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Plane waves and problems of steady vibrations in the theory of viscoelasticity for Kelvin–Voigt materials with double porosity

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In the present paper the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity is considered. Some basic properties of plane harmonic waves are established and the boundary value problems (BVPs) of steady vibrations are investigated. Indeed on the basis of this theory three longitudinal and two transverse plane harmonic waves propagate through a Kelvin–Voigt material with double porosity and these waves are attenuated. The basic properties of the singular integral operators and potentials (surface and volume) are presented. The uniqueness and existence theorems for regular (classical) solutions of the BVPs of steady vibrations are proved by using the potential method (boundary integral equations method) and the theory of singular integral equations.

Key words: viscoelasticity, double porosity, plane harmonic waves, uniqueness and existence theorems.
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1. Introduction

The intensive investigation of the theories of viscoelasticity and thermoviscoelasticity of continua arise due to the wide use of viscoelastic materials in engineering, technology and biomechanics. Viscoelastic materials, such as amorphous polymers, semicrystalline polymers, and biopolymers, can be modeled in order to determine their stress or strain interactions as well as their temporal dependencies (see Lakes [1], Christensen [2], Brinson and Brinson [3]). Various theories of viscoelastic materials have been proposed and studied in the series of works (for details, see Truesdell and Noll [4]; Amendola *et al.* [5], Eringen [6], Fabrizio and Morro [7], Fabrizio *et al.* [8], De Cicco and Nappa [9] and references therein).

In the beginning of the 21st century, there has been interest in formulation of the theories of viscoelasticity and thermoviscoelasticity of differential type for materials with microstructures. In connection with this has been notic, a theory of thermoviscoelasticity for a composite that is a mixture of porous elastic solid and Kelvin–Voigt material is presented in [10]. A nonlinear theory of heat conducting mixtures is introduced by Ieşan and Nappa [11], where the individual components are modelled as Kelvin–Voigt viscoelastic materials. The theory of thermoviscoelastic composites modelled as interacting Cosserat continua is developed in [12]. A theory of thermoviscoelastic mixtures is presented by Ieşan and Scalia [13] with the help of an entropy production inequality proposed by Green and Laws. The theory of porous thermoviscoelastic mixtures is developed by Ieşan and Quintanilla [14]. A mixture theory for microstretch thermoviscoelastic solids is introduced by Chiriţă and Galeş [15]. The basic equations of the theory of thermoviscoelasticity for Kelvin–Voigt materials with voids are established by Ieşan [16]. A theory of thermoviscoelasticity for Kelvin–Voigt microstretch composite materials is presented by Passarella *et al.* [17] and a theory of viscoelasticity for Kelvin–Voigt materials with double porosity is introduced in [18]. Recently, a linear first-strain gradient theory of non-simple thermoviscoelastic solids has been developed by Ieşan [19]. Much of the theoretical progress in the above mentioned theories of differential type for materials with microstructures is discussed in the series of papers [20–34].

In the present paper, the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity is considered. This paper is articulated as follows. In Section 2, the basic dynamical equations of this theory are introduced. In Section 3, the dispersion equations of the plane harmonic waves are studied. On the basis of these equations, it is established that three longitudinal and two transverse plane harmonic waves propagate through a Kelvin–Voigt material with double porosity and these waves are attenuated. In Section 4, the basic internal and external BVPs of steady vibrations are formulated and the uniqueness theorems are given. In Section 5, the properties of the singular integral operators and potentials (surface and volume) are established. Finally, in Section 6, the existence theorems for regular (classical) solutions of the internal and external BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$, let t denote the time variable, $t \geq 0$ and a dot denotes differentiation with respect to t. In what follows, we consider an isotropic and homogeneous viscoelastic Kelvin–Voigt material with double porosity. Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement vector, $\mathbf{u} = (u_1, u_2, u_3)$, and $p_1(\mathbf{x}, t)$ and $p_2(\mathbf{x}, t)$ are the pore and fissure fluid pressures, respectively.

The system of homogeneous dynamical equations in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity expressed in terms of the displacement vector $\mathbf{u}$ and the pressures p_1 and p_2 has the following form [18]:

$$\begin{aligned} &\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\nabla\,\mathrm{div}\,\mathbf{u} + \mu^*\Delta\dot{\mathbf{u}} + (\lambda^*+\mu^*)\,\nabla\,\mathrm{div}\,\dot{\mathbf{u}} \\ &\qquad\qquad - \beta_1\,\nabla p_1 - \beta_2\,\nabla p_2 = \rho\,\ddot{\mathbf{u}}, \\ &(k_1\Delta-\gamma)p_1 + (k_3\Delta+\gamma)p_2 - \alpha_1\dot{p}_1 - \alpha_3\dot{p}_2 - \beta_1\,\mathrm{div}\,\dot{\mathbf{u}} = 0, \\ &(k_3\Delta+\gamma)p_1 + (k_2\Delta-\gamma)p_2 - \alpha_3\dot{p}_1 - \alpha_2\dot{p}_2 - \beta_2\,\mathrm{div}\,\dot{\mathbf{u}} = 0, \end{aligned} \tag{2.1}$$

where Δ is the Laplacian operator, ρ is the reference mass density, $\rho > 0$; β_1 and β_2 are the effective stress parameters, and γ is the internal transport coefficient (leakage parameter) that corresponds to a fluid transfer rate with respect to the intensity of flow between the pores and fissures, $\gamma \geq 0$, α_1 and α_2 measure the compressibilities of the pore and fissure systems, respectively, α_3 is the cross-coupling compressibility for fluid flow at the interface between the two pore systems at a microscopic level, λ and μ are the Lamé constants, λ^* and μ^* are the viscosity constants; k_1 and k_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosities, respectively, and k_3 is the cross-coupling permeability for fluid flow at the interface between the matrix and fissure phases.

We suppose that $\beta_1^2 + \beta_2^2 > 0$. The case $\beta_1 = \beta_2 = 0$ is too simple to be considered (see remark 2).

3. Plane waves

In this section we assume that

$$\begin{aligned} &\mu^* > 0, \qquad k_1 > 0, \qquad k_1k_2 - k_3^2 > 0, \\ &\alpha_1 > 0, \qquad \alpha_1\alpha_2 - \alpha_3^2 > 0 \end{aligned} \tag{3.1}$$

and

$$\lambda^* + 2\mu^* > 0. \tag{3.2}$$

Let us suppose that the plane harmonic waves corresponding to a wave number τ and to an angular frequency ω propagate in the x_1-direction through the Kelvin–Voigt material with double porosity. Then,

$$u_l(\mathbf{x},t) = C_l\,e^{i(\tau x_1-\omega t)}, \qquad p_j(\mathbf{x},t) = C_{j+3}\,e^{i(\tau x_1-\omega t)}, \tag{3.3}$$

where $C_1, C_2, \ldots, C_5$ are constants, $\omega > 0$, $l = 1,2,3$, and $j = 1,2$.

Keeping in mind (3.3) from (2.1) it follows that

$$\begin{aligned}&\left\{\left[\mu_1+(\lambda_1+\mu_1)\,\delta_{1l}\right]\tau^2-\rho\omega^2\right\}C_l+i\tau\left(\beta_1C_4+\beta_2C_5\right)\delta_{1l}=0,\\ &\omega\tau\beta_1C_1+\left(k_1\tau^2-a_1\right)C_4+\left(k_3\tau^2-a_3\right)C_5=0,\\ &\omega\tau\beta_2C_1+\left(k_3\tau^2-a_3\right)C_4+\left(k_2\tau^2-a_2\right)C_5=0,\end{aligned}\tag{3.4}$$

where δ_{jl} is the Kronecker delta, $l=1,2,3$ and

$$\begin{aligned}&\lambda_1=\lambda-i\omega\lambda^*,\qquad \mu_1=\mu-i\omega\mu^*,\qquad a_1=i\omega\alpha_1-\gamma,\\ &a_2=i\omega\alpha_2-\gamma,\qquad a_3=i\omega\alpha_3+\gamma.\end{aligned}\tag{3.5}$$

From (3.4) for C_1, C_4 and C_5 we have

$$\begin{aligned}\left(\mu_0\tau^2-\rho\omega^2\right)C_1+i\tau\left(\beta_1C_4+\beta_2C_5\right)&=0,\\ \omega\tau\beta_1C_1+\left(k_1\tau^2-a_1\right)C_4+\left(k_3\tau^2-a_3\right)C_5&=0,\\ \omega\tau\beta_2C_1+\left(k_3\tau^2-a_3\right)C_4+\left(k_2\tau^2-a_2\right)C_5&=0,\end{aligned}\tag{3.6}$$

where $\mu_0=\lambda_1+2\mu_1$. If τ is the solution of equation

$$L(\tau^2)=0,\tag{3.7}$$

where

$$L(\tau^2)=b_1\tau^6+b_2\tau^4+b_3\tau^2+b_4,$$

then the system (3.6) has a non-trivial solution. Here,

$$\begin{aligned}&b_1=\mu_0k,\qquad b_2=-\mu_0r-\rho\omega^2k-i\omega r_1,\qquad b_3=\mu_0a+\rho\omega^2r+i\omega r_2,\\ &b_4=-\rho\omega^2a,\qquad k=k_1k_2-k_3^2,\qquad r=k_1a_2+k_2a_1-2k_3a_3,\\ &a=a_1a_2-a_3^2,\qquad r_1=\beta_1^2k_2+\beta_2^2k_1-2\beta_1\beta_2k_3,\\ &r_2=\beta_1^2a_2+\beta_2^2a_1-2\beta_1\beta_2a_3.\end{aligned}\tag{3.8}$$

On the other hand, from (3.4) for C_2 and C_3 we have

$$T(\tau^2)C_j=0,\tag{3.9}$$

where $T(\tau^2)=\mu_1\tau^2-\rho\omega^2$ and $j=2,3$. If τ is the solution of equation

$$T(\tau^2)=0,\tag{3.10}$$

then (3.9) has a non-trivial solution.

The relations (3.7) and (3.10) will be called *the dispersion equations* of longitudinal and transverse plane harmonic waves in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity, respectively. It is obvious that if $\tau > 0$, then the corresponding plane wave has the constant amplitude, and if τ is complex with $\operatorname{Im}\tau > 0$, then the plane wave is attenuated as $x_1 \to +\infty$.

We are now in a position to establish some properties of the roots of the dispersion equations (3.7) and (3.10). In what follows we used the following:

LEMMA 1. *If the conditions* (3.1) *and* (3.2) *are satisfied, then*

a)

$$\begin{aligned} &\mu_2 > 0, \qquad k > 0, \qquad k_0 > 0, \qquad \alpha > 0,\\ &\alpha_0 > 0, \qquad r_1 > 0, \qquad r_3 > 0, \qquad \beta_3 > 0,\\ &k_1^2\alpha_2^2 + k_2^2\alpha_1^2 \pm 2k_3^2\alpha_1\alpha_2 > 0; \end{aligned} \tag{3.11}$$

b)

$$\begin{aligned} &r_3^2 - 2k\alpha > 0, \qquad \beta_3 r_3 - \alpha r_1 > 0,\\ &r_3\alpha_0 - k_0\alpha > 0, \qquad \beta_3\alpha_0 - \alpha(\beta_1+\beta_2)^2 > 0, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} &\mu_2 = \lambda^* + 2\mu^*, \qquad k_0 = k_1 + k_2 + 2k_3, \qquad \alpha = \alpha_1\alpha_2 - \alpha_3^2,\\ &\alpha_0 = \alpha_1 + \alpha_2 + 2\alpha_3, \qquad r_3 = k_1\alpha_2 + k_2\alpha_1 - 2k_3\alpha_3,\\ &\beta_3 = \beta_1^2\alpha_2 + \beta_2^2\alpha_1 - 2\beta_1\beta_2\alpha_3. \end{aligned} \tag{3.13}$$

P r o o f. It is easy to see that conditions (3.1) and (3.2) imply the inequalities (3.11). On the basis of (3.8) and (3.13) we have

$$r_3^2 - 2k\alpha = 2\alpha_3^2(k_1k_2 + k_3^2) - 4\alpha_3k_3(k_1\alpha_2 + k_2\alpha_1) + (k_1^2\alpha_2^2 + k_2^2\alpha_1^2 + 2k_3^2\alpha_1\alpha_2),$$

where the right side is a quadratic function (with respect to α_3) and has the following negative discriminant:

$$\mathcal{D} = -8k[(k_1\alpha_2 - k_2\alpha_1)^2 + 2k\alpha_1\alpha_2] < 0.$$

Hence, the first inequality of (3.12) is valid.

On the other hand, (3.8) and (3.13) imply

$$\begin{aligned} \beta_3 r_3 - \alpha r_1 = {}& k_1(\beta_1\alpha_2 - \beta_2\alpha_3)^2 + k_2(\beta_1\alpha_3 - \beta_2\alpha_1)^2\\ &- 2k_3(\beta_1\alpha_2 - \beta_2\alpha_3)(\beta_1\alpha_3 - \beta_2\alpha_1). \end{aligned} \tag{3.14}$$

Keeping in mind (3.1) and (3.2) from (3.14) the second inequality of (3.12) is obtained.

Similarly, we can prove the third and fourth inequalities of (3.12). □

Let $\xi=\tau^2$, then (3.7) and (3.10) can be written as

$$b_1\xi^3+b_2\xi^2+b_3\xi+b_4=0 \tag{3.15}$$

and

$$\mu_1\xi-\rho\omega^2=0, \tag{3.16}$$

respectively.

LEMMA 2. *If conditions* (3.1) *and* (3.2) *are satisfied, then Eq.* (3.15) *with respect to* ξ *does not have positive root.*

P r o o f. Let ξ be a real root of Eq. (3.15). Separating real and imaginary parts in (3.15), on the basis of relations (3.5) and

$$\begin{aligned}
b_1&=k\lambda_2-i\omega k\mu_2,\\
b_2&=-\rho\omega^2k-\omega^2r_3\mu_2+\lambda_2\gamma k_0-i\omega(\lambda_2r_3+\mu_2\gamma k_0+r_1),\\
b_3&=-\omega^2(\alpha\lambda_2+\mu_2\gamma\alpha_0+\rho\gamma k_0+\beta_3)\\
&\quad-i\omega\left[\lambda_2\gamma\alpha_0-\omega^2\alpha\mu_2-\rho\omega^2r_3+\gamma(\beta_1+\beta_2)^2\right],\\
b_4&=\rho\omega^4\alpha+i\rho\omega^3\gamma\alpha_0
\end{aligned}$$

we obtain

$$\begin{aligned}
&k\lambda_2\xi^3-(\rho\omega^2k+\omega^2r_3\mu_2-\lambda_2\gamma k_0)\xi^2\\
&\qquad-\omega^2(\alpha\lambda_2+\mu_2\gamma\alpha_0+\rho\gamma k_0+\beta_3)\xi+\omega^4\rho\alpha=0,\\
&k\mu_2\xi^3+(\lambda_2r_3+\mu_2\gamma k_0+r_1)\xi^2\\
&\qquad+\left[\lambda_2\gamma\alpha_0-\omega^2\alpha\mu_2-\rho\omega^2r_3+\gamma(\beta_1+\beta_2)^2\right]\xi-\rho\omega^2\gamma\alpha_0=0,
\end{aligned} \tag{3.17}$$

where $\lambda_2=\lambda+2\mu$. As one may easily verify, the system (3.17) may be written in the form

$$\begin{aligned}
X_1X_2&=\omega^2\xi\left(\mu_2X_3+\beta_3\right),\\
X_1X_3&=-\xi\left[\mu_2X_2+r_1\xi+\gamma(\beta_1+\beta_2)^2\right],
\end{aligned} \tag{3.18}$$

where

$$X_1=\lambda_2\xi-\rho\omega^2,\qquad X_2=k\xi^2+\gamma k_0\xi-\alpha\omega^2,\qquad X_3=r_3\xi+\alpha_0\gamma. \tag{3.19}$$

Obviously, the system (3.18) implies

$$\mu_2X_2^2+\left[r_1\xi+\gamma(\beta_1+\beta_2)^2\right]X_2+\omega^2X_3\left(\mu_2X_3+\beta_3\right)=0. \tag{3.20}$$

By virtue of (3.19) from (3.20) it follows that

$$c_1\xi^4 + c_2\xi^3 + c_3\xi^2 + c_4\xi + c_5 = 0, \tag{3.21}$$

where

$$\begin{aligned} &c_1 = \mu_2 k^2, \qquad c_2 = k(2\mu_2 k_0 \gamma + r_1),\\ &c_3 = \mu_2\gamma^2 k_0^2 + \gamma\left[r_1 k_0 + k(\beta_1+\beta_2)^2\right] + \omega^2\mu_2(r_3^2 - 2k\alpha),\\ &c_4 = \omega^2(\beta_3 r_3 - \alpha r_1) + 2\omega^2\mu_2\gamma(r_3\alpha_0 - k_0\alpha) + \gamma^2 k_0(\beta_1+\beta_2)^2,\\ &c_5 = \omega^2\mu_2(\alpha^2\omega^2 + \alpha_0^2\gamma^2) + \omega^2\gamma\left[\beta_3\alpha_0 - \alpha(\beta_1+\beta_2)^2\right]. \end{aligned} \tag{3.22}$$

On the basis of inequalities (3.11) and (3.12) from (3.22) we obtain $c_j > 0$ for $j = 1, 2, \ldots, 5$. Consequently, Eq. (3.21) does not have a positive root and, hence, Eq. (3.15) with respect to ξ does not have a positive root. □

We denote the roots of the equation (3.15) (with respect to ξ) by ξ_1, ξ_2 and ξ_3. Clearly, $\xi_4 = \rho\omega^2\mu_1^{-1}$ is the complex root of (3.16).

Let $\tau_j^2 = \xi_j$ and $\mathrm{Im}\tau_j > 0$ for $j = 1, 2, 3, 4$. Obviously, λ_1, λ_2, λ_3 and λ_4 are the wave numbers of longitudinal and transverse plane harmonic waves in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity, respectively.

We have thereby proved the following:

Theorem 1. *If conditions* (3.1) *and* (3.2) *are satisfied, then five plane harmonic waves propagate through a Kelvin–Voigt material with double porosity: three longitudinal plane waves with wave numbers τ_1, τ_2, τ_3 and two transverse (horizontal and vertical) plane waves with wave number τ_4; these are attenuated waves as $x_1 \to +\infty$.*

Remark 1. It is obvious that if plane harmonic waves propagate in an arbitrary direction through a Kelvin–Voigt material, then we obtain the same result as the one given in Theorem 1.

Remark 2. If $\beta_1 = \beta_2 = 0$, then Eq. (3.15) with respect to ξ is reduced to the following equation

$$(\mu_0\xi - \rho\omega^2)(k\xi^2 - r\xi + a) = 0.$$

Obviously, this equation with respect to ξ does not have positive root, and consequently, theorem 1 is valid in the case $\beta_1 = \beta_2 = 0$ too.

4. Boundary value problems and uniqueness theorems

If the displacement vector $\mathbf{u}$ and the pressures p_1 and p_2 are postulated to have a harmonic time variation, that is,

$$\{\mathbf{u}, p_1, p_2\}(\mathbf{x}, t) = \mathrm{Re}\left[\{\mathbf{v}, q_1, q_2\}(\mathbf{x})\, e^{-i\omega t}\right],$$

then from the system of dynamical equations (2.1) we obtain the following system of equations of steady vibrations:

$$\begin{aligned} &\mu_1\,\Delta\mathbf{v} + (\lambda_1+\mu_1)\,\nabla\,\mathrm{div}\,\mathbf{v} - \beta_1\,\nabla q_1 - \beta_2\,\nabla q_2 + \rho\,\omega^2\mathbf{v} = 0,\\ &(k_1\Delta + a_1)q_1 + (k_3\Delta + a_3)q_2 + i\omega\beta_1\,\mathrm{div}\,\mathbf{v} = 0,\\ &(k_3\Delta + a_3)q_1 + (k_2\Delta + a_2)q_2 + i\omega\beta_2\,\mathrm{div}\,\mathbf{v} = 0. \end{aligned} \tag{4.1}$$

Next, we introduce the matrix differential operator

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}}) = (A_{lj}(\mathbf{D}_{\mathbf{x}}))_{5\times 5}\,,$$

where

$$\begin{aligned} &A_{lj}(\mathbf{D}_{\mathbf{x}}) = (\mu_1\,\Delta + \rho\omega^2)\delta_{lj} + (\lambda_1+\mu_1)\frac{\partial^2}{\partial x_l\partial x_j},\\ &A_{l4}(\mathbf{D}_{\mathbf{x}}) = -\beta_1\frac{\partial}{\partial x_l}, \qquad A_{l5}(\mathbf{D}_{\mathbf{x}}) = -\beta_2\frac{\partial}{\partial x_l},\\ &A_{4l}(\mathbf{D}_{\mathbf{x}}) = i\omega\beta_1\frac{\partial}{\partial x_l}, \qquad A_{5l}(\mathbf{D}_{\mathbf{x}}) = i\omega\beta_2\frac{\partial}{\partial x_l},\\ &A_{44}(\mathbf{D}_{\mathbf{x}}) = k_1\,\Delta + a_1, \qquad A_{45}(\mathbf{D}_{\mathbf{x}}) = A_{54}(\mathbf{D}_{\mathbf{x}}) = k_3\,\Delta + a_3,\\ &A_{55}(\mathbf{D}_{\mathbf{x}}) = k_2\,\Delta + a_2, \qquad \mathbf{D}_{\mathbf{x}} = \left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right), \qquad l,j = 1,2,3. \end{aligned}$$

Clearly, the system of equations of steady vibrations (4.1) can be rewritten as

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\mathbf{V}(\mathbf{x}) = \mathbf{0}, \tag{4.2}$$

where $\mathbf{V} = (\mathbf{v}, q_1, q_2)$ and $\mathbf{x} \in \mathbb{R}^3$.

Let S be the smooth closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$. We denote by $\mathbf{n}(\mathbf{z})$ the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$.

Definition 1. A vector function $\mathbf{V} = (V_1, V_2, \cdots, V_5)$ is called *regular* in Ω^- (or Ω^+) if

1)
$$V_j \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \qquad (\text{or } \ V_j \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})),$$

2)
$$V_j(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \frac{\partial}{\partial x_l}V_j(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \qquad \text{for} \qquad |\mathbf{x}| \gg 1, \tag{4.3}$$

where $j = 1, 2, \ldots, 5$, $l = 1, 2, 3$.

In what follows, we assume that the constitutive coefficients satisfy the conditions (3.1) and

$$3\lambda^* + 2\mu^* > 0. \tag{4.4}$$

Obviously, the inequalities (4.4) and $\mu^* > 0$ imply (3.2).

In the sequel, we use the matrix differential operator

$$\mathbf{P}(\mathbf{D_x}, \mathbf{n}) = (P_{lj}(\mathbf{D_x}, \mathbf{n}))_{5\times5},$$

where

$$P_{lj}(\mathbf{D_x}, \mathbf{n}) = \mu_1 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \mu_1 n_j \frac{\partial}{\partial x_l} + \lambda_1 n_l \frac{\partial}{\partial x_j}, \qquad P_{l;m+3}(\mathbf{D_x}, \mathbf{n}) = -\beta_m n_l,$$
$$P_{m+3;l}(\mathbf{D_x}, \mathbf{n}) = 0, \qquad P_{44}(\mathbf{D_x}, \mathbf{n}) = k_1 \frac{\partial}{\partial \mathbf{n}},$$
$$P_{45}(\mathbf{D_x}, \mathbf{n}) = P_{54}(\mathbf{D_x}, \mathbf{n}) = k_3 \frac{\partial}{\partial \mathbf{n}}, \quad P_{55}(\mathbf{D_x}, \mathbf{n}) = k_2 \frac{\partial}{\partial \mathbf{n}},$$
$$m = 1, 2, \;\; l, j = 1, 2, 3,$$

where $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity are formulated as follows.

Find a regular (classical) solution $\mathbf{V} = (\mathbf{v}, q_1, q_2)$ to system

$$\mathbf{A}(\mathbf{D_x})\mathbf{V}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \tag{4.5}$$

for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \ \mathbf{z} \in S} \mathbf{V}(\mathbf{x}) \equiv \{\mathbf{V}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+ \ni \mathbf{x} \to \ \mathbf{z} \in S} \mathbf{P}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{V}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{V}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are known smooth vector five-component functions.

Find a regular (classical) solution $\mathbf{V} = (\mathbf{v}, q_1, q_2)$ to system (4.5) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \ \mathbf{z} \in S} \mathbf{V}(\mathbf{x}) \equiv \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external Problem $(\mathrm{I})^-_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^-\ni\mathbf{x}\to\ \mathbf{z}\in S} \mathbf{P}(\mathbf{D}_\mathbf{x},\mathbf{n}(\mathbf{z}))\mathbf{V}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D}_\mathbf{z},\mathbf{n}(\mathbf{z}))\mathbf{V}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external Problem $(\mathrm{II})^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are known smooth vector five-component functions, and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

The following uniqueness theorems are valid.

THEOREM 2. *If conditions* (3.1) *and* (4.4) *are satisfied, then the internal BVP* $(K)^+_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution, where* $K = I, II$.

THEOREM 3. *If conditions* (3.1) *and* (4.4) *are satisfied, then the external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution, where* $K = I, II$.

Theorems 2 and 3 can be proved in a similar manner as the uniqueness theorems of the single porosity viscoelasticity (for details see [35]).

5. Basic properties of potentials and singular integral operators

We introduce the following notation:

$$\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) = \int_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_\mathbf{y}S$$

is the single-layer potential,

$$\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) = \int_S [\widetilde{\mathbf{P}}(\mathbf{D}_\mathbf{y},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top\mathbf{g}(\mathbf{y})d_\mathbf{y}S$$

is the double-layer potential, and

$$\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{h},\Omega^\pm) = \int_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y},$$

is the volume potential, where $\mathbf{g}$ and $\mathbf{h}$ are vector five-component functions, $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D}_\mathbf{x})$, the operator $\widetilde{\mathbf{P}}(\mathbf{D}_\mathbf{x},\mathbf{n})$ may be obtained from the operator $\mathbf{P}(\mathbf{D}_\mathbf{x},\mathbf{n})$ by replacing β_j with $i\omega\beta_j$ $(j = 1, 2)$ and vice versa, the superscript $\top$ denotes transposition.

REMARK 3. The matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed explicitly by means of elementary functions and its basic properties are established in [18].

We have the following results.

THEOREM 4. *If* $S \in C^{2,p}$, $\mathbf{g} \in C^{1,p'}(S)$, $0 < p' < p \le 1$, *then:*

a) $\mathbf{Q}^{(1)}(\cdot,\mathbf{g}) \in C^{0,p'}(\mathbb{R}^3) \cap C^{2,p'}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm)$,

b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) = \mathbf{0}$,
c) $\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})$ *is a singular integral,*
d)

$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})\}^{\pm} = \mp\tfrac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}), \tag{5.1}$$

where $\mathbf{x}\in\Omega^{\pm}$ *and* $\mathbf{z}\in S$.

Theorem 5. *If* $S\in C^{2,p}$, $\mathbf{g}\in C^{1,p'}(S)$, $0<p'<p\le 1$, *then:*
a) $\mathbf{Q}^{(2)}(\cdot,\mathbf{g})\in C^{1,p'}(\overline{\Omega^{\pm}})\cap C^{\infty}(\Omega^{\pm})$,
b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) = \mathbf{0}$,
c) $\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral,*
d)

$$\{\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} = \pm\tfrac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \tag{5.2}$$

e) $\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{+} = \{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{-}$, *where* $\mathbf{x}\in\Omega^{\pm}$ *and* $\mathbf{z}\in S$.

Theorem 6. *If* $S\in C^{1,p}$, $\mathbf{h}\in C^{0,p'}(\Omega^{+})$, $0<p'<p\le 1$, *then:*
a) $\mathbf{Q}^{(3)}(\cdot,\mathbf{h},\Omega^{+})\in C^{1,p'}(\mathbb{R}^3)\cap C^{2}(\Omega^{+})\cap C^{2,p'}(\overline{\Omega_0^{+}})$,
b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{h},\Omega^{+}) = \mathbf{h}(\mathbf{x})$, *where* $\mathbf{x}\in\Omega^{+}$, Ω_0^{+} *is a domain in* $\mathbb{R}^3$ *and* $\Omega_0^{+}\subset\Omega^{+}$.

Theorem 7. *If* $S\in C^{1,p}$, $\operatorname{supp}\mathbf{h} = \Omega\subset\Omega^{-}$, $\mathbf{h}\in C^{0,p'}(\Omega^{-})$, $0<p'<p\le 1$, *then:*
a) $\mathbf{Q}^{(3)}(\cdot,\mathbf{h},\Omega^{-})\in C^{1,p'}(\mathbb{R}^3)\cap C^{2}(\Omega^{-})\cap C^{2,p'}(\overline{\Omega_0^{-}})$,
b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{h},\Omega^{-}) = \mathbf{h}(\mathbf{x})$, *where* $\mathbf{x}\in\Omega^{-}$, Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{-}}\subset\Omega^{-}$.

The following notation is introduced:

$$\begin{aligned}
\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(4)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}_{\varsigma}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \zeta\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \qquad \text{for} \quad \mathbf{z}\in S,
\end{aligned} \tag{5.3}$$

where ς is a complex parameter. On the basis of Theorems 4 and 5, $\mathcal{K}^{(j)}$ $(j = 1,2,3,4)$ and $\mathcal{K}_\varsigma$ are singular integral operators.

In the sequel we need the following lemmas.

LEMMA 3. *If conditions* (3.1) *and* (4.4) *are satisfied, then the singular integral operator* $\mathcal{K}^{(j)}$ *is of the normal type, where* $j = 1,2,3,4$.

P r o o f. Let $\boldsymbol{\sigma}^{(j)} = (\sigma_{lm}^{(j)})_{5\times 5}$ be the symbol (symbolic matrix) of the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1,2,3,4)$ (the method for calculating the symbol of singular integral operator is introduced in [36]). Taking into account (5.3) for $\det \boldsymbol{\sigma}^{(j)}$ we find

$$\det \boldsymbol{\sigma}^{(1)} = -\det \boldsymbol{\sigma}^{(2)} = -\det \boldsymbol{\sigma}^{(3)} = \det \boldsymbol{\sigma}^{(4)} = \frac{1}{32}\left[1 - \frac{\mu_1^2}{(\lambda_1 + 2\mu_1)^2}\right]. \tag{5.4}$$

Keeping in mind the relations (3.1) and (4.4) obtained from (5.4) we get $\det \boldsymbol{\sigma}^{(j)} \neq 0$, which proves that the singular integral operator $\mathcal{K}^{(j)}$ is of the normal type, where $j = 1,2,3,4$. □

THEOREM 8. *If conditions* (3.1) *and* (4.4) *are satisfied, then the Fredholm's theorems are valid for the singular integral operator* $\mathcal{K}^{(j)}$ *, where* $j = 1,2,3,4$.

P r o o f. Let $\boldsymbol{\sigma}_\varsigma$ and $\operatorname{ind}\mathcal{K}_\varsigma$ be the symbol and the index of the operator $\mathcal{K}_\varsigma$, respectively. It may be easily shown that

$$\det \boldsymbol{\sigma}_\varsigma = \frac{1}{32}\left[1 - \frac{\mu_1^2\varsigma^2}{(\lambda_1 + 2\mu_1)^2}\right]$$

and $\det \boldsymbol{\sigma}_\varsigma$ vanishes only at two points ς_1 and ς_2 of the complex plane. By virtue of (5.4) and $\det \boldsymbol{\sigma}_1 = \det \boldsymbol{\sigma}^{(1)}$ we get $\varsigma_j \neq 1$ for $j = 1,2,3,4$, and consequently (see Mikhlin [37]), we obtain

$$\operatorname{ind}\mathcal{K}^{(1)} = \operatorname{ind}\mathcal{K}_1 = 0.$$

The relation $\operatorname{ind}\mathcal{K}^{(2)} = 0$ is proved in a quite similar manner. Obviously, the operators $\mathcal{K}^{(3)}$ and $\mathcal{K}^{(4)}$ are the adjoint operators for $\mathcal{K}^{(2)}$ and $\mathcal{K}^{(1)}$, respectively. Evidently,

$$\operatorname{ind}\mathcal{K}^{(3)} = -\operatorname{ind}\mathcal{K}^{(2)} = 0, \qquad \operatorname{ind}\mathcal{K}^{(4)} = -\operatorname{ind}\mathcal{K}^{(1)} = 0.$$

Thus, the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1,2,3,4)$ is of the normal type with an index equal to zero and the Fredholm theorems are valid for $\mathcal{K}^{(j)}$. □

REMARK 4. The definitions of a normal type singular integral operator, the symbol and the index of operator are given in [36, 37].

6. Existence theorems

We are now in a position to prove the existence theorems for regular (classical) solutions of the BVPs of steady vibrations in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity.

Obviously, according to Theorems 6 and 7 the volume potential $\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{F},\Omega^{\pm})$ is a partial regular solution of the nonhomogeneous equation (4.5), where $\mathbf{F} \in C^{0,p'}(\Omega^{\pm})$, $0 < p' \leq 1$ and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^{-}. Therefore, further we will consider problem $(K)^{\pm}_{\mathbf{0},\mathbf{f}}$ for $K = I, II$.

PROBLEM $(I)^{+}_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to Problem $(\mathrm{I})^{+}_{\mathbf{0},\mathbf{f}}$ in the form of double-layer potential

(6.1) $$\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) \qquad \text{for} \qquad \mathbf{x} \in \Omega^{+},$$

where $\mathbf{g}$ is the required vector five-component function. According to Theorem 5 the vector function $\mathbf{V}$ is a solution of (4.2) for $\mathbf{x} \in \Omega^{+}$. Keeping in mind the boundary condition and using (5.2), for determining the unknown vector $\mathbf{g}$, a singular integral equation is obtained from (6.1)

(6.2) $$\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S.$$

By Theorem 8 the Fredholm theorems are valid for operator $\mathcal{K}^{(1)}$. We prove that (6.2) is always solvable for an arbitrary vector $\mathbf{f}$. Let us consider the adjoint homogeneous equation

(6.3) $$\mathcal{K}^{(4)}\mathbf{h}_0(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S,$$

where $\mathbf{h}_0$ is the required vector five-component function.

Now we prove that (6.3) has only the trivial solution. Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (6.3). On the basis of Theorem 4 and (6.3) the vector function $\mathbf{W}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h}_0)$ is a regular solution of Problem $(\mathrm{II})^{-}_{\mathbf{0},\mathbf{0}}$. Using Theorem 3, the Problem $(\mathrm{II})^{-}_{\mathbf{0},\mathbf{0}}$ has only the trivial solution that is

(6.4) $$\mathbf{W}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for} \qquad \mathbf{x} \in \Omega^{-}.$$

On the other hand, by Theorem 4 and (6.4) we get

$$\{\mathbf{W}(\mathbf{z})\}^{+} = \{\mathbf{W}(\mathbf{z})\}^{-} = \mathbf{0} \qquad \text{for } \mathbf{z} \in S,$$

i.e., the vector $\mathbf{W}(\mathbf{x})$ is a regular solution of Problem $(\mathrm{I})^{+}_{\mathbf{0},\mathbf{0}}$. Using Theorem 2, the Problem $(\mathrm{I})^{+}_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

(6.5) $$\mathbf{W}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^{+}.$$

By virtue of (6.4), (6.5) and identity (5.1) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{W}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{W}(\mathbf{z})\}^+ \equiv \mathbf{0} \qquad \text{for } \mathbf{z}\in S.$$

Thus, the homogeneous equation (6.3) has only the trivial solution and therefore (6.2) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

THEOREM 9. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{1,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of Problem* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, it is unique and is represented by double-layer potential* (6.1)*, where* $\mathbf{g}$ *is a solution of the singular integral equation* (6.2) *which is always solvable for an arbitrary vector* $\mathbf{f}$*.*

PROBLEM $(\mathrm{II})^-_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to Problem $(\mathrm{II})^-_{\mathbf{0},\mathbf{f}}$ in the form of single-layer potential

$$\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h}) \qquad \text{for } \mathbf{x}\in \Omega^-, \tag{6.6}$$

where $\mathbf{h}$ is the required vector five-component function. Obviously, by Theorem 4 the vector function $\mathbf{V}$ is a solution of (4.2) for $\mathbf{x}\in\Omega^-$. Keeping in mind the boundary condition and using (5.1), for determining the unknown vector $\mathbf{h}$, a singular integral equation is obtained from (6.6)

$$\mathcal{K}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z}\in S. \tag{6.7}$$

It has been proved above that the corresponding homogeneous equation (6.3) has only the trivial solution. Hence, it follows that (6.7) is always solvable.

We have thereby proved

THEOREM 10. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{0,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of Problem* $(\mathrm{II})^-_{\mathbf{0},\mathbf{f}}$ *exists, it is unique and is represented by single-layer potential* (6.6)*, where* $\mathbf{h}$ *is a solution of the singular integral equation* (6.7) *which is always solvable for an arbitrary vector* $\mathbf{f}$*.*

Quite similarly, we can prove the following results.

THEOREM 11. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{0,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of Problem* $(\mathrm{II})^+_{\mathbf{0},\mathbf{f}}$ *exists, it is unique and is represented by single-layer potential*

$$\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) \qquad \text{for } \mathbf{x}\in \Omega^+,$$

where $\mathbf{g}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z}\in S$$

which is always solvable for an arbitrary vector $\mathbf{f}$*.*

THEOREM 12. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{1,p'}(S)$, $0 < p' < p \leq 1$, *then a regular solution of Problem* $(\mathrm{I})^-_{\mathbf{0},\mathbf{f}}$ *exists, it is unique and is represented by double-layer potential*

$$\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{h}) \qquad \text{for } \mathbf{x} \in \Omega^-,$$

where $\mathbf{h}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(3)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for } \mathbf{z} \in S,$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

7. Concluding remarks

1. In this paper, the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity is considered and the following results are obtained:

i) three longitudinal and two transverse plane harmonic waves propagate through a Kelvin–Voigt material with double porosity and these waves are attenuated;

ii) the basic properties of the singular integral operators and potentials (surface and volume) are established;

iii) the existence theorems for regular (classical) solutions of the basic internal and external BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

2. On the basis of this paper results it is possible:

i) to study the plane harmonic waves in the linear theory of thermoviscoelasticity for Kelvin–Voigt materials with double porosity;

ii) to prove the uniqueness and existence theorems for classical solutions of the BVPs of steady vibrations in the theory of thermoviscoelasticity for Kelvin–Voigt materials with double porosity by using the potential method and the theory of singular integral equations.

3. The BVPs of the classical theories of elasticity and theormoelasticity are investigated by using the potential method in [36, 38, 39]. An extensive review of works on this method can be found in [40]. The potential method is developed in the theories of viscoelasticity and thermoviscoelasticity for Kelvin–Voigt materials with voids in [41–43].

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