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On 3D symmetrical thermoelastic anticrack problems

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A potential theory method is developed to solve a symmetrical thermoelastic problem of a cooling temperature field applied over the faces of a rigid sheet-like inclusion (an anticrack) in an elastic space. The governing boundary two-dimensional (2D) singular integral equations for an arbitrarily shaped anticrack are derived in terms of unknown thermal shear stress jumps. As an illustration, a complete solution expressed in elementary functions to the problem of a circular rigid inclusion subjected to a uniform temperature is presented and interpreted from the point of view of fracture theory.

Key words: three-dimensional anticrack, symmetrical temperature problem, potential theory method, thermal stress singularity.

1. Introduction

The investigation of the redistribution of stresses due to the presence of different kinds of inhomogeneities in elastic bodies is important in various engineering applications. Cracks and rigid lamellar inclusions (known as anticracks) are two extreme cases that result in the concentration of their border stresses and further affect the behavior of material structures. As shown in [1–5], rigid inhomogeneities have wide applications in the fields of materials science, the mechanics of composites and geomechanics. Although the problem of cracks has been studied extensively over the past 50 years, research on the corresponding anticrack problems has been rather limited and mainly concentrated on 2D problems connected with rigid line inclusions (see for example, monographs by Ting [6] and Berezhnitskii *et al.* [7]).

The increasing development of high-strength engineering structures containing cracks or anticracks has driven researchers to focus on the effects of thermal loads. The main achievements of space crack problems in this field are documented in monographs by Kassir and Sih [8] and Kit and Khay [9]. However, because of their mathematical complexity, truly three-dimensional problems involving anticracks subjected to thermal activity have not yet been studied sufficiently well.

An anticrack is basically a through slit crack filled with a rigid lamella, which, unlike a crack, transmits tractions, but prevents displacement discontinuity. In the thermoelastic case, both cracks and anticracks are thermally insulated, thermally conductive or thermally active. To deal with the thermoelastic problems involving planes of discontinuities of some kind, two types of stress systems can be distinguished. One system is symmetric with reference to these planes and the other is characterized by antisymmetry. It is the purpose of this work to present a method of solution for the symmetric system, in which the prescribed temperatures on the upper surface of the anticrack are identical to those on the lower surface. The features of antisymmetry have already been presented by considering the thermal-stress problem of vertically uniform heat flow disturbed by an arbitrarily shaped heat-insulated anticrack in an elastic isotropic space in Kaczyński and Kozłowski [10] and in a transversely isotropic space in Kaczyński [11]. It should be noted that the corresponding problems involving penny-shaped cracks have been analyzed in the symmetric case in [12–15] and in the antisymmetric case in [16–19]; see also the results of a generalization for flat elliptical cracks given in [20–23].

This paper is arranged as follows. Section 2 outlines the fundamental equations of linear thermoelasticity in an uncoupled static setting. The thermal problem dealing with anticracks under symmetric temperatures is first analysed in Section 3. Once the temperature potential for solving this problem has been found, the induced problem of thermal stresses is investigated in Section 4. It is reduced to a symmetric mechanical analog developed in Kaczyński [24]. Consequently, the governing boundary integral equations for a planar anticrack of arbitrary shape are obtained in terms of shear stress discontinuities. A typical application of a circular anticrack under uniform-temperature load is presented and discussed in Section 5. It is noteworthy that the analytical solution obtained can serve as a benchmark for various approximate analyses and numerical codes. Finally, Section 6 concludes the article.

2. Fundamental equations of thermoelastostatics

The best developed theory that is widely used in practice is the theory of static uncoupled homogeneous isotropic thermoelasticity when the temperature field does not depend on the field of elastic displacements and when the inertial terms can be ignored. The governing equations for this theory include the equation of equilibrium, the heat conduction equation and the constitutive relations for stresses and fluxes. In the absence of body forces and heat sources, the system of these equations in Cartesian coordinates (x_1, x_2, x_3) is (Nowacki [25])

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - \beta T_{,i} = 0, \tag{2.1}$$

$$T_{,ii} = 0, \tag{2.2}$$

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \beta T \delta_{ij}, \tag{2.3}$$

$$q_i = -kT_{,i}, \tag{2.4}$$

where u_i, T, σ_{ij}, q_i are the displacement, temperature increment, stress and heat flux fields, respectively, λ, μ are the Lamé constants, $\beta = \alpha(3\lambda + 2\mu)$ with α being the linear coefficient of thermal expansion, k is the thermal conductivity and δ_{ij} is Kronecker's delta. Latin lower case indices range over 1, 2, 3 and a comma denotes partial differentiation with respect to coordinate variables. The usual summation convention for repeated indices holds.

A traditional two-staged method of solution will be used. Using the symmetry conditions, first we need to solve a mixed boundary-value problem of heat conduction in a half space governed by Eq. (2.2) with a temperature applied over the anticrack surface. Secondly, we search for the solution to the thermoelastic equation (2.1) at the already known temperature field and with some mechanical anticrack boundary conditions.

3. Symmetrical thermal anticrack problem

Consider a thermoelastic isotropic space with an arbitrarily shaped rigid sheet-like inclusion (anticrack) occupying a region S in the plane $x_3 = 0$. A given temperature $-T_0(x_1, x_2)$ (negative with respect to the stress-free state) is applied symmetrically to the anticrack faces. Similarly to the symmetrical thermal crack problem posed in [20, 21], we may consider the problem to be that of finding a harmonic function $T(x_1, x_2, x_3)$ in $x_3 \geq 0$, vanishing at infinity and satisfying the following mixed conditions on $x_3 = 0$:

$$\begin{aligned} &T(x_1, x_2, 0) = -T_0(x_1, x_2), \qquad (x_1, x_2) \in S, \\ &T_{,3}|_{x_3=0^+} = 0, \qquad (x_1, x_2) \in R^2 - S. \end{aligned} \tag{3.1}$$

Let us introduce a temperature potential $\omega(x_1, x_2, x_3)$ such that (Kaczyński [26])

$$T = -\omega_{,33}, \qquad \omega_{,ii} = 0. \tag{3.2}$$

The conditions (3.1) suggest taking the sought harmonic function T as the Newtonian potential of a single layer of intensity $\varphi(x_1, x_2)$ distributed over the inclusion region S (Kellogg [27]). Therefore, the solution to this thermal problem can be written as

$$
(3.3)\qquad
\begin{aligned}
\omega(x_1,x_2,x_3) &= \iint\limits_S [x_3 \ln(R_\xi + x_3) - R_\xi]\varphi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2,\\
T(x_1,x_2,x_3) &= -\iint\limits_S \frac{\varphi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2}{R_\xi},
\end{aligned}
$$

where $R_\xi = |\mathbf{x} - \boldsymbol{\xi}| = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}$ is the distance between the field point $\mathbf{x} = (x_1,x_2,x_3)$ and the integration point $\boldsymbol{\xi} = (\xi_1,\xi_2,0) \in S$, and the unknown density $\varphi(x_1,x_2)$ must satisfy, in view of $(3.1)_1$, the following singular integral equation (similar to the one for elastic contact mechanics [28, 29]):

$$
(3.4)\qquad \iint\limits_S \frac{\varphi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} = T_0(x_1,x_2), \qquad (x_1,x_2)\in S.
$$

It should be mentioned here that a closed-form solution to this equation was obtained by RAHMAN [30] for the case where S is an ellipse and T_0 is a polynomial.

Moreover, from the simple layer potential properties it follows that when density function φ is continuous and the region S belongs to the Lyapunov class, then the sought temperature T is a continuous function in R^3 and has a normal derivative jump $[\![T_{,3}]\!]_S$ for $(x_1,x_2)\in S$:

$$
(3.5)\qquad [\![T_{,3}]\!]_S \equiv T_{,3}(x_1,x_2,0^+) - T_{,3}(x_1,x_2,0^-) = 4\pi\varphi(x_1,x_2).
$$

4. Symmetrical thermal stress anticrack problem

Now we pass to the associated thermoelastic rigid inclusion problem due to the foregoing symmetric thermal loadings. This problem may be reduced to a half-space $x_3 \geq 0$ with the following mechanical conditions resulting from displacement-free anticrack surfaces, the symmetry of the temperature and deformation state and the possible rigid motion of the inclusion (see KACZYŃSKI [24]):

$$
(4.1)\qquad u_\alpha(x_1,x_2,0) = \varepsilon_\alpha + (-1)^\alpha \omega_3 x_{3-\alpha}, \qquad (x_1,x_2)\in S, \qquad \alpha = 1,2,
$$

$$
(4.2)\qquad u_3(x_1,x_2,0) = 0, \qquad (x_1,x_2)\in R^2,
$$

$$
(4.3)\qquad \sigma_{3\alpha}(x_1,x_2,0^+) = 0, \qquad (x_1,x_2)\in R^2 - S, \qquad \alpha = 1,2,
$$

$$
(4.4)\qquad u_i = O\left(\frac{1}{|\mathbf{x}|}\right) \qquad \text{as } |\mathbf{x}| = \sqrt{x_1^2+x_2^2+x_3^2} \to \infty.
$$

Here, the unknown small constants ε_α (ω_3) stand for the corresponding displacements (angle of rotation) of the inclusion as a rigid whole. These parameters will be determined in the course of solving the problem in hand from the equilibrium conditions of zero resultant forces (moments), expressed by the jumps of shear stresses on S denoted by $[\![\sigma_{3\alpha}(x_1,x_2)]\!] = \sigma_{3\alpha}(x_1,x_2,0^+) - \sigma_{3\alpha}(x_1,x_2,0^-)$, $\alpha = 1,2$ as follows:

$$\begin{aligned}&\iint\limits_S [\![\sigma_{3\alpha}(x_1,x_2)]\!]\,dx_1\,dx_2 = 0, \qquad \alpha = 1,2,\\ &\iint\limits_S \big[x_2[\![\sigma_{31}\alpha(x_1,x_2)]\!] - x_1[\![\sigma_{32}(x_1,x_2)]\!]\big]\,dx_1\,dx_2 = 0.\end{aligned} \tag{4.5}$$

The general solution of the governing displacement equation (2.1) for three-dimensional stationary problems with geometric discontinuities on the plane $x_3 = 0$ may be expressed by three harmonic functions $\phi_i(x_1,x_2,x_3)$ and the temperature potential $\omega(x_1,x_2,x_3)$ as follows (Kaczyński [26]):

$$\begin{aligned}u_\alpha &= (\phi_1 + x_3F + c\,\omega)_{,\alpha} + (-1)^\alpha\phi_{3,3-\alpha}, \qquad \alpha = 1,2,\\ u_3 &= \phi_{1,3} + x_3F_{,3} - \frac{\lambda+3\mu}{\lambda+\mu}\,\phi_2,\end{aligned} \tag{4.6}$$

in which

$$F = \phi_2 - c\,\omega_{,3}, \qquad c = \frac{\beta}{2(\lambda+2\mu)}. \tag{4.7}$$

To reduce the anticrack stress problem to one in the potential theory, we construct potential functions well suited to the boundary conditions (4.1)–(4.3). Let us introduce two harmonic functions $G(x_1,x_2,x_3)$, $H(x_1,x_2,x_3)$ and denote $g = G_{,3}$, $h = H_{,3}$. The potentials appearing in the displacement representation (4.6) are selected according to

$$\begin{aligned}\phi_1 &= -G_{,1} - H_{,2},\\ \phi_2 &= -\frac{\lambda+\mu}{\lambda+3\mu}(G_{,13} + H_{,23}),\\ \phi_3 &= G_{,2} - H_{,1}.\end{aligned} \tag{4.8}$$

Upon substitution of Eqs. (4.8) into (4.6), the displacement components become

$$\begin{aligned}u_1 &= g_{,3} + c\,\omega_{,1} + F_{,1}x_3,\\ u_2 &= h_{,3} + c\,\omega_{,2} + F_{,2}x_3,\\ u_3 &= F_{,3}x_3\end{aligned} \tag{4.9}$$

with

$$F = -\frac{\lambda+\mu}{\lambda+3\mu}(g_{,1} + h_{,2}) - c\,\omega_{,3}. \tag{4.10}$$

The corresponding stress components are found from Eq. (2.3) as

$$\frac{\sigma_{31}}{\mu} = C[g_{,33} + \kappa(g_{,2} - h_{,1})_{,2}] + 2F_{,13}x_3, \tag{4.11}$$

$$\frac{\sigma_{32}}{\mu} = C[h_{,33} - \kappa(g_{,2} - h_{,1})_{,1}] + 2F_{,23}x_3, \tag{4.12}$$

$$\frac{\sigma_{33}}{\mu} = -D(g_{,31} + h_{,32}) + 2c\,\omega_{,33} + 2F_{,33}x_3, \tag{4.13}$$

$$\sigma_{11} = D[(2\lambda + 3\mu)g_{,13} + \lambda h_{,23}] + 2c\mu(2\omega_{,33} + \omega_{,11}) + 2\mu F_{,11}x_3, \tag{4.14}$$

$$\sigma_{22} = D[\lambda g_{,13} + (2\lambda + 3\mu)h_{,23}] + 2c\mu(2\omega_{,33} + \omega_{,22}) + 2\mu F_{,22}x_3, \tag{4.15}$$

$$\sigma_{12} = \mu[g_{,32} + h_{,31} + 2c\,\omega_{,12} + 2F_{,12}x_3], \tag{4.16}$$

where

$$C = \frac{2(\lambda + 2\mu)}{\lambda + 3\mu}, \qquad D = \frac{2\mu}{\lambda + 3\mu}, \qquad \kappa = \frac{\lambda + \mu}{2(\lambda + 2\mu)}. \tag{4.17}$$

Note that on the plane $x_3 = 0$ the displacement $u_3 = 0$. In addition, the quantities of interest u_1, u_2 and σ_{31}, σ_{32} can be expressed in terms of the partial derivatives of potentials h, g and ω evaluated at $x_3 = 0$, i.e.,

$$\begin{gathered} u_1 = g_{,3} + c\,\omega_{,1}, \qquad u_2 = h_{,3} + c\,\omega_{,2}, \\ \sigma_{31} = C\mu[g_{,33} + \kappa(g_{,2} - h_{,1})_{,2}], \qquad \sigma_{32} = C\mu[h_{,33} - \kappa(g_{,2} - h_{,1})_{,1}]. \end{gathered} \tag{4.18}$$

Application of the conditions (4.1)–(4.4) leads now to the boundary-value problem for the determination of two harmonic functions g and h in the half space $x_3 \geq 0$ such that their partial derivatives up to the third order vanish at infinity and satisfy the following mixed conditions on the plane boundary:

$$\left.\begin{aligned} [g_{,3}(x_1, x_2, x_3)]_{x_3 = 0^+} &= -c[\omega_{,1}]_{x_3 = 0^+} + \varepsilon_1 - \omega_3 x_2 \\ [h_{,3}(x_1, x_2, x_3)]_{x_3 = 0^+} &= -c[\omega_{,2}]_{x_3 = 0^+} + \varepsilon_2 + \omega_3 x_1 \end{aligned}\right\} \quad (x_1, x_2) \in S, \tag{4.19}$$

$$\left.\begin{aligned} g_{,33} + \kappa(g_{,2} - h_{,1})_{,2} = 0 \\ h_{,33} - \kappa(g_{,2} - h_{,1})_{,1} = 0 \end{aligned}\right\} \quad (x_1, x_2, 0^+) \in R^2 - S, \tag{4.20}$$

in which the values of functions $-c\,\omega_{,\alpha}(x_1, x_2, 0^+)$, $\alpha = 1, 2$ can be found from the solution of the temperature problem.

It is noteworthy here that the same boundary-value problem is observed if we consider the symmetrical part of the problem involving an anticrack under mechanical loadings. Thus, the results of Kaczyński [24] are used to formulate the symmetrical thermoelastic anticrack problems in terms of the governing pair of singular 2D integral equations for the unknown interface shear stresses $\sigma_{3\alpha}^{+}|_S \equiv \sigma_{3\alpha}(x_1, x_2, 0^+)$, $(x_1, x_2) \in S$, $\alpha = 1, 2$ as follows:

$$
(4.21)\quad
\begin{aligned}
&\frac{1}{2\pi\mu}\iint_S \left\{ \frac{\sigma_{31}^{+}(\xi_1,\xi_2)}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} \left[1-\kappa\frac{(x_2-\xi_2)^2}{(x_1-\xi_1)^2+(x_2-\xi_2)^2}\right]\right.\\
&\qquad\left. +\,\kappa\frac{\sigma_{32}^{+}(\xi_1,\xi_2)(x_1-\xi_1)(x_2-\xi_2)}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}\right\} d\xi_1\,d\xi_2 = c[\omega_{,1}]_{x_3=0^+} - \varepsilon_1 + \omega_3 x_2,\\
&\frac{1}{2\pi\mu}\iint_S \left\{ \frac{\sigma_{32}^{+}(\xi_1,\xi_2)}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} \left[1-\kappa\frac{(x_1-\xi_1)^2}{(x_1-\xi_1)^2+(x_2-\xi_2)^2}\right]\right.\\
&\qquad\left. +\,\kappa\frac{\sigma_{31}^{+}(\xi_1,\xi_2)(x_1-\xi_1)(x_2-\xi_2)}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}\right\} d\xi_1\,d\xi_2 = c[\omega_{,2}]_{x_3=0^+} - \varepsilon_2 - \omega_3 x_1,
\end{aligned}
$$

Note that analytical solutions of the above equations are available, when their right sides are polynomials and region (of anticrack) S is elliptical (Rahman [31]).

Knowning the stresses σ_{31}^{+}, σ_{32}^{+} from the solution to governing equations (4.21), the derivatives of the main potentials giving the full-space stress-displacement field are expressed as

$$
(4.22)\quad
\begin{aligned}
g_{,3}(\mathbf{x}) &= -\frac{1}{2\pi\mu}\iint_S \left\{ \frac{\sigma_{31}^{+}(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\left[1-\kappa\frac{(x_2-\xi_2)^2}{|\mathbf{x}-\boldsymbol{\xi}|^2}\right]\right.\\
&\qquad\left. +\,\kappa\frac{\sigma_{32}^{+}(\boldsymbol{\xi})(x_1-\xi_1)(x_2-\xi_2)}{|\mathbf{x}-\boldsymbol{\xi}|^3}\right\} dS_\xi,\\
h_{,3}(\mathbf{x}) &= -\frac{1}{2\pi\mu}\iint_S \left\{ \frac{\sigma_{32}^{+}(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\left[1-\kappa\frac{(x_1-\xi_2)^2}{|\mathbf{x}-\boldsymbol{\xi}|^2}\right]\right.\\
&\qquad\left. +\,\kappa\frac{\sigma_{31}^{+}(\boldsymbol{\xi})(x_1-\xi_1)(x_2-\xi_2)}{|\mathbf{x}-\boldsymbol{\xi}|^3}\right\} dS_\xi,
\end{aligned}
$$

5. Uniform-temperature load applied on the surface of a circular anticrack

To illustrate the use of the general method presented above, consider the posed problem for the particular case of a circular (penny-shaped) anticrack of radius a subjected to a constant cooling temperature T_0 as shown in Fig. 1, i.e.,

$$
(5.1)\qquad S \equiv S_a = \left\{(x_1,x_2,0) : r \equiv \sqrt{x_1^2+x_2^2} \le a\right\}, \qquad T(x_1,x_2) \equiv T_0.
$$

A well-known solution to Eq. (3.4) for a circular domain S_a is

$$
(5.2)\qquad \varphi(x_1,x_2) = \hat{\varphi}(r) = \frac{T_0}{\pi^2\sqrt{a^2-r^2}}.
$$

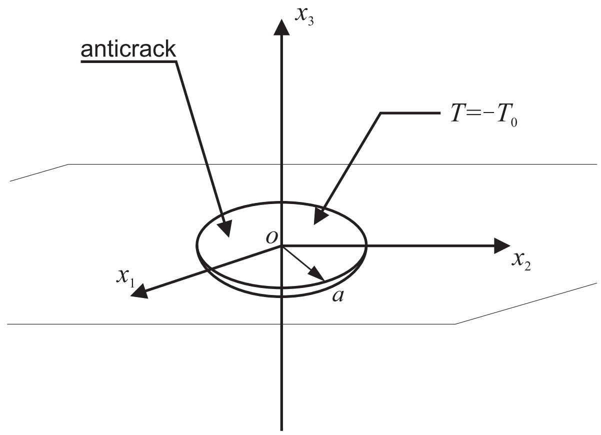

FIG. 1. A circular anticrack in an elastic isotropic space under a uniform-temperature load.

Substituting Eq. (5.2) into Eqs. (3.3) yields the following formulas for thermal potential ω and temperature T in terms of elementary functions by virtue of FABRIKANT's results [28]:

$$
\begin{aligned}
\omega(x_1,x_2,x_3) &= \frac{T_0}{\pi}\left[\left(x_3^2-a^2-\frac{r^2}{2}\right)\sin^{-1}\frac{a}{l_2}\right.\\
&\qquad \left. -\frac{3(2a^2-l_1^2)}{2a}\sqrt{l_2^2-a^2}+2ax_3\ln\left(l_2+\sqrt{l_2^2-r^2}\right)\right],\\
T(x_1,x_2,x_3) &= \hat{T}(r,x_3) = -\frac{2T_0}{\pi}\sin^{-1}\frac{a}{l_2}\\
&= -\frac{2T_0}{\pi}\sin^{-1}\frac{2a}{\sqrt{(r+a)^2+x_3^2}+\sqrt{(r-a)^2+x_3^2}},
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
l_1 \equiv l_1(a,r,x_3) &= \frac{1}{2}\left[\sqrt{(r+a)^2+x_3^2}-\sqrt{(r-a)^2+x_3^2}\right],\\
l_2 \equiv l_2(a,r,x_3) &= \frac{1}{2}\left[\sqrt{(r+a)^2+x_3^2}+\sqrt{(r-a)^2+x_3^2}\right].
\end{aligned}
\tag{5.4}
$$

Furthermore, the fluxes q_i are found from Eq. (2.4) as

$$
\begin{aligned}
q_\alpha(x_1,x_2,x_3) &= -\frac{2T_0k}{\pi}\frac{x_\alpha l_1\sqrt{l_2^2-a^2}}{rl_2^2(l_2^2-l_1^2)}, \qquad \alpha=1,2,\\
q_3(x_1,x_2,x_3) &= -\frac{2T_0k}{\pi}\frac{\sqrt{a^2-l_1^2}}{l_2^2-l_1^2}.
\end{aligned}
\tag{5.5}
$$

In particular, in the inclusion plane $x_3 = 0$ (where $l_1 = \min(a, r)$, $l_2 = \max(a, r)$) we find that

$$\hat{T}(r, 0^{\pm}) = \begin{cases} -T_0 & 0 \le r \le a, \\ -\dfrac{2T_0}{\pi} \sin^{-1} \dfrac{a}{r} & r > a, \end{cases} \tag{5.6}$$

$$\begin{aligned} q_r(r, 0^{\pm}) &= -k[\hat{T}_{,r}]_{x_3=0^{\pm}} = \begin{cases} 0 & 0 \le r \le a, \\ -\dfrac{2T_0 ak}{\pi r\sqrt{r^2 - a^2}} & r > a, \end{cases} \\ q_3(r, 0^{\pm}) &= -k[\hat{T}_{,3}]_{x_3=0^{\pm}} = \begin{cases} \mp\dfrac{2T_0 k}{\pi\sqrt{a^2 - r^2}} & 0 \le r \le a, \\ 0 & r > a. \end{cases} \end{aligned} \tag{5.7}$$

Proceeding now to the solution of governing equations (4.21), we find from Eq. $(5.3)_1$ that

$$c[\omega_{,\alpha}]_{x_3=0^+} = -\frac{1}{2} cT_0 x_\alpha, \qquad \alpha = 1, 2. \tag{5.8}$$

At this stage, it is noteworthy that Eqs. (4.21) are identical to the governing equations for the symmetrical perturbed anticrack problem of triaxial tension considered by Kaczyński in [24] by letting there $D_1 = D_2 \equiv \frac{1}{2} cT_0$. Taking advantage of this fact (see the details of the solution process therein), the sought shear stresses are obtained as follows:

$$\sigma_{31}^{+}(x_1, x_2) = -\frac{\beta_r T_0 x_1}{\pi\sqrt{a^2 - r^2}}, \qquad \sigma_{32}^{+}(x_1, x_2) = -\frac{\beta_r T_0 x_2}{\pi\sqrt{a^2 - r^2}}, \qquad (x_1, x_2) \in \operatorname{int} S, \tag{5.9}$$

where

$$\beta_r = \frac{2\mu\beta}{\lambda + 3\mu} = \frac{2(1 - 2\nu)\beta}{3 - 4\nu} \tag{5.10}$$

with $\nu = \lambda/2(\lambda + \mu)$ being Poisson's ratio.

As can be easily seen, the problem in hand is axially symmetric with respect to the x_3-axis, and the singular thermal shear stress inside the region of the anticrack is

$$\sigma_{3r}(r, 0^{\pm}) = \frac{\mp\beta_r r}{\pi\sqrt{a^2 - r^2}}, \qquad 0 \le r < a. \tag{5.11}$$

Moreover, the use of the equilibrium conditions (4.5) leads to $\varepsilon_1 = \varepsilon_2 = \omega_3 = 0$ as might be expected.

The full elastic field is determined if we find the main potentials g and h (see Eqs. (4.9)–(4.17)). When Eqs. (5.9) are inserted into (4.22), it is found that

$$
\begin{aligned}
g_{,3} &= \frac{\beta_r}{2\pi^2\mu}(\psi_1 + \kappa x_2\psi_{1,2} - \kappa x_1\psi_{2,2}),\\
h_{,3} &= \frac{\beta_r}{2\pi^2\mu}(\psi_2 + \kappa x_1\psi_{2,2} - \kappa x_2\psi_{1,1}).
\end{aligned} \tag{5.12}
$$

Here ψ_1 and ψ_2 are the potentials of simple layers defined as

$$
\begin{aligned}
\psi_1(\mathbf{x}) &= \iint\limits_{S_a} \frac{\xi_1\, d\xi_1\, d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2}\sqrt{a^2-\xi_1^2-\xi_2^2}},\\
\psi_2(\mathbf{x}) &= \iint\limits_{S_a} \frac{\xi_2\, d\xi_1\, d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2}\sqrt{a^2-\xi_1^2-\xi_2^2}},
\end{aligned} \tag{5.13}
$$

for which the method by Fabrikant [28] yields the explicit results in elementary functions as follows:

$$
\psi_\alpha(\mathbf{x}) = \pi x_\alpha\left(\sin^{-1}\frac{a}{l_2} - \frac{a\sqrt{l_2^2-a^2}}{l_2^2}\right), \qquad \alpha = 1,2. \tag{5.14}
$$

Having the explicit expressions for the harmonic potentials g, h and ω, the thermoelastic field can be obtained simply by differentiation or integration with the use of Appendix 5 in the book by Fabrikant [29]. It is obvious that the results will be in terms of elementary functions. Since the derivation is straightforward, it is omitted here to save space. In order to examine the singular behavior of the thermal border stresses, however, normal stress $\sigma_{33}(r,0^\pm)$ is found from Eq. (4.13) with the help of Eqs. (5.12) and (5.14) as

$$
\sigma_{33}(r,0^\pm) = \begin{cases} \dfrac{\mu\beta T_0}{\lambda+3\mu} & 0 \le r < a,\\[2ex] \dfrac{\beta_3 T_0}{\pi}\,\dfrac{a}{\sqrt{r^2-a^2}} + \dfrac{2\mu\beta T_0}{\pi(\lambda+3\mu)}\sin^{-1}\dfrac{a}{r} & r > a, \end{cases} \tag{5.15}
$$

where

$$
\beta_3 = \frac{2\mu^2\beta}{(\lambda+3\mu)(\lambda+2\mu)} = \frac{\beta(1-2\nu)^2}{(3-4\nu)(1-\nu)}. \tag{5.16}
$$

The obtained results reveal that the thermal stresses near the anticrack front $r = a$ have the classical singularity $r^{-1/2}$ as in the fracture mechanics of conventional elastic materials. Strictly speaking, singularities in σ_{3r} occur at the points on the edge of the disc where $r = a^-$, and in σ_{33} – at the points exterior to the disc where $r = a^+$. This indicates that there are two major mechanisms controlling the material cracking around the inclusion front:

– exfoliation of the material from the surface of the inclusion described by the stress singularity coefficients

$$S_{\mathrm{II}}^{\pm} = \lim_{r\to a^-} \sqrt{2\pi(a-r)}\sigma_{3r}(r,0^{\pm}) = \mp\beta_r T_0\sqrt{a/\pi}, \tag{5.17}$$

– a mode I fracture in the immediate vicinity of the edge of the disc characterized by the stress intensity factor

$$K_{\mathrm{I}} = \lim_{r\to a^+} \sqrt{2\pi(r-a)}\sigma_{33}(r,0) = \beta_3 T_0\sqrt{a/\pi}. \tag{5.18}$$

These parameters may be used in conjunction with a suitable failure criterion for initiating fractures at the rim of the inclusion.

In order to compare the amplitudes of the local stress components in the foregoing mechanisms, let us calculate the ratio

$$\frac{K_{\mathrm{I}}}{|S_{\mathrm{II}}^{\pm}|} = \frac{\beta_3}{\beta_r} = \frac{\mu}{\lambda+2\mu} = \frac{1-2\nu}{2-2\nu}. \tag{5.19}$$

It decreases from 0.5 to 0 for $\nu \in (0, 1/2)$. Hence, one would expect that the most critical state of fracture is the mode II of separation of material from the anticrack edge.

Finally, it is also interesting to compare the SIF of mode I given by Eq. (5.18) with the corresponding SIF for a penny-shaped crack, which is denoted by $K_{\mathrm{I}}^{\mathrm{crack}}$ and can be obtained using the results from [8, 20] as follows:

$$K_{\mathrm{I}}^{\mathrm{crack}} = \frac{2\mu}{\lambda+2\mu}\beta T_0\sqrt{a/\pi} = \frac{1-2\nu}{1-\nu}\beta T_0\sqrt{a/\pi}. \tag{5.20}$$

The ratio of stress intensity factors for the two problems is the function of Poisson's ratio $\nu \in (0, 1/2)$ given as

$$\frac{K_{\mathrm{I}}}{K_{\mathrm{I}}^{\mathrm{crack}}} = \frac{1-2\nu}{3-4\nu}, \tag{5.21}$$

which decreases from 1/3 to 0. This means that the crack is more dangerous than the anticrack in the symmetric thermal stress problem under study.

6. Conclusions

A space thermal stress problem in which a prescribed cooling temperature field is symmetrically distributed over the faces of a rigid sheet-like inclusion (anticrack) has been investigated. A general potential method of solving the resulting boundary value problems was developed. By constructing the appropriate harmonic functions, the thermoelastic problem was reduced to the mixed problem of potential theory that appears in the corresponding anticrack symmetrical problem under mechanical loads (Kaczyński [24]). The governing 2D singular integral equations were established for a planar anticrack of arbitrary shape in terms of the shear stress discontinuities across the inclusion. The problem of a prescribed constant temperature applied over the faces of a rigid circularly shaped inclusion was considered as an illustrative example. In this case a complete solution was obtained and analyzed from the point of view of initiating fractures near the edge of the inclusion. A comparison with the corresponding penny-shaped crack problem was also made. The results obtained are completely new to the literature.

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