Arch. Mech., **69**, 4–5, pp. 347–370, Warszawa 2017

# Boundary value problems of steady vibrations in the theory of thermoelasticity for materials with a double porosity structure

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The purpose of the present paper is to develop the classical potential method in the linear theory of thermoelasticity for materials with a double porosity structure based on the mechanics of materials with voids. The fundamental solution of the system of equations of steady vibrations is constructed explicitly by means of elementary functions and its basic properties are established. The Sommerfeld-Kupradze type radiation conditions are established. The basic internal and external boundary value problems (BVPs) are formulated and the uniqueness theorems of these problems are proved. The basic properties of the surface (single-layer and double-layer) and volume potentials are established and finally, the existence theorems for regular (classical) solutions of the internal and external BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

**Key words:** thermoelasticity, double porosity, fundamental solution, steady vibrations, uniqueness and existence theorems.
**Mathematics subject classification:** 74F05, 74F10, 74G25, 74G30, 35E05.



## 1. Introduction

The prediction of the thermal conductivity of materials with multiple porosity has been one of hot topics in the area of porous media for more than one hundred years. Heat transfer in single- and multi-porosity materials has received much attention in science and engineering, for instance, heat extraction in hot dry rock, heat transfer in biological tissue, geothermal and oil-gas reservoirs (see Straughan [1, 2], Miao *et al.* [3]).

There are two sets of the mathematical models for single and double porosity materials, where: (i) fluid flow in pore system is described by using Darcy's law and (ii) the behavior of porous solids is described by using the mechanics of materials with voids (vacuous pores).

The first model for single porosity deformable solid by using the classical Darcy's law is presented by Biot [4]. This law has been extended to describe fluid flow through undeformable double porosity materials in Barenblatt *et al.* [5], Warren and Root [6]. The theory for deformable materials with double porosity by using the extended Darcy's law developed by Wilson and Aifantis [7]. This theory unifies the earlier proposed models of Barenblatt *et al.* [5] for porous media with double porosity and Biot [4] for porous media with single porosity. In the last three decades more general models of the theories of elasticity and thermoelasticity for double porosity materials by using the extended Darcy's law have been introduced in [8–13] and studied by several authors [14–20]. The mathematical models of elasticity and thermoelasticity are presented by using the extended Darcy's law for media with multiple porosity in Bai *et al.* [21], Moutsopoulos *et al.* [22] and Straughan [2]. In addition, in these models the dependent variables are the displacement vector, the pressures in the pore networks and the variation of temperature.

Nunziato and Cowin [23, 24] have established a theory for the behavior of single porous deformable materials in which the skeletal or matrix materials are elastic and the interstices are voids (vacuous pores). This theory of deformable materials with voids has been extensively studied by several authors (see [25] and references therein). Recently, Ieşan and Quintanilla [25] have developed the theory of Nunziato and Cowin [24] for thermoelastic deformable materials with double porosity structure by using the mechanics of materials with voids. The basic BVPs of this theory are studied in [26–29]. In addition, in these models the dependent variables are the displacement vector, the volume fractions of pores and fissures and the variation of temperature.

In this paper the linear theory of thermoelasticity of Ieşan and Quintanilla [25] is considered. The purpose of the present paper is to develop the classical potential method in the linear theory of thermoelasticity for materials with a double porosity structure with voids (vacuous pores) based on the mechanics of materials with voids (see [25]). This work is articulated as follows. In Section 2 the governing field equations of the considered theory are given. In Section 3 the fundamental solution of the system of equations of steady vibrations is constructed explicitly by means of elementary functions and its basic properties are established. In Section 4 the Sommerfeld–Kupradze type radiation conditions are established and the basic internal and external BVPs are formulated. In Section 5 the uniqueness theorems for these problems are proved. In Section 6 the basic properties of the surface (single-layer and double-layer) and volume potentials are established and finally, the existence theorems for regular (classical) solutions of the internal and external BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

## 2. Governing equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$, let $t$ denote the time variable, $t \geq 0$. In what follows, we consider an isotropic and homogeneous elastic material with a double porosity structure with voids (vacuous pores) that occupies the region $\Omega$ of $\mathbb{R}^3$. $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ denotes the displacement vector, $\hat{\varphi}(\mathbf{x}, t)$ and $\hat{\psi}(\mathbf{x}, t)$ are the changes of volume fractions from the reference configuration corresponding to pores and fissures, respectively; $\hat{\theta}$ is the temperature measured from the constant absolute temperature $T_0$ ($T_0 > 0$).

We assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range $(1, 2, 3)$, and the dot denotes differentiation with respect to $t$.

The governing field equations in the linear dynamical theory of elastic materials with a double porosity structure based on the mechanics of materials with voids have the following form (see [25]):

- *Constitutive equations*

$$\begin{aligned} t_{lj} &= \lambda e_{rr}\delta_{lj} + 2\mu e_{lj} + (b\hat{\varphi} + d\hat{\psi} - \gamma_0\hat{\theta})\delta_{lj}, \\ \hat{\sigma}_l^{(1)} &= \alpha\hat{\varphi}_{,l} + \beta\hat{\psi}_{,l}, \qquad \hat{\sigma}_l^{(2)} = \beta\hat{\varphi}_{,l} + \gamma\hat{\psi}_{,l}, \\ \rho\eta &= \gamma_0 e_{rr} + \gamma_1\hat{\varphi} + \gamma_2\hat{\psi} + a\hat{\theta}, \qquad q_l = k\hat{\theta}_{,l}; \end{aligned} \tag{2.1}$$

- *Equations of motions*

$$\begin{aligned} t_{lj,j} &= \rho(\ddot{\hat{u}}_l - \hat{F}_l), \\ \hat{\sigma}_{l,l}^{(1)} + \hat{\xi}^{(1)} &= \rho_1\ddot{\hat{\varphi}} - \rho\hat{F}_4, \\ \hat{\sigma}_{l,l}^{(2)} + \hat{\xi}^{(2)} &= \rho_2\ddot{\hat{\psi}} - \rho\hat{F}_5; \end{aligned} \tag{2.2}$$

- *Equation of energy*

$$\rho T_0\dot{\eta} = q_{l,l} + \rho\hat{F}_6. \tag{2.3}$$

Here $\lambda, \mu, b, d, \alpha, \beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \gamma_0, \gamma_1, \gamma_2, a, k$ are constitutive coefficients, $t_{lj}$ is the component of the total stress tensor; $\hat{\sigma}_l^{(1)}$, $\hat{\sigma}_l^{(2)}$ and $q_l$ are the components of the equilibrated stress and the heat flux vectors, respectively; $\eta$ is the entropy per unit mass, $\rho$ is the reference mass density, $\rho > 0$; $\rho_1$ and $\rho_2$ are the coefficients of the equilibrated inertia, $\rho_1 > 0$, $\rho_2 > 0$; $\hat{\mathbf{F}}^{(1)} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ is the body force per unit mass, $\hat{F}_4$ and $\hat{F}_5$ are the extrinsic equilibrated body forces per unit mass associated to macro pores and fissures, respectively; $\hat{F}_6$ is the heat supply per unit mass, $\delta_{lj}$ is the Kronecker's delta, $e_{lj}$ are the components of the strain tensor,

$$e_{lj} = \tfrac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \qquad l, j = 1, 2, 3, \tag{2.4}$$

the functions $\hat{\xi}^{(1)}$ and $\hat{\xi}^{(2)}$ are the intrinsic equilibrated body forces and defined by

$$\hat{\xi}^{(1)} = -be_{rr} - \alpha_1\hat{\varphi} - \alpha_3\hat{\psi} + \gamma_1\hat{\theta}, \qquad \hat{\xi}^{(2)} = -de_{rr} - \alpha_3\hat{\varphi} - \alpha_2\hat{\psi} + \gamma_2\hat{\theta}. \tag{2.5}$$

Remark 1. Clearly, the last two equations of (2.1) and equation (2.3) imply that in the considered model the heat transport through porous materials is based on the classical Fourier's law of heat conduction (parabolic type equation of heat conduction).

Remark 2. Obviously, in the considered model the temperatures of the solid, pore and fissure systems are in a local thermal equilibrium, i.e. the solid, pore and fissure networks have the same temperature. On the other hand, it is very interesting for engineering and geomechanics the mathematical models of thermoelasticity of double porosity materials based on the Maxwell–Cattaneo law of heat conduction (the hyperbolic type equation of heat conduction) or/and based on the local thermal non-equilibrium, i.e. the solid, pore and fissure networks are endowed with their own temperatures.

Substituting Eqs. (2.1), (2.4) and (2.5) into (2.2) and (2.3) we obtain the following system of equations of motion in the full coupled linear theory of thermoelasticity for materials with a double porosity structure expressed in terms of the displacement vector $\hat{\mathbf{u}}$, the pressures $\hat{\varphi}, \hat{\psi}$ and the temperature $\hat{\theta}$:

$$\begin{gathered}
\mu\Delta\hat{\mathbf{u}} + (\lambda+\mu)\nabla\operatorname{div}\hat{\mathbf{u}} + b\nabla\hat{\varphi} + d\nabla\hat{\psi} - \gamma_0\nabla\hat{\theta} = \rho(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}^{(1)}),\\
\alpha\Delta\hat{\varphi} + \beta\Delta\hat{\psi} - \alpha_1\hat{\varphi} - \alpha_3\hat{\psi} - b\operatorname{div}\hat{\mathbf{u}} + \gamma_1\nabla\hat{\theta} = \rho_1\ddot{\hat{\varphi}} - \hat{F}_4,\\
\beta\Delta\hat{\varphi} + \gamma\Delta\hat{\psi} - \alpha_3\hat{\varphi} - \alpha_2\hat{\psi} - d\operatorname{div}\hat{\mathbf{u}} + \gamma_2\nabla\hat{\theta} = \rho_2\ddot{\hat{\psi}} - \hat{F}_5,\\
k\Delta\hat{\theta} - T_0(a\dot{\hat{\theta}} + \gamma_0\operatorname{div}\dot{\hat{\mathbf{u}}} + \gamma_1\dot{\hat{\varphi}} + \gamma_2\dot{\hat{\psi}}) = -\rho\hat{F}_6,
\end{gathered} \tag{2.6}$$

where $\Delta$ is the Laplacian operator.

Remark 3. It is well-known that in solid mechanics one encounters two types of dynamical problems; on the one hand, there are the problems in which the laws of motion as functions of time are known in advance and usually have a sinusoidal character; on the other hand, there are the problems in which the character of the dependence of time is unknown and has to be determined from the solution itself. The problems of the first type describe the steady-state or the steady vibrations. The problems of the second type describe the nonstationary motions, unrestricted with respect to the time. A sufficiently complete bibliography for the most important classical investigations of this kind is given in Nowacki [30] and Kupradze *et al.* [31].

In the follows we study the steady vibrations problems of the full coupled linear theory of thermoelasticity for materials with a double porosity structure.

Usually, as in the classical theory of thermoelasticity (see e.g. [30] and [31]), the steady vibrations case of the dynamic equations means, that all the dependent variables (displacement vector, temperature, etc.) are postulated to have a harmonic time variation.

Consequently, if the displacement vector $\hat{\mathbf{u}}$, the volume fractions $\hat{\varphi}$, $\hat{\psi}$, the temperature $\hat{\theta}$ and the components of body forces $\hat{F}_j$ $(j = 1, \ldots, 6)$ are postulated to have a harmonic time variation, that is,

$$\{\hat{\mathbf{u}}, \hat{\varphi}, \hat{\psi}, \hat{\theta}, \hat{F}_j\}(\mathbf{x}, t) = \mathrm{Re}[\{\mathbf{u}, \varphi, \psi, \theta, F_j\}(\mathbf{x})\, e^{-i\omega t}],$$

then from (2.6) we obtain the following system of equations of steady vibrations in the linear theory of elasticity for materials with a double porosity structure

$$\begin{aligned}
(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u} + b\nabla\varphi + d\nabla\psi - \gamma_0\nabla\theta &= -\rho\mathbf{F}^{(1)},\\
(\alpha\Delta + \eta_1)\varphi + (\beta\Delta - \alpha_3)\psi - b\,\mathrm{div}\,\mathbf{u} + \gamma_1\theta &= -\rho F_4,\\
(\beta\Delta - \alpha_3)\varphi + (\gamma\Delta + \eta_2)\psi - d\,\mathrm{div}\,\mathbf{u} + \gamma_2\theta &= -\rho F_5,\\
(k\Delta + a')\theta + \gamma_0'\,\mathrm{div}\,\mathbf{u} + \gamma_1'\varphi + \gamma_2'\psi &= -\rho F_6,
\end{aligned} \tag{2.7}$$

where $\omega$ is the oscillation frequency, $\mathbf{F}^{(1)} = (F_1, F_2, F_3)$, $\eta_l = \rho_l\omega^2 - \alpha_l$, $a' = i\omega a T_0$, $\gamma_j' = i\omega\gamma_j T_0$, $l = 1, 2$, $j = 1, 2, 3$.

We introduce the matrix differential operator $\mathbf{A}(\mathbf{D}_\mathbf{x}) = (A_{lj}(\mathbf{D}_\mathbf{x}))_{6\times 6}$, where

$$\begin{aligned}
&A_{lj}(\mathbf{D}_\mathbf{x}) = (\mu\Delta + \rho\omega^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},\\
&A_{l4}(\mathbf{D}_\mathbf{x}) = b\frac{\partial}{\partial x_l}, \qquad A_{l5}(\mathbf{D}_\mathbf{x}) = d\frac{\partial}{\partial x_l}, \qquad A_{l6}(\mathbf{D}_\mathbf{x}) = -\gamma_0\frac{\partial}{\partial x_l},\\
&A_{4l}(\mathbf{D}_\mathbf{x}) = -b\frac{\partial}{\partial x_l}, \qquad A_{5l}(\mathbf{D}_\mathbf{x}) = -d\frac{\partial}{\partial x_l}, \qquad A_{6l}(\mathbf{D}_\mathbf{x}) = \gamma_0'\frac{\partial}{\partial x_l},\\
&A_{44}(\mathbf{D}_\mathbf{x}) = \alpha\Delta + \eta_1, \qquad A_{45}(\mathbf{D}_\mathbf{x}) = A_{54}(\mathbf{D}_\mathbf{x}) = \beta\Delta - \alpha_3,\\
&A_{55}(\mathbf{D}_\mathbf{x}) = \gamma\Delta + \eta_2, \qquad A_{46}(\mathbf{D}_\mathbf{x}) = \gamma_1, \qquad A_{56}(\mathbf{D}_\mathbf{x}) = \gamma_2,\\
&A_{64}(\mathbf{D}_\mathbf{x}) = \gamma_1', \qquad A_{65}(\mathbf{D}_\mathbf{x}) = \gamma_2', \qquad A_{66}(\mathbf{D}_\mathbf{x}) = k\Delta + a',\\
&\mathbf{D}_\mathbf{x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad l, j = 1, 2, 3.
\end{aligned}$$

It is easily seen that the system (2.7) we can rewritten in the following matrix form

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{2.8}$$

where $\mathbf{U} = (\mathbf{u}, \varphi, \psi, \theta)$ and $\mathbf{F} = (-\rho F_1, -\rho F_2, \ldots, -\rho F_6)$ are six-component vector functions.

The purpose of this paper is to investigate the internal and external BVPs of steady vibrations for the system (2.7) by means of the potential method and the theory of integral equations. For this we need some basic properties of the fundamental solution of the system (2.7) and the surface and volume potentials.

## 3. Fundamental solution

In this section the fundamental solution of the system (2.7) (the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$) is constructed explicitly by means of elementary functions and its basic properties are established.

DEFINITION 1. The fundamental solution of the system (2.7) is the matrix $\mathbf{\Gamma}(\mathbf{x}) = (\varGamma_{lj}(\mathbf{x}))_{6\times 6}$ satisfying the following equation in the class of generalized functions

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{3.1}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{6\times 6}$ is the unit matrix, $\mathbf{x} \in \mathbb{R}^3$.

We introduce the notation:

1)

$$\varLambda_1(\Delta) = \frac{1}{k\alpha_0\mu_0} \det \mathbf{B}(\Delta),$$

where

$$\mathbf{B}(\Delta) = \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & b\Delta & d\Delta & -\gamma_0\Delta \\ -b & \alpha\Delta + \eta_1 & \beta\Delta - \alpha_3 & \gamma_1 \\ -d & \beta\Delta - \alpha_3 & \gamma\Delta + \eta_2 & \gamma_2 \\ \gamma_0' & \gamma_1' & \gamma_2' & k\Delta + a' \end{pmatrix}_{4\times 4}$$

and $\alpha_0 = \alpha\gamma - \beta^2$, $\mu_0 = \lambda + 2\mu$. We can consider $\varLambda_1(-\xi) = 0$ as an algebraic equation of the fourth degree, which admits four roots $\lambda_1^2$,$\lambda_2^2$, $\lambda_3^2$ and $\lambda_4^2$ (with respect to $\xi$). Then we have

$$\varLambda_1(\Delta) = \prod_{j=1}^{4}(\Delta + \lambda_j^2).$$

We assume that the values $\lambda_1^2, \lambda_2^2, \ldots, \lambda_5^2$ are distinct and different from zero, where $\lambda_5^2 = \rho\omega^2/\mu$.

2)

$$n_{j1}(\Delta) = -\frac{1}{k\alpha_0\mu\mu_0}[(\lambda+\mu)B_{j1}^*(\Delta) - bB_{j2}^*(\Delta) - dB_{j3}^*(\Delta) + \gamma_0' B_{j4}^*(\Delta)],$$

$$n_{jl}(\Delta) = \frac{1}{k\alpha_0\mu_0} B_{jl}^*(\Delta), \qquad j = 1,2,3,4, \qquad l = 2,3,4,$$

where $B_{lj}^*$ is the cofactor of the element $B_{lj}$ of matrix $\mathbf{B}$.

3)

$$
\begin{aligned}
&\mathbf{L}(\mathbf{D_x}) = (L_{lj}(\mathbf{D_x}))_{6\times 6}, \quad L_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\Lambda_1(\Delta)\delta_{lj} + n_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j},\\
&L_{l;m+2}(\mathbf{D_x}) = n_{1m}(\Delta)\frac{\partial}{\partial x_l}, \qquad L_{m+2;l}(\mathbf{D_x}) = n_{m1}(\Delta)\frac{\partial}{\partial x_l},\\
&L_{m+2;p+2}(\mathbf{D_x}) = n_{mp}(\Delta), \qquad l,j=1,2,3, \qquad m,p=2,3,4.
\end{aligned} \tag{3.2}
$$

4)

$$
\begin{aligned}
&\mathbf{Y}(\mathbf{x}) = (Y_{lm}(\mathbf{x}))_{6\times 6}, \quad Y_{11}(\mathbf{x}) = Y_{22}(\mathbf{x}) = Y_{33}(\mathbf{x}) = \sum_{j=1}^{5}\eta_{2j}\gamma^{(j)}(\mathbf{x}),\\
&Y_{44}(\mathbf{x}) = Y_{55}(\mathbf{x}) = Y_{66}(\mathbf{x}) = \sum_{j=1}^{4}\eta_{1j}\gamma^{(j)}(\mathbf{x}),\\
&Y_{lm}(\mathbf{x}) = 0, \qquad l \neq m, \qquad l,m = 1,2,\cdots,6,
\end{aligned} \tag{3.3}
$$

where

$$
\gamma^{(j)}(\mathbf{x}) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi|\mathbf{x}|} \tag{3.4}
$$

and

$$
\eta_{1m} = \prod_{l=1,\, l\neq m}^{4}(\lambda_l^2 - \lambda_m^2)^{-1}, \quad \eta_{2j} = \prod_{l=1,\, l\neq j}^{5}(\lambda_l^2 - \lambda_j^2)^{-1}, \quad m=1,\dots,4,\ j=1,\dots,5.
$$

We have the following

THEOREM 1. *If*

$$
k\alpha_0\mu\mu_0 \neq 0, \tag{3.5}
$$

*then the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *defined by*

$$
\mathbf{\Gamma}(\mathbf{x}) = \mathbf{L}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) \tag{3.6}
$$

*is the fundamental solution of the system* (2.7), *where the matrices* $\mathbf{L}(\mathbf{D_x})$ *and* $\mathbf{Y}(\mathbf{x})$ *are given by* (3.2) *and* (3.3), *respectively.*

*Proof.* On the basis of identities

$$
\mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta), \qquad \mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},
$$

where

$$
\begin{aligned}
&\mathbf{\Lambda}(\Delta) = (\mathit{\Lambda}_{lj}(\Delta))_{6\times 6},\\
&\mathit{\Lambda}_{11}(\Delta) = \mathit{\Lambda}_{22}(\Delta) = \mathit{\Lambda}_{33}(\Delta) = \mathit{\Lambda}_1(\Delta)(\Delta + \lambda_5^2),
\end{aligned}
$$

$$\Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) = \Lambda_{66}(\Delta) = \Lambda_1(\Delta),$$
$$\Lambda_{lj}(\Delta) = 0, \qquad l \neq j, \qquad l, j = 1, \dots, 6,$$

it follows (3.1). □

Hence, the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed by 5 metaharmonic functions (solutions of the Helmholtz equation) $\gamma^{(j)}$ $(j = 1, \dots, 5)$ (see (3.4)).

Theorem 1 directly leads to the following basic properties of $\mathbf{\Gamma}(\mathbf{x})$.

THEOREM 2. *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of the homogeneous equation*

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{3.7}$$

*at every point* $\mathbf{x} \in \mathbb{R}^3$ *except the origin.*

THEOREM 3. *If the condition* (3.5) *is satisfied, then the fundamental solution of the system*

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} = \mathbf{0},$$
$$\alpha\Delta\varphi + \beta\Delta\psi = 0, \qquad \beta\Delta\varphi + \gamma\Delta\psi = 0, \qquad k\,\Delta\theta = 0$$

*is the matrix* $\mathbf{\Psi}(\mathbf{x}) = (\Psi_{lj}(\mathbf{x}))_{6\times 6}$, *where*

$$\begin{aligned}
&\Psi_{lj}(\mathbf{x}) = \frac{1}{\mu}\left(\Delta\delta_{lj} - \frac{\lambda+\mu}{\mu_0}\frac{\partial^2}{\partial x_l \partial x_j}\right)\gamma^{(6)}(\mathbf{x}) = \lambda'\frac{\delta_{lj}}{|\mathbf{x}|} + \mu'\frac{x_l x_j}{|\mathbf{x}|^3},\\
&\Psi_{44}(\mathbf{x}) = \frac{\gamma}{\alpha_0}\gamma^{(7)}(\mathbf{x}), \qquad \Psi_{45}(\mathbf{x}) = \Psi_{45}(\mathbf{x}) = -\frac{\beta}{\alpha_0}\gamma^{(7)}(\mathbf{x}),\\
&\Psi_{55}(\mathbf{x}) = \frac{\alpha}{\alpha_0}\gamma^{(7)}(\mathbf{x}), \qquad \Psi_{66}(\mathbf{x}) = \frac{1}{k}\gamma^{(7)}(\mathbf{x}),\\
&\Psi_{l;j+3}(\mathbf{x}) = \Psi_{l+3;j}(\mathbf{x}) = \Psi_{m6}(\mathbf{x}) = \Psi_{6m}(\mathbf{x}) = 0,\\
&\gamma^{(6)}(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \qquad \gamma^{(7)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|},\\
&\lambda' = -\frac{\lambda+3\mu}{8\pi\mu\mu_0}, \qquad \mu' = -\frac{\lambda+\mu}{8\pi\mu\mu_0}, \qquad l, j = 1, 2, 3,\ m = 4, 5.
\end{aligned} \tag{3.8}$$

Obviously, (3.6) and (3.8) imply the following results.

COROLLARY 1. *The relations*

$$\Psi_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Psi_{mm}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad (\textit{no sum}) \tag{3.9}$$

*hold in the neighborhood of the origin, where* $l, j = 1, 2, 3$ *and* $m = 4, 5, 6$.

Theorem 4. *The relations*

$$\varGamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \varGamma_{mr}(\mathbf{x}) = O(|\mathbf{x}|^{-1}),$$
$$\varGamma_{66}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \varGamma_{l;j+3}(\mathbf{x}) = O(1),$$
$$\varGamma_{l+3;j}(\mathbf{x}) = O(1), \qquad \varGamma_{m6}(\mathbf{x}) = O(1), \qquad \varGamma_{6m}(\mathbf{x}) = O(1)$$

*hold in the neighborhood of the origin, where* $l, j = 1, 2, 3$, $m, r = 4, 5$.

On the basis of Theorem 4 and Corollary 1 we can prove the following

Theorem 5. *The relations*

$$\varGamma_{lj}(\mathbf{x}) - \varPsi_{lj}(\mathbf{x}) = \text{const.} + O(|\mathbf{x}|) \tag{3.10}$$

*hold in the neighborhood of the origin, where* $l, j = 1, \dots, 6$.

Thus, in view of (3.9) and (3.10), matrix $\mathbf{\Psi}(\mathbf{x})$ gives the singular part of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin.

## 4. Boundary value problems

In what follows we assume that the constitutive coefficients satisfy the inequalities

$$\begin{aligned} &\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \alpha > 0, \qquad \alpha_0 > 0, \\ &\alpha_1 > 0, \qquad (3\lambda + 2\mu)\beta_0 > 3\beta_1, \qquad a > 0, \qquad k > 0. \end{aligned} \tag{4.1}$$

where $\beta_0 = \alpha_1\alpha_2 - \alpha_3^2$, $\beta_1 = \alpha_1 d^2 - 2\alpha_3 bd + \alpha_2 b^2$.

Obviously, the condition (4.1) implies

$$\alpha_2 > 0, \qquad \beta_0 > 0, \qquad \gamma > 0, \qquad \mu_0 > 0, \qquad \mu_0\beta_0 > \beta_1 \geq 0.$$

Let $S$ be the closed surface surrounding the finite domain $\varOmega^+$ in $\mathbb{R}^3$, $S \in C^{1,\nu}$, $0 < \nu \leq 1$, $\overline{\varOmega^+} = \varOmega^+ \cup S$; $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to $S$ at $\mathbf{z}$. The scalar product of two vectors $\mathbf{U} = (u_1, u_2, \dots, u_6)$ and $\mathbf{V} = (v_1, v_2, \dots, v_6)$ is denoted by $\mathbf{U} \cdot \mathbf{V} = \sum_{j=1}^{6} u_j \bar{v}_j$, where $\bar{v}_j$ is the complex conjugate of $v_j$.

Definition 2. A vector function $\mathbf{U} = (\mathbf{u}, \varphi, \psi, \theta) = (U_1, U_2, \dots, U_6)$ is called regular in $\varOmega^-$ (or $\varOmega^+$) if

(i) $U_l \in C^2(\varOmega^-) \cap C^1(\overline{\varOmega^-})$ (or $U_l \in C^2(\varOmega^+) \cap C^1(\overline{\varOmega^+})$),

(ii) $U_l = \displaystyle\sum_{j=1}^{5} U_l^{(j)}, \quad U_l^{(j)} \in C^2(\varOmega^-) \cap C^1(\bar{\varOmega}^-)$,

(iii) $(\Delta+\lambda_j^2)U_l^{(j)}(\mathbf{x})=0$ and

$$\left(\frac{\partial}{\partial|\mathbf{x}|}-i\lambda_j\right)U_l^{(j)}(\mathbf{x})=e^{i\lambda_j|\mathbf{x}|}o(|\mathbf{x}|^{-1})\qquad\text{for } |\mathbf{x}|\gg 1, \tag{4.2}$$

where $U_4^{(5)}=U_5^{(5)}=U_6^{(5)}=0$, $j=1,\dots,5$, $l=1,\dots,6$.

In VEKUA [32] it is proved that the relation (4.2) implies

$$U_l^{(j)}(\mathbf{x})=e^{i\lambda_j|\mathbf{x}|}O(|\mathbf{x}|^{-1})\qquad\text{for } |\mathbf{x}|\gg 1, \tag{4.3}$$

where $j=1,\dots,5$, $l=1,\dots,6$.

Relations (4.2) and (4.3) are the Sommerfeld-Kupradze type radiation conditions in the full coupled linear theory of thermoelasticity for solids with a double porosity structure.

In the sequel, we use the matrix differential operator

$$\mathbf{P}(\mathbf{D_x},\mathbf{n})=(P_{lj}(\mathbf{D_x},\mathbf{n}))_{6\times 6},$$

where

$$\begin{aligned}
&P_{lj}(\mathbf{D_x},\mathbf{n})=\mu\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+\mu n_j\frac{\partial}{\partial x_l}+\lambda n_l\frac{\partial}{\partial x_j},\qquad P_{l4}(\mathbf{D_x},\mathbf{n})=bn_l,\\
&P_{l5}(\mathbf{D_x},\mathbf{n})=dn_l,\qquad P_{l6}(\mathbf{D_x},\mathbf{n})=-\gamma_0 n_l,\qquad P_{44}(\mathbf{D_x},\mathbf{n})=\alpha\frac{\partial}{\partial\mathbf{n}},\\
&P_{45}(\mathbf{D_x},\mathbf{n})=P_{54}(\mathbf{D_x},\mathbf{n})=\beta\frac{\partial}{\partial\mathbf{n}},\quad P_{55}(\mathbf{D_x},\mathbf{n})=\gamma\frac{\partial}{\partial\mathbf{n}},\\
&P_{66}(\mathbf{D_x},\mathbf{n})=k\frac{\partial}{\partial\mathbf{n}},\\
&P_{l+3;j}(\mathbf{D_x},\mathbf{n})=P_{m6}(\mathbf{D_x},\mathbf{n})=P_{6m}(\mathbf{D_x},\mathbf{n})=0,\quad l,j=1,2,3,\ m=4,5
\end{aligned} \tag{4.4}$$

and $\partial/\partial\mathbf{n}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the theory of thermoelasticity for materials with double porosity structure are formulated as follows.

Find a regular (classical) solution to (2.8) for $\mathbf{x}\in\Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^+=\mathbf{f}(\mathbf{z}) \tag{4.5}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{P}(\mathbf{D_x},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x})\equiv\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+=\mathbf{f}(\mathbf{z}) \tag{4.6}$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed six-component vector functions.

Find a regular (classical) solution to (2.8) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}(\mathbf{D}_\mathbf{x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{4.7}$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed six-component vector functions, $\operatorname{supp} \mathbf{F}$ is a finite domain in $\Omega^-$.

## 5. Uniqueness theorems

In this section the uniqueness of regular solutions of the BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$ is studied, where $K = I, II$. In the sequel we use the matrix differential operators:

1) $$\mathbf{A}^{(0)}(\mathbf{D}_\mathbf{x}) = (A^{(0)}_{lj}(\mathbf{D}_\mathbf{x}))_{3\times 3}, \qquad A^{(0)}_{lj}(\mathbf{D}_\mathbf{x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},$$
$$\mathbf{A}^{(1)}(\mathbf{D}_\mathbf{x}) = (A^{(1)}_{lr}(\mathbf{D}_\mathbf{x}))_{3\times 6}, \qquad A^{(1)}_{lr}(\mathbf{D}_\mathbf{x}) = A_{lr}(\mathbf{D}_\mathbf{x}),$$
$$\mathbf{A}^{(m)}(\mathbf{D}_\mathbf{x}) = (A^{(m)}_{1r}(\mathbf{D}_\mathbf{x}))_{1\times 6}, \qquad A^{(m)}_{1r}(\mathbf{D}_\mathbf{x}) = A_{m+2;r}(\mathbf{D}_\mathbf{x});$$

2) $$\mathbf{P}^{(0)}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (P^{(0)}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{3\times 3}, \qquad P^{(0)}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}),$$
$$\mathbf{P}^{(1)}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (P^{(1)}_{lr}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{3\times 6}, \qquad P^{(1)}_{lr}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{lr}(\mathbf{D}_\mathbf{x}, \mathbf{n}),$$

where $l, j = 1, 2, 3$, $m = 2, 3, 4$ and $r = 1, \dots, 6$.

We introduce the notation

$$\begin{aligned}
W^{(0)}(\mathbf{u}) &= \frac{1}{3}(3\lambda + 2\mu)|\operatorname{div} \mathbf{u}|^2 + \frac{\mu}{2}\sum_{l,j=1,\, l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right|^2 \\
&\quad + \frac{\mu}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2, \\
W^{(1)}(\mathbf{U}) &= W^{(0)}(\mathbf{u}) - \rho\omega^2|\mathbf{u}|^2 + (b\varphi + d\psi - \gamma_0\theta)\operatorname{div} \overline{\mathbf{u}}, \\
W^{(2)}(\mathbf{U}) &= \alpha|\nabla\varphi|^2 + \beta\nabla\psi\cdot\nabla\varphi - (\eta_1\varphi - \alpha_3\psi - b\operatorname{div}\mathbf{u} + \gamma_1\theta)\overline{\varphi}, \\
W^{(3)}(\mathbf{U}) &= \beta\nabla\varphi\cdot\nabla\psi + \gamma|\nabla\psi|^2 - (-\alpha_3\varphi + \eta_2\psi - d\operatorname{div}\mathbf{u} + \gamma_2\theta)\overline{\psi}, \\
W^{(4)}(\mathbf{U}) &= k|\nabla\theta|^2 - (a'\theta + \gamma'_0\operatorname{div}\mathbf{u} + \gamma'_1\varphi + \gamma'_2\psi)\overline{\theta}.
\end{aligned} \tag{5.1}$$

We have the following

LEMMA 1. *If* $\mathbf{U} = (\mathbf{u}, \varphi, \psi, \theta)$ *is a regular vector in* $\Omega^+$, *then*

$$\begin{gathered}
\int\limits_{\Omega^+} [\mathbf{A}^{(1)}(\mathbf{D}_\mathbf{x})\mathbf{u}(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) + W^{(1)}(\mathbf{U})]\, d\mathbf{x} \\
= \int\limits_S \mathbf{P}^{(1)}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \cdot \mathbf{u}(\mathbf{z})\, d_\mathbf{z}S,
\end{gathered}$$

$$\begin{aligned}
\int\limits_{\Omega^+} [\mathbf{A}^{(2)}(\mathbf{D}_\mathbf{x})\mathbf{u}(\mathbf{x})\overline{\varphi(\mathbf{x})} + W^{(2)}(\mathbf{U})]\, d\mathbf{x} &= \int\limits_S \left(\alpha\frac{\partial\varphi}{\partial\mathbf{n}} + \beta\frac{\partial\psi}{\partial\mathbf{n}}\right)\overline{\varphi(\mathbf{z})}\, d_\mathbf{z}S, \\
\int\limits_{\Omega^+} [\mathbf{A}^{(3)}(\mathbf{D}_\mathbf{x})\mathbf{u}(\mathbf{x})\overline{\psi(\mathbf{x})} + W^{(3)}(\mathbf{U})]\, d\mathbf{x} &= \int\limits_S \left(\beta\frac{\partial\varphi}{\partial\mathbf{n}} + \gamma\frac{\partial\psi}{\partial\mathbf{n}}\right)\overline{\psi(\mathbf{z})}\, d_\mathbf{z}S, \\
\int\limits_{\Omega^+} [\mathbf{A}^{(4)}(\mathbf{D}_\mathbf{x})\mathbf{u}(\mathbf{x})\overline{\theta(\mathbf{x})} + W^{(4)}(\mathbf{U})]\, d\mathbf{x} &= \int\limits_S k\frac{\partial\theta}{\partial\mathbf{n}}\overline{\theta(\mathbf{z})}\, d_\mathbf{z}S.
\end{aligned} \tag{5.2}$$

*Proof.* On the basis of the divergence theorem the following identities are proved (see [31])

$$\begin{aligned}
\int\limits_{\Omega^+} [\mathbf{A}^{(0)}(\mathbf{D}_\mathbf{x})\mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + W^{(0)}(\mathbf{u})]\, d\mathbf{x} &= \int\limits_S \mathbf{P}^{(0)}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{u}(\mathbf{z}) \cdot \mathbf{u}(\mathbf{z})\, d_\mathbf{z}S, \\
\int\limits_{\Omega^+} [\Delta\varphi(\mathbf{x})\overline{\psi(\mathbf{x})} + \nabla\varphi(\mathbf{x}) \cdot \nabla\psi(\mathbf{x})]\, d\mathbf{x} &= \int\limits_S \frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\overline{\psi(\mathbf{z})}\, d_\mathbf{z}S, \\
\int\limits_{\Omega^+} [\nabla\varphi(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + \varphi(\mathbf{x})\operatorname{div}\overline{\mathbf{u}(\mathbf{x})}] d\mathbf{x} &= \int\limits_S \varphi(\mathbf{z})\mathbf{n}(\mathbf{z}) \cdot \mathbf{u}(\mathbf{z})\, d_\mathbf{z}S.
\end{aligned} \tag{5.3}$$

Keeping in mind (5.1), from (5.3) we obtain the identities (5.2). □

We are now in a position to study the uniqueness of regular solutions of the BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. We have the following results.

THEOREM 6. *Two regular solutions of the internal BVP* $(I)^+_{\mathbf{F},\mathbf{f}}$, *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, \varphi, \psi, \theta)$, *where*

$$\theta(\mathbf{x}) = 0 \qquad \textit{for } \mathbf{x} \in \Omega^+ \tag{5.4}$$

*and the five-component vector* $\mathbf{V} = (\mathbf{u}, \varphi, \psi)$ *is a regular solution of the following system*

$$\begin{aligned}
(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda + \mu)\nabla\operatorname{div}\mathbf{u} + b\nabla\varphi + d\nabla\psi = \mathbf{0}, \\
(\alpha\Delta + \eta_1)\varphi + (\beta\Delta - \alpha_3)\psi - b\operatorname{div}\mathbf{u} = 0, \\
(\beta\Delta - \alpha_3)\varphi + (\gamma\Delta + \eta_2)\psi - d\operatorname{div}\mathbf{u} = 0, \\
\gamma_0\operatorname{div}\mathbf{u} + \gamma_1\varphi + \gamma_2\psi = 0
\end{aligned} \tag{5.5}$$

*satisfying the boundary condition*

$$\{\mathbf{V}(\mathbf{z})\}^+ = 0 \qquad \textit{for } \mathbf{z} \in S. \tag{5.6}$$

*In addition, problems* $(I)^+_{\mathbf{0},\mathbf{0}}$ *and* (5.5), (5.6) *have the same eigenfrequencies.*

*Proof.* Suppose that there are two regular solutions of problem $(I)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous system of equations (3.7) in $\Omega^+$ satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = 0 \qquad \text{for } \mathbf{z} \in S. \tag{5.7}$$

On the basis of (3.7) and (5.7), from (5.2) we obtain

$$\int\limits_{\Omega^+} W^{(j)}(\mathbf{U})\, d\mathbf{x} = 0, \qquad j = 1, 2, 3, 4. \tag{5.8}$$

Clearly, from (5.1) we have

$$\operatorname{Re} W^{(4)}(\mathbf{U}) - \omega T_0 \operatorname{Im}\left[W^{(1)}(\mathbf{U}) + W^{(2)}(\mathbf{U}) + W^{(3)}(\mathbf{U})\right] = k|\nabla\theta|^2$$

and from (5.8) it follows that

$$\int\limits_{\Omega^+} |\nabla\theta(\mathbf{x})|^2\, d\mathbf{x} = 0.$$

Hence, $\nabla\theta(\mathbf{x}) \equiv 0$ in $\Omega^+$ and consequently,

$$\theta(\mathbf{x}) = c = \text{const} \qquad \text{for } \mathbf{x} \in \Omega^+. \tag{5.9}$$

On the basis of the homogeneous boundary condition (5.7) from (5.9) we obtain the relation (5.4). By virtue of (5.4) from (3.7) we get the system (5.5). Obviously, in view of the condition (5.7) the five-component vector $\mathbf{V} = (\mathbf{u}, \varphi, \psi)$ satisfies the boundary condition (5.6).

Finally, it is easy to see that the homogeneous boundary value problems $(I)^+_{\mathbf{0},\mathbf{0}}$ and (5.5), (5.6) have the same eigenfrequencies. □

Remark 4. In the particular case, when porosities are neglected in the isotropic elastic solid (i.e. $\varphi = \psi \equiv 0$), the BVP (5.5), (5.6) is reduced to the following BVP

$$(\mu\Delta + \rho\omega^2)\mathbf{u}(\mathbf{x}) = \mathbf{0}, \qquad \operatorname{div}\mathbf{u}(\mathbf{x}) = 0, \qquad \{\mathbf{u}(\mathbf{z})\}^+ = \mathbf{0} \tag{5.10}$$

for $\mathbf{x} \in \Omega^+$, $\mathbf{z} \in S$. On the other hand, in the classical theory of thermoelasticity, the first internal homogeneous BVP of steady vibrations is reduced to the BVP

(5.10) (see [31]). In DAFERMOS [33], the existence of eigenfrequencies of the homogeneous BVP (5.10) is proved.

Let $\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))$ be the following matrix differential operator

$$\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z})) = (R_{lj}(\mathbf{D_z}, \mathbf{n}(\mathbf{z})))_{5\times 5}, \qquad R_{lj} = P_{lj}, \qquad l, j = 1, \dots, 5.$$

where $P_{lj}$ is given by (4.4).

THEOREM 7. *Two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$, *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, \varphi, \psi, \theta)$, *where* $\theta$ *satisfies the condition* (5.4), *the vector* $\mathbf{V} = (\mathbf{u}, \varphi, \psi)$ *is a regular solution of the system* (5.5) *satisfying the boundary condition*

$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{V}(\mathbf{z})\}^+ = 0 \qquad \textit{for } \mathbf{z} \in S. \tag{5.11}$$

*In addition, problems* $(I)^+_{\mathbf{0},\mathbf{0}}$ *and* (5.5), (5.11) *have the same eigenfrequencies.*

*Proof.* Suppose that there are two regular solutions of problem $(II)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous system of equations (3.7) in $\Omega^+$ satisfying the homogeneous boundary condition

$$\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = 0 \qquad \text{for } \mathbf{z} \in S. \tag{5.12}$$

Quite similarly as in Theorem 6, we obtain the relation (5.9). On the other hand, from (3.7) it follows that

$$\Lambda_1(\Delta)\theta(\mathbf{x}) = 0. \tag{5.13}$$

By virtue of (5.9) and the relation $\lambda_j \neq 0$ $(j = 1, \dots, 5)$ from (5.13) we have (5.4) and consequently, the system (3.7) implies (5.5). Obviously, in view of the condition (5.12) the vector $\mathbf{V}$ satisfies the boundary condition (5.11).

Finally, it is easy to see that the homogeneous boundary value problems $(II)^+_{\mathbf{0},\mathbf{0}}$ and (5.5), (5.11) have the same eigenfrequencies. □

REMARK 5. In the particular case, when porosities are neglected in the isotropic elastic solid (i.e. $\varphi = \psi \equiv 0$), the BVP (5.5), (5.11) is reduced to the following BVP

$$\begin{gathered} (\mu\Delta + \rho\omega^2)\mathbf{u}(\mathbf{x}) = \mathbf{0}, \qquad \operatorname{div}\mathbf{u}(\mathbf{x}) = 0, \\ \left\{ 2\frac{\partial \mathbf{u}(\mathbf{z})}{\partial \mathbf{n}(\mathbf{z})} + [\mathbf{n}(\mathbf{z}) \times \operatorname{curl}\mathbf{u}(\mathbf{z})] \right\}^+ = \mathbf{0} \end{gathered} \tag{5.14}$$

for $\mathbf{x} \in \Omega^+$, $\mathbf{z} \in S$, where $[\mathbf{n} \times \operatorname{curl}\mathbf{u}]$ is the vector product of the vectors $\mathbf{n}$ and $\operatorname{curl}\mathbf{u}$. On the other hand, in the classical theory of thermoelasticity, the second internal homogeneous BVP of steady vibrations is reduced to the BVP (5.14) (see [31]). In [33], the existence of eigenfrequencies of the homogeneous BVP (5.14) is proved.

THEOREM 8. *The external BVP* $(K)^{-}_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Theorem 8 can be proved similarly to Theorems 6 and 7 using the radiation conditions (4.2) and (4.3).

## 6. Existence theorems

In the sequel we use the matrix differential operator

$$\tilde{\mathbf{P}}(\mathbf{D_x}, \mathbf{n}) = (\tilde{P}_{lj}(\mathbf{D_x}, \mathbf{n}))_{6\times 6},$$

where

$$\tilde{P}_{lm}(\mathbf{D_x}, \mathbf{n}) = P_{lm}(\mathbf{D_x}, \mathbf{n}), \qquad P_{l6}(\mathbf{D_x}, \mathbf{n}) = -\gamma_0'\, n_l,$$
$$\tilde{P}_{l+3;j}(\mathbf{D_x}, \mathbf{n}) = P_{l+3;j}(\mathbf{D_x}, \mathbf{n}), \qquad l = 1,2,3,\ m = 1,\dots,5,\ j = 1,\dots,6.$$

It is easy to verify that the operator $\tilde{\mathbf{P}}(\mathbf{D_x}, \mathbf{n})$ may be obtained from the operator $\mathbf{P}(\mathbf{D_x}, \mathbf{n})$ by replacing $\gamma_0$ by $\gamma_0'$ and vice versa.

We introduce the following notation:

1) $\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \int_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})\, d_{\mathbf{y}}S$ (single-layer potential),

2) $\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \int_S [\tilde{\mathbf{P}}(\mathbf{D_y}, \mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top \mathbf{g}(\mathbf{y})\, d_{\mathbf{y}}S$ (double-layer potential),

3) $\mathbf{Z}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^\pm) = \int_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y})\, d\mathbf{y}$ (volume potential),

where $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{6\times 6}$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$ defined by (3.6); $\mathbf{g}$ and $\boldsymbol{\phi}$ are six-component vector functions; $\mathbf{\Gamma}^\top$ is the transpose of the matrix $\mathbf{\Gamma}$.

On the basis of properties of the matrix $\mathbf{\Gamma}(\mathbf{x})$ (see section 3) we have the following results.

THEOREM 9. *If* $S \in C^{m+1,\nu}$, $\mathbf{g} \in C^{m,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *and* $m$ *is a non-negative integer, then:*

a) $\mathbf{Z}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\nu'}(\mathbb{R}^3) \cap C^{m+1,\nu'}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm)$,

b) $\mathbf{A}(\mathbf{D_x})\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}$,

(6.1) c) $\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})\}^\pm = \mp\frac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})$,

d) $\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})$

*is a singular integral, where* $\mathbf{z} \in S$, $\mathbf{x} \in \Omega^\pm$.

THEOREM 10. *If* $S \in C^{m+1,\nu}$, $\mathbf{g} \in C^{m,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *then:*

a) $\mathbf{Z}^{(2)}(\cdot, \mathbf{g}) \in C^{m,\nu'}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm})$,

b) $\mathbf{A}(\mathbf{D_x})\, \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}$,

$$\text{c) } \{\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{\pm} = \pm\tfrac{1}{2}\, \mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}) \tag{6.2}$$

*for the non-negative integer* $m$,

d) $\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})$ *is a singular integral, where* $\mathbf{z} \in S$,

e) $\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{+} = \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{-}$, *for the natural number* $m$, *where* $\mathbf{z} \in S$, $\mathbf{x} \in \Omega^{\pm}$.

THEOREM 11. *If* $S \in C^{1,\nu}$, $\boldsymbol{\phi} \in C^{0,\nu'}(\Omega^{+})$, $0 < \nu' < \nu \leq 1$, *then:*

a) $\mathbf{Z}^{(3)}(\cdot, \boldsymbol{\phi}, \Omega^{+}) \in C^{1,\nu'}(\mathbb{R}^3) \cap C^2(\Omega^{+}) \cap C^{2,\nu'}(\overline{\Omega_0^{+}})$,

b) $\mathbf{A}(\mathbf{D_x})\, \mathbf{Z}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^{+}) = \boldsymbol{\phi}(\mathbf{x})$,

*where* $\mathbf{x} \in \Omega^{+}$, $\Omega_0^{+}$ *is a domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{+}} \subset \Omega^{+}$.

THEOREM 12. *If* $S \in C^{1,\nu}$, $\operatorname{supp} \boldsymbol{\phi} = \Omega \subset \Omega^{-}$, $\boldsymbol{\phi} \in C^{0,\nu'}(\Omega^{-})$, $0<\nu'<\nu\leq 1$, *then:*

a) $\mathbf{Z}^{(3)}(\cdot, \boldsymbol{\phi}, \Omega^{-}) \in C^{1,\nu'}(\mathbb{R}^3) \cap C^2(\Omega^{-}) \cap C^{2,\nu'}(\overline{\Omega_0^{-}})$,

b) $\mathbf{A}(\mathbf{D_x})\, \mathbf{Z}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^{-}) = \boldsymbol{\phi}(\mathbf{x})$,

*where* $\mathbf{x} \in \Omega^{-}$, $\Omega$ *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{-}} \subset \Omega^{-}$.

We introduce the notation

$$\begin{aligned}
\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) &\equiv \tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) &\equiv -\tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z}) &\equiv -\tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}^{(4)}\mathbf{g}(\mathbf{z}) &\equiv \tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}_{\chi}\mathbf{g}(\mathbf{z}) &\equiv \tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \chi\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})
\end{aligned} \tag{6.3}$$

for $\mathbf{z} \in S$, where $\chi$ is a complex number. Obviously, on the basis of Theorems 9 and 10, $\mathcal{K}_j$ and $\mathcal{K}_\chi$ are the singular integral operators $(j = 1, 2, 3, 4)$.

Let $\boldsymbol{\sigma}^{(j)} = (\sigma_{lm}^{(j)})_{6\times 6}$ be the symbol of the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1, 2, 3, 4)$ (see [31]). Taking into account (6.3) we find

$$\det \boldsymbol{\sigma}^{(j)} = \frac{1}{64}\left[1 - \frac{\mu^2}{(\lambda + 2\mu)^2}\right] = \frac{(\lambda+\mu)(\lambda+3\mu)}{64(\lambda+2\mu)^2} > 0. \tag{6.4}$$

Hence, the operator $\mathcal{K}^{(j)}$ is of the normal type, where $j = 1, 2, 3, 4$.

Let $\boldsymbol{\sigma}_\chi$ and $\operatorname{ind}\mathcal{K}_\chi$ be the symbol and the index of the operator $\mathcal{K}_\chi$, respectively. It may be easily shown that

$$\det \boldsymbol{\sigma}_\chi = \frac{(\lambda+2\mu)^2 - \mu^2\chi^2}{64(\lambda+2\mu)^2}$$

and $\det \boldsymbol{\sigma}_\chi$ vanishes only at two points $\chi_1$ and $\chi_2$ of the complex plane. By virtue of (6.4) and $\det \boldsymbol{\sigma}_1 = \det \boldsymbol{\sigma}^{(1)}$ we get $\chi_j \neq 1$ $(j = 1, 2)$ and

$$\operatorname{ind}\mathcal{K}_1 = \operatorname{ind}\mathcal{K}^{(1)} = \operatorname{ind}\mathcal{K}_0 = 0.$$

Quite similarly we obtain $\operatorname{ind}\mathcal{K}^{(2)} = -\operatorname{ind}\mathcal{K}^{(3)} = 0$ and $\operatorname{ind}\mathcal{K}^{(4)} = -\operatorname{ind}\mathcal{K}^{(1)} = 0$.

Thus, the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1, 2, 3, 4)$ is of the normal type with an index equal to zero. Consequently, Fredholm's theorems are valid for $\mathcal{K}^{(j)}$.

Remark 6. The definitions of a normal type singular integral operator, the symbol and the index of operator, and Fredholm's theorems for the singular integral equations are given in Kupradze *et al.* [31] and Mikhlin [34].

By Theorems 11 and 12 the volume potential $\mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^\pm)$ is a regular solution of (2.8), where $\mathbf{F} \in C^{0,\nu'}(\Omega^\pm)$, $0 < \nu' \leq 1$; $\operatorname{supp}\mathbf{F}$ is a finite domain in $\Omega^-$. Therefore, further we will consider problems $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$, where $K = I, II$. Now, we prove the existence theorems of a regular (classical) solution of these BVPs.

*Problem* $(I)^+_{\mathbf{0},\mathbf{f}}$. Let us assume that $\omega$ is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. We seek a regular solution to this problem in the form of the double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) \qquad \text{for } \mathbf{x} \in \Omega^+, \tag{6.5}$$

where $\mathbf{g}$ is the required six-component vector function.

Obviously, by Theorem 10 the vector function $\mathbf{U}$ is a solution of (3.7) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition (4.5) and using (6.2), from (6.5) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S. \tag{6.6}$$

We prove that the equation (6.6) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the associate homogeneous equation

$$\mathcal{K}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{0} \qquad \text{for } \mathbf{z} \in S, \tag{6.7}$$

where $\mathbf{h}$ is the required six-component vector function. Now, we prove that (6.7) has only the trivial solution.

Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (6.7). On the basis of Theorem 9 and Eq. (6.1) the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of the external homogeneous BVP $(II)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 8, the problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^-. \tag{6.8}$$

On the other hand, by Theorem 9 and (6.8) we get

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \qquad \text{for } \mathbf{z} \in S,$$

i.e., on the basis of Theorem 9 the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 6 and the assumption that $\omega$ is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$, the problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^+. \tag{6.9}$$

By virtue of (6.8), (6.9) and the identity (6.1) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for } \mathbf{z} \in S.$$

Thus, the homogeneous equation (6.7) has only the trivial solution and therefore on the basis of Fredholm's theorem the integral equation (6.6) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved

THEOREM 13. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, and $\omega$ is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$, then a regular solution of the internal BVP $(I)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the double-layer potential* (6.5), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (6.6) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

*Problem* $(II)^+_{\mathbf{0},\mathbf{f}}$. Let us assume that $\omega$ is not an eigenfrequency of the BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. We seek a regular solution to this problem in the form of the single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) \qquad \text{for } \mathbf{x} \in \Omega^+, \tag{6.10}$$

where $\mathbf{g}$ is the required six-component vector function.

Obviously, by Theorem 9 the vector function $\mathbf{U}$ is a solution of (3.7) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition (4.6) and using (6.1), from (6.10) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S. \tag{6.11}$$

We prove that the equation (6.11) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the homogeneous equation

$$-\frac{1}{2}\mathbf{g}_0(\mathbf{z}) + \mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}_0) = \mathbf{0} \qquad \text{for } \mathbf{z} \in S, \tag{6.12}$$

where $\mathbf{g}_0$ is the required six-component vector function. Now we prove that (6.12) has only the trivial solution. On the basis of Theorem 9 and Eq. (6.12) the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}_0)$ is a regular solution of the internal homogeneous BVP $(\mathrm{II})^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 7 and the assumption that $\omega$ is not an eigenfrequency of the problem $(\mathrm{II})^+_{\mathbf{0},\mathbf{0}}$, this problem has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^+. \tag{6.13}$$

On the other hand, by Theorem 9 and (6.13) we get

$$\{\mathbf{V}(\mathbf{z})\}^- = \{\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for } \mathbf{z} \in S,$$

i.e., on the basis of Theorem 9 the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 8 the problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^-. \tag{6.14}$$

By virtue of (6.13), (6.14) and the identity (6.1) we obtain

$$\mathbf{g}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for } \mathbf{z} \in S.$$

Thus, the homogeneous equation (6.12) has only the trivial solution and therefore on the basis of Fredholm's theorem the integral equation (6.11) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

THEOREM 14. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, and $\omega$ is not an eigenfrequency of the BVP $(II)^+_{\mathbf{0},\mathbf{0}}$, then a regular solution of the internal BVP $(II)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the single-layer potential* (6.10), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (6.11) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

*Problem* $(I)^-_{\mathbf{0},\mathbf{f}}$. Quite similarly the following theorem is proved.

THEOREM 15. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then a regular solution* $\mathbf{U}$ *of the external BVP $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the sum of double-layer and single-layer potentials*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) + (1 - i)\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) \qquad \text{for } \mathbf{x} \in \Omega^-,$$

*where* $\mathbf{g}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(3)}\,\mathbf{g}(\mathbf{z}) + (1-i)\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S.$$

*which is always solvable for an arbitrary vector* $\mathbf{f}$.

*Problem* $(II)^-_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to this problem in the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x},\mathbf{h}) + \mathbf{U}^*(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega^-, \tag{6.15}$$

where $\mathbf{h}$ is the required six-component vector function and the six-component vector function $\mathbf{U}^*$ is a regular solution of the equation

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}^*(\mathbf{x}) = \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^-. \tag{6.16}$$

Keeping in mind the boundary condition (4.7) and using (6.1), from (6.15) we obtain the following singular integral equation for determining the unknown vector $\mathbf{h}$

$$\mathcal{K}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{f}^*(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S, \tag{6.17}$$

where

$$\mathbf{f}^*(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}^*(\mathbf{z})\}^-. \tag{6.18}$$

Now, we prove that the equation (6.17) is always solvable for an arbitrary vector $\mathbf{f}$. We assume that the homogeneous equation

$$\mathcal{K}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{0} \tag{6.19}$$

has $m$ linearly independent solutions $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^m$ that are assumed to the orthonormal. By Fredholm's theorem the solvability condition of Eq. (6.17) can be written as

$$\int_S \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}^*(\mathbf{z})\}^- \cdot \boldsymbol{\psi}^{(l)}(\mathbf{z})d_{\mathbf{z}}S = N_l, \tag{6.20}$$

where

$$N_l = \int_S \mathbf{f}(\mathbf{z}) \cdot \boldsymbol{\psi}^{(l)}(\mathbf{z})d_{\mathbf{z}}S$$

and $\{\boldsymbol{\psi}^{(l)}(\mathbf{z})\}_{l=1}^m$ is a complete system of solutions of the homogeneous associated equation of (6.19), i.e.

$$\mathcal{K}^{(1)}\,\boldsymbol{\psi}^{(l)} = \mathbf{0}, \qquad l = 1,\ldots,m.$$

It is easy to see that the condition (6.20) takes the form (for details see [31])

$$\int_S \mathbf{h}^{(l)}(\mathbf{z}) \cdot \{\mathbf{U}^*(\mathbf{z})\}^- d_{\mathbf{z}}S = -N_l, \qquad l = 1, \dots, m. \tag{6.21}$$

Let the vector $\mathbf{U}^*$ be a solution of (6.16) and satisfies the boundary condition

$$\{\mathbf{U}^*(\mathbf{z})\}^- = \hat{\mathbf{f}}(\mathbf{z}), \tag{6.22}$$

where

$$\hat{\mathbf{f}}(\mathbf{z}) = \sum_{l=1}^{m} N_l \mathbf{h}^{(l)}(\mathbf{z}). \tag{6.23}$$

By virtue of Theorem 15 the BVP (6.16), (6.22) is always solvable. Because of the orthonormalization of $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^m$, the condition (6.21) is fulfilled automatically and the solvability of (6.17) is proved. Consequently, the existence of a regular solution of the problem $(II)^-_{\mathbf{0},\mathbf{f}}$ is proved too. Thus, the following theorem has been proved.

Theorem 16. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, then a regular solution $\mathbf{U}$ of the external BVP $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the sum* (6.15)*, where $\mathbf{h}$ is a solution of the singular integral equation* (6.17) *which is always solvable, $\mathbf{U}^*$ is the solution of BVP* (6.16),(6.22) *which is always solvable; and the vector functions $\mathbf{f}^*$ and $\hat{\mathbf{f}}$ are determined by* (6.18) *and* (6.23)*, respectively.*

## 7. Concluding remarks

1. In this paper the linear theory of thermoelasticity for materials with a double porosity structure based on the mechanics of materials with voids is considered and the following results are obtained:

i) the fundamental solution of the system of equations of steady vibrations is constructed explicitly by means of elementary functions and its basic properties are established;

ii) the Sommerfeld-Kupradze type radiation conditions are established;

iii) the uniqueness theorems of the basic internal and external BVPs of steady vibrations are proved;

iv) the basic properties of the surface (single-layer and double-layer) and volume potentials are established;

v) the existence theorems for regular (classical) solutions of the above mentioned BVPs are proved by using the potential method and the theory of singular integral equations.

2. On the basis of results of this paper it is possible to construct the fundamental solution and to prove the uniqueness and existence theorems in the linear theories of elasticity and thermoelasticity for materials with a multiple porosity structure by using the potential method and the theory of singular integral equations.

3. The BVPs of the classical theories of elasticity and theormoelasticity are investigated by using the potential method in KUPRADZE *et al.* [31], KUPRADZE [35], BURCHULADZE and GEGELIA [36]. An extensive review of works on this method can be found in GEGELIA and JENTSCH [37].

## Acknowledgments

The author is very grateful to the anonymous reviewers for their valuable comments concerning this work.

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*Received November 29, 2016; revised version April 23, 2017.*