

## Exact solution to structural instability of a parallel array of mutually attracting identical simply-supported plates

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BY MEANS OF THE THEORY OF ELASTICITY, we investigate the structural instability of a parallel array of identical simply-supported plates. One plate interacts with the neighboring plates through surface attractive forces. The proposed method is based on the  $2 \times 2$  transfer matrix for a plate and on the solution of the generalized eigenvalue problem for the plate array. Analytical expressions of the critical interaction coefficients for two, three and four interacting plates are obtained when the end-effect of the plates at the ends of the parallel array and the surface energy of the plates are ignored. The influence of the end-effect and surface energy on the critical interaction coefficient is also numerically studied. Our solution is valid whether the plates are thin or extremely thick.

**Key words:** structural instability, surface forces, critical interaction coefficient, transfer matrix, generalized eigenvalue problem, exact solution.

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### 1. Introduction

STRUCTURAL INSTABILITY OF A PARALLEL ARRAY of microbeams or microplates due to surface attractive forces, such as van der Waals force, electrostatic force, capillary force or Casimir force, has become an intensive research topic in microelectrical mechanical systems (MEMS) in the last decade [1–6]. In the structural instability analysis of interacting microbeams or microplates, the majority of previous investigations have adopted the simplified Euler–Bernoulli beam bending model.

In this work, we endeavor to employ the rigorous elasticity theory to carry out a linear perturbation analysis of the structural instability of an arbitrary number of mutually attracting simply-supported identical plates under plane strain deformations. Based on the general solution by HE and JIANG [7], the  $2 \times 2$  transfer matrix for a plate can be finally derived by considering the fact that the shear stress on the two surfaces of the plate is zero. Consequently, a generalized eigenvalue problem for  $N$  interacting plates can be obtained. When the

surface energy of the plates is absent, the eigenvalue is just the normalized interaction coefficient and the eigenvector is composed of the normal displacement and the normal traction on the upper surfaces of the  $N$  plates. The critical interaction coefficient is then determined by the smallest of the  $(N - 1)$  positive and nonzero eigenvalues. The critical interaction coefficient for any number of mutually attracting plates without or with the end-effect is determined. Finally, the influence of surface energy of the plates on the structural instability is discussed.

## 2. Analysis

As shown in Fig. 1, we consider the plane strain deformations of a parallel array of  $N$  equally spaced and mutually interacting identical plates with equal shear modulus  $\mu$ , Poisson's ratio  $\nu$ , length  $L$  and thickness  $h$ . The plates are simply-supported at  $x_1 = 0$  and  $x_1 = L$ , and they are numbered sequentially from 1 for the bottom plate to  $N$  for the top plate. Let  $\sigma_{11}$ ,  $\sigma_{22}$ ,  $\sigma_{12}$  be the stresses and  $u_1$ ,  $u_2$  be the displacements. If the end-effect of the plates at the ends of the parallel array and the surface energy of the plates are ignored [1], the perturbed normal displacement and normal traction on the lower surface of of

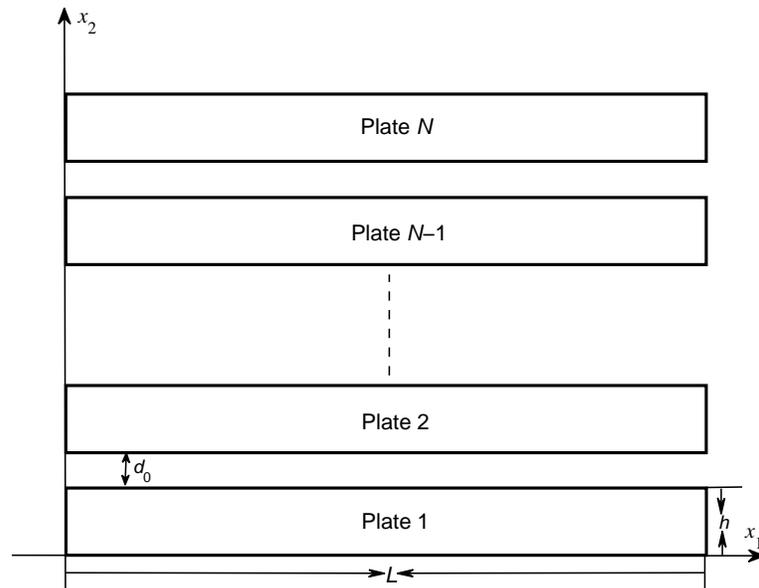


FIG. 1. A parallel array of  $N$  equally spaced and mutually interacting identical simply-supported plates.

plate  $j$  are related to those on the upper surface of plate  $j - 1$  through

$$(2.1) \quad \sigma_{22}^{L(j)} = \sigma_{22}^{U(j-1)} = Y(u_2^{U(j-1)} - u_2^{L(j)}),$$

where the superscripts  $L$  and  $U$  are used to distinguish the quantities on the lower and upper surfaces, the superscripts  $(j)$  and  $(j - 1)$  refer to plate  $j$  and plate  $j - 1$ ,  $Y (> 0)$  is the interaction coefficient that is determined by the first-order derivative of the surface attractive forces (van der Waals force, electrostatic force, capillary force or Casimir force) with respect to the relative normal displacement between the two surfaces [1]. For example, if the attractive stress between two parallel flat surfaces is described by  $F = c/d^n$ , where  $c$  is a coefficient and  $d$  is the distance between two flat surfaces and  $n$  is an integer index, and the end-effect is ignored, the interaction coefficient is simply  $Y = nc/d_0^{n+1} > 0$  with  $d_0$  being the initial separation between the two surfaces (see Fig. 1). It is seen from Eq. (2.1) that the interaction between the two surfaces acts as a uniformly distributed linear spring with negative spring constant, and is the driving force for structural instability.

According to HE and JIANG [7], the perturbed displacements and tractions in a simply-supported plate are given by

$$(2.2) \quad \begin{aligned} u_1 &= [e^{\alpha x_2}(C_1 + x_2 C_2) + e^{-\alpha x_2}(C_3 + x_2 C_4)] \cos \alpha x_1, \\ u_2 &= \left\{ e^{\alpha x_2} \left[ C_1 + \left( x_2 - \frac{3 - 4\nu}{\alpha} \right) C_2 \right] \right. \\ &\quad \left. - e^{-\alpha x_2} \left[ C_3 + \left( x_2 + \frac{3 - 4\nu}{\alpha} \right) C_4 \right] \right\} \sin \alpha x_1, \\ \sigma_{12} &= 2\mu \{ e^{\alpha x_2} [\alpha C_1 + (\alpha x_2 - 1 + 2\nu) C_2] \\ &\quad - e^{-\alpha x_2} [\alpha C_3 + (\alpha x_2 + 1 - 2\nu) C_4] \} \cos \alpha x_1, \\ \sigma_{22} &= 2\mu \{ e^{\alpha x_2} [\alpha C_1 + (\alpha x_2 - 2 + 2\nu) C_2] \\ &\quad + e^{-\alpha x_2} [\alpha C_3 + (\alpha x_2 + 2 - 2\nu) C_4] \} \sin \alpha x_1, \end{aligned}$$

where  $C_1, C_2, C_3, C_4$  are four constants to be determined, and

$$(2.4) \quad \alpha = \frac{n\pi}{L}, \quad n = 1, 2, \dots$$

For a certain plate, the displacements and tractions on its lower surface can be expressed in terms of those on its upper surface as follows [8]:

$$(2.5) \quad \begin{bmatrix} u_1^L & u_2^L & \frac{\sigma_{12}^L}{2\alpha\mu} & \frac{\sigma_{22}^L}{2\alpha\mu} \end{bmatrix}^T = \mathbf{Q}(\beta, \nu) \begin{bmatrix} u_1^U & u_2^U & \frac{\sigma_{12}^U}{2\alpha\mu} & \frac{\sigma_{22}^U}{2\alpha\mu} \end{bmatrix}^T,$$

where  $\beta = \alpha h$ , and

$$(2.6) \quad \mathbf{Q}(\beta, \nu) = \cosh(\beta) \begin{bmatrix} \mathbf{I} - \frac{\beta}{2(1-\nu)} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \\ - \frac{\sinh(\beta)}{2(1-\nu)} \begin{bmatrix} -\beta & -(1-2\nu) & 3-4\nu & \beta \\ -(1-2\nu) & \beta & -\beta & 3-4\nu \\ 1 & \beta & -\beta & 1-2\nu \\ -\beta & 1 & 1-2\nu & \beta \end{bmatrix} \end{bmatrix}.$$

REMARK. The factors  $\sin \alpha x_1$  and  $\cos \alpha x_1$  in the displacements and tractions have been discarded in Eq. (2.5).

In view of the fact that the shear stress on the two surfaces of the plate is zero, we can further deduce from Eq. (2.5) that

$$(2.7) \quad \begin{bmatrix} u_2^L & \frac{\sigma_{22}^L(1-\nu)}{\alpha\mu} \end{bmatrix}^T = \mathbf{R}(\beta) \begin{bmatrix} u_2^U & \frac{\sigma_{22}^U(1-\nu)}{\alpha\mu} \end{bmatrix}^T,$$

where  $\mathbf{R}(\beta)$  is a  $2 \times 2$  transfer matrix defined by

$$(2.8) \quad \mathbf{R}(\beta) = \frac{1}{2[\beta \cosh(\beta) + \sinh(\beta)]} \begin{bmatrix} 2\beta + \sinh(2\beta) & 1 - \cosh(2\beta) \\ 1 + 2\beta^2 - \cosh(2\beta) & 2\beta + \sinh(2\beta) \end{bmatrix}.$$

It is seen from Eq. (2.7) that the normal displacement and normal traction on the lower surface of the plate are related to those on its upper surface through the transfer matrix  $\mathbf{R}(\beta)$ , which is independent of the shear modulus and Poisson's ratio of the plates and is only dependent on the thickness parameter  $\beta$ . For a parallel array composed of  $N$  interacting plates, the following set of equations can then be expediently arrived at:

$$(2.9) \quad \begin{aligned} \mathbf{x}_{j-1}^U - \mathbf{R}(\beta)\mathbf{x}_j^U &= \frac{\beta}{\lambda} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_{j-1}^U, & j = 2, 3, \dots, N, \\ [0 \ 1] \mathbf{R}(\beta)\mathbf{x}_1^U &= 0, & [0 \ 1] \mathbf{x}_N^U = 0, \end{aligned}$$

where

$$(2.10) \quad \begin{aligned} \lambda &= \frac{Yh(1-\nu)}{\mu}, \\ \mathbf{x}_j^U &= \begin{bmatrix} u_2^{U(j)} & \frac{\sigma_{22}^{U(j)}(1-\nu)}{\alpha\mu} \end{bmatrix}^T. \end{aligned}$$

Equation (2.9) is equivalent to the following generalized eigenvalue problem:

$$(2.11) \quad \mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v},$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are  $2N \times 2N$  matrices,  $\lambda$  is the eigenvalue and  $\mathbf{v}$  defined below is the associated eigenvector

$$(2.12) \quad \mathbf{v} = \begin{bmatrix} \mathbf{x}_1^U \\ \mathbf{x}_2^U \\ \vdots \\ \mathbf{x}_N^U \end{bmatrix}.$$

It is seen that the eigenvalue defined by Eq. (2.10)<sub>1</sub> is just the normalized interaction coefficient, while the  $2N$ -dimensional eigenvector defined by Eq. (2.12) is composed of the normal displacement and normal traction on the upper surfaces of the  $N$  plates. There are in total  $2N$  eigenvalues of Eq. (2.11). Apparently, the values of all these eigenvalues are independent of the elastic property of the plates. Among these eigenvalues,  $(N - 1)$  eigenvalues are positive and the rest  $(N + 1)$  eigenvalues are zero in view of the fact that the rank of  $\mathbf{A}$  is  $(N - 1)$  while that of  $\mathbf{B}$  is  $2N$ . The critical interaction coefficient  $Y = Y_c$  is determined by the smallest of the  $(N - 1)$  positive and nonzero eigenvalues at  $\beta = \pi h/L$ . The reason for this is the fact that the critical interaction coefficient is the lowest interaction coefficient at the discrete values of  $\alpha$  given by Eq. (2.4) and the fact that the smallest positive and nonzero eigenvalue is always an increasing function of  $\beta$ .

### 3. Discussions

#### 3.1. $N = 2$

First, we consider two interacting plates ( $N = 2$ ). In this case, the critical interaction coefficient is

$$(3.1) \quad \frac{Y_c h}{\mu} = \frac{\beta_c [\cosh(2\beta_c) - 1 - 2\beta_c^2]}{2(1 - \nu)[2\beta_c + \sinh(2\beta_c)]},$$

where

$$(3.2) \quad \beta_c = \alpha_c h = \frac{\pi h}{L}.$$

The critical interaction coefficient is just half of that for a plate interacting with a rigid body (or rigid plate).

When  $\beta_c \rightarrow 0$  for thin plates, Eq. (3.1) becomes

$$(3.3) \quad \frac{Y_c h}{\mu} \cong \frac{\beta_c^4}{12(1-\nu)} + O(\beta_c^6),$$

or equivalently

$$(3.4) \quad Y_c = \frac{E^* \pi^4 h^3}{24L^4},$$

where  $E^*$  is the plane strain elastic modulus,  $E/(1-\nu^2)$  with  $E$  being the Young modulus. Let us remember that the Euler–Bernoulli beam model adopted by ZHU *et al.* [1] is valid for plane stress. When considering a beam with width  $b$  under plane stress condition, Eq. (3.4) becomes

$$(3.5) \quad Y_c b = \frac{E(\frac{1}{12}bh^3)\pi^4}{2L^4},$$

which is just the result presented by ZHU *et al.* [1].

When  $\beta_c \rightarrow \infty$  for extremely thick plates, Eq. (3.1) becomes

$$(3.6) \quad \frac{Y_c h}{\mu} \cong \frac{\beta_c}{2(1-\nu)} + O(\beta_c^{-1}).$$

If a simply-supported plate with shear modulus  $\mu_1$ , Poisson's ratio  $\nu_1$ , length  $L$  and thickness  $h_1$  interacts with another simply-supported plate with shear modulus  $\mu_2$ , Poisson's ratio  $\nu_2$ , length  $L$  and thickness  $h_2$ , the critical interaction coefficient can be derived as

$$(3.7) \quad \frac{Y_c h_1}{\mu_1} = \frac{\beta_1}{\frac{(1-\nu_1)[\sinh(2\beta_1)+2\beta_1]}{\cosh(2\beta_1)-1-2\beta_1^2} + \frac{\mu_1(1-\nu_2)[\sinh(2\beta_2)+2\beta_2]}{\mu_2 \cosh(2\beta_2)-1-2\beta_2^2}},$$

where  $\beta_1 = \pi h_1/L$  and  $\beta_2 = \pi h_2/L$ . When the two plates are identical, Eq. (3.7) reduces to Eq. (3.1). When  $\mu_2 \rightarrow \infty$ , Eq. (3.7) reduces to that of a plate interacting with a rigid body.

### 3.2. $N = 3$

Next, we consider three interacting plates. In this case, the critical interaction coefficient can be analytically given by

$$(3.8) \quad \frac{Y_c h}{\mu} = \frac{\beta_c [\cosh(2\beta_c) - 1 - 2\beta_c^2] [2\beta_c + \sinh(2\beta_c) - \sqrt{\beta_c^2 + \beta_c \sinh(2\beta_c) + (1 + \beta_c^2) \sinh^2(\beta_c)}]}{(1-\nu) [7\beta_c^2 + 6\beta_c \sinh(2\beta_c) - (1 + \beta_c^2) \cosh(2\beta_c) + \cosh(4\beta_c)]},$$

where  $\beta_c$  is given by Eq. (3.2).

When  $\beta_c \rightarrow 0$  for thin plates, Eq. (3.8) becomes

$$(3.9) \quad \frac{Y_c h}{\mu} \cong \frac{\beta_c^4}{18(1-\nu)} + O(\beta_c^6).$$

When  $\beta_c \rightarrow \infty$  for extremely thick plates, Eq. (3.8) becomes

$$(3.10) \quad \frac{Y_c h}{\mu} \cong \frac{\beta_c}{2(1-\nu)} + O(\beta_c^{-1}).$$

### 3.3. $N = 4$

We now consider four interacting plates. In this case, we have to solve a cubic equation. It is found that a root of the cubic equation is just the critical interaction coefficient in Eq. (3.1) for  $N=2$ . As a result, the critical interaction coefficient for  $N=4$  can still be analytically determined as

$$(3.11) \quad \frac{Y_c h}{\mu} = \frac{\beta_c [\cosh(2\beta_c) - 1 - 2\beta_c^2] [2\beta_c + \sinh(2\beta_c) - \sqrt{\beta_c^2 - 1 + 2\beta_c \sinh(2\beta_c) + (1 + \beta_c^2) \cosh(2\beta_c)}]}{2(1-\nu) [3\beta_c^2 + 2\beta_c \sinh(2\beta_c) - (1 + \beta_c^2) \cosh(2\beta_c) + \cosh^2(2\beta_c)]},$$

where  $\beta_c$  is also given by Eq. (3.2).

When  $\beta_c \rightarrow 0$  for thin plates, Eq. (3.11) becomes

$$(3.12) \quad \frac{Y_c h}{\mu} \cong \frac{(2 - \sqrt{2})\beta_c^4}{12(1-\nu)} + O(\beta_c^6).$$

When  $\beta_c \rightarrow \infty$  for extremely thick plates, Eq. (3.11) becomes

$$(3.13) \quad \frac{Y_c h}{\mu} \cong \frac{\beta_c}{2(1-\nu)} + O(\beta_c^{-1}).$$

### 3.4. $N > 4$

When  $N > 4$ , the generalized eigenvalue problem in Eq. (2.11) has to be numerically solved. We illustrate in Fig. 2 the normalized critical interaction coefficient  $Y_c h(1-\nu)/\mu$  as a function of  $\beta_c = \pi h/L$  for different values of  $N$ .  $N = 1$  is for the case of a single plate interacting with a rigid body. It is seen from Fig. 2 that  $Y_c h(1-\nu)/\mu$  is an increasing function of  $\beta_c$  and a decreasing function of  $N$ . When  $\beta_c \rightarrow \infty$  for extremely thick plates,  $Y_c h(1-\nu)/\mu$  for  $N=1$  approaches  $\beta_c$  while those for  $N = 2, \dots, \infty$  approach the same value  $\beta_c/2$ . To see more clearly the influence of  $N$  on the critical interaction coefficient, we list in Table 1  $Y_c h(1-\nu)/\mu$  for different values of  $N$  when  $h/L = 1$ .

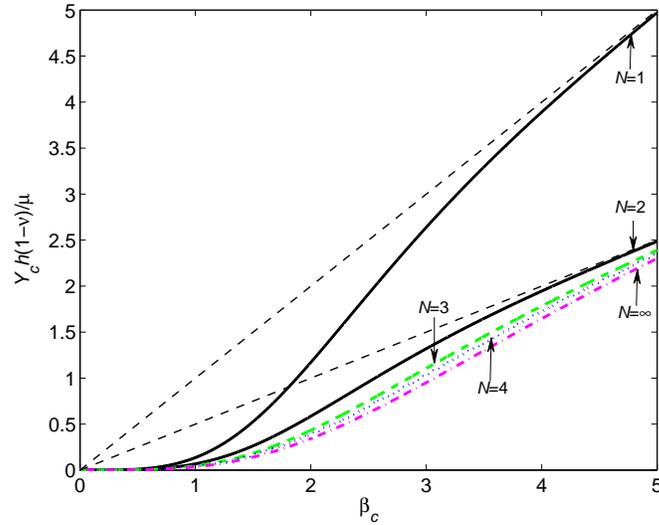


FIG. 2. The normalized critical interaction coefficient  $Y_c h(1 - \nu)/\mu$  as a function of  $\beta_c = \pi h/L$  for different values of  $N$ .

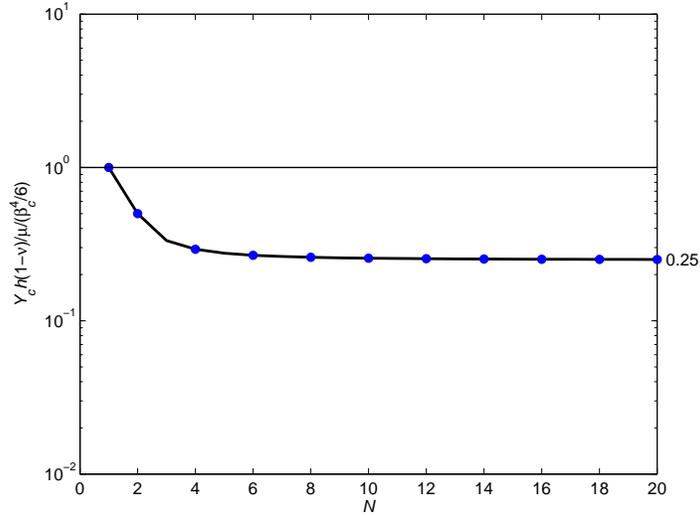


FIG. 3. The critical interaction coefficient for different values of  $N$  when  $\beta_c \rightarrow 0$ .

It is seen from the above table that the critical interaction coefficient for  $N = 100$  has become extremely close to that for  $N = \infty$ .

In order to numerically validate the correctness of the present solution, we present in Fig. 3 the critical interaction coefficient for different values of  $N$

**Table 1.**  $Y_c h(1 - \nu)/\mu$  for different values of  $N$  when  $h/L = 1$ .

$N$	$Y_c h(1 - \nu)/\mu$
1	2.831816
2	1.415908
4	1.134956
10	1.062239
20	1.052117
50	1.049295
100	1.048892
150	1.048818
200	1.048792
$\infty$	1.048759

when  $\beta_c \rightarrow 0$ . Apparently, the present results for thin plates just recover those presented by ZHU *et al.* [1] by using the Euler–Bernoulli beam bending model. Roughly speaking, the relative errors between the results using the beam model and those using the rigorous theory of elasticity are less than 10% when  $h/L < 0.2$ . The errors caused by using the beam model are unacceptable when  $h/L > 0.5$ . In this case, we have to resort to the present solution based on the theory of elasticity.

### 3.5. The end-effect

In the above analysis and discussions, the end-effect of the plates at the ends of the parallel array is ignored by assuming that the equilibrium deformations of all the plates are negligibly small [1]. The end-effect can be roughly taken into account by replacing the interaction coefficient  $Y$  in Eq. (2.1) by  $\rho Y$  ( $\rho > 1$ ) for  $j = 2$  and  $N$  with  $\rho$  being the amplified factor while Eq. (2.1) should still be adopted for other values of  $j$  [1] to reflect the fact that the equilibrium deformations of the two end plates are unignorable. Figure 4 shows the normalized critical interaction coefficient  $Y_c h(1 - \nu)/\mu$  as a function of  $\beta_c = \pi h/L$  for different values of the amplified factor with  $N \rightarrow \infty$ . It is observed from Fig. 4 that the end-effect will the lower critical interaction coefficient and that  $Y_c h(1 - \nu)/\mu \cong \beta_c/2\rho$  as  $\beta_c \rightarrow \infty$ . As  $\rho$  increases,  $Y_c h(1 - \nu)/\mu$  approaches its asymptotic value of  $\beta_c/2\rho$  at a smaller value of  $\beta_c$ . To make a direct comparison with Fig. 6 by ZHU *et al.* [1], we illustrate in Fig. 5 the end-effect factor  $\varepsilon$  as a function of the amplified factor  $\rho$  for different values of  $\beta_c = \pi h/L$ . The end-effect factor  $\varepsilon$  is defined as the ratio between the critical interaction coefficient when  $N \rightarrow \infty$  with end-effect and that without end-effect. It is observed from Fig. 5 that:

(i) When  $\beta_c \rightarrow 0$  for thin plates, the results just recover those by ZHU *et al.* [1].

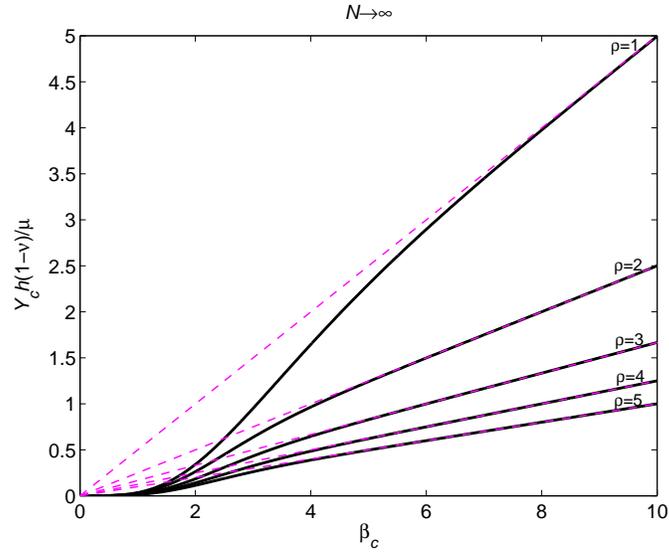


FIG. 4. The normalized critical interaction coefficient  $Y_c h(1 - \nu)/\mu$  as a function of  $\beta_c = \pi h/L$  for different values of the amplified factor with  $N \rightarrow \infty$ .

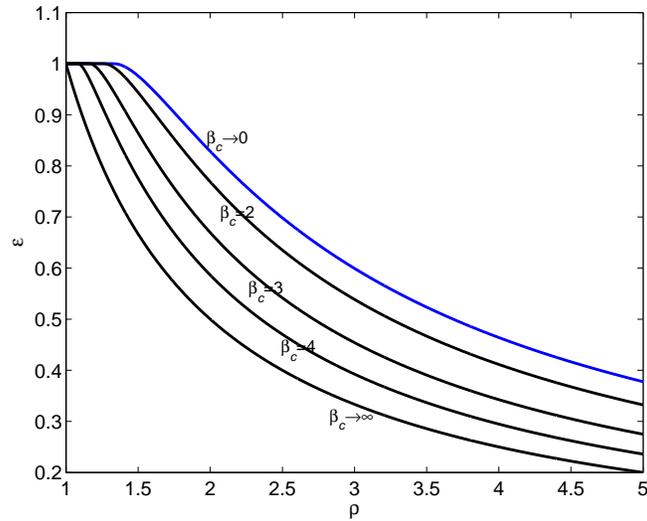


FIG. 5. The end-effect factor  $\varepsilon$  as a function of the amplified factor  $\rho$  for different values of  $\beta_c = \pi h/L$ .

(ii) The end-effect factor decreases with increasing  $\beta_c$  for thicker plates. For example,  $\varepsilon = 1/3$  when  $\rho = 3$  and  $\beta_c \rightarrow \infty$  for extremely thick plates. This value is only about half of the value  $\varepsilon = 0.5993$  when  $\rho = 3$  and  $\beta_c \rightarrow 0$  for thin

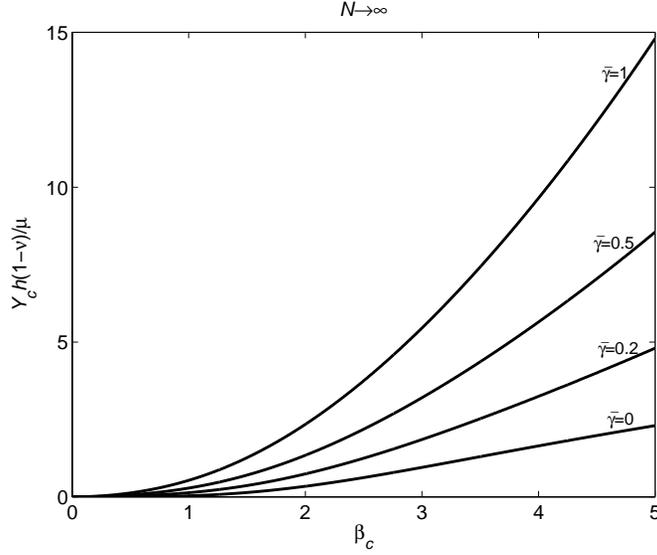


FIG. 6. The influence of surface energy on the critical interaction coefficient.

plates. This fact implies that the critical interaction coefficient for interacting thick plates is more sensitive to the end-effect.

### 3.6. The influence of surface energy

In this section, we discuss the influence of surface energy on the critical interaction coefficient. If the end-effect is ignored, the perturbed normal displacement and normal traction on the lower surface of plate  $j$  are related to those on the upper surface of plate  $j-1$  through

$$(3.14) \quad \begin{aligned} \sigma_{22}^{L(j)} &= Y(u_2^{U(j-1)} - u_2^{L(j)}) - \gamma u_{2,11}^{L(j)}, \\ \sigma_{22}^{U(j-1)} &= Y(u_2^{U(j-1)} - u_2^{L(j)}) + \gamma u_{2,11}^{U(j-1)}, \end{aligned}$$

where  $\gamma$  is the surface energy of the plates. Equation (3.14) can be equivalently written into

$$(3.15) \quad \begin{aligned} \sigma_{22}^{L(j)} + \sigma_{22}^{U(j-1)} &= (2Y - \alpha^2 \gamma)(u_2^{U(j-1)} - u_2^{L(j)}), \\ \sigma_{22}^{L(j)} - \sigma_{22}^{U(j-1)} &= \alpha^2 \gamma(u_2^{U(j-1)} + u_2^{L(j)}). \end{aligned}$$

For a parallel array composed of  $N$  interacting plates, the following set of equations can then be expediently arrived at

$$\begin{aligned}
(3.16) \quad & \begin{bmatrix} 1 & 0 \\ \beta\bar{\gamma} & 1 \end{bmatrix} \mathbf{x}_{j-1}^U - \begin{bmatrix} 1 & 0 \\ -\beta\bar{\gamma} & 1 \end{bmatrix} \mathbf{R}(\beta)\mathbf{x}_j^U \\
& = \frac{\beta}{2\lambda} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} [\mathbf{R}(\beta)\mathbf{x}_j^U + \mathbf{x}_{j-1}^U], \quad j = 2, \dots, N, \\
& [-\beta\bar{\gamma} \ 1] \mathbf{R}(\beta)\mathbf{x}_1^U = 0, \quad [\beta\bar{\gamma} \ 1] \mathbf{x}_N^U = 0,
\end{aligned}$$

where  $\mathbf{R}(\beta)$  and  $\mathbf{x}_j^U$  have been defined in Eqs. (2.8) and (2.10), and

$$(3.17) \quad \bar{\gamma} = \frac{\gamma(1-\nu)}{\mu h}, \quad \lambda = \frac{Yh(1-\nu)}{\mu} - \frac{\beta^2\bar{\gamma}}{2}.$$

The last two equations in Eq. (3.16) imply that the surface energy on the lower surface of the bottom plate and that on the upper surface of the top plate have been taken into account. It is seen from Eq. (3.17) that when the surface energy is taken into account, the eigenvalue is a linear function of the normalized interaction coefficient and the normalized surface energy. Equation (3.16) can also be written into the generalized eigenvalue problem in Eq. (2.11) with  $\text{rank}(\mathbf{A}) = N - 1$  and  $\text{rank}(\mathbf{B}) = 2N$ . Among the  $2N$  eigenvalues,  $(N - 1)$  eigenvalues are positive and the rest  $(N + 1)$  eigenvalues are zero. The normalized critical interaction coefficient  $Y_c h(1 - \nu)/\mu$  is just the sum of  $\frac{1}{2}\beta_c^2\bar{\gamma}$  and the smallest of the  $(N - 1)$  positive eigenvalues. We illustrate in Fig. 6 the influence of surface energy on the critical interaction coefficient for an infinite number of interacting plates. Apparently, the surface energy will increase the magnitude of the critical interaction coefficient. We can see that the surface energy plays a stabilizing role. Our results also indicate that the surface energy on the lower surface of the bottom plate and that on the upper surface of the top plate exert minimal influence on the critical interaction coefficient. Furthermore, the critical interaction coefficient for an infinite number of interacting plates can be accurately given by the following formula with the relative errors below 0.025%

$$(3.18) \quad \frac{Y_c h(1-\nu)}{\mu} = \frac{Y_c^0 h(1-\nu)}{\mu} + \frac{\beta_c^2 \bar{\gamma}}{2},$$

where  $Y_c^0$  is the critical interaction coefficient for an infinite number of interacting plates in the absence of surface energy. The reason for the relationship in Eq. (3.18) is due to the fact that  $\sigma_{22}^{L(j)} \approx \sigma_{22}^{U(j-1)}$  when the number  $N$  of the interacting plates is sufficient large. The results in Sections 3.1 to 3.4 indicate that  $Y_c^0 h(1 - \nu)/\mu$  is proportional to  $\beta_c^4$  when  $\beta_c \rightarrow 0$  for thin plates and is proportional to  $\beta_c$  when  $\beta_c \rightarrow \infty$  for extremely thick plates. Consequently, it is seen from Eq. (3.18) that the contribution from surface energy cannot be ignored for both thin and thick plates.

#### 4. Conclusions

We rigorously study the structural instability of  $N$  mutually interacting identical simply-supported plates by using the general solution for an elastic strip by HE and JIANG [7]. Through a  $2 \times 2$  transfer matrix  $\mathbf{R}(\beta)$ , the normal displacement and normal traction on the lower surface of a plate are related to those on its upper surface. By using this relationship, we obtain a generalized eigenvalue problem for the plate array. The critical interaction coefficient is determined by the smallest of the  $(N - 1)$  positive eigenvalues. Analytical expressions of the critical interaction coefficients for  $N = 2, 3, 4$  are derived when the end-effect and surface energy are ignored. It is confirmed that the results obtained by ZHU *et al.* [1] are valid for thin plates ( $h/L < 0.2$ ). For interacting thick plates ( $h/L > 0.5$ ), the present solution based on the exact elasticity theory should be adopted. For example, the critical interaction coefficients for different numbers of interacting extremely thick plates are the same.

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