparison of pull-in voltages versus different values of fractional parameter ( $\alpha$ ) and nonlocal parameter ( $\mu/L$ ); a) C-C micro beam, b) C-F micro beam.

The effects of the non-dimensional nonlocal parameter ( $\mu/L$ ) on the pull-in voltages in the FNT have been shown in Figs. 5, 6 and Figs. 7, 8 for C-C and C-F micro silicon beams, respectively. These figures have been plotted to show the effect of a nonlocal parameter where the inhomogeneity orders are integer and non-integer. Figure 5 shows that when  $\alpha = 1.92$ , the increase in the non-dimensional nonlocal parameter leads to increase of pull-in voltage but as it can be seen from Fig. 6 in the ENT ( $\alpha = 2$ ) increasing of  $\mu/L$  causes decrease in pull-in voltage. For the C-F beam in both cases of parameter  $\alpha =$  integer and  $\alpha =$  non-integer (Fig. 7 and Fig. 8, respectively) an increase of  $\mu/L$  leads to increasing of pull-in voltage. Notice that the results for  $\alpha = 2$  (ENT) for both C-C and C-F beams are in agreement with MOSAVI *et al.* [42]. As it could be seen effects of a dimensionless nonlocal parameter on pull-in voltage for C-C beam are greater than for C-F beam in both cases of  $\alpha$  (integer and non-integer).

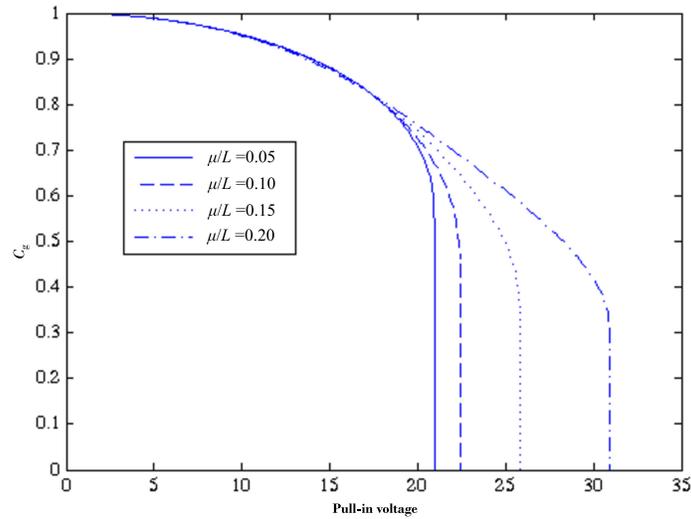


FIG. 5. Effect of the non-dimensional nonlocal parameter on the pull-in of micro C-C beam. ( $\alpha = 1.92$ ).

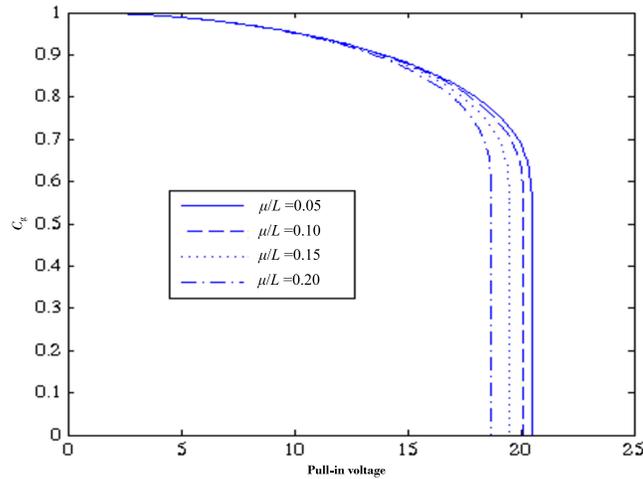


FIG. 6. Effect of the non-dimensional nonlocal parameter on the pull-in of micro C-C beam. ( $\alpha = 2$  (ENT)).

In Fig. 9 for a large interval of  $\mu/L$ , variation of pull-in has been plotted. The diagram has been plotted for both integer ( $\alpha = 2$ ) and non-integer ( $\alpha = 1.95, 1.85$ ) values of an inhomogeneity parameter. Their effects are more visible here and as presented in Fig. 9a when the values of the  $\alpha$  parameter are non-integer ( $\alpha = 1.95, 1.85$ ) the increase of the  $\mu/L$  causes increase in pull-in voltage but when it is integer ( $\alpha = 2$ ) the increase of  $\mu/L$  causes a decrease in pull-in

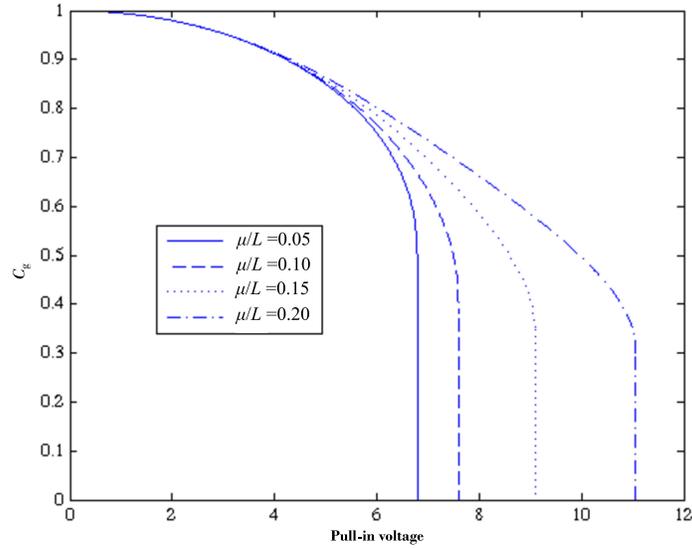


FIG. 7. Effect of the non-dimensional nonlocal parameter on the pull-in of micro C-F beam. ( $\alpha = 1.92$ ).

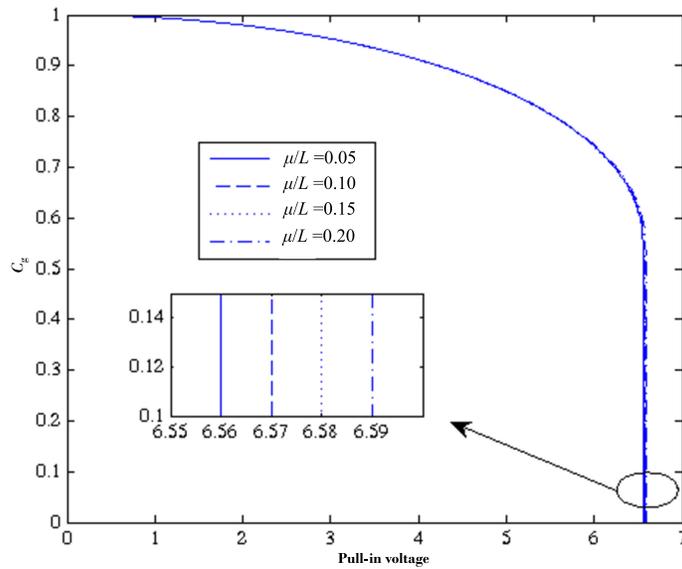


FIG. 8. Effect of the non-dimensional nonlocal parameter on the pull-in of micro C-F beam. ( $\alpha = 2$  (ENT)).

voltage. From Fig. 9b, it can be seen that for the C-F micro beam, the increase of the  $\mu/L$  causes an increase in pull-in voltages in the both integer and non-integer values of the inhomogeneity parameter.

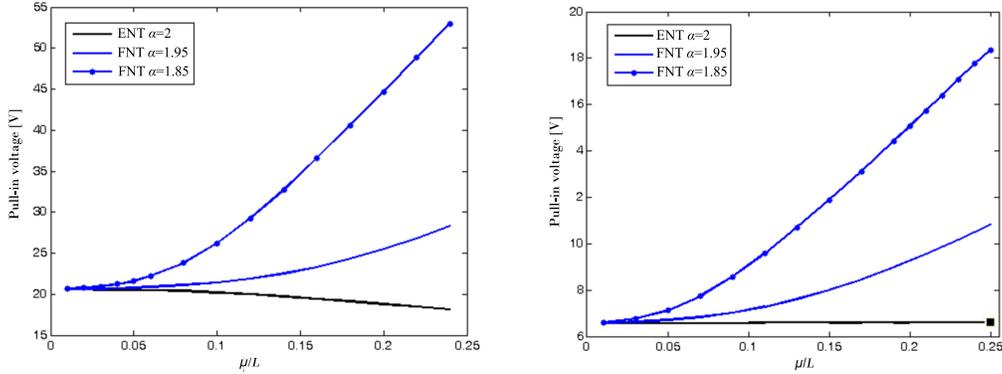


FIG. 9. Comparison of pull-in voltages versus different values of  $\mu/L$  where the fractional parameter is  $\alpha = 2, 1.95, 1.85$ . a) C-C micro beam, b) C-F micro beam.

In Fig. 10, the calculated results from CT, ENT and the FNT have been compared with experimental results of Osterberg [45]. As it can be seen the difference of measured pull-in voltages with calculated ones decreases as  $\alpha$  goes from 2 to 1.84. It can be concluded from Fig. 10 that the FNT could predict pull-in voltage better than ENT and CT.

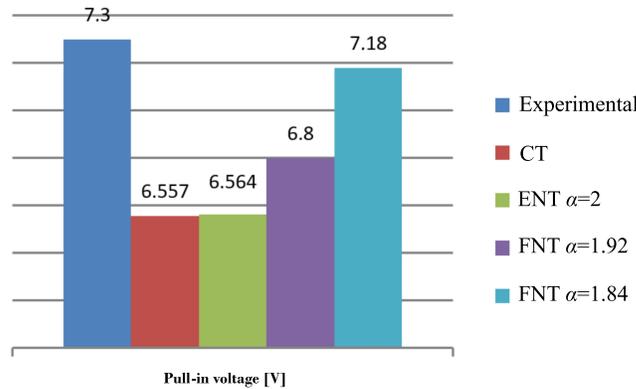


FIG. 10. Comparison of experimental pull-in voltage with results of different theories. (From left side: first column is experimental data [45], second column is the result of classic theory, third column is the result of Eringen theory, fourth and fifth columns are result of inhomogeneous nonlocal theory).

For C-C beam the experimental pull-in voltage is 20.3 V [45] and as mentioned in Table 2 the value which was calculated based on CT is 20.06 V, also the value which has been calculated based on ENT is 20.47 V. According to Fig. 9a the calculated pull-in voltage based on the FNT is higher than ENT so the pull-in voltage based on the FNT is larger than 20.47 and the difference be-

tween the experimental value and calculated one, is be greater. This conclusion does not mean that the FNT is not suitable for C-C micro beam, rather the inhomogeneity parameter should be different than between 1–2.

#### 4. Conclusion

A FNT that is a generalization of ENT by using the CFD definition was presented. The resulting model is analogous to Eringen’s stress gradient with inhomogeneous length scale effects. This theory has two free parameters: 1) the inhomogeneity parameter (which controls the order of a stress gradient in the stress-strain constitutive equation that gives a general form to the theory and this general form makes it more flexible than the integer derivatives based model to describe behaviour of materials) and 2) nonlocal parameter (to consider size dependence effects in micron and sub-micron scales).

The equation of a micro/nano beam under a distributed variable pressure force (electrostatic force, electromagnetic force and so on) in the presence of von-Karman non-linearity was obtained to study the static critical point instability. Then as a case study, static pull-in instability of micro C-C and C-F beam under an electrostatic force was analysed. The governing equation was solved by the Galerkin method. The effects of the inhomogeneity parameter (integer and non-integer number) and the nonlocal parameter on the pull-in instability was shown. It was demonstrated that in both C-C and C-F beams decreasing of the inhomogeneity parameter leads to increase of pull-in voltage and calculated pull-in voltage based on the FNT are larger than ENT and CT result. In C-C beam increasing of non-dimensional nonlocal parameter leads to an increase of the pull-in voltage when  $\alpha = 2$  the increase of  $\mu/L$  causes a decrease in pull-in voltage. For the C-F beam in both cases  $\alpha = \text{integer}$  and  $\alpha = \text{non-integer}$  increase of  $\mu/L$  leads to an increase of pull-in voltage.

To show that the FNT could be effective, the calculated results from CT, ENT and the FNT of C-F were compared with experimental results. It was shown that the difference of measured pull-in voltages with calculated ones is smaller for FNT.

#### Appendix

Conformable FDs for multi-variables functions:

Assume the function  $f(x, y)$ , we have:

$$(A.1) \quad \begin{aligned} f_x(x, y) &= \frac{df(x, y)}{dx} = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}, \\ f_y(x, y) &= \frac{df(x, y)}{dy} = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}. \end{aligned}$$

Based on CFDD we have [13, 41]:

$$(A.2) \quad \begin{aligned} f_x^\alpha(x, y) &= \frac{d^\alpha f(x, y)}{dx^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon x^{1-\alpha}, y) - f(x, y)}{\varepsilon}, \\ f_y^\alpha(x, y) &= \frac{d^\alpha f(x, y)}{dy^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f(x, y + \varepsilon y^{1-\alpha}) - f(x, y)}{\varepsilon}. \end{aligned}$$

If  $0 < \alpha \leq 1$ , let  $h = \varepsilon x^{\alpha-1}$ ,  $h = \varepsilon y^{\alpha-1}$  then Eq. (2.2) is:

$$(A.3) \quad \begin{aligned} f_x^\alpha(x, y) &= \frac{d^\alpha f(x, y)}{dx^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon x^{1-\alpha}, y) - f(x, y)}{\varepsilon} \\ &= x^{1-\alpha} \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h} = x^{1-\alpha} \frac{df(x, y)}{dx}, \\ f_y^\alpha(x, y) &= \frac{d^\alpha f(x, y)}{dy^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f(x, y + \varepsilon y^{1-\alpha}) - f(x, y)}{\varepsilon} \\ &= y^{1-\alpha} \lim_{h \rightarrow 0} \frac{f(x, y) - f(x, y + h)}{h} = y^{1-\alpha} \frac{df(x, y)}{dy}. \end{aligned}$$

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