

## On the well-posedness of the time-differential three-phase-lag thermoelasticity model

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THIS PAPER ANALYZES THE TIME DIFFERENTIAL THREE-PHASE-LAG MODEL of coupled thermoelasticity. The uniqueness and continuous dependence results are established for the solutions of the corresponding initial boundary value problems associated with the model in concern. The key tool of the method is to associate with the basic initial boundary value problem of the model an appropriate auxiliary initial boundary value problem and then to establish an identity of Lagrange type. This last identity is used to analyze the uniqueness of solutions under appropriate mild restrictions assumed upon the constitutive coefficients and upon the delay times. Uniqueness question is also discussed for a set of models of thermoelasticity developed in literature. Further, for the continuous dependence problem an appropriate estimate of the solution is obtained in terms of the given data. This expresses the continuous dependence of solution with respect to the initial data and with respect to the given supply loads, provided some appropriate constitutive assumptions are considered. These results give information upon the well-posedness of the time differential three-phase-lag model of coupled thermoelasticity.

**Key words:** three-phase-lag thermoelastic model, Lagrange identity, uniqueness of solutions, continuous data dependence, well-posedness.

**2000 Mathematics Subject Classification:** 74F05, 74H25, 74H55.

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### 1. Introduction

TZOU [1] HAS PROPOSED THE DUAL-PHASE-LAG MODEL of heat conduction based on the following constitutive equation for the heat supply

$$(1.1) \quad q_i(\mathbf{x}, t + \tau_q) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x}, t + \tau_T),$$

where  $q_i$  are the components of the heat flux vector,  $T$  is the temperature variation from the constant ambient temperature  $T_0 > 0$ ,  $k_{ij}$  are the components of the conductivity tensor at the position  $\mathbf{x}$ ,  $t$  is the current time,  $\tau_q$  is the phase lag of the heat flux and  $\tau_T$  is the phase lag of the gradient of temperature. The phase lags  $\tau_T$  and  $\tau_q$  are treated as two additional intrinsic thermal properties characterizing the energy-bearing capacity of the material. The thermodynamic consistency, as well as the stability issues, of the related time differential models (obtained considering the Taylor series expansions of both sides of the equation (1.1) and retaining terms up to suitable orders in  $\tau_q$  and  $\tau_T$ ) have been widely investigated in literature (see, for example, [2], [3] and [4]). Such models have been used in literature in order to study the transient heat transfer in some real applications, see e.g. [5], [6], [7].

On the other hand, GREEN and NAGHDI [8]–[10] developed a model of thermoelasticity, which includes temperature gradient and thermal displacement gradient among the constitutive variables and proposed a heat conduction law as

$$(1.2) \quad q_i(\mathbf{x}, t) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x}, t) - K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x}, t),$$

where  $T = \dot{\alpha}$ ,  $\alpha$  is the thermal displacement and the heat conductivity tensor  $k_{ij}$  satisfies the dissipation inequality

$$(1.3) \quad k_{ij}\dot{\alpha}_{,i}\dot{\alpha}_{,j} \geq 0.$$

We have to say here that the thermal displacement has been first introduced by HELMHOLTZ [11], [12]. A historical account of the fortunes of thermal displacement can be found in the Appendix of the paper by PODIO-GUIDUGLI [13]. It is outlined there the important role of the concept of thermal displacement for stating the basic balance laws of thermomechanics, but a physical interpretation is still wanted, especially if consistent with the statistical-mechanics interpretation for temperature.

Introducing three-phase-lags to the heat flux vector  $q_i$ , the temperature gradient  $T_{,j}$  and the thermal displacement gradient  $\alpha_{,i}$ , ROY CHOUDHURI [14] proposed the following generalized constitutive equation for heat flux for describing the lagging behavior

$$(1.4) \quad q_i(\mathbf{x}, t + \tau_q) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x}, t + \tau_T) - K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x}, t + \tau_\alpha).$$

The third delay time  $\tau_\alpha$  may be interpreted, following TZOU [1], as the phase-lag of the thermal displacement gradient. The thermal lagging is explained in the Chapter 2 of the book by TZOU [15]. For such model one can consider several kind of Taylor approximations to recover (in particular) the GREEN and

NAGHDI models. Great interest was developed to study these equations and the different Taylor approximations (see, for example, QUINTANILLA [16], [17] and QUINTANILLA and RACKE [18]).

Retaining terms up to second order in  $\tau_q$  in the Taylor's expansion of the heat flux vector into generalized conduction law (1.4) and by taking Taylor's series expansion of (1.4) up to the first-order terms in  $\tau_T$  and  $\tau_\alpha$  leads to the following generalized heat conduction law valid at a position  $\mathbf{x}$  at time  $t$ ,

$$(1.5) \quad q_i(\mathbf{x}, t) + \tau_q \dot{q}_i(\mathbf{x}, t) + \frac{1}{2} \tau_q^2 \ddot{q}_i(\mathbf{x}, t) = -(k_{ij}(\mathbf{x}) + \tau_\alpha K_{ij}(\mathbf{x})) T_{,j}(\mathbf{x}, t) \\ - \tau_T k_{ij}(\mathbf{x}) \dot{T}_{,j}(\mathbf{x}, t) - K_{ij}(\mathbf{x}) \alpha_{,j}(\mathbf{x}, t).$$

The restrictions that make the constitutive equation (1.5) thermodynamically consistent have been established by CHIRIȚĂ et al. [19]. The model is reformulated by means of the fading memory theory, in which the heat flux vector depends on the history of the thermal displacement gradient: the thermodynamic principles are then applied to obtain suitable restrictions involving the delay times, namely the following tensors  $\varkappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q) K_{ij}$  and  $\kappa_{ij} = \tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij})$  are positive semi-definite. The three-phase-lag model of heat conduction based on the constitutive equation (1.5) has been used to study the thermal responses of multilayered systems, functionally graded solid media and porous materials (see, for example, [20], [21], [22]).

As it is well-known in literature, a mathematical model of thermoelasticity is understood to be well posed in the sense of Hadamard if the corresponding boundary-initial value problem possesses a unique solution that depends continuously on the prescribed data. Knowing whether or not a solution is unique is important for numerical evaluation or for completeness of constructed by semi-inverse or similar methods. While the continuous data dependence is of practical and numerical importance. Physical measurements introduce unavoidable errors and these small errors have to influence the real solution in little measure. A problem is called improperly posed (or ill-posed) if it fails to have a global solution, or if it fails to have a unique solution, or if the solution does not depend continuously on the data.

In the present paper we search for the well-posed problem concerning the time-differential three-phase-lag model of thermoelasticity. In fact, we derive uniqueness results for the solutions of the initial boundary value problems associated with the model of the linear theory of coupled thermoelasticity with time differential three-phase-lag based on the constitutive equation (1.5) for the heat flux vector. The proof uses a modified initial boundary value problem to construct a Lagrange identity associated with the solutions in concern. On this basis we establish an identity that allows us to analyze the uniqueness of so-

lutions under mild restrictions upon the constitutive coefficients and upon the delay times. Uniqueness question is also discussed for a set of models of thermoelasticity developed in literature. Further, for the continuous dependence problem we establish an appropriate estimate of the solution in terms of the given data that expresses the continuous dependence of solution with respect to the initial data and with respect to the given supply loads, provided some appropriate constitutive assumptions are considered. Thus, we can conclude that the time-differential three-phase-lag model of thermoelasticity is well-posed, provided appropriate thermodynamic and constitutive assumptions are considered.

## 2. Basic equations

Throughout this paper, we refer the motion of a continuum to a fixed system of rectangular Cartesian axes  $Ox_k$  ( $k = 1, 2, 3$ ). We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers  $(1, 2, 3)$ , summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate and a superposed dot denotes time differentiation. Throughout this section we suppose that a regular region  $B$  is filled by an inhomogeneous and anisotropic thermoelastic material. Considering the linear coupled theory of thermoelastic materials with time differential three-phase-lag and assuming that the initial body is free from stresses and has zero entropy, the system of field equations consists of

– the equations of motion

$$(2.1) \quad t_{ji,j} + \varrho f_i = \varrho \ddot{u}_i, \quad \text{in } B \times (0, \infty),$$

– the equation of energy

$$(2.2) \quad \varrho T_0 \dot{\eta} = -q_{i,i} + \varrho r, \quad \text{in } B \times (0, \infty),$$

– the constitutive equations

$$(2.3) \quad \begin{aligned} t_{ij} &= C_{ijkl} e_{kl} - M_{ij} \dot{\alpha}, \\ \varrho \eta &= M_{ij} e_{ij} + a \dot{\alpha}, \\ q_i + \tau_q \dot{q}_i + \frac{1}{2} \tau_q^2 \ddot{q}_i &= -(k_{ij} + \tau_\alpha K_{ij}) \dot{\beta}_j - \tau_T k_{ij} \ddot{\beta}_j - K_{ij} \beta_j, \quad \text{in } \overline{B} \times [0, \infty), \end{aligned}$$

and

– the geometrical relations

$$(2.4) \quad \begin{aligned} e_{ij} &= \frac{1}{2} (u_{i,j} + u_{j,i}), \\ \beta_j &= \alpha_{,j}, \quad \text{in } \overline{B} \times [0, \infty). \end{aligned}$$

Here we have used the following notations:  $u_i$  are the components of the displacement vector,  $\alpha$  is the thermal displacement,  $T = \dot{\alpha}$  is the change in temperature from the constant ambient temperature  $T_0 > 0$ ,  $e_{ij}$  are the components of the strain tensor,  $\beta_i$  are the components of the thermal displacement gradient vector,  $t_{ij}$  are the components of the stress tensor,  $q_i$  are the components of the heat flux vector,  $\eta$  is the entropy per unit mass,  $\rho$  is the mass density of the medium,  $f_i$  represent the components of the external body force vector per unit mass and  $r$  is the external rate of supply of heat per unit mass. Furthermore,  $C_{ijkl}$ ,  $M_{ij}$ ,  $a$ ,  $k_{ij}$  and  $K_{ij}$  are the constitutive coefficients depending on the spatial variables  $x_i$ , continuously differentiable on  $B$  and satisfying the following symmetries

$$(2.5) \quad C_{ijkl} = C_{klij} = C_{jikl}, \quad M_{ij} = M_{ji}, \quad k_{ij} = k_{ji}, \quad K_{ij} = K_{ji}.$$

The components of the surface traction and the heat flux at regular points of  $\partial B$  can be expressed in the form

$$(2.6) \quad t_i = t_{ji}n_j, \quad q = q_in_i,$$

where  $n_i$  are the components of the unit outward normal vector to  $\partial B$ .

To the field equations we adjoin initial and boundary conditions. The initial conditions are

$$(2.7) \quad \begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}), \quad \alpha(\mathbf{x}, 0) = 0, \quad \dot{\alpha}(\mathbf{x}, 0) = T^0(\mathbf{x}), \\ q_i(\mathbf{x}, 0) &= q_i^0(\mathbf{x}), \quad \dot{q}_i(\mathbf{x}, 0) = \dot{q}_i^0(\mathbf{x}), \quad \mathbf{x} \in \bar{B}, \end{aligned}$$

where  $u_i^0(\mathbf{x})$ ,  $\dot{u}_i^0(\mathbf{x})$ ,  $T^0(\mathbf{x})$ ,  $q_i^0(\mathbf{x})$  and  $\dot{q}_i^0(\mathbf{x})$  are prescribed functions on  $\bar{B}$ . In the above relation we have set  $\alpha(\mathbf{x}, 0) = 0$  because we have taken

$$(2.8) \quad \alpha(\mathbf{x}, t) = \int_0^t T(\mathbf{x}, s) ds.$$

The boundary conditions are

$$(2.9) \quad \begin{aligned} u_i(\mathbf{x}, t) &= \tilde{u}_i(\mathbf{x}, t) \quad \text{on } \bar{\Sigma}_1 \times [0, \infty), \quad t_{ji}(\mathbf{x}, t)n_j = \tilde{t}_i(\mathbf{x}, t) \quad \text{on } \Sigma_2 \times [0, \infty), \\ \alpha(\mathbf{x}, t) &= \tilde{\alpha}(\mathbf{x}, t) \quad \text{on } \bar{\Sigma}_3 \times [0, \infty), \quad q_i(\mathbf{x}, t)n_i = \tilde{q}(\mathbf{x}, t) \quad \text{on } \Sigma_4 \times [0, \infty), \end{aligned}$$

where  $\tilde{u}_i(\mathbf{x}, t)$ ,  $\tilde{t}_i(\mathbf{x}, t)$ ,  $\tilde{\alpha}(\mathbf{x}, t)$  and  $\tilde{q}(\mathbf{x}, t)$  are prescribed continuous functions and  $\Sigma_1 \cup \bar{\Sigma}_2 = \Sigma_3 \cup \bar{\Sigma}_4 = \partial B$  and  $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$ .

Throughout this paper we consider the initial boundary value problem  $\mathcal{P}$  defined by the basic equations (2.1)–(2.4) and the initial conditions (2.7) and the boundary conditions (2.9). Furthermore, we denote by  $\mathcal{P}_0$  the initial boundary

value problem  $\mathcal{P}$  corresponding to the given data  $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; \tilde{u}_i, \tilde{t}_i, \tilde{\alpha}, \tilde{q}\} = 0$ .

By a solution of the initial boundary value problem  $\mathcal{P}$  corresponding to the given data  $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; \tilde{u}_i, \tilde{t}_i, \tilde{\alpha}, \tilde{q}\}$  we mean the ordered array  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  defined on  $\overline{B} \times [0, \infty)$  with the properties that

$$\begin{aligned} u_i(\mathbf{x}, t) &\in C^{2,2}(B \times (0, \infty)), & \alpha(\mathbf{x}, t) &\in C^{2,2}(B \times (0, \infty)), \\ e_{ij}(\mathbf{x}, t) = e_{ji}(\mathbf{x}, t) &\in C^{0,0}(B \times (0, \infty)), & \beta_j(\mathbf{x}, t) &\in C^{0,0}(B \times (0, \infty)), \\ t_{ij}(\mathbf{x}, t) = t_{ji}(\mathbf{x}, t) &\in C^{1,0}(B \times (0, \infty)), & \eta(\mathbf{x}, t) &\in C^{0,1}(B \times (0, \infty)), \\ q_i(\mathbf{x}, t) &\in C^{1,2}(B \times (0, \infty)) \end{aligned}$$

and which satisfy the field equations (2.1) to (2.4), the initial conditions (2.7) and the boundary conditions (2.9). Concerning the initial boundary value problem  $\mathcal{P}$  we will treat the corresponding uniqueness question. In view of the linearity of the problem, the uniqueness question is equivalent to prove that the initial boundary value problem  $\mathcal{P}_0$  admits only the zero solution. So in the next four sections we will study the initial boundary value problem  $\mathcal{P}_0$  and we will establish uniqueness results under mild restrictions upon the constitutive coefficients and upon the delay times. Finally, in Section 7, we establish an estimate describing the continuous dependence of solution of the initial boundary value problem  $\mathcal{P}$  with respect to the initial data and with respect to the given loads.

### 3. Two auxiliary operators and some their properties

For further convenience we introduce some auxiliary operators. Thus, for any continuous function of time variable  $f(t)$ , we denote by  $f'(t)$  the integral over  $[0, t]$  of that function, that is

$$(3.1) \quad f'(t) = \int_0^t f(z) dz, \quad f''(t) = \int_0^t \int_0^s f(z) dz ds, \quad \text{and so on;}$$

moreover, for any continuous function  $g(t)$  we will denote by  $g^*(t)$  the following function

$$(3.2) \quad g^*(t) = g''(t) + \tau_q g'(t) + \frac{1}{2} \tau_q^2 g(t).$$

Furthermore, we note that

$$(3.3) \quad g^*(0) = \frac{1}{2} \tau_q^2 g(0), \quad \frac{dg^*}{dt}(0) = \tau_q g(0) + \frac{1}{2} \tau_q^2 \dot{g}(0).$$

Concerning these concepts we can establish the following properties.

LEMMA 1. *Suppose that  $g$  is twice continuously differentiable. Then, we have*

$$(3.4) \quad \begin{aligned} \left(\frac{dg}{dt}\right)'(t) &= \frac{dg'}{dt}(t) - g(0), \\ \left(\frac{dg}{dt}\right)^*(t) &= \frac{dg^*}{dt}(t) - [\tau_q g(0) + tg(0)], \\ \left(\frac{d^2g}{dt^2}\right)^*(t) &= \frac{d^2g^*}{dt^2}(t) - [g(0) + \tau_q \dot{g}(0) + t\dot{g}(0)]. \end{aligned}$$

LEMMA 2. *Suppose that  $g$  is twice continuously differentiable with  $g(0) = 0$  and  $\dot{g}(0) = 0$ . Then, we have*

$$(3.5) \quad \begin{aligned} \left(\frac{dg}{dt}\right)'(t) &= \frac{dg'}{dt}(t), \\ \left(\frac{dg}{dt}\right)^*(t) &= \frac{dg^*}{dt}(t), \\ \left(\frac{d^2g}{dt^2}\right)^*(t) &= \frac{d^2g^*}{dt^2}(t). \end{aligned}$$

LEMMA 3. *Suppose that  $g$  is twice continuously differentiable and satisfies*

$$(3.6) \quad g^*(t) = 0, \quad \text{for all } t > 0,$$

and

$$(3.7) \quad g^*(0) = 0, \quad \dot{g}^*(0) = 0.$$

Then, we have

$$(3.8) \quad g(t) = 0, \quad \text{for all } t \geq 0.$$

**Proof.** By a twice derivative with respect to time variable in (3.2), from (3.6) we deduce that

$$(3.9) \quad \frac{1}{2} \tau_q^2 \ddot{g}(t) + \tau_q \dot{g}(t) + g(t) = 0, \quad \text{for all } t > 0,$$

while from (3.3) and (3.7) we have

$$(3.10) \quad g(0) = 0, \quad \dot{g}(0) = 0.$$

Now it is easy to see that the Cauchy problem defined by the differential equation (3.9) and the initial conditions (3.10) has only the zero solution and hence we get the conclusion expressed by relation (3.8) and the proof is complete.

#### 4. An auxiliary initial boundary value problem

Let us now consider that  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  is a solution of the initial boundary value problem  $\mathcal{P}$ . Then it follows that  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\}$  is a solution of the initial boundary value problem  $\mathcal{P}^*$  defined by the following basic equations

$$(4.1) \quad t_{j,i}^*(t) + F_i(t) = \varrho \frac{\partial^2 u_i^*}{\partial t^2}(t),$$

$$(4.2) \quad \varrho \frac{\partial \eta^*}{\partial t}(t) = -\frac{1}{T_0} q_{i,i}^*(t) + R(t),$$

in  $B \times (0, \infty)$ ,

$$(4.3) \quad t_{ij}^*(t) = C_{ijkl} e_{kl}^*(t) - M_{ij} \frac{\partial \alpha^*}{\partial t}(t),$$

$$(4.4) \quad \varrho \eta^*(t) = M_{ij} e_{ij}^*(t) + a \frac{\partial \alpha^*}{\partial t}(t),$$

$$(4.5) \quad q_i^*(t) = -K_{ij} \beta_j''(t) - (k_{ij} + \tau_\alpha K_{ij}) \beta_j'(t) - \tau_T k_{ij} \beta_j(t) \\ + \frac{1}{2} \tau_q^2 q_i^0 + t \left( \tau_q q_i^0 + \tau_T k_{ij} T_{,j}^0 + \frac{1}{2} \tau_q^2 \dot{q}_i^0 \right),$$

and

$$(4.6) \quad e_{ij}^*(t) = \frac{1}{2} (u_{i,j}^*(t) + u_{j,i}^*(t)),$$

$$\beta_j^*(t) = \alpha_{,j}^*(t),$$

in  $\bar{B} \times [0, \infty)$  and the initial conditions

$$(4.7) \quad u_i^*(\mathbf{x}, 0) = \frac{1}{2} \tau_q^2 u_i^0(\mathbf{x}), \quad \frac{\partial u_i^*}{\partial t}(\mathbf{x}, 0) = \tau_q u_i^0(\mathbf{x}) + \frac{1}{2} \tau_q^2 \dot{u}_i^0(\mathbf{x}), \\ \alpha^*(\mathbf{x}, 0) = 0, \quad \frac{\partial \alpha^*}{\partial t}(\mathbf{x}, 0) = \frac{1}{2} \tau_q^2 T^0(\mathbf{x}),$$

$$q_i^*(\mathbf{x}, 0) = \frac{1}{2} \tau_q^2 q_i^0(\mathbf{x}), \quad \frac{\partial q_i^*}{\partial t}(\mathbf{x}, 0) = \tau_q q_i^0(\mathbf{x}) + \frac{1}{2} \tau_q^2 \dot{q}_i^0(\mathbf{x}), \quad \text{on } \bar{B},$$

and the boundary conditions

$$(4.8) \quad u_i^*(\mathbf{x}, t) = \tilde{u}_i^*(\mathbf{x}, t) \quad \text{on } \bar{\Sigma}_1 \times [0, \infty), \\ t_{ji}^*(\mathbf{x}, t) n_j = \tilde{t}_i^*(\mathbf{x}, t) \quad \text{on } \Sigma_2 \times [0, \infty), \\ \alpha^*(\mathbf{x}, t) = \tilde{\alpha}^*(\mathbf{x}, t) \quad \text{on } \bar{\Sigma}_3 \times [0, \infty), \\ q_i^*(\mathbf{x}, t) n_i = \tilde{q}^*(\mathbf{x}, t) \quad \text{on } \Sigma_4 \times [0, \infty).$$

In the above relations we have used the following notation

$$(4.9) \quad \varrho \eta^0(\mathbf{x}) = \frac{1}{2} M_{ij} (u_{i,j}^0(\mathbf{x}) + u_{j,i}^0(\mathbf{x})) + aT^0(\mathbf{x}),$$

and

$$(4.10) \quad \begin{aligned} F_i(\mathbf{x}, t) &= \varrho f_i^*(\mathbf{x}, t) + \varrho [(t + \tau_q) \dot{u}_i^0(\mathbf{x}) + u_i^0(\mathbf{x})], \\ R(\mathbf{x}, t) &= \frac{1}{T_0} \varrho r^*(\mathbf{x}, t) + \varrho (t + \tau_q) \eta^0(\mathbf{x}). \end{aligned}$$

REMARK 1. Let us now consider that  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  is a solution of the initial boundary value problem  $\mathcal{P}_0$ . Then it follows that  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\}$  is a solution of the initial boundary value problem  $\mathcal{P}_0^*$  defined by the following basic equations

$$(4.11) \quad t_{ji,j}^*(t) = \varrho \frac{\partial^2 u_i^*}{\partial t^2}(t),$$

$$(4.12) \quad \varrho \frac{\partial \eta^*}{\partial t}(t) = -\frac{1}{T_0} q_{i,i}^*(t),$$

in  $B \times (0, \infty)$ ,

$$(4.13) \quad t_{ij}^*(t) = C_{ijkl} e_{kl}^*(t) - M_{ij} \frac{\partial \alpha^*}{\partial t}(t),$$

$$(4.14) \quad \varrho \eta^*(t) = M_{ij} e_{ij}^*(t) + a \frac{\partial \alpha^*}{\partial t}(t),$$

$$(4.15) \quad q_i^*(t) = -K_{ij} \beta_j''(t) - (k_{ij} + \tau_\alpha K_{ij}) \beta_j'(t) - \tau_T k_{ij} \beta_j(t),$$

and

$$(4.16) \quad \begin{aligned} e_{ij}^*(t) &= \frac{1}{2} (u_{i,j}^*(t) + u_{j,i}^*(t)), \\ \beta_j^*(t) &= \alpha_{i,j}^*(t), \end{aligned}$$

in  $\overline{B} \times [0, \infty)$  and the initial conditions

$$(4.17) \quad \begin{aligned} u_i^*(\mathbf{x}, 0) &= 0, \quad \frac{\partial u_i^*}{\partial t}(\mathbf{x}, 0) = 0, \quad \alpha^*(\mathbf{x}, 0) = 0, \quad \frac{\partial \alpha^*}{\partial t}(\mathbf{x}, 0) = 0, \\ q_i^*(\mathbf{x}, 0) &= 0, \quad \frac{\partial q_i^*}{\partial t}(\mathbf{x}, 0) = 0, \quad \text{on } \overline{B}, \end{aligned}$$

and the boundary conditions

$$(4.18) \quad \begin{aligned} u_i^*(\mathbf{x}, t) &= 0 \quad \text{on } \bar{\Sigma}_1 \times [0, \infty), \\ t_{ji}^*(\mathbf{x}, t)n_j &= 0 \quad \text{on } \Sigma_2 \times [0, \infty), \\ \alpha^*(\mathbf{x}, t) &= 0 \quad \text{on } \bar{\Sigma}_3 \times [0, \infty), \\ q_i^*(\mathbf{x}, t)n_i &= 0 \quad \text{on } \Sigma_4 \times [0, \infty). \end{aligned}$$

REMARK 2. In view of the Lemma 3, we can conclude that when  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\} = 0$  then we have  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\} = 0$ . So in order to prove the uniqueness of solutions to the initial boundary value problem  $\mathcal{P}$  it is sufficient to prove that the unique solution of the initial boundary value problem  $\mathcal{P}_0^*$  is the banal solution  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\} = 0$ .

## 5. A Lagrange identity for the initial boundary value problem $\mathcal{P}_0^*$

In this section we will establish an identity of Lagrange type for the solutions of the initial boundary value problem  $\mathcal{P}_0^*$ . Such identity is useful to establish uniqueness of solutions to the initial boundary value problem  $\mathcal{P}$  under mild assumptions upon the thermoelastic characteristic coefficients. Thus, we have

THEOREM 1. *For any solution  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\}$  of the initial boundary value problem  $\mathcal{P}_0^*$ , we have the following Lagrange identity*

$$(5.1) \quad \begin{aligned} &2 \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv \\ &= \int_0^t \int_B \frac{1}{T_0} \left[ K_{ij} \beta_j'''(t-s) \dot{\beta}_i^*(t+s) - K_{ij} \beta_j'''(t+s) \dot{\beta}_i^*(t-s) \right] dv ds \\ &\quad + \int_0^t \int_B \frac{1}{T_0} \left[ (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t-s) \dot{\beta}_i^*(t+s) \right. \\ &\quad \quad \quad \left. - (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t+s) \dot{\beta}_i^*(t-s) \right] dv ds \\ &\quad + \int_0^t \int_B \frac{\tau_T}{T_0} \left[ k_{ij} \beta_j'(t-s) \dot{\beta}_i^*(t+s) - k_{ij} \beta_j'(t+s) \dot{\beta}_i^*(t-s) \right] dv ds, \end{aligned}$$

for all  $t \geq 0$ .

**Proof.** Let  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\}$  be a solution of the initial boundary value problem  $\mathcal{P}_0^*$ . We start with the identity

$$(5.2) \quad \frac{\partial}{\partial s} [\varrho u_i^*(t+s) \dot{u}_i^*(t-s) + \varrho u_i^*(t-s) \dot{u}_i^*(t+s)] \\ = \varrho [\ddot{u}_i^*(t+s) u_i^*(t-s) - \ddot{u}_i^*(t-s) u_i^*(t+s)],$$

which, integrated with respect to  $(s, \mathbf{x}) \in (0, t) \times B$  and by using the initial conditions (4.17), gives

$$(5.3) \quad 2 \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv \\ = \int_0^t \int_B \varrho [\ddot{u}_i^*(t-s) u_i^*(t+s) - \ddot{u}_i^*(t+s) u_i^*(t-s)] dv ds, \quad t \geq 0.$$

Further, we use the basic equations (4.11), (4.13), (4.14), (4.17) and the boundary conditions (4.18) into (5.3) to obtain

$$(5.4) \quad 2 \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv \\ = \int_0^t \int_B [\dot{\alpha}^*(t-s) \varrho \eta^*(t+s) - \dot{\alpha}^*(t+s) \varrho \eta^*(t-s)] dv ds, \quad t \geq 0.$$

Now we integrate with respect to time variable the equation (4.12) to obtain

$$(5.5) \quad \varrho \eta^*(t) = -\frac{1}{T_0} \int_0^t q_{i,i}^*(s) ds,$$

which, when replaced into (5.4) and by using the divergence theorem and the boundary conditions (4.18), implies

$$(5.6) \quad 2 \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv \\ = \int_0^t \int_B \frac{1}{T_0} \left[ \dot{\beta}_i^*(t-s) \int_0^{t+s} q_i^*(z) dz - \dot{\beta}_i^*(t+s) \int_0^{t-s} q_i^*(z) dz \right] dv ds, \quad t \geq 0.$$

Finally, by replacing the equation (4.15) into relation (5.6), we are led to the identity (5.1) and the proof is complete.

## 6. Uniqueness result

In this section we use the identity (5.1) in order to prove that the unique solution of the initial boundary value problem  $\mathcal{P}_0^*$  is  $\mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\} = 0$ . That means we can establish the following uniqueness result.

**THEOREM 2.** *Suppose that  $\varrho > 0$ ,  $\tau_T > 0$ ,  $0 < \tau_q \leq \tau_\alpha$ ,  $K_{ij}$  is a positive semi-definite tensor and  $k_{ij}$  is a positive definite tensor. Moreover, we assume that  $\text{meas } \Sigma_3 \neq 0$  or  $a > 0$ . Then the initial boundary value problem  $\mathcal{P}$  has at most one solution.*

**Proof.** We use the notation (3.2) to write

$$(6.1) \quad \begin{aligned} & K_{ij}\beta_j''''(t-s)\dot{\beta}_i^*(t+s) - K_{ij}\beta_j''''(t+s)\dot{\beta}_i^*(t-s) \\ &= K_{ij}\beta_j''''(t-s)\beta_i'(t+s) - K_{ij}\beta_j''''(t+s)\beta_i'(t-s) \\ &\quad + \tau_q [K_{ij}\beta_j''''(t-s)\beta_i(t+s) - K_{ij}\beta_j''''(t+s)\beta_i(t-s)] \\ &\quad + \frac{1}{2}\tau_q^2 [K_{ij}\beta_j''''(t-s)\dot{\beta}_i(t+s) - K_{ij}\beta_j''''(t+s)\dot{\beta}_i(t-s)], \end{aligned}$$

and note that a straightforward calculation proves that

$$(6.2) \quad \begin{aligned} & K_{ij}\beta_j''''(t-s)\dot{\beta}_i^*(t+s) - K_{ij}\beta_j''''(t+s)\dot{\beta}_i^*(t-s) \\ &= \frac{\partial}{\partial s} \{ [K_{ij}\beta_j''''(t-s)\beta_i''(t+s) + K_{ij}\beta_j''''(t+s)\beta_i''(t-s)] \\ &\quad + \tau_q [K_{ij}\beta_j''''(t-s)\beta_i'(t+s) + K_{ij}\beta_j''''(t+s)\beta_i'(t-s) + K_{ij}\beta_j''(t-s)\beta_i''(t+s)] \\ &\quad + \frac{1}{2}\tau_q^2 [K_{ij}\beta_j''''(t-s)\beta_i(t+s) + K_{ij}\beta_j''''(t+s)\beta_i(t-s) \\ &\quad + K_{ij}\beta_j''(t-s)\beta_i'(t+s) + K_{ij}\beta_j''(t+s)\beta_i'(t-s)] \}. \end{aligned}$$

In a similar way, we get

$$(6.3) \quad \begin{aligned} & (k_{ij} + \tau_\alpha K_{ij})\beta_j''(t-s)\dot{\beta}_i^*(t+s) - (k_{ij} + \tau_\alpha K_{ij})\beta_j''(t+s)\dot{\beta}_i^*(t-s) \\ &= \frac{\partial}{\partial s} \{ (k_{ij} + \tau_\alpha K_{ij})\beta_j''(t-s)\beta_i''(t+s) + \tau_q [(k_{ij} + \tau_\alpha K_{ij})\beta_j''(t-s)\beta_i'(t+s) \\ &\quad + (k_{ij} + \tau_\alpha K_{ij})\beta_j''(t+s)\beta_i'(t-s)] + \frac{1}{2}\tau_q^2 [(k_{ij} + \tau_\alpha K_{ij})\beta_j''(t-s)\beta_i(t+s) \\ &\quad + (k_{ij} + \tau_\alpha K_{ij})\beta_j''(t+s)\beta_i(t-s) + (k_{ij} + \tau_\alpha K_{ij})\beta_j'(t-s)\beta_i'(t+s)] \}, \end{aligned}$$

and

$$(6.4) \quad \begin{aligned} & k_{ij}\beta_j'(t-s)\dot{\beta}_i^*(t+s) - k_{ij}\beta_j'(t+s)\dot{\beta}_i^*(t-s) \\ &= \frac{\partial}{\partial s} \{ \tau_q k_{ij}\beta_j'(t-s)\beta_i'(t+s) \\ &\quad + \frac{1}{2}\tau_q^2 [k_{ij}\beta_j'(t-s)\beta_i(t+s) + k_{ij}\beta_j'(t+s)\beta_i(t-s)] \}. \end{aligned}$$

We now substitute the relations (6.2) to (6.4) into the Lagrange identity (5.1) and take into account the zero initial conditions (4.17). Thus, we obtain

$$\begin{aligned}
 (6.5) \quad & 2 \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv + \frac{1}{T_0} \int_B [2K_{ij} \beta_j'''(t) \beta_i''(t) + (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i''(t)] dv \\
 & + \frac{\tau_q}{T_0} \int_B [2K_{ij} \beta_j'''(t) \beta_i'(t) + K_{ij} \beta_j''(t) \beta_i''(t) + 2(k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i'(t)] dv \\
 & + \frac{\tau_q^2}{2T_0} \int_B [2K_{ij} \beta_j'''(t) \beta_i(t) + 2K_{ij} \beta_j''(t) \beta_i'(t) + 2(k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i(t) \\
 & + (k_{ij} + \tau_\alpha K_{ij}) \beta_j'(t) \beta_i'(t)] dv + \frac{\tau_T \tau_q}{T_0} \int_B [k_{ij} \beta_j'(t) \beta_i'(t) + \tau_q k_{ij} \beta_j'(t) \beta_i(t)] dv = 0,
 \end{aligned}$$

for all  $t \geq 0$ . Furthermore, we can write it under the following form

$$\begin{aligned}
 (6.6) \quad & \frac{d}{dt} \left\{ \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv + \frac{1}{T_0} \int_B K_{ij} \beta_j'''(t) \beta_i'''(t) dv + \frac{\tau_q}{T_0} \int_B [2K_{ij} \beta_j'''(t) \beta_i''(t) \right. \\
 & \quad \left. + (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i''(t)] dv + \frac{\tau_q^2}{T_0} \int_B [K_{ij} \beta_j'''(t) \beta_i'(t) \right. \\
 & \quad \left. + (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i'(t)] dv + \frac{\tau_T \tau_q^2}{2T_0} \int_B k_{ij} \beta_j'(t) \beta_i'(t) dv \right\} \\
 & + \frac{1}{T_0} \int_B [(k_{ij} + \tau_\alpha K_{ij}) - \tau_q K_{ij}] \beta_j''(t) \beta_i''(t) dv \\
 & + \frac{\tau_q}{2T_0} \int_B [2\tau_T k_{ij} - \tau_q (k_{ij} + \tau_\alpha K_{ij})] \beta_j'(t) \beta_i'(t) dv = 0,
 \end{aligned}$$

so that, by means of an integration with respect to time variable and by using the initial conditions (4.17), we deduce

$$\begin{aligned}
 (6.7) \quad & \int_B \varrho u_i^*(t) \dot{u}_i^*(t) dv + \frac{1}{T_0} \int_B K_{ij} \beta_j'''(t) \beta_i'''(t) dv + \frac{\tau_q}{T_0} \int_B [2K_{ij} \beta_j'''(t) \beta_i''(t) \\
 & \quad + (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i''(t)] dv + \frac{\tau_q^2}{T_0} \int_B [K_{ij} \beta_j'''(t) \beta_i'(t) \\
 & \quad + (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i'(t)] dv + \frac{\tau_T \tau_q^2}{2T_0} \int_B k_{ij} \beta_j'(t) \beta_i'(t) dv
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{T_0} \int_0^t \int_B [(k_{ij} + \tau_\alpha K_{ij}) - \tau_q K_{ij}] \beta_j''(s) \beta_i''(s) dv ds \\
& + \frac{\tau_q}{2T_0} \int_0^t \int_B [2\tau_T k_{ij} - \tau_q (k_{ij} + \tau_\alpha K_{ij})] \beta_j'(s) \beta_i'(s) dv ds = 0.
\end{aligned}$$

On the other side, we note that

$$\begin{aligned}
(6.8) \quad & \frac{\tau_q}{T_0} \int_B 2K_{ij} \beta_j'''(t) \beta_i''(t) dv + \frac{\tau_q^2}{T_0} \int_B [K_{ij} \beta_j'''(t) \beta_i'(t) + (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i'(t)] dv \\
& = \frac{d}{dt} \left\{ \frac{\tau_q}{T_0} \int_B K_{ij} \beta_j'''(t) \beta_i'''(t) dv + \frac{\tau_q^2}{2T_0} \int_B (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(t) \beta_i''(t) dv \right\} \\
& + \frac{d^2}{dt^2} \left[ \frac{\tau_q^2}{2T_0} \int_B K_{ij} \beta_j'''(t) \beta_i'''(t) dv \right] - \frac{\tau_q^2}{T_0} \int_B K_{ij} \beta_j''(t) \beta_i''(t) dv,
\end{aligned}$$

so that the relation (6.7), after twice integrated with respect to time variable, implies

$$\begin{aligned}
(6.9) \quad & \int_0^t \int_0^s \int_B \varrho u_i^*(z) u_i^*(z) dv dz ds + \frac{1}{T_0} \int_0^t \int_0^s \int_B K_{ij} \beta_j'''(z) \beta_i'''(z) dv dz ds \\
& + \frac{\tau_q}{T_0} \int_0^t \int_B K_{ij} \beta_j'''(s) \beta_i'''(s) dv ds + \frac{\tau_q^2}{2T_0} \int_B K_{ij} \beta_j'''(t) \beta_i'''(t) dv \\
& + \frac{1}{T_0} \int_0^t \int_0^s \int_0^z \int_B [k_{ij} + (\tau_\alpha - \tau_q) K_{ij}] \beta_j''(r) \beta_i''(r) dv dr dz ds \\
& + \frac{\tau_q}{T_0} \int_0^t \int_0^s \int_B [k_{ij} + (\tau_\alpha - \tau_q) K_{ij}] \beta_j''(z) \beta_i''(z) dv dz ds \\
& + \frac{\tau_q^2}{2T_0} \int_0^t \int_B (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(s) \beta_i''(s) dv ds + \frac{\tau_T \tau_q^2}{2T_0} \int_0^t \int_0^s \int_B k_{ij} \beta_j'(z) \beta_i'(z) dv dz ds \\
& + \frac{\tau_q}{T_0} \int_0^t \int_0^s \int_0^z \int_B [\tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij})] \beta_j'(r) \beta_i'(r) dv dr dz ds = 0,
\end{aligned}$$

for all  $t \geq 0$ .

At this time, we can observe that our constitutive hypotheses make all the integral terms in (6.9) to be positive, excepting the last integral whose sign remains indefinite. On this basis we can conclude that

$$(6.10) \quad \frac{\tau_T \tau_q^2}{2T_0} \int_0^t \int_0^s \int_B k_{ij} \beta'_j(z) \beta'_i(z) dv dz ds$$

$$+ \frac{\tau_q}{T_0} \int_0^t \int_0^s \int_0^z \int_B \left[ \tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij}) \right] \beta'_j(r) \beta'_i(r) dv dr dz ds \leq 0, \quad t \geq 0,$$

and hence we are lead to the following Gronwall inequality

$$(6.11) \quad \Phi(t) \leq M \int_0^t \Phi(s) ds, \quad t \geq 0,$$

where

$$(6.12) \quad \Phi(t) = \int_0^t \int_0^s \int_B k_{ij} \beta'_j(z) \beta'_i(z) dv dz ds,$$

$$(6.13) \quad M = \frac{2}{\tau_T \tau_q} \sup_{\bar{B}} \left\{ \frac{([\tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij})][\tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij})])^{1/2}}{k_0(\mathbf{x})} \right\},$$

and  $k_0$  is related to the lowest eigenvalue of the positive definite tensor  $k_{ij}$ . By means of the Gronwall lemma, it follows from (6.11) that

$$(6.14) \quad \Phi(t) = \int_0^t \int_0^s \int_B k_{ij} \beta'_j(z) \beta'_i(z) dv dz ds = 0 \quad \text{for all } t \geq 0,$$

which implies that

$$(6.15) \quad \beta'_i(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \bar{B} \times [0, \infty),$$

and hence we have

$$(6.16) \quad \beta_i(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \bar{B} \times [0, \infty).$$

Then, the relation (6.9) becomes

$$(6.17) \quad \int_0^t \int_0^s \int_B \varrho u_i^*(z) u_i^*(z) dv dz ds = 0 \quad \text{for all } t \geq 0,$$

that implies

$$(6.18) \quad u_i^*(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty).$$

In view of the Lemma 3 we can conclude that

$$(6.19) \quad u_i(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty).$$

Let us first consider that  $meas \Sigma_3 \neq 0$ . Then the relations (6.16), (4.16)<sub>2</sub> and (4.18)<sub>3</sub> imply that

$$(6.20) \quad \alpha^*(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty),$$

and on the basis of the Lemma 3 we conclude that

$$(6.21) \quad \alpha(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty).$$

Let us now consider that  $a > 0$ . It follows from the relations (6.16) and (4.15) that

$$(6.22) \quad q_i^*(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty),$$

and then the equation (4.12) gives

$$(6.23) \quad \eta^*(\mathbf{x}, t) = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty).$$

Finally, the relations (4.14), (6.19) and (6.23) imply again that (6.21) holds true.

Concluding, we have obtained that

$$(6.24) \quad \mathcal{S}^* = \{u_i^*, \alpha^*, e_{ij}^*, \beta_j^*, t_{ij}^*, \eta^*, q_i^*\} = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty),$$

and then

$$(6.25) \quad \mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\} = 0 \quad \text{for all } (\mathbf{x}, t) \in \overline{B} \times [0, \infty).$$

This proves the uniqueness of solution of the initial boundary value problem  $\mathcal{P}$  and the proof is complete.

## 7. Continuous dependence of solutions of the problem $\mathcal{P}$ with respect to the given data

In this section we study the continuous dependence of solutions of the initial boundary value problem  $\mathcal{P}$  with respect to the initial data and with respect to the supply terms. To this aim, throughout this section, we consider that  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  is a solution of the initial boundary value problem

$\mathcal{P}$  corresponding to the given data  $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; 0, 0, 0, 0\}$ . With  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  we associate the following functional

$$\begin{aligned}
 (7.1) \quad \mathcal{E}(t) = & \\
 = & \frac{1}{2} \int_0^t \int_0^s \int_B \left[ \rho \frac{\partial u_i^*}{\partial z}(z) \frac{\partial u_i^*}{\partial z}(z) + C_{ijkl} e_{ij}^*(z) e_{kl}^*(z) + a \left( \frac{\partial \alpha^*}{\partial z}(z) \right)^2 \right] dv dz ds \\
 & + \int_B \frac{\tau_q^2}{4T_0} K_{ij} \beta_j''(t) \beta_i''(t) dv + \int_0^t \int_B \frac{\tau_q^2}{4T_0} (k_{ij} + \tau_\alpha K_{ij}) \beta_j'(s) \beta_i'(s) dv ds \\
 & + \int_0^t \int_B \frac{\tau_q}{2T_0} K_{ij} \beta_j''(s) \beta_i''(s) dv ds + \int_0^t \int_0^s \int_B \frac{1}{2T_0} K_{ij} \beta_j''(z) \beta_i''(z) dv dz ds \\
 & + \int_0^t \int_0^s \int_B \frac{1}{2T_0} \left[ (\tau_T + \tau_q) k_{ij} + \tau_q \left( \tau_\alpha - \frac{3}{2} \tau_q \right) K_{ij} \right] \beta_j'(z) \beta_i'(z) dv dz ds \\
 & + \int_0^t \int_0^s \int_B \frac{\tau_T \tau_q^2}{4T_0} k_{ij} \beta_j(z) \beta_i(z) dv dz ds \\
 & + \int_0^t \int_0^s \int_0^z \int_B \frac{1}{T_0} [k_{ij} + (\tau_\alpha - \tau_q) K_{ij}] \beta_j'(r) \beta_i'(r) dv dr dz ds \\
 & + \int_0^t \int_0^s \int_0^z \int_B \frac{\tau_q}{T_0} \left[ \tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij}) \right] \beta_j(r) \beta_i(r) dv dr dz ds, \quad t \geq 0.
 \end{aligned}$$

Concerning this functional we can establish the following conservation law.

LEMMA 4. *Suppose that  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  is a solution of the initial boundary value problem  $\mathcal{P}$  corresponding to the given data  $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; 0, 0, 0, 0\}$ . Then the following identity holds true*

$$(7.2) \quad \mathcal{E}(t) = \frac{t^2}{2} \mathcal{E}(0) + \int_0^t \int_0^s \int_0^z \int_B \left[ F_i(r) \frac{\partial u_i^*}{\partial r}(r) + \tilde{R}(r) \frac{\partial \alpha^*}{\partial r}(r) \right] dv dr dz ds, \quad t \geq 0,$$

where

$$(7.3) \quad \tilde{R}(\mathbf{x}, t) = R(\mathbf{x}, t) - \frac{1}{T_0} \left[ \frac{1}{2} \tau_q^2 q_{i,i}^0(\mathbf{x}) + t \left( \tau_q q_{i,i}^0(\mathbf{x}) + \tau_T (k_{ij}(\mathbf{x}) T_{,j}^0(\mathbf{x}))_{,i} + \frac{1}{2} \tau_q^2 \dot{q}_{i,i}^0(\mathbf{x}) \right) \right],$$

and  $F_i$  and  $R$  are defined by means of relation (4.10).

Throughout in the remaining of this section we assume the following constitutive assumptions:

- (H1)  $\varrho > 0$ ;  $a > 0$ ;  $C_{ijkl}$  is a positive semi-definite tensor;
- (H2)  $k_{ij}$  is a positive definite tensor;  $K_{ij}$  is a positive semi-definite tensor;
- (H3) the tensors  $\varkappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q) K_{ij}$  and  $\kappa_{ij} = \tau_T k_{ij} - \frac{\tau_q}{2} (k_{ij} + \tau_\alpha K_{ij})$  are positive semi-definite.

It can be easily seen the following result.

LEMMA 5. *Suppose that the hypotheses (H1)–(H3) hold true. Then  $\mathcal{E}(t)$  is a measure associated with the solution  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$ .*

THEOREM 3. *Suppose that the hypotheses (H1)–(H3) hold true. Let  $\mathcal{S} = \{u_i, \alpha, e_{ij}, \beta_j, t_{ij}, \eta, q_i\}$  be a solution of the initial boundary value problem  $\mathcal{P}$  corresponding to the given data  $\mathcal{D} = \{f_i, r; u_i^0, \dot{u}_i^0, T^0, q_i^0, \dot{q}_i^0; 0, 0, 0, 0\}$ . Then, for any fixed time  $S \in (0, \infty)$ , the following inequality holds true*

$$(7.4) \quad \sqrt{\mathcal{E}(t)} \leq S\sqrt{\mathcal{E}(0)} + \frac{1}{\sqrt{2}} \int_0^t G(s) ds, \quad t \in [0, S],$$

where

$$(7.5) \quad G(t) = \sqrt{\int_0^t \int_0^s \int_B \left[ \frac{1}{\varrho} F_i(z) F_i(z) + \frac{1}{a} \tilde{R}(z) \tilde{R}(z) \right] dv dz ds}.$$

**Proof.** By means of the Cauchy-Schwarz inequality, from the identity (7.2), we obtain

$$(7.6) \quad \mathcal{E}(t) \leq S^2 \mathcal{E}(0) + \int_0^t G(s) \sqrt{\int_0^s \int_0^z \int_B \left[ \varrho \frac{\partial u_i^*}{\partial r}(r) \frac{\partial u_i^*}{\partial r}(r) + a \left( \frac{\partial \alpha^*}{\partial r}(r) \right)^2 \right] dv dr dz ds},$$

for all  $t \in [0, S]$ . By using the constitutive hypotheses (H1)–(H3) and the relation (7.1), from (7.6) we deduce the following Gronwall type inequality

$$(7.7) \quad \mathcal{E}(t) \leq S^2 \mathcal{E}(0) + \int_0^t G(s) \sqrt{2\mathcal{E}(s)} ds, \quad t \in [0, S],$$

which, integrated as in DAFERMOS [28], furnishes the estimate (7.4) and the proof is complete.

## 8. Concluding remarks

We investigated the well posedness of the time differential three-phase-lag model of thermoelasticity as proposed by CHOUDHURI [14]. The results concluded from the above analysis can be summarized as follows:

- (1) Our uniqueness result is established under the following assumptions upon the delay times:  $\tau_T > 0$ ,  $0 < \tau_q \leq \tau_\alpha$ ;
- (2) When  $\tau_q = 0$  then the identity (6.9) becomes

$$(8.1) \quad \int_0^t \int_0^s \int_B \rho u_i^*(z) u_i^*(z) dv dz ds + \frac{1}{T_0} \int_0^t \int_0^s \int_B K_{ij} \beta_j'''(z) \beta_i'''(z) dv dz ds \\ + \frac{1}{T_0} \int_0^t \int_0^s \int_0^z \int_B (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(r) \beta_i''(r) dv dr dz ds = 0,$$

for all  $t \geq 0$ . This last identity implies the uniqueness result without any restrictions upon the other two delay times  $\tau_T$  and  $\tau_\alpha$ . This includes the uniqueness result when we have  $\tau_T = 0$  and  $\tau_\alpha = 0$ , that is for the model considered by GREEN and NAGHDI [23], [24].

- (3) Moreover, when  $\tau_q = 0$  we can replace the assumptions upon  $k_{ij}$  and  $K_{ij}$  with the hypotheses that  $K_{ij}$  to be a positive definite tensor and  $k_{ij}$  to be a positive semi-definite tensor.
- (4) It can be seen from the identity (6.9) that the uniqueness result remains valid when we replace the assumption  $0 < \tau_q \leq \tau_\alpha$  by the hypothesis that  $k_{ij} + (\tau_\alpha - \tau_q)K_{ij}$  to be a positive semi-definite tensor.
- (5) For the well known law of the Jeffrey's-type heat flux equation [25], [26], the terms containing  $K_{ij}$  and  $\tau_q^2$  are neglected and then the above identity (6.9) reduces to

$$(8.2) \quad \int_0^t \int_0^s \int_B \rho u_i^*(z) u_i^*(z) dv dz ds + \frac{1}{T_0} \int_0^t \int_0^s \int_0^z \int_B k_{ij} \beta_j''(r) \beta_i''(r) dv dr dz ds \\ + \frac{\tau_q}{T_0} \int_0^t \int_0^s \int_B k_{ij} \beta_j''(z) \beta_i''(z) dv dz ds \\ + \frac{\tau_q \tau_T}{T_0} \int_0^t \int_0^s \int_0^z \int_B k_{ij} \beta_j'(r) \beta_i'(r) dv dr dz ds = 0,$$

for all  $t \geq 0$ , and it implies obvious the uniqueness result without any restrictions upon the delay times  $\tau_q \geq 0$  and  $\tau_T \geq 0$ . In particular, for the

LORD-SHULMAN theory of heat conduction [27], when  $\tau_T = 0$  and  $\tau_q \geq 0$ , the uniqueness result ready follows from the identity (8.2).

- (6) When the term  $\tau_q^2$  is neglected into the constitutive law (1.5), that is we retain only the terms of first order in the Taylor's expansion of (1.4), we have the constitutive law

$$(8.3) \quad q_i(\mathbf{x}, t) + \tau_q \dot{q}_i(\mathbf{x}, t) \\ = -(k_{ij}(\mathbf{x}) + \tau_\alpha K_{ij}(\mathbf{x}))T_{,j}(\mathbf{x}, t) - \tau_T k_{ij}(\mathbf{x})\dot{T}_{,j}(\mathbf{x}, t) - K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x}, t).$$

In that case the identity (6.9) has to be replaced by the following one

$$(8.4) \quad \int_0^t \int_B \varrho u_i^*(s) u_i^*(s) dv ds + \frac{1}{T_0} \int_0^t \int_B K_{ij} \beta_j''(s) \beta_i''(s) dv ds \\ + \frac{\tau_q}{T_0} \int_B K_{ij} \beta_j''(t) \beta_i''(t) dv \\ + \frac{\tau_q \tau_T}{T_0} \int_0^t \int_0^s \int_B k_{ij} \beta_j(z) \beta_i(z) dv dz ds \\ + \frac{\tau_q}{T_0} \int_0^t \int_B [k_{ij} + \tau_\alpha K_{ij}] \beta_j'(s) \beta_i'(s) dv ds \\ + \frac{1}{T_0} \int_0^t \int_0^s \int_B [k_{ij} + (\tau_\alpha - \tau_q) K_{ij}] \beta_j'(z) \beta_i'(z) dv dz ds = 0,$$

where now

$$(8.5) \quad u_i^*(t) = u_i'(t) + \tau_q u_i(t).$$

Then it can be seen that the uniqueness result follows, provided we assume the following hypotheses:  $\varrho > 0$ ,  $\tau_T \geq 0$ ,  $\tau_q \geq 0$ ,  $\tau_\alpha \geq 0$ ,  $k_{ij}$  (or  $K_{ij}$ ) is a positive definite tensor and  $K_{ij}$  (or  $k_{ij}$ ) is a positive semi-definite tensor.

- (7) When we consider the Taylor series expansion of second order for the both members of the constitutive equation (1.4), that is the constitutive equation (1.5) is replaced by the following one

$$(8.6) \quad q_i(\mathbf{x}, t) + \tau_q \dot{q}_i(\mathbf{x}, t) + \frac{1}{2} \tau_q^2 \ddot{q}_i(\mathbf{x}, t) \\ = -K_{ij}(\mathbf{x}) \beta_j(\mathbf{x}, t) - (k_{ij}(\mathbf{x}) + \tau_\alpha K_{ij}(\mathbf{x})) \dot{\beta}_j(\mathbf{x}, t) \\ - \left( \tau_T k_{ij}(\mathbf{x}) + \frac{1}{2} \tau_\alpha^2 K_{ij}(\mathbf{x}) \right) \ddot{\beta}_j(\mathbf{x}, t) - \frac{1}{2} \tau_T^2 k_{ij}(\mathbf{x}) \ddot{\beta}_j(\mathbf{x}, t),$$

then the identity (6.9) becomes

$$\begin{aligned}
 (8.7) \quad & \int_0^t \int_0^s \int_B \varrho u_i^*(z) u_i^*(z) dv dz ds + \frac{1}{T_0} \int_0^t \int_0^s \int_B K_{ij} \beta_j'''(z) \beta_i'''(z) dv dz ds \\
 & + \frac{\tau_q}{T_0} \int_0^t \int_B K_{ij} \beta_j'''(s) \beta_i'''(s) dv ds + \frac{\tau_q^2}{2T_0} \int_B K_{ij} \beta_j'''(t) \beta_i'''(t) dv \\
 & + \frac{1}{T_0} \int_0^t \int_0^s \int_0^z \int_B [k_{ij} + (\tau_\alpha - \tau_q) K_{ij}] \beta_j''(r) \beta_i''(r) dv dr dz ds \\
 & + \frac{\tau_q}{T_0} \int_0^t \int_0^s \int_B [k_{ij} + (\tau_\alpha - \tau_q) K_{ij}] \beta_j''(z) \beta_i''(z) dv dz ds \\
 & + \frac{\tau_q^2}{2T_0} \int_0^t \int_B (k_{ij} + \tau_\alpha K_{ij}) \beta_j''(s) \beta_i''(s) dv ds \\
 & + \frac{\tau_q^2}{2T_0} \int_0^t \int_0^s \int_B \left( \tau_T k_{ij} + \frac{1}{2} \tau_\alpha^2 K_{ij} \right) \beta_j'(z) \beta_i'(z) dv dz ds \\
 & + \frac{1}{T_0} \int_0^t \int_0^s \int_0^z \int_B \left[ \left( \tau_q \tau_T - \frac{1}{2} \tau_T^2 - \frac{1}{2} \tau_q^2 \right) k_{ij} \right. \\
 & \quad \left. + \frac{1}{2} \tau_\alpha \tau_q (\tau_\alpha - \tau_q) K_{ij} \right] \beta_j'(r) \beta_i'(r) dv dr dz ds \\
 & + \frac{1}{4T_0} \tau_q^2 \tau_T^2 \int_0^t \int_0^s \int_0^z \int_B k_{ij} \beta_j(r) \beta_i(r) dv dr dz ds = 0.
 \end{aligned}$$

It can be easily seen that the uniqueness result follows, provided we assume the following hypotheses:  $\varrho > 0$ ,  $\tau_T \geq 0$ ,  $\tau_q \geq 0$ ,  $\tau_\alpha \geq 0$ ,  $k_{ij}$  (or  $K_{ij}$ ) is a positive definite tensor and  $K_{ij}$  (or  $k_{ij}$ ) is a positive semi-definite tensor, and  $k_{ij} + (\tau_\alpha - \tau_q) K_{ij}$  is a positive semi-definite tensor.

- (8) We established the estimate (7.4) describing the continuous dependence of solution of the initial boundary value problem  $\mathcal{P}$  with respect to the given data, provided the thermodynamic restrictions  $\varkappa_{ij} \xi_i \xi_j \geq 0$  and  $\kappa_{ij} \xi_i \xi_j \geq 0$ , for all  $\xi_i$ , are fulfilled. Analog estimates can be established for the models considered within the above two points.
- (9) The above considerations allow us to conclude that the time differential three-phase-lag models of thermoelasticity as proposed by CHOUDHURI [14] are well posed.

## Acknowledgments

The authors are very grateful to the anonymous reviewers for their valuable comments, which have led to an improvement of the present work.

## References

1. D.Y. TZOU, *A unified approach for heat conduction from macro to micro-scales*, ASME Journal of Heat Transfer, **117**, 8–16, 1995.
2. M. FABRIZIO, F. FRANCHI, *Delayed thermal models: Stability and thermodynamics*, Journal of Thermal Stresses, **37**, 160–173, 2014.
3. M. FABRIZIO, B. LAZZARI, *Stability and second law of thermodynamics in dual-phase-lag heat conduction*, International Journal of Heat and Mass Transfer, **74**, 484–489, 2014.
4. R. QUINTANILLA, *Exponential stability in the dual-phase-lag heat conduction theory*, Journal of Non-Equilibrium Thermodynamics, **27**, 217–227, 2002.
5. K. RAMADAN, *Semi-analytical solutions for the dual phase lag heat conduction in multi-layered media*, International Journal of Thermal Sciences, **48**, 14–25, 2009.
6. A.H. AKBARZADEH, Z.T. CHEN, *Transient heat conduction in a functionally graded cylindrical panel based on the dual phase lag theory*, International Journal of Thermophysics, **33**, 1100–1125, 2012.
7. N. AFRIN, Y. ZHANG, J.K. CHEN, *Dual-phase lag behavior of a gas-saturated porous-medium heated by a short-pulsed laser*, International Journal of Thermal Sciences, **75**, 21–27, 2014.
8. A.E. GREEN, P.M. NAGHDI, *On undamped heat waves in an elastic solid*, Journal of Thermal Stresses, **15**, 253–264, 1992.
9. A.E. GREEN, P.M. NAGHDI, *Thermoelasticity without energy dissipation*, Journal of Elasticity, **31**, 189–208, 1993.
10. A.E. GREEN, P.M. NAGHDI, *A unified procedure for construction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua*, Proceedings of the Royal Society London A, **448**, 335–356, 357–377, 379–388, 1995.
11. H. VON HELMHOLTZ, *Prinzipien der Statik monocyklischer Systeme*, Borchardt-Crelle's Journal für Die Reine Und Angewandte Mathematik, **97**, 111–140, 1884.
12. H. VON HELMHOLTZ, *Studien zur Statik monocyklischer Systeme*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, I, 159–177, 1884.
13. P. PODIO-GUIDUGLI, *A virtual power format for thermomechanics*, Continuum Mechanics and Thermodynamics, **20**, 479–487, 2009.
14. S.K. ROY CHOUDHURI, *On a thermoelastic three-phase-lag model*, Journal of Thermal Stresses, **30**, 231–238, 2007.
15. D.Y. TZOU, *Macro- to Microscale Heat Transfer: The Lagging Behavior*, John Wiley and Sons, Chichester, 2015.

16. R. QUINTANILLA, *A well posed problem for the dual-phase-lag heat conduction*, Journal of Thermal Stresses, **31**, 260–269, 2008.
17. R. QUINTANILLA, *A well-posed problem for the three-dual-phase-lag heat conduction*, Journal of Thermal Stresses, **32**, 1270–1278, 2009.
18. R. QUINTANILLA, R. RACKE, *A note on stability in three-phase-lag heat conduction*, International Journal of Heat and Mass Transfer, **58**, 24–29, 2008.
19. S. CHIRIȚĂ, C. D'APICE, V. ZAMPOLI, *The time differential three-phase-lag heat conduction model: Thermodynamic compatibility and continuous dependence*, International Journal of Heat and Mass Transfer, **102**, 226–232, 2016.
20. M. WANG, N. YANG, Z.Y. GUO, *Non-Fourier heat conduction in nanomaterials*, Journal of Applied Physics, **110**, 064310-1-7, 2011.
21. A.H. AKBARZADEH, D. PASINI, *Phase-lag heat conduction in multilayered cellular media with imperfect bonds*, International Journal of Heat and Mass Transfer, **75**, 656–667, 2014.
22. A.H. AKBARZADEH, J.W. FU, Z.T. CHEN, *Three-phase-lag heat conduction in a functionally graded hollow cylinder*, Transactions of the Canadian Society for Mechanical Engineering, **38**, 155–171, 2014.
23. A.E. GREEN, P.M. NAGHDI, *On undamped heat waves in an elastic solid*, Journal of Thermal Stresses, **15**, 253–264, 1992.
24. A.E. GREEN, P.M. NAGHDI, *A re-examination of the basic postulates of thermomechanics*, Proceedings of the Royal Society London A, **432**, 171–194, 1991.
25. D.D. JOSEPH, L. PREZIOSI, *Heat waves*, Reviews of Modern Physics, **61**, 41–73, 1989.
26. D.D. JOSEPH, L. PREZIOSI, *Addendum to the paper “Heat waves”*, Reviews of Modern Physics, **62**, 375–391, 1990.
27. H.W. LORD, Y.H. SHULMAN, *A generalized dynamical theory of thermoelasticity*, Journal of the Mechanics and Physics of Solids, **15**, 299–309, 1967.
28. C.M. DAFERMOS, *The second law of thermodynamics and stability*, Archive for Rational Mechanics and Analysis, **70**, 167–179, 1979.

Received February 17, 2016; revised version August 31, 2016.

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