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Antiplane strain (shear) of orthotropic non-homogeneous prismatic shell-like bodies

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Antiplane strain (shear) of orthotropic non-homogeneous prismatic shell-like bodies are considered when the shear modulus depending on the body projection (i.e., on a domain lying in the plane of interest) variables may vanish either on a part or on the entire boundary of the projection. We study the dependence of the well-posedness of the boundary conditions (BCs) on the character of the vanishing of the shear modulus. The case of vibration is considered as well.

Key words: antiplane strain, degenerate elliptic equations, weighted spaces, Hardy's inequality.

1. Introduction

The antiplane shear (strain) is a special state of strain in a body. This state is achieved when the displacements in the body are zero in the plane of interest but nonzero in the direction perpendicular to the plane. If the plane Ox_1x_2 of the rectangular Cartesian frame $Ox_1x_2x_3$ is the plane of interest, then

$$\begin{aligned} &u_\alpha(x_1, x_2, x_3) \equiv 0, \quad \alpha = 1, 2; \\ &u_3(x_1, x_2, x_3) = u_3(x_1, x_2), \qquad (x_1, x_2) \in \omega, \end{aligned} \tag{1.1}$$

where u_j, $j = 1, 2, 3$, are the displacements, ω is a projection of the prismatic shell-like body Ω on the plane Ox_1x_2, correspondingly $\partial\omega$ is a projection of the lateral boundary S of Ω. The relations (1.1) mean that all the sections of the body parallel to the plane of interest Ox_1x_2 will be bent as its section by the plane Ox_1x_2. Ω may have either Lipschitz (see Figs. 1–4) or non-Lipschitz boundary (see Fig. 5), ω has Lipschitz boundary (see Figs. 6, 7). Below Einstein's summation convention is used. A bar under one of the repeated indices means that this convention is not used.

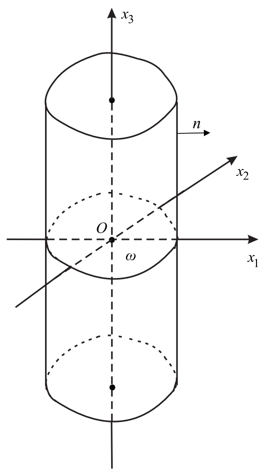

Fig. 1. A non-homogeneous elastic cylinder.

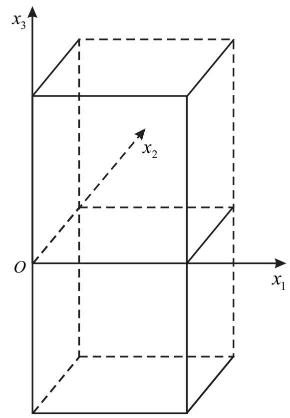

Fig. 2. Ω with a Lipschitz boundary.

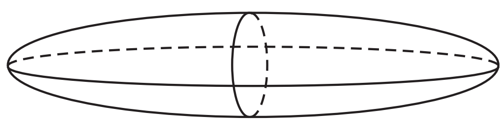

Fig. 3. Ω with a smooth boundary.

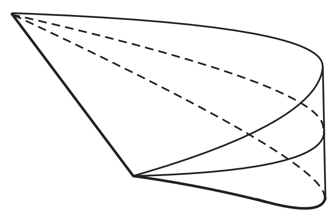

Fig. 4. Ω with a Lipschitz boundary.

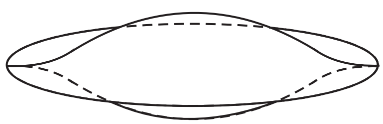

Fig. 5. Ω with a non-Lipschitz boundary.

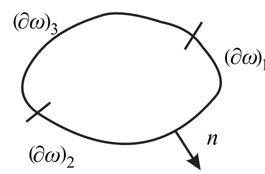

Fig. 6. General case of ω.

For an orthotropic linear elastic material the strain e_{kj} and stress X_{kj}, $k, j = 1, 2, 3$, tensors resulting from a state of antiplane shear can be expressed as

$$(1.2) \quad e_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad e_{33} \equiv 0; \qquad e_{\alpha 3} = \frac{1}{2} u_{3,\alpha}(x_1, x_2) \not\equiv 0, \quad \alpha = 1, 2,$$

where the comma after the index means differentiation with respect to the variable corresponding to the index indicated after the comma, and

$$(1.3) \quad \begin{aligned} &X_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \qquad X_{33} \equiv 0; \\ &X_{3\alpha} = X_{\alpha 3} = \mu_{\underline{\alpha}}(x_1, x_2) u_{3,\alpha}(x_1, x_2), \qquad \alpha = 1, 2, \end{aligned}$$

since for non-homogeneous body with the shear moduli $\mu_\alpha(x_1, x_2)$, $\alpha = 1, 2$, Hooke's law is expressed in the following way

$$X_{\alpha 3} = 2\mu_{\underline{\alpha}} e_{\alpha 3} = \mu_{\underline{\alpha}}(x_1, x_2) u_{3,\alpha}(x_1, x_2), \qquad \alpha = 1, 2. \tag{1.4}$$

From (1.3), (1.4) it follows that at any point $x := (x_1, x_2, x_3)$ the stress vector components are

$$X_{n\alpha} = X_{j\alpha} n_j = X_{3\alpha} n_3 = \mu_{\underline{\alpha}} u_{3,\alpha} n_3, \qquad \alpha = 1, 2; \tag{1.5}$$

$$X_{n3} = X_{j3} n_j = X_{\alpha 3} n_\alpha = \sum_{\alpha=1}^{2} \mu_\alpha u_{3,\alpha} n_\alpha, \tag{1.6}$$

where n is the unit normal of a surface element passing through x.

The equilibrium equations are reduced to

$$\Phi_\alpha \equiv 0, \quad \alpha = 1, 2, \qquad X_{\alpha 3,\alpha} + \Phi_3 = 0, \tag{1.7}$$

where Φ_j, $j = 1, 2, 3$, are the components of the volume force.

Let $u_3 \in C^2(\omega)$, $\mu \in C^1(\omega)$, and $\Phi_3 \in C(\overline{\omega})$. Substituting (1.3) into (1.7) we get only one governing equation

$$\sum_{\alpha=1}^{2} (\mu_\alpha(x_1, x_2) u_{3,\alpha}(x_1, x_2))_{,\alpha} + \Phi_3(x_1, x_2) = 0, \qquad (x_1, x_2) \in \omega. \tag{1.8}$$

In the dynamical case we have

$$\sum_{\alpha=1}^{2} (\mu_\alpha(x_1, x_2) u_{3,\alpha}(x_1, x_2, t))_{,\alpha} + \Phi_3(x_1, x_2, t) = \rho \ddot{u}_3(x_1, x_2, t), \qquad (x_1, x_2) \in \omega,\ t \geq t_0. \tag{1.9}$$

The aim of the present paper is to investigate static and dynamical problems for the symmetric prismatic shell-like body Ω (see [1, 2]), in particular, of constant thickness (which (body) may also be infinite) when the shear moduli

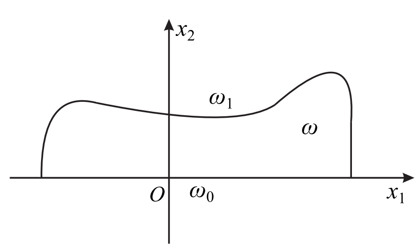

FIG. 7. A finite ω.

may vanish either on a part or on the entire boundary of the projection ω on the plane of interest Ox_1x_2 (see Figs. 6, 7).

In [3] antiplane strain (shear) of an isotropic prismatic shell-like body is considered when the shear modulus depending on the body projection variables vanishes either on a part or on the entire boundary of the projection. The dependence of well-posedeness of boundary conditions on the character of vanishing of the shear modulus is studied. When the above-mentioned domain is either the half-plane or the half-disk and the shear modulus is a power function with respect to the variable along the perpendicular to the linear boundary, the basic boundary value problems are solved explicitly in quadratures. In [4] the dynamical problem of antiplane strain (shear) of an isotropic non-homogeneous prismatic shell-like body is considered.

2. Investigation of BVPs. General case

In this section the antiplane deformation of an orthotropic non-homogeneous elastic cylinders and prismatic shell-like bodies (see Figs. 1–5, for other examples see [2]) are studied.

Evidently (see (1.5), (1.6)), on the lateral boundary S of the cylinder Ω

$$X_{n\alpha} = 0, \quad \alpha = 1,2, \qquad X_{n3} = \sum_{\alpha=1}^{2} \mu_\alpha u_{3,\alpha} n_\alpha, \tag{2.1}$$

where n is the outward normal to S.

If the cylinder is finite, on the upper and lower bases of the cylinder Ω (see (1.5), (1.6))

$$X_{\overset{(+)}{n}\alpha} = X_{3\alpha} = \mu_{\underline{\alpha}} u_{3,\alpha}, \quad \alpha = 1,2, \qquad X_{n3} = X_{33} = 0,$$

and

$$X_{\overset{(-)}{n}\alpha} = -X_{3\alpha} = -\mu_{\underline{\alpha}} u_{3,\alpha}, \quad \alpha = 1,2, \qquad X_{n3} = -X_{33} = 0,$$

respectively (in the case of prismatic shell-like bodies they are given by formulas (1.5), (1.6) on the face surfaces).

Let the shear moduli $\mu_\alpha \in C^1(\overline{\omega}\backslash(\partial\omega)_2) \cap C(\overline{\omega})$, $\alpha = 1,2$, as has already been assumed, be independent of x_3 and $\mu_\alpha(x_1,x_2) > 0$ in $\omega \cup (\partial\omega)_3$ (see Figure 6), $\mu_\alpha(x_1,x_2) = 0$ on $(\partial\omega)_0$, where the boundary $(\partial\omega)_0$ is divided in $(\partial\omega)_1$ and $(\partial\omega)_2$, i.e., $\partial\omega = \overline{(\partial\omega)_1} \cup \overline{(\partial\omega)_2} \cup \overline{(\partial\omega)_3}$ (correspondingly $S = \overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3$). If, moreover, $\mu_\alpha(x_1,x_2) = \mu_0^\alpha \mu(x_1,x_2)$, $\mu_0^\alpha = \text{const} > 0$, and

$$\left.\frac{\partial\mu}{\partial n}\right|_{(\partial\omega)_2} = +\infty, \tag{2.2}$$

$$\left.\frac{\partial\mu}{\partial n}\right|_{(\partial\omega)_1} \geq 0, \tag{2.3}$$

then (compare with [3]): if $(\partial\omega)_1 \neq \emptyset$, a solution u_3 of equation (1.8) is determined uniquely by its values prescribed only on $(\partial\omega)_2 \cup (\partial\omega)_3$ (Problem E: $u_3 \in C^2(\omega) \cap C(\bar{\omega} \setminus (\partial\omega)_1) \cap C^b(\omega)$, $C^b(\omega)$ means a class of bounded functions); if $(\partial\omega)_1 = \emptyset$, for unique solvability of the BVP the values of u_3 should be prescribed on the whole boundary $\partial\omega$ (Problem D: $u_3 \in C^2(\Omega) \cap C(\overline{\Omega})$).

The criteria (2.2) and (2.3) can be replaced by the equivalent criteria in the integral form (see [5], formulas (13), (14)).

If $X_{n3} = \varphi$ is prescribed on $\partial\omega$, then on $(\partial\omega)_0$ we have to consider the weighted boundary condition (BC) (Problem W: $u_3 \in C^2(\Omega)$, $\sum_{\alpha=1}^{2} \mu_\alpha u_{3,\alpha} n_\alpha \in C(\overline{\Omega})$)

$$\lim_{(x_1,x_2)\to(\partial\omega)_0} \sum_{\alpha=1}^{2} \mu_\alpha u_{3,\alpha} n_\alpha = \varphi. \tag{2.4}$$

The above mentioned problems are well-posed under some restrictions on classes of functions, where solutions are sought (for classical solutions see below; for $\mathcal{H}$-weak solutions [6, 7] of Problems D and E see Appendix 1 in [5] which can be reformulated for our case; see also [8–10]).

As it is seen in the case of Problem D on the cylindrical boundary S deflections $u_3(x_1, x_2)$ are prescribed, while in the case of Problem E deflections $u_3(x_1, x_2)$ should be prescribed only on $S_2 \cup S_3$. BC (2.4) means that at S shear stresses $X_{\alpha 3}(x_1, x_2)$, $\alpha = 1, 2$, are applied. In all the cases at face surfaces one should apply stresses calculated by formulas (1.5), (1.6) in order to maintain the antiplane state in the body. Note that if the thickness of the prismatic shell-like body vanishes on a part of $\partial\omega$ or on the entire $\partial\omega$, then by the antiplane shear the character of the thickness vanishing does not affect well-posedness of BVPs at cusped edges. This is in contrast to cusped prismatic shells for which it is the case and depends (see [2]) on the sharpening geometry.

Let

$$\mu_\alpha(x_1, x_2) = \mu_0^\alpha x_2^{\kappa_\alpha}, \quad \mu_0^\alpha = \text{const} > 0, \quad \kappa_\alpha \geq 0,\ \alpha = 1, 2,\ (x_1, x_2) \in \omega. \tag{2.5}$$

In this case equation (1.8) has the form

$$\mu_0^1 x_2^{\kappa_1} u_{3,11} + \mu_0^2 x_2^{\kappa_2} u_{3,22} + \kappa_2 \mu_0^2 x_2^{\kappa_2 - 1} u_{3,2} = -\Phi_3(x_1, x_2). \tag{2.6}$$

The partial differential operator in the left-hand side of equation (1.8) degenerates on the x_1-axis, provided at least one of κ_1 and κ_2 is not zero.

If $\mu_0^1 = \mu_0^2 =: \mu_0$, then (2.6) can be rewritten as follows

$$x_2^{\kappa_1} u_{3,11} + x_2^{\kappa_2} u_{3,22} + \kappa_2 x_2^{\kappa_2 - 1} u_{3,2} = -\frac{1}{\mu_0} \Phi_3(x_1, x_2). \tag{2.7}$$

If $\kappa_1 = \kappa_2 =: \kappa$, then

$$x_2(\mu_0^1 u_{3,11} + \mu_0^2 u_{3,22}) + \kappa\mu_0^2 u_{3,2} = -x_2^{1-\kappa}\Phi_3(x_1, x_2). \tag{2.8}$$

Problem D and Problem E are uniquely solvable for equation (2.7) for $\kappa_2 < 1$ and $\kappa_2 \geq 1$, correspondingly. This follows from the following theorem (see [11]).

THEOREM 2.1. If the coefficients a_α, $\alpha = 1, 2$ and c of the equation

$$x_2^{\kappa_\alpha} u_{,\alpha\alpha} + a_\alpha(x_1, x_2)u_{,\alpha} + c(x_1, x_2)u = 0, \qquad c \leq 0,\ \kappa_\alpha = \text{const} \geq 0,\ \alpha = 1, 2,$$

are analytic in $\overline{\omega}$ bounded by a sufficiently smooth arc $(\partial\omega \setminus \omega_0)$ lying in the half plane $x_2 \geq 0$ and by a segment ω_0 of the x_1-axis, then

(i) if either $\kappa_2 < 1$, or $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) < x_2^{\kappa_2 - 1} \tag{2.9}$$

in $\overline{I}_\delta$ for some $\delta = \text{const} > 0$, where

$$I_\delta := \{(x_1, x_2) \in \omega : 0 < x_2 < \delta\},$$

the Dirichlet problem is well-posed;

(ii) if $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) \geq x_2^{\kappa_2 - 1} \tag{2.10}$$

in I_δ and $a_1(x_1, x_2) = O(x_2^{\kappa_1})$, $x_2 \to 0_+$ (O is the Landau symbol), the Keldysh problem is well-posed.

REMARK. If $1 < \kappa_2 < 2$, $a_2(x, 0) \leq 0$, the Dirichlet problem is correct.

Using the method applied in [12] (see pages 58, 68–74), it is not difficult to verify that the theorem is also true for Hölder continuous c and a_α, $\alpha = 1, 2$, on $\overline{\omega}$, provided:

(i) $\lim\limits_{x_2 \to 0_+} x_2^{1-\kappa_2} a_2(x_1, x_2) = a_0 = const < 1$ for $(x_1, 0) \in \omega_0$ when $0 \leq \kappa_2 < 1$;

(ii) if $a_2(x_1^0, 0) = 0$ for a fixed $(x_1^0, 0) \in \omega_0$ when $1 < n < 2$, then there exists such a $\delta = const > 0$ that $a_2(x_1^0, x_2) = \kappa(x_1^0, x_2) \cdot x_2$ with bounded $\kappa(x_1^0, x_2)$ for $0 \leq x_2 < \delta$.

Here ω_0 is a part of $\partial\omega$ lying on the x_1-axis.

Indeed, for (2.7) the conditions (2.9) and (2.10) mean

$$\kappa_2 x_2^{\kappa_2 - 1} < x_2^{\kappa_2 - 1}, \quad \text{i.e., } \kappa_2 < 1,$$

and

$$\kappa_2 x_2^{\kappa_2 - 1} \geq x_2^{\kappa_2 - 1}, \quad \text{i.e., } \kappa_2 \geq 1,$$

respectively.

3. Harmonic vibration

Let us consider the case (2.5) for $\kappa_1 = \kappa_2 =: \kappa < 1$.

In the case of harmonic vibration

$$u_3(x_1, x_2, t) = e^{-i\vartheta t} v(x_1, x_2), \quad \Phi_3(x_1, x_2, t) = e^{-i\vartheta t}\Phi(x_1, x_2), \quad i^2 = -1, \ \vartheta = \text{const} > 0,$$

from equation (1.9) we get

$$\mu_0^1 x_2^\kappa v_{,11}(x_1, x_2) + \mu_0^2 x_2^\kappa v_{,22}(x_1, x_2) + \kappa\mu_0^2 x_2^{\kappa-1} v_{,2}(x_1, x_2) + \Phi(x_1, x_2) = -\rho\vartheta^2 v(x_1, x_2), \tag{3.1}$$

$$(x_1, x_2) \in \omega, \ 0 < x_2 \leq l, \ l = \text{const} > 0.$$

Let

$$v, v^* \in C^2(\omega) \cap C^1(\overline{\omega}), \quad \Phi \in C(\overline{\omega}),$$

Green's formula

$$\int\limits_\omega Av \cdot v^* d\omega = J(v, v^*) - \int\limits_{\partial\omega} T_n v \cdot v^* d\partial\omega = \int\limits_\omega \Phi \cdot v^* d\omega, \tag{3.2}$$

where

$$A := -x_2^\kappa \left(\mu_0^1 \frac{\partial^2}{\partial x_1^2} + \mu_0^2 \frac{\partial^2}{\partial x_2^2}\right) - \kappa\mu_0^2 x_2^{\kappa-1}\frac{\partial}{\partial x_2},$$

$$\begin{aligned} J(v, v^*) &:= \int\limits_\omega [\mu_1 v_{,1} v^*_{,1} + \mu_2 v_{,2} v^*_{,2} - \rho\vartheta^2 v v^*] d\omega \\ &= \int\limits_\omega 4[\mu_1 e_{13}(v) e_{13}(v^*) + \mu_2 e_{23}(v) e_{23}(v^*) - \rho\vartheta^2 v v^*] d\omega \\ &= \int\limits_\omega [X_{3\alpha} e_{\alpha 3}(v^*) - \rho\vartheta^2 v v^*] d\omega, \end{aligned} \tag{3.3}$$

$n := (n_1, n_2)$ is the inward normal to $\partial\omega$:

$$T_n := X_{n3} = X_{3\alpha 0} n_\alpha,$$

is valid.

If we consider BVPs for equation (3.1) with a homogeneous boundary condition

$$v = 0 \ \text{ on } \ \partial\omega \tag{3.4}$$

for which the curvilinear integral along $\partial\omega$ in (3.2) disappears, we obtain

$$J(v,v^*)=\int\limits_{\omega}\Phi\cdot v^* d\omega.$$

Note that throughout the paper, for smooth classical solutions, equation (3.1) and boundary condition (3.4) are understood in the classical point-wise sense, while for generalized weak solution equation (3.1) is understood in the distributional sense and the boundary condition (3.4) is understood in the usual trace sense ([13], [14]).

Denote by $\mathcal{D}(\omega)$ a space of infinitely differentiable functions with compact support in ω. The bilinear form and norm are introduced by the following formulas:

$$(v,v^*)_{X^\kappa}:=\int\limits_{\omega}x_2^\kappa[v_{,1}v^*_{,1}+v_{,2}v^*_{,2}]d\omega$$

and

$$\|v\|^2_{X^\kappa}:=\int\limits_{\omega}x_2^\kappa[v_{,1}^2+v_{,2}^2]d\omega.$$

The last is the norm because of the well-known Hardy-type inequality (see [15], p. 69; [16]). So, X^κ is a Hilbert space.

The classical and weak setting of the homogeneous Dirichlet problem can be formulated as follows:

Problem 3.1. Find $v\in C^2(\omega)\cap C^1(\overline{\omega})$ satisfying equation (3.1) in ω and the homogeneous Dirichlet boundary condition (3.4).

Problem 3.2. Find $v\in X^\kappa$ satisfying the equality

$$J(v,v^*)=\langle\Phi,v^*\rangle\qquad \textit{for all } v^*\in X^\kappa, \tag{3.5}$$

here Φ belongs to the adjoint space $[X^\kappa]^*$, and $\langle\cdot,\cdot\rangle$ denotes duality brackets between the spaces $[X^\kappa]^*$ and X^κ.

Lemma 3.3. The bilinear form $J(\cdot,\cdot)$ is bounded and strictly coercive in the space $X^\kappa(\omega)$, i.e., there are positive constant C_0 and C_1 such that

$$|J(v,v^*)|\le C_1\|v\|_{X^\kappa}\|v^*\|_{X^\kappa}, \tag{3.6}$$

$$J(v,v)\ge C_0\|v\|^2_{X^\kappa} \tag{3.7}$$

for all $v,v^*\in X^\kappa$, if

$$\vartheta^2<\frac{\mu_0^2}{4\rho l^{2-\kappa}}. \tag{3.8}$$

Proof. (3.7) immediately follows from (3.3) and Hardy's inequality (see [15, p. 69]; [16]), namely,

$$\begin{aligned}
J(v,v) &= \int_\omega [x_2^\kappa(\mu_0^1 v,_1^2 + \mu_0^2 v,_2^2) - \rho\vartheta^2 v^2]\, d\omega \\
&\geq \int_\omega [x_2^\kappa(\mu_0^1 v,_1^2 + \mu_0^2 v,_2^2) - 4\rho\vartheta^2 x^2 v,_2^2]\, d\omega \\
&\geq \int_\omega x_2^\kappa(\mu_0^1 v,_1^2 + \mu_0^2 v,_2^2 - 4\rho\vartheta^2 l^{2-\kappa} v,_2^2)\, d\omega \\
&\quad \text{(taking into account of (3.8) } \mu_0^2 - 4\rho\vartheta^2 l^{2-\kappa} > 0) \\
&\geq C_0 \int_\omega x_2^\kappa(v,_1^2 + v,_2^2) = C_0 \|v\|^2_{X^\kappa}, \qquad C_0 := \min\{\mu_0^1, \mu_0^2 - 4\rho\vartheta^2 l^{2-\kappa}\}.
\end{aligned}$$

Now, we have to prove (3.6). From (3.3) we get

$$\begin{aligned}
|J(v,v^*)|^2 &= \Big|\int_\omega [x_2^\kappa(\mu_0^1 v,_1 v,_1^* + \mu_0^2 v,_2 v,_2^*) - \rho\vartheta^2 v v^*]\, d\omega\Big|^2 \\
&= \Big|\int_\omega x_2^\kappa(\mu_0^1 v,_1 v,_1^* + \mu_0^2 v,_2 v,_2^*)\, d\omega\Big|^2 + \Big|\int_\omega \rho\vartheta^2 v v^*\, d\omega\Big|^2 \\
&\quad + 2\Big|\int_\omega x_2^\kappa(\mu_0^1 v,_1 v,_1^* + \mu_0^2 v,_2 v,_2^*)\, d\omega\Big|\Big|\int_\omega \rho\vartheta^2 v v^* d\omega\Big| \\
&\leq C_2\Big|\int_\omega x_2^\kappa(v,_1 v,_1^* + v,_2 v,_2^*)\, d\omega\Big|^2 + \Big|\int_\omega \rho\vartheta^2 v v^*\, d\omega\Big|^2 \\
&\quad + 2C_3\Big|\int_\omega x_2^\kappa(v,_1 v,_1^* + v,_2 v,_2^*)\, d\omega\Big|\Big|\int_\omega \rho\vartheta^2 v v^*\, d\omega\Big| \\
&:= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
C_2 &:= \max\{(\mu_0^1)^2, (\mu_0^2)^2\}, \qquad C_3 := \max\{\mu_0^1, \mu_0^2\}, \\
I_1 &:= C_2\Big|\int_\omega x_2^\kappa(v,_1 v,_1^* + v,_2 v,_2^*)\, d\omega\Big|^2 + \Big|\int_\omega \rho\vartheta^2 v v^*\, d\omega\Big|^2, \\
I_2 &:= 2C_3\Big|\int_\omega x_2^\kappa(v,_1 v,_1^* + v,_2 v,_2^*)\, d\omega\Big|\Big|\int_\omega \rho\vartheta^2 v v^*\, d\omega\Big|.
\end{aligned}$$

Taking into account Hardy's inequality (see [15, p. 69]; [16]), we have (all the norms bellow are of the space X^κ)

$$\begin{aligned}
I_1 &\leq C_2\|v\|^2\|v^*\|^2 + 16\rho^2\vartheta^4 \int\limits_\omega x_2^\kappa x_2^{2-\kappa} v,_2^2\, d\omega \int\limits_\omega x_2^\kappa x_2^{2-\kappa}(v,_2^*)^2 d\omega \\
&\leq C_2\|v\|^2\|v^*\|^2 + 16\rho^2\vartheta^4 l^{4-2\kappa} \int\limits_\omega x_2^\kappa v,_2^2\, d\omega \int\limits_\omega x_2^\kappa (v,_2^*)^2 d\omega \\
&\leq C_2\|v\|^2\|v^*\|^2 + 16\rho^2\vartheta^4 l^{4-2\kappa} \int\limits_\omega x_2^\kappa (v,_1^2 + v,_2^2) d\omega \int\limits_\omega x_2^\kappa [(v,_1^*)^2 + (v,_1^*)^2] d\omega \\
&= C_2\|v\|^2\|v^*\|^2 + 16\rho^2\vartheta^4 l^{4-2\kappa}\|v\|^2\|v^*\|^2 \leq C_4\|v\|^2\|v^*\|^2, \\
I_2 &= 2C_3\rho\vartheta^2\Big[\Big|\int\limits_\omega x_2^\kappa (v,_1 v,_1^* + v,_2 v,_2^*) d\omega\Big|^2 \Big|\int\limits_\omega v v^* d\omega\Big|^2\Big]^{1/2} \\
&\leq 2C_3\rho\vartheta^2\bigg[\|v\|^2\|v^*\|^2 \int\limits_\omega x_2^\kappa x_2^{2-\kappa} v,_2^2\, d\omega \int\limits_\omega x_2^\kappa x_2^{2-\kappa} (v,_2^*)^2\, d\omega\bigg]^{1/2} \\
&\leq 2C_3\rho\vartheta^2 l^{2-\kappa}[\|v\|^2\|v^*\|^2\|v\|^2\|v^*\|^2]^{1/2} = C_5\|v\|^2\|v^*\|^2.
\end{aligned}$$

The last two inequalities prove (3.6). □

Remark 3.4. If $J(v,v) = 0$, then $v \equiv 0$ by (3.7).

Theorem 3.5. Let Φ be a bounded linear functional from $[X^\kappa]^*$. Then the variational problem (3.5) has a unique solution $v \in X^\kappa$ for an arbitrary value of the parameter κ and

$$\|v\|_{X^\kappa} \leq \frac{1}{C_0}\|\Phi\|_{[X^\kappa]^*}.$$

Proof. Taking into account Lemma 3.3, the proof immediately follows from the Lax-Milgram theorem. □

Remark 3.6. It can be easily shown that if $\Phi \in L(\omega)$ and $\operatorname{supp}\Phi \cap \overline{\gamma}_0 = \emptyset$, then $\Phi \in [X^\kappa]^*$ and

$$\langle \Phi,\, v^* \rangle = \int\limits_\omega \Phi(x)\, v^*(x)\, d\omega,$$

since $v^* \in H^1(\omega_\varepsilon)$, where ε is sufficiently small positive number such that $\operatorname{supp}\Phi \subset \omega_\varepsilon = \omega \cap \{x_2 > \varepsilon\}$. Therefore,

$$\begin{aligned}
|\langle \Phi,\, v^* \rangle| &= \Big|\int\limits_\omega \Phi(x)\, v^*(x)\, d\omega\Big| \leq \|\Phi\|_{L_2(\omega)}\, \|v^*\|_{L_2(\omega_\varepsilon)} \\
&\leq \|\Phi\|_{L_2(\omega)}\, \|v^*\|_{H^1(\omega_\varepsilon)} \leq C_\varepsilon \|\Phi\|_{L_2(\omega)}\, \|v^*\|_{X^\kappa}.
\end{aligned}$$

In this case, we obtain the estimate

$$\|v\|_{X^\kappa} \leq \frac{C_\varepsilon}{C_0} \|\Phi\|_{L_2(\omega)}.$$

REMARK 3.7. The space X^κ is the weighted Sobolev space.

COROLLARY 3.8. v has the zero trace on $\partial\omega$ if $\kappa < 1$.

Proof. It follows directly from the trace theorems (see [15] and [16]).

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