

## Unified fractional derivative models of magneto-thermo-viscoelasticity theory

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A UNIFIED MATHEMATICAL MODEL of fractional magneto-thermo-viscoelasticity for isotropic perfectly conducting media involving fractional relaxation operator is given. Some essential theorems on the linear coupled and generalized theories of thermo-viscoelasticity can be easily obtained. The new fractional model is applied to a half-space subjected to two different forms of time-dependent thermal shock in, the presence of a transverse magnetic field. The Laplace transform techniques are used. Numerical computation is performed by using a numerical inversion technique and the resulting quantities are shown graphically. The effects of the fractional orders on viscoelastic material are discussed.

**Key words:** magneto-thermo-viscoelasticity, generalized theories of thermoelasticity, fractional relaxation operator, Laplace transforms, fractional calculus, numerical results.

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### Notations

$\lambda, \mu$  Lamé's constants,  
 $C_E$  specific heat at constant strains,  
 $K = \lambda + (2/3)\mu$ , bulk modulus,  
 $C_0^2 = K/\rho$ , longitudinal wave speed,  
 $\varepsilon_{ij}$  components of strain tensor,  
 $e_{ij}$  components of strain deviator tensor,  
 $\sigma_{ij}$  components of stress tensor,  
 $S_{ij}$  components of stress deviator tensor,  
 $e = \varepsilon_{ii}$ , dilatation,  
 $\mathbf{H}$  strength of the applied magnetic field,  
 $\mathbf{J}$  electric current density,  
 $\mathbf{B}$  magnetic induction vector,

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$\mathbf{D}$	electric induction vector,
$\mathbf{h}$	induced magnetic field,
$\mathbf{E}$	induced electric field,
$\mu_0$	magnetic permeability,
$\varepsilon_0$	electric permeability,
$\sigma_0$	electrical conductivity,
$k$	thermal conductivity,
$R(t, \beta)$	relaxation functions,
$T$	absolute temperature,
$u_i$	components of displacement vector,
$\alpha_T$	coefficient of linear thermal expansion,
$\gamma$	$= 3K\alpha_T$ ,
$\delta_{ij}$	Kronecker's delta,
$T_0$	reference temperature,
$\varepsilon$	$= \gamma^2 T_0 / (k\eta_0 \rho C_0^2)$ , thermal coupling parameter,
$\Theta$	$= T - T_0$ , such that $ \Theta/T_0  \ll 1$ ,
$\rho$	mass density,
$\tau, \nu$	two relaxation times,
$\tau_q$	phase lag of heat flux,
$\tau_\Theta$	phase lag of temperature gradient,
$\alpha, \beta$	fractional orders,
$t$	time,
$\Gamma(\cdot)$	gamma function,
$n, t_1, t_2$	constants.

## 1. Introduction

Due to the recent large-scale development and utilization of polymers and composite materials, the linear-viscoelasticity remains an important area of research. Linear viscoelastic materials are rheological materials that exhibit time-temperature rate-of-loading dependence. A general overview of time-dependent material properties has been presented by TSCHOEGL [1]. The mechanical model representation of linear viscoelastic behavior results was investigated by GROSS [2]. One can refer to ATKINSON and CRASTER [3] for a review of fracture mechanics and generalizations of the viscoelastic materials, and to RAJAGOPAL and SACCOMANDI for non-linear theory [4]. The solutions of some boundary value problems of thermo-viscoelasticity were investigated by ILIOUSHIN and POBEDRIA [5]. The works of BIOT [6], MORLAND and LEE [7], TANNER [8], and HUILGOL and PHAN-THIEN [9] have made great strides in the last decade in finding solutions to boundary value problems for linear viscoelastic materials including temperature variations in both quasi-static and dynamic problems.

The classical uncoupled theory [11] of thermoelasticity predicts two phenomena that are not compatible with the physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; second, the heat equation is of a parabolic type, predicting infinite speeds of propagation

for heat waves. The coupled theory of thermoelasticity (CTE) [12] was proposed to overcome the first shortcoming. The equations of elasticity and heat conduction for this theory are both coupled, which eliminates the first paradox of the classical uncoupled theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic.

Cattaneo's theory [13] allows for the existence of thermal waves which propagate at finite speeds. Starting from Maxwell's idea [14] and from the paper [13], an extensive amount of the literature [15–16] has contributed to the elimination of the paradox of instantaneous propagation of thermal disturbances. The approach used is known as extended irreversible thermodynamics, which introduces time derivative of the heat flux vector, and Cauchy stress tensor and its trace into the classical Fourier law by preserving the entropy principle. A history of heat conduction also appears in the review article [17], in which the authors discussed the low temperature heat propagation in dielectric solids in which second sound effects are present.

Several generalizations to the coupled theory of thermoelasticity are introduced. The mathematical aspects of LORD and SHULMAN'S [18] theory (LS) are explained and illustrated in detail in the work of IGNACZAK and OSTOJA-STARZEWSKI [19]. JOSEPH and PREZIOSI [20] stated that the Cattaneo heat conduction law [13] is the most obvious and the simplest generalization of the Fourier law that gives rise to a finite propagation speed. One can refer to IGNACZAK [21] and CHANDRASEKHARAI AH [22] for the review and presentation of generalized theories. HETNARSKI and IGNACZAK [23], in their survey article, examined five generalizations of the coupled theory and obtained a number of important analytical results. HETNARSKI and ESLAMI [24] introduced a unified generalized thermoelasticity theory and presented the advanced theory and applications of classical thermoelasticity, generalized thermoelasticity, and mathematical and mechanical background of thermodynamics and theory of elasticity as well. The uniqueness theorem for generalized thermoelasticity with one relaxation time under different conditions was proved by many researchers (e.g., see [25, 26]). The fundamental solution to this theory was obtained in [27]. EL-KARAMANY and EZZAT [28] introduced a formulation of the boundary integral equation method for generalized thermoviscoelasticity with one relaxation time. The propagation of discontinuities of solutions in this theory was investigated in [29, 30].

The second generalization of the coupled theory of elasticity is what is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. MÜLLER [31], in a review of the thermodynamics of thermoelastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by GREEN

and LAWS [32]. GREEN and LINDSAY obtained an explicit version of the constitutive equations in [33]. These equations were also obtained independently by SUHUBI [34]. They contain two constants that act as relaxation times and modify all the equations of the coupled theory, not only the heat equation. The classical Fourier law of heat conduction is not violated if the medium under consideration has a center of symmetry. SHERIEF [35] and EZZAT [36] obtained the fundamental solution for this theory. EZZAT and EL-KARAMANY [37] proved the uniqueness and reciprocity theorems for thermo-viscoelastic anisotropic media, and EL-KARAMANY and EZZAT [38] introduced a formulation of the boundary integral equation method for generalized thermo-viscoelasticity with two relaxation times. ROYCHOUDHURI and MUKHOPADHYAY [39] studied the propagation of harmonically time-dependent thermo-viscoelastic plane waves of assigned frequency in an infinite visco-elastic solid of Kelvin-Voigt type, when the entire medium rotates with a uniform angular velocity. EZZAT [40] introduced the model of the equations of generalized thermoviscoelasticity with one and two relaxation times, when the relaxation effects of the volume properties of the material are taken into account, respectively, and solved some problems by using state-space approach [41].

Another hyperbolic thermoelasticity theory was proposed by TZOU [42], where a dual-phase-lag (DPL) heat conduction law was proposed, in which two different phase-lags, one for the heat flux vector  $\tau_q$  and another for the temperature gradient  $\tau_\theta$ , have been introduced into the Fourier law to capture the microstructural effects of heat transport mechanism into the delayed response in time in the macroscopic formulation. The dual-phase-lag model, which reduces to Fourier's law in the limit of  $\tau_q - \tau_\theta \rightarrow 0$ , describes the process in which a temperature gradient that is established across the material volume at time  $t + \tau_\theta$  will not produce the thermal flux at point  $\mathbf{x}$  within that volume until the later time  $t + \tau_q$ . More information on the dual-phase-lag model can be found in ANTAKI [44], HORGAN and QUINTANILLA [45], JOU and CRIADO-SANCHO [46], and EL-KARAMANY and EZZAT [47, 48].

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. The most important advantage of using fractional differential equations in the above mentioned and other applications is their non-local property. It is well known that the integer-order differential operator is a local operator, however the fractional-order differential operator is non-local. This means that the next state of a system depends not only on its current state but also on all of its historical states. This is more realistic, and it is one reason why fractional calculus has become more and more popular. Although the tools of fractional calculus were available and applicable to various fields of study for some time, the investigation into the theory

of fractional differential equations began quite recently. The differential equations involving the Riemann–Liouville differential operators of fractional order  $0 < \alpha < 1$  appear to be important in modeling of several physical phenomena [49–51]; therefore, they seem to deserve an independent study of their theory, parallel to the well-known theory of ordinary differential equations. CAPUTO [52] found a good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity. One can refer to PODLUBNY for a survey of fractional calculus applications [53].

Recently, a considerable research effort has been expended to study the anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation presented by KIMMICH in [54]:

$$(1.1) \quad \rho C = -\lambda I^\alpha \nabla^2 C, \quad 0 < \alpha \leq 2,$$

where  $C$  is the concentration and the notion  $I^\alpha$  is the Riemann–Liouville fractional integral, which was introduced as a natural generalization of the well-known  $n$ -fold repeated integral  $I^n$  written in a convolution-type form [55]:

$$(1.2) \quad I^\alpha f(y, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi) f(\xi) d\xi, \quad I^0 f(t) = f(t), \quad 0 < \alpha \leq 2.$$

According to KIMMICH [54], Eq. (1.1) describes the different cases of diffusion where  $0 < \alpha < 1$  corresponds to weak diffusion (subdiffusion),  $\alpha = 1$  corresponds to normal diffusion,  $0 < \alpha < 2$  corresponds to strong diffusion (superdiffusion), and  $\alpha = 2$  corresponds to ballistic diffusion. It should be noted that the term diffusion is often used in a more generalized sense including various transport phenomena. Equation (1.1) is a mathematical model with a wide range of important physical phenomena; for example, the subdiffusive transport occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous and random media. Superdiffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, and in transport of organic molecules and atomic clusters on the surface. One might expect the anomalous heat conduction in the media in which the anomalous diffusion is observed.

FUJITA [56] considered the constitutive equation for the heat flux in the form:

$$(1.3) \quad q_i = -k I^{\alpha-1} \nabla T, \quad 1 < \alpha \leq 2.$$

POVSTENKO [57] used the Caputo heat wave equation to define the fractional heat conduction equation in the form:

$$(1.4) \quad q_i = -k I^{\alpha-1} \nabla T, \quad 0 < \alpha \leq 2.$$

SHERIEF *et al.* [58] introduced the formula of heat conduction as

$$(1.5) \quad q_i + \tau_0 \frac{\partial^\alpha q_i}{\partial t^\alpha} = -k \nabla T, \quad 0 < \alpha \leq 1.$$

And by taking into account the following consideration:

$$(1.6) \quad \frac{\partial^\alpha}{\partial t^\alpha} f(y, t) = \begin{cases} f(y, t) - f(y, 0) & \alpha \rightarrow 0, \\ I^{\alpha-1} \frac{\partial f(y, t)}{\partial t} & 0 < \alpha < 1, \\ \frac{\partial f(y, t)}{\partial t} & \alpha = 1, \end{cases}$$

they proved the uniqueness theorem and derived the reciprocity relation and the variational principle.

YOUSSEF [59] introduced another formula of heat conduction by taking into consideration (1.3)–(1.5):

$$(1.7) \quad q_i + \tau_0 \frac{\partial^\alpha q_i}{\partial t^\alpha} = -k I^{\alpha-1} \nabla T, \quad 0 < \alpha \leq 2.$$

Thus, the uniqueness theorem has been proved.

EZZAT [60–62] established a new model of fractional heat conduction equation using the Taylor–Riemann series expansion of time-fractional order which was developed by JUMARIE [62]. EL.-KARAMANY and EZZAT [63, 64] introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. Uniqueness and reciprocal theorems were proved and the convolutional variational principle was established and used to prove the uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors, except symmetry conditions. One can refer to EZZAT *at el.* [65–70] for a survey of fractional calculus applications in magneto-thermo-elasticity and viscoelasticity.

The purpose of the present research is to introduce the unified model for the linear theory of fractional thermo-viscoelasticity based on the Lord–Shulman theory, the Green–Lindsay theory, and the Tzou theory, as well as introduce the unified model to the coupled theory. We shall use the CAPUTO [52] definition of fractional derivatives of order  $\alpha \in (0, 1]$  of the absolutely continuous function  $f(t)$  given by

$$(1.8) \quad \frac{d^\alpha}{dt^\alpha} f(t) = I^{1-\alpha} f'(t),$$

where  $I^\beta$  is the fractional integral of the function  $f(t)$  of order  $\beta$  [55]. The resulting formulation is applied to one-dimensional problems for a half-space

of perfectly conducting electric solid. The bounding surface is assumed to be traction free and subjected to a time-dependent thermal shock in the presence of magnetic field. The Laplace transform technique is used. The inversion of the Laplace transforms is carried out using a numerical approach. Numerical results for the temperature, stress and displacement distributions are given and illustrated graphically for given problems further in this paper.

## 2. The mathematical model

The system of governing equations of the fractional linear magneto-thermo-viscoelasticity theory in a perfectly conducting medium consists of the following:

1. Maxwell's equations valid for slowly moving media:

$$(2.1) \quad \text{curl } \mathbf{h} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

$$(2.2) \quad \text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t},$$

$$(2.3) \quad \text{div } \mathbf{h} = 0, \quad \text{div } \mathbf{E} = 0,$$

$$(2.4) \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{h}), \quad \mathbf{D} = \varepsilon_0 \mathbf{E}.$$

These equations are supplemented by Ohm's law:

$$(2.5) \quad \mathbf{J} = \sigma_0[\mathbf{E} + \mu_0 \mathbf{V} \times \mathbf{H}].$$

2. The constitutive equation:

$$(2.6) \quad S_{ij}(\mathbf{x}, t) = \int_0^t R_\beta(t - \xi) \frac{\partial e_{ij}(\mathbf{x}, \xi)}{\partial \xi} d\xi = \hat{\mathbf{R}}_\beta(e_{ij}), \quad 0 < \beta \leq 1,$$

where  $R(t)$  is the relaxation modulus function such that  $R(\infty) > 0$ ,

$$(2.7) \quad \hat{\mathbf{R}}_\beta(f) = \frac{R_0 \tau^\beta}{\Gamma(1 - \beta)} \int_0^t (t - \xi)^{-\beta} \frac{\partial f(\mathbf{x}, \xi)}{\partial \xi} d\xi, \quad 0 < \beta \leq 1,$$

$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij}$ ,  $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ ,  $e_{ij} = \varepsilon_{ij} - \frac{e}{3} \delta_{ij}$ ,  $e = \varepsilon_{kk}$ ,  $\mathbf{x} = (x_1, x_2, x_3)$ , and

$$(2.8) \quad R(t, \beta) = \frac{R_0}{\Gamma(1 - \beta)} (t/\tau)^{-\beta}, \quad 0 < \beta \leq 1.$$

3. The stress-strain temperature relationship:

$$(2.9) \quad \sigma_{ij} = \hat{\mathbf{R}}_\beta \left( \varepsilon_{ij} - \frac{e}{3} \delta_{ij} \right) + K e \delta_{ij} - \gamma \left( 1 + \frac{v^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \Theta \delta_{ij}.$$

4. The equation of motion:

$$(2.10) \quad \rho \ddot{u}_i = \hat{\mathbf{R}}_\beta \left( \frac{1}{2} \nabla^2 u_i + \frac{1}{6} e_{,i} \right) + K e_{,i} - \gamma \left[ \left( 1 + \frac{v^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \Theta \right]_{,i} + \mu_0 (\mathbf{J} \wedge \mathbf{H})_i.$$

5. The heat equation:

$$(2.11) \quad k \left( 1 + \frac{\tau_\theta^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \Theta_{,ii} = \rho C_E \frac{\partial}{\partial t} \left( 1 + \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \right) \Theta + \gamma T_0 \frac{\partial}{\partial t} \left( 1 + n \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \right) e.$$

Equations (2.9)–(2.11) represent the unified model of fractional magneto-thermo-viscoelasticity involving fractional relaxation operator.

#### Limiting cases

##### *Coupled thermoelasticity theory*

1. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = \tau = v = n = 0$ ,  $\alpha = 0$ ,  $R_0 = 2\mu$ , transforms to the one given by Biot (CTE) in [12].

##### *Coupled thermo-viscoelasticity theory*

2. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = \tau = v = n = \alpha = \beta = 0$ , transforms to the one given by GROSS [2] and ATKINSON and CRASTER [3].

3. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = \tau = v = n = \alpha = 0$ ,  $\beta = 1$ , transforms to the one presented by of KOLTUNOV [10].

##### *Generalized thermoelasticity theory*

4. The model of equations (2.9)–(2.11) in the limiting case,  $\alpha \rightarrow 1$ ,  $\tau_\theta = \tau_q = v = 0$ ,  $\tau > 0$ ,  $n = 1$ ,  $R_0 = 2\mu$ , transforms to the one given by LORD and SHULMAN (LS) [18], GLASS and VICK [15], JOSEPH and PREZIOSI [20], IGNACZAK [21], and SHERIEF [27] for thermoelasticity with one relaxation time.

5. The model of equations (2.9)–(2.11) in the limiting case,  $\alpha \rightarrow 1$ ,  $\tau_q = \tau_\theta = n = 0$ ,  $v \geq \tau > 0$ ,  $R_0 = 2\mu$ , transforms to the one given by GREEN and LINDSAY (GL) [34] and EZZAT [36] for thermoelasticity with two relaxation times.

6. The model of equations (2.9)–(2.11) in the limiting case,  $\alpha \rightarrow 1$ ,  $\tau_\theta > \tau_q > 0$ ,  $\tau = \tau_q$ ,  $v = 0$ ,  $n = 1$ ,  $R_0 = 2\mu$ , transforms to the one presented by TZOU (DFL) [42], QUINTANILLA and RACKE [43], ABBAS and ZENKOUR [47], and EL-KARAMANY and EZZAT [46] for dual-phase-lag thermoelasticity.

##### *Generalized thermo-viscoelasticity theory*

7. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = v = \alpha = \beta = 0$ ,  $\tau > 0$ ,  $n = 1$ , transforms to the one presented by EL-KARAMANY and EZZAT [28, 29].

8. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = \alpha = \beta = n = 0$ ,  $v \geq \tau > 0$ , transforms to the one given by EZZAT [36], EZZAT and EL-KARAMANY [37], EL-KARAMANY and EZZAT [38], and ROYCHOUDHURI and MUKHOPADHYAY.

9. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = \alpha = v = 0$ ,  $n = \beta = 1$ ,  $\tau > 0$ , transforms to the one by EL-KARAMANY and EZZAT [29].

10. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = \alpha = n = 0$ ,  $\beta = 1$ ,  $v \geq \tau > 0$ , transforms to the one by EZZAT [40].

11. The model of equations (2.9)–(2.11) in the limiting case,  $\alpha \rightarrow 1$ ,  $\tau_\theta > \tau_q > 0$ ,  $\tau = \tau_q$ ,  $\beta = v = 0$ ,  $n = 1$  transforms to the one presented by EL-KARAMANY and EZZAT [29].

*Fractional thermoelasticity theory*

12. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = v = 0$ ,  $n = 1$ ,  $\tau_q = \tau > 0$ ,  $1 > \alpha > 0$ ,  $R_0 = 2\mu$ , transforms to the one by of SHERIEF *et al.* [58], EZZAT [60–62], EL-KARAMANY, EZZAT [63, 64] and EZZAT *et al.* [67, 70].

13. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta > \tau_q > 0$ ,  $\tau = \tau_q$ ,  $\beta = v = 0$ ,  $n = 1$ ,  $1 > \alpha > 0$ ,  $R_0 = 2\mu$ , transforms to the one by EZZAT *et al.* [70].

14. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = 0$ ,  $n = 0$ ,  $v \geq \tau > 0$ ,  $1 > \alpha > 0$ ,  $R_0 = 2\mu$ , transforms to the one by HAMZA [71].

*Fractional thermo-viscoelasticity theory*

15. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = v = \beta = 0$ ,  $n = 1$ ,  $\tau > 0$ ,  $1 > \alpha > 0$ , transforms to the one by EZZAT and EL-KARAMANY [72].

16. The model of equations (2.9)–(2.11) in the limiting case,  $\tau_\theta = \tau_q = v = 0$ ,  $n = 1$ ,  $\tau > 0$ ,  $1 > \beta > 0$ ,  $1 > \alpha > 0$ , transforms to the one by EZZAT *et al.* [73].

### 3. The analytical solutions in the Laplace-transform domain

Now, we shall consider an infinite homogeneous isotropic perfectly conducting thermo-viscoelastic medium permeated by the initial magnetic field  $\mathbf{H} \equiv (0, 0, H_0)$  occupying the region  $x \geq 0$ , which is initially quiescent and where all the state functions depend only on the dimension  $x$  and the time  $t$ . The  $x$ -axis is taken perpendicular to the bounding plane pointing inwards. Due to the effect of this magnetic field, the induced magnetic field  $\mathbf{h} \equiv (0, 0, h)$  and the induced electric field  $\mathbf{E} \equiv (0, E, 0)$  appear in the conducting medium. Also, the force  $\mathbf{F}$  (the Lorentz force) appears. Due to the effect of this force, the points on the medium undergo the displacement  $\mathbf{u} \equiv (u, 0, 0)$ , which increases temperature. The sys-

tem of fractional magneto-thermo-viscoelasticity with two relaxation times and fractional relaxation operator for a medium with a perfect electric conductivity can be written [47] as follows.

The displacement vector has the following components:

$$(3.1) \quad u_x = u(x, t), \quad u_y = u_z = 0.$$

The strain component takes the form:

$$(3.2) \quad e = \varepsilon_{xx} = \frac{\partial u}{\partial x}.$$

The linearized equations of electromagnetism for a perfectly conducting medium are:

$$(3.3) \quad J = -\left(\frac{\partial h}{\partial x} + \varepsilon_0 \mu_0 H_0 \frac{\partial^2 u}{\partial t^2}\right),$$

$$(3.4) \quad h = -H_0 \frac{\partial u}{\partial x},$$

$$(3.5) \quad E = \mu_0 H_0 \frac{\partial u}{\partial t}.$$

The equation of motion takes the form:

$$(3.6) \quad (\rho + \varepsilon_0 \mu_0^2 H_0^2) \frac{\partial^2 u}{\partial t^2} = \left(\frac{2}{3} \hat{\mathbf{R}}_\beta + K\right) \frac{\partial^2 u}{\partial x^2} + \mu_0 H_0^2 \frac{\partial^2 u}{\partial x^2} \\ - \gamma \frac{\partial}{\partial x} \left(1 + \frac{v^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \Theta, \quad 0 < \beta \leq 1, 0 < \alpha \leq 1.$$

The constitutive equation yields:

$$(3.7) \quad \sigma = \left(\frac{2}{3} \hat{\mathbf{R}}_\beta + K\right) \frac{\partial u}{\partial x} - \gamma \left(\Theta + \frac{v^\alpha}{\alpha!} \frac{\partial^\alpha \Theta}{\partial t^\alpha}\right), \quad 0 < \beta \leq 1, 0 < \alpha \leq 1.$$

The heat conduction equation is given by:

$$(3.8) \quad k \left(1 + \frac{\tau_\theta^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial^2 \Theta}{\partial x^2} = \rho C_E \frac{\partial}{\partial t} \left(1 + \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) \Theta \\ + \gamma T_0 \frac{\partial}{\partial t} \left(1 + n \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) e, \quad 0 < \alpha \leq 1.$$

The previous equations constitute a complete system of fractional magneto-thermo-viscoelasticity involving fractional relaxation operator in the absence of heat sources in perfectly conducting medium.

Let us introduce the following non-dimensional variables:

$$\begin{aligned}
 x^* &= C_0 \eta_0 x, \quad u^* = C_0 \eta_0 u, \quad t^* = C_0^2 \eta_0 t, \\
 \tau^* &= C_0^2 \eta_0 \tau, \quad v^* = C_0^2 \eta_0 v, \quad \varepsilon = \frac{\delta_0 \gamma}{\rho C_E}, \quad \Theta^* = \frac{\gamma \Theta}{\rho C_0^2}, \\
 C_0^2 &= \frac{K}{\rho}, \quad \eta_0 = \frac{\rho C_E}{K}, \quad \sigma^* = \frac{\sigma}{K}, \quad R_0^* = \frac{2}{3K} R_0, \quad h^* = \frac{h}{H_0}, \quad E^* = \frac{E}{\mu_0 H_0 C_0}.
 \end{aligned}$$

In terms of these non-dimensional variables, we have (dropping asterisks for convenience):

$$(3.9) \quad h = -\frac{\partial u}{\partial x},$$

$$(3.10) \quad E = \frac{\partial u}{\partial t},$$

$$\begin{aligned}
 (3.11) \quad \left(1 + \frac{\tau_\theta^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial^2 \Theta}{\partial x^2} &= \frac{\partial}{\partial t} \left(1 + \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) \Theta \\
 &+ \varepsilon \frac{\partial^2}{\partial x \partial t} \left(1 + n \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) u, \quad 0 < \alpha \leq 1,
 \end{aligned}$$

$$\begin{aligned}
 (3.12) \quad a \frac{\partial^2 u}{\partial t^2} &= b \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} \left(1 + \frac{v^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \Theta \\
 &+ \frac{R_0 \tau^\beta}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} \frac{\partial}{\partial \xi} \left(\frac{\partial^2 u(x, \xi)}{\partial x^2}\right) d\xi, \quad 0 < \beta \leq 1, 0 < \alpha \leq 1,
 \end{aligned}$$

$$\begin{aligned}
 (3.13) \quad \sigma &= \frac{\partial u}{\partial x} - \left(1 + \frac{v^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \Theta + \frac{R_0 \tau^\beta}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} \frac{\partial}{\partial \xi} \left(\frac{\partial u(x, \xi)}{\partial x}\right) d\xi, \\
 &0 < \beta \leq 1, 0 < \alpha \leq 1,
 \end{aligned}$$

where  $a = 1 + (a_0/c)^2$ ,  $a_0 = [\mu_0 H_0^2 / \rho]^{1/2}$  is the Alfvén velocity,  $c = 1/\sqrt{\varepsilon_0 \mu_0}$  is the light speed and  $b = 1 + a_0^2$ .

We assume that the boundary conditions have the form:

$$(3.14) \quad \Theta(0, t) = f(t), \quad \Theta(\infty, t) = 0, \quad \sigma(0, t) = \sigma(\infty, t) = 0,$$

and the initial state of the medium is quiescent, i.e.,

$$(3.15) \quad u(x, 0) = \dot{u}(x, 0) = \sigma(x, 0) = \dot{\sigma}(x, 0) = \Theta(x, 0) = \dot{\Theta}(x, 0) = 0.$$

Using the Laplace transform, defined by the following relationship:

$$\bar{g}(s) = \int_0^\infty e^{-st} g(t) dt,$$

on both sides Eqs. (3.6)–(3.13), we obtain

$$(3.16) \quad \bar{h} = -D\bar{u},$$

$$(3.17) \quad \bar{E} = s\bar{u},$$

$$(3.18) \quad (D^2 - c)\bar{\Theta} = \varepsilon ds D\bar{u},$$

$$(3.19) \quad (D^2 - a\varpi s^2)\bar{u} = \varpi m D\bar{\Theta},$$

$$(3.20) \quad \bar{\sigma} = \frac{1}{\varpi} D\bar{u} - m\bar{\Theta},$$

where

$$\begin{aligned} c &= s \left( 1 + \frac{\tau^\alpha}{\alpha!} s^\alpha + \frac{\tau_q^{2\alpha}}{2\alpha!} s^{2\alpha} \right) / p, \\ d &= \left( 1 + n \frac{\tau^\alpha}{\alpha!} s^\alpha + \frac{\tau_q^{2\alpha}}{2\alpha!} s^{2\alpha} \right) / p, \quad p = 1 + \frac{\tau_\theta^\alpha}{\alpha!} s^\alpha, \\ m &= \left( 1 + \frac{v^\alpha}{\alpha!} s^\alpha \right), \quad \varpi = \frac{1}{(b + M_\beta s^\beta)}, \quad M_\beta = R_0 \tau^\beta, \end{aligned}$$

since  $L\{t^{-\beta}\} = \Gamma(1-\beta)/s^{1-\beta}$ , the Laplace transform of the relaxation modulus can be written in the form:

$$(3.21) \quad L\{R(t, \beta)\} = R_0 (s\tau)^\beta \left( \frac{1}{s} \right), \quad 0 < \beta \leq 1.$$

And the boundary conditions take the form:

$$(3.22) \quad \bar{\Theta}(0, s) = \bar{f}(s), \quad \bar{\sigma}(0, s) = 0.$$

Eliminating  $\bar{\Theta}$  in Eqs. (3.18) and (3.19), we get

$$(3.23) \quad \{D^4 - [a\varpi s^2 + c + \varepsilon\varpi s dm]D^2 + ca\varpi s^2\}\bar{u} = 0.$$

The general solution of Eq. (3.23), which is bounded at infinity, can be written as

$$(3.24) \quad \bar{u} = -k_1 C_1 e^{-k_1 x} - k_2 C_2 e^{-k_2 x},$$

where  $C_1$  and  $C_2$  are the parameters depending on  $s$  only, and  $k_1$  and  $k_2$  are the roots with positive real parts of the characteristic equation

$$(3.25) \quad k^4 - [a\varpi s^2 + c + \varepsilon\varpi ds n]k^2 + a\varpi s^2 c = 0.$$

From Eqs. (3.25) and (3.30), we get

$$(3.26) \quad \bar{\Theta} = \frac{1}{m\varpi} [C_1(k_1^2 - a\varpi s^2)e^{-k_1 x} + C_2(k_2^2 - a\varpi s^2)e^{-k_2 x}].$$

Substituting Eqs. (3.24) and (3.26) into Eq. (3.20), we obtain

$$(3.27) \quad \bar{\sigma} = \frac{as^2}{\varpi} (C_1 e^{-k_1 x} + C_2 e^{-k_2 x}).$$

In order to determine  $C_1$  and  $C_2$  we shall use the Laplace transform of the boundary conditions (3.30) to obtain

$$C_1 = -C_2 = \frac{m\varpi}{k_1^2 - k_2^2} \bar{f}(s).$$

Equations (3.16)–(3.20) become

$$(3.28) \quad \bar{u}(x, s) = \frac{-m\varpi(k_1 e^{-k_1 x} - k_2 e^{-k_2 x})}{k_1^2 - k_2^2} \bar{f}(s),$$

$$(3.29) \quad \bar{\Theta}(x, s) = \frac{[(k_1^2 - a\varpi s^2)e^{-k_1 x} - (k_2^2 - a\varpi s^2)e^{-k_2 x}]}{k_1^2 - k_2^2} \bar{f}(s),$$

$$(3.30) \quad \bar{\sigma}(x, s) = \frac{ams^2(e^{-k_1 x} - e^{-k_2 x})}{k_1^2 - k_2^2} \bar{f}(s),$$

$$(3.31) \quad \bar{h}(x, s) = \frac{m\varpi(k_1^2 e^{-k_1 x} - k_2^2 e^{-k_2 x})}{k_1^2 - k_2^2} \bar{f}(s),$$

$$(3.32) \quad \bar{E}(x, s) = \frac{-m\varpi s(k_1 e^{-k_1 x} - k_2 e^{-k_2 x})}{k_1^2 - k_2^2} \bar{f}(s).$$

These complete the solutions in the Laplace transform domain.

#### 4. Inversion of the Laplace transforms

We shall now outline the method used to invert the Laplace transforms in the above equations. Let  $\bar{f}(s)$  be the Laplace transform of a function  $f(t)$ . The inversion formula for Laplace transforms can be written as HONIG and HIRDES presented in [76]

$$f(t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{ity} \bar{f}(d + iy) dy,$$

where  $d$  is an arbitrary real number greater than all the real parts of the singularities of  $\bar{f}(s)$ .

Expanding the function  $h(t) = \exp(-dt)f(t)$  into a Fourier series in the interval  $[0, 2L]$ , we obtain the approximate formula presented in [76]

$$(4.1) \quad f(t) \approx f_N(t) = \frac{1}{2}c_0 + \sum_{k=1}^N c_k, \quad \text{for } 0 \leq t \leq 2L,$$

where

$$(4.2) \quad c_k = \frac{e^{dt}}{L} \operatorname{Re}[e^{ik\pi t/L} \bar{f}(d + ik\pi/L)].$$

Two methods are used to reduce the total error. First, the ‘Korrektur’ method is used to reduce the discretization error. Next, the  $\varepsilon$ -algorithm is used to reduce the truncation error and therefore to accelerate convergence.

The Korrektur method uses the following formula to evaluate the function  $f(t)$ :

$$(4.3) \quad f(t) = f_{NK}(t) = f_N(t) - e^{-2dL} f'_N(2L + t).$$

We shall now describe the  $\varepsilon$ -algorithm that is used to accelerate the convergence of the series in (4.1). Let  $N$  be an odd natural number and let  $s_m = \sum_{k=1}^m c_k$  be the sequence of partial sums of (4.1). We define the  $\varepsilon$ -sequence as

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m, \quad m = 1, 2, \dots$$

and  $\varepsilon_{n+1,m} = \varepsilon_{n-1,m+1} + 1/(\varepsilon_{n,m+1} - \varepsilon_{n,m})$ ,  $n, m = 1, 2, \dots$

It is shown in [76] that the sequence  $\varepsilon_{1,1}, \varepsilon_{3,1}, \dots, \varepsilon_{N,1}, \dots$  converges to  $f(t) - c_0/2$  faster than the sequence of partial sums.

## 5. Numerical results and discussion

In this section, we aim to illustrate the numerical results of the analytical expressions obtained in the previous section and explain the influence of fractional orders and relaxation times on the behavior of the field quantities. In order to interpret the numerical computations, we consider material properties of a polymethyl methacrylate (Plexiglas) material. The following values of physical constants are shown in Table 1 [77].

**Table 1.**

$\rho = 1.2 \times 10^3 \text{ kg/m}^3$	$k = 0.55 \text{ J/m} \cdot \text{sec} \cdot \text{K}$	$E = 525 \times 10^7 \text{ N/m}^2$
$C_E = 1.4 \times 10^3 \text{ J/kg} \cdot \text{K}$	$\lambda = 453.7 \times 10^7 \text{ N/m}^2$	$\mu = 194 \times 10^7 \text{ N/m}^2$
$\gamma = 210 \times 10^4 \text{ N/m}^2 \cdot \text{K}$	$\eta_0 = 3.36 \times 10^6 \text{ sec/m}^2$	$C_0 = 2200 \text{ m/sec}$
$H_0 = 1 \text{ Tesla}$	$T_0 = 293 \text{ K}$	$\alpha_T = 13 \times 10^{-5} \text{ K}^{-1}$
$\varepsilon = 0.12$	$a = 1.005$	$b = 1.01$

The calculations were carried out for two cases of arbitrary function  $f(t)$ , as follows:

Case 1: Periodically varying heat sources:

$$f(t) = \begin{cases} \sin\left(\frac{\pi t}{\ell}\right) & 0 \leq t \leq \ell \\ 0 & \text{otherwise.} \end{cases} \quad \text{or} \quad \bar{f}(s) = \frac{\pi\ell(1 + e^{-\ell s})}{\ell^2 s^2 + \pi^2},$$

Case 2: A thermal shock problem subjected to a ramp-type heating:

$$f(t) = \begin{cases} 0 & 0 \leq t, \\ \Theta_0 \frac{t}{t_0} & 0 \leq t \leq t_0, \\ \Theta_0 & t > 0 \end{cases} \quad \text{or} \quad \bar{f}(s) = \frac{\Theta_0(1 - e^{-st_0})}{t_0 s^2},$$

where  $t_0$  indicates the length of time to the heat.

For each case, we apply the following procedure:

The computations are carried out for one value of time:  $t = 0.1$ , two different values of relaxation times:  $\tau = 0.02$  and  $\nu = 0.04$ , and the orders of relaxation operator are  $\beta = 0.0, 0.5$  and  $1.0$ , while the differential fractional orders are  $\alpha = 0.0, 1.0, 0.5$ . The temperature, stress and displacement distributions are obtained and plotted. Case 1 is shown in Figs. 1–3, while Case 2 is shown in Figs. 4–8. In the first group of figures, the solid lines represent the solution obtained in the frame of the new unified model of magneto-thermo-viscoelasticity

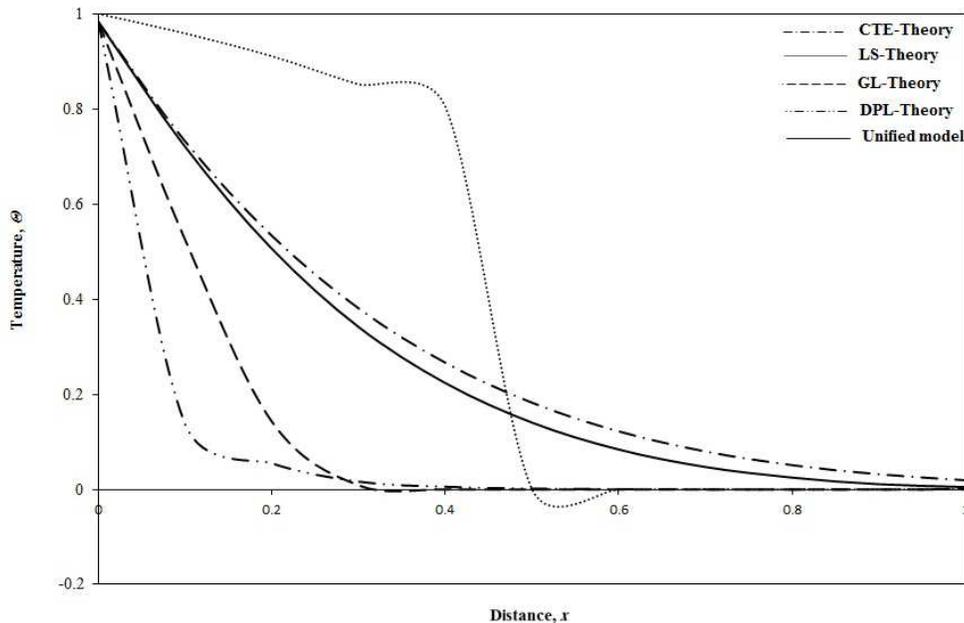


FIG. 1. The variation of temperature for different theories for  $\alpha = \beta = 0.5$ .

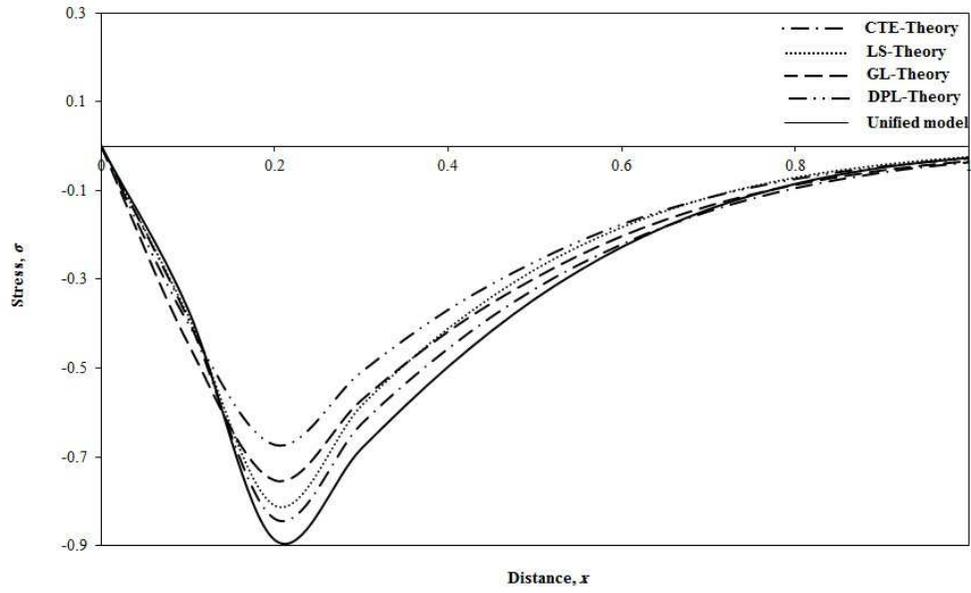


FIG. 2. The variation of stress for different theories for  $\alpha = \beta = 0.5$ .

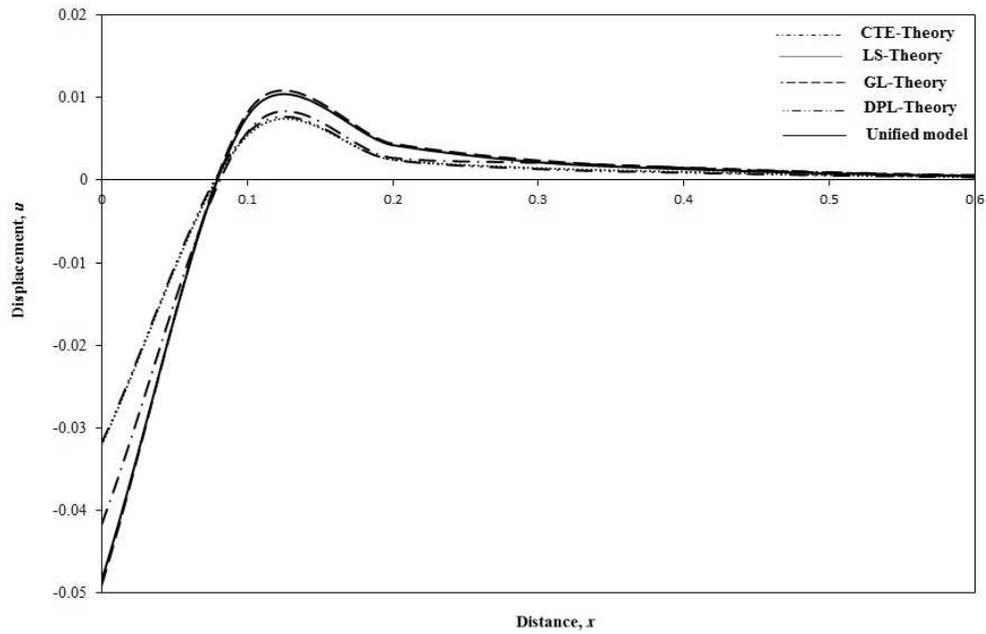


FIG. 3. The variation of displacement for different theories for  $\alpha = \beta = 0.5$ .

and other lines represent the different theories. The effects of the Alfvén velocity and ramping parameter on all fields are shown in the second group of

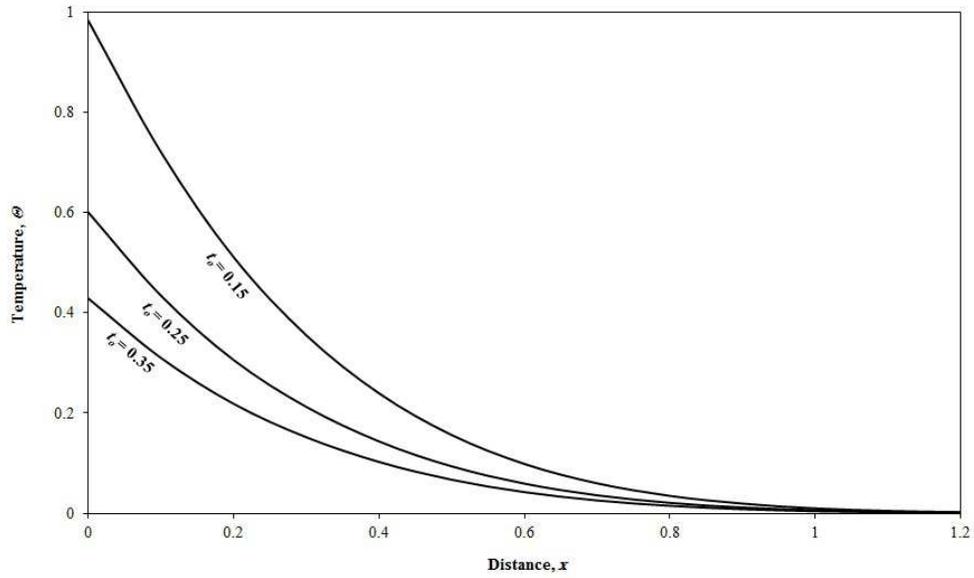


FIG. 4. The variation of temperature for different values of ramping parameter  $t_0$ .

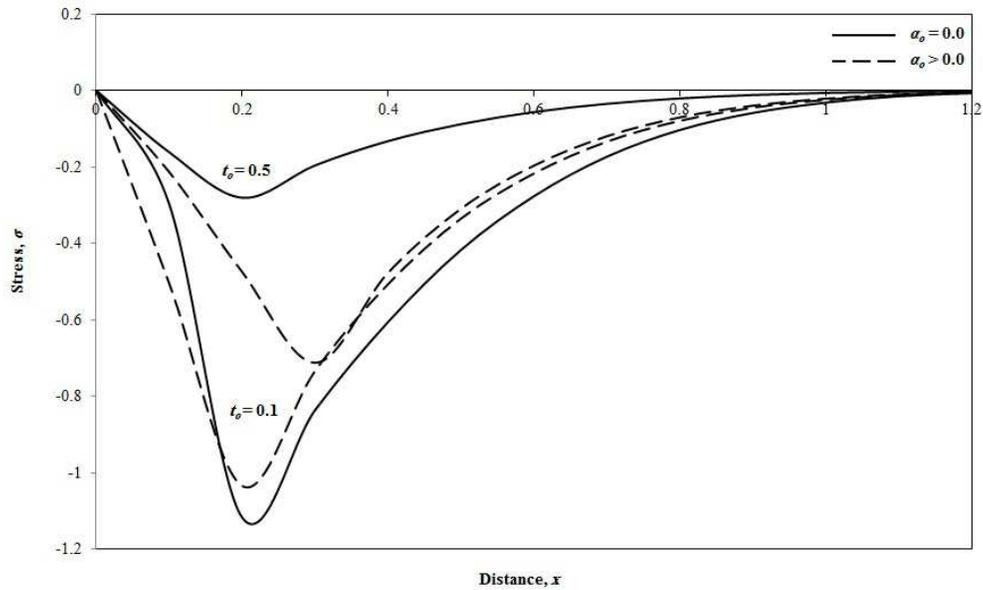


FIG. 5. The variation of stress for different values of Alfvén velocity  $\alpha_0$  and ramping parameter  $t_0$ .

figures. The numerical results are obtained with the help of Mathematica software (Version 6). Subsequently, a comparative study of analytical and numerical

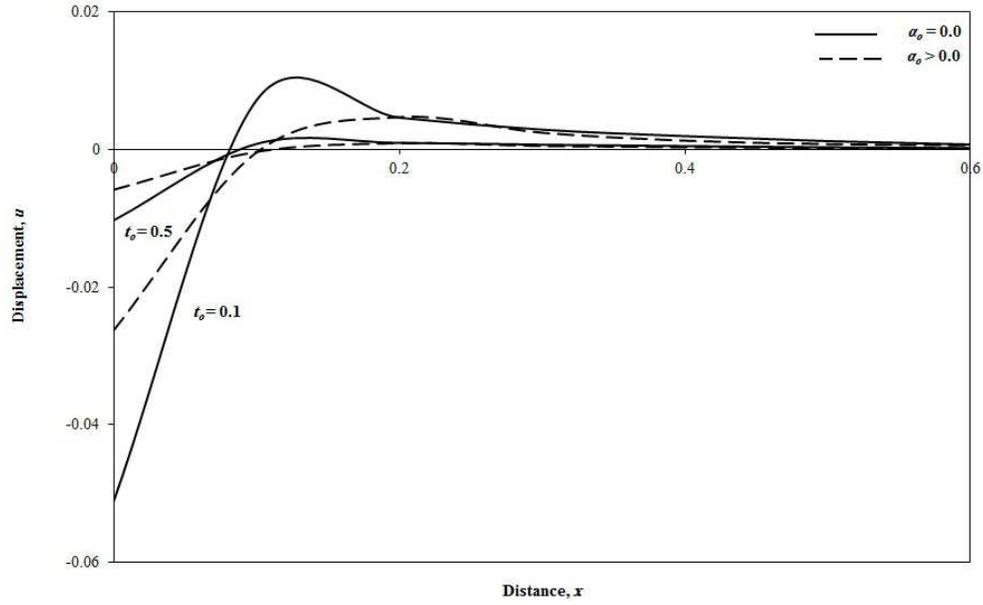


FIG. 6. The variation of displacement for different values of Alfvén velocity  $\alpha_0$  and ramping parameter  $t_0$ .

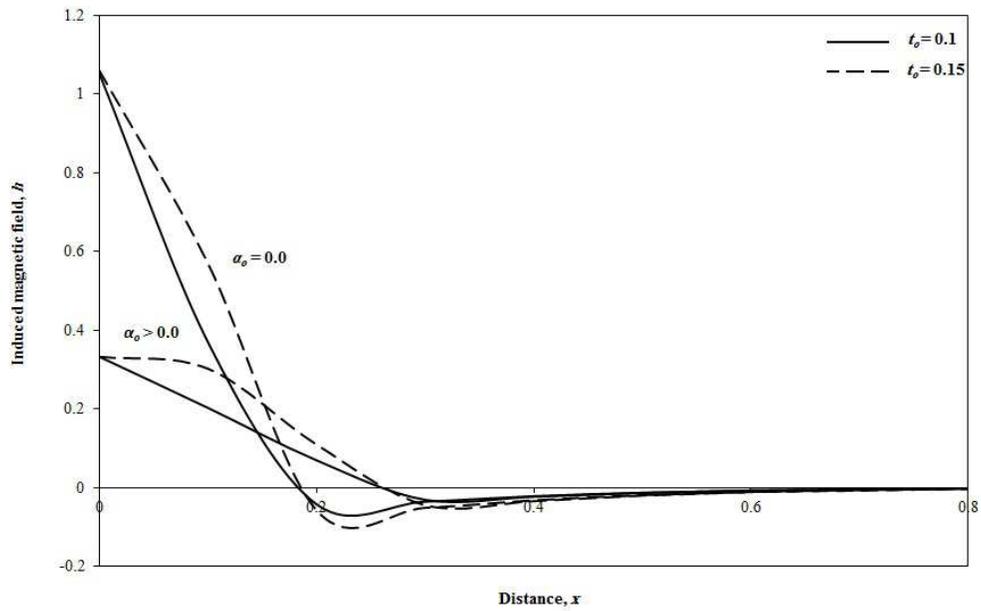


FIG. 7. The variation of induced magnetic field for different values of Alfvén velocity  $\alpha_0$  and ramping parameter  $t_0$ .

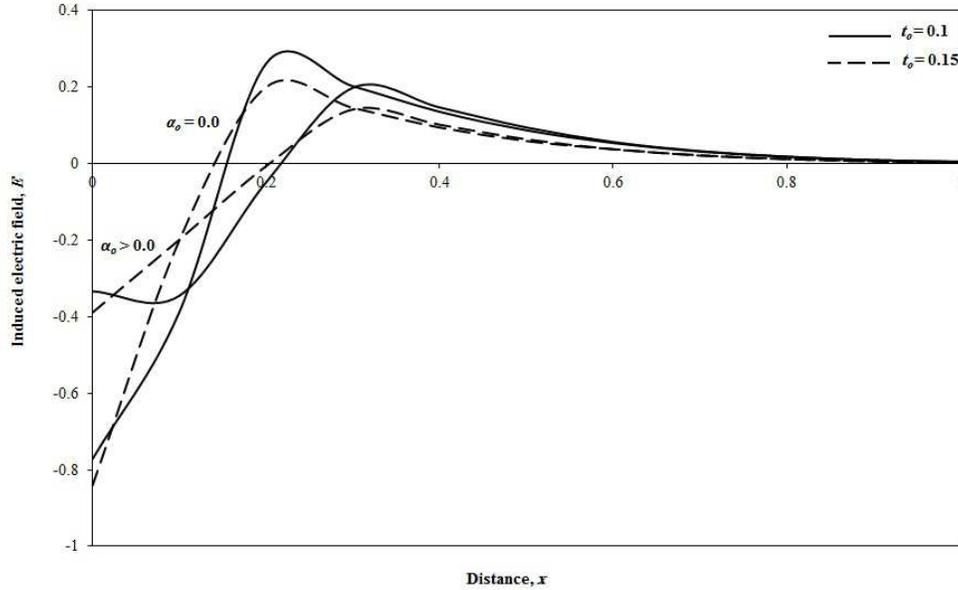


FIG. 8. The variation of induced electric field for different values of Alfvén velocity  $\alpha_0$  and ramping parameter  $t_0$ .

results is conducted to analyze the effect of fractional order parameters in details. While conducting this analysis, we found the following highlighted results:

In all figures, we noticed that the speed of the wave propagation of fractional thermoelastic variable fields is finite and coincides with the physical behaviors of elastic materials.

1. The speed of the wave propagation in fractional thermoelastic variable fields according to the new model is finite like in the generalized theories and coincides with the physical behaviors of elastic materials.

2. The response to the thermal and mechanical effects does not reach infinity instantaneously but remains in the bounded region of space that expands with the passing of time.

3. It is noticed that the fractional orders  $\alpha$  ( $0 < \alpha \leq 1$ ) and  $\beta$  ( $0 < \beta \leq 1$ ) have a significant effect on all fields.

4. In the new unified framework, it is observed that the thermal waves are continuous smooth functions and they reach the steady state depending on the values of  $\alpha$  and  $\beta$ , which means that the particles transport the heat to the other particles easily and this makes the decreasing rate of the temperature greater than the other ones.

5. The effect of ramp-type heating parameter  $t_0$  on temperature field is studied in Fig. 4. A significant difference in the value of temperature is noticed for

different values of the ramping time parameter  $t_0$  in the context of the unified model and it increases when this parameter decreases.

6. The effects of Alfvén velocity  $a_0$  and ramp-type heating parameter  $t_0$  on stress, displacement and induced magnetic field as well as induced electric field are shown in Figs. 5–8. Their effects are more noticeable as it is shown in these figures. The magnetic field causes a decrease of the fields. This is mainly due to the fact that the magnetic field corresponds to the term signifying a positive force that tends to accelerate the charge carriers. The ramping parameter  $t_0$  has the same effect on these fields.

7. The changes of magnetic parameters  $M = \sigma_0 \mu_0^2 H_0^2 / (\rho C_0^2 \eta_0)$ , where  $\sigma_0$  is the electric conductivity that is caused by different technological processes that have been conducted in many laboratories, and which affects stress and deformation in thermoelastic materials [74].

8. This model enables us to improve the efficiency of a thermoelectric material figure of merit ( $ZT = \frac{\sigma_0 S^2}{k} T$ ), where  $S$  is the thermoelectric power coefficient or Seebeck coefficient [75]. It is known that in order to achieve a high thermoelectric material figure of merit, a low thermal conductivity is required. This can occur for small values of  $\alpha$ .

## 6. Conclusions

1. The main goal of this work is to introduce a unified generalized model for the Fourier law with fractional derivative for heat conduction law so that some essential theorems for the linear coupled (CTE) and generalized theories of thermo-viscoelasticity (LS theory, GL theory and DOL theory) can be easily obtained. The results of all the functions for the new unified model are distinctly different from those obtained for coupled and generalized theory.

2. It is clear from the obtained results that the results for fractional generalized thermo-viscoelasticity are distinctly different from those of coupled and generalized thermoelasticity. The solution of any of the considered functions for the fractional generalized theories vanishes identically outside the bounded region of space. This demonstrates clearly the difference between the coupled and the generalized theories of thermoelasticity. In the first and latter theory, the waves propagate with infinite speeds, so the value of any function is not identically zero (though it may be very small) for any large value of  $x$ . In the fractional generalized theories, the response to thermal and mechanical effects does not reach infinity instantaneously but remains in the bounded region of the surface.

3. The advantage of the considered unified model consists in:

- (i) Disappearance of the discontinuities in the temperature distribution.
- (ii) Disappearance of the negative values in the temperature distribution that usually appeared in the previously generalized theories of thermoelasticity.

4. The method used in this article is applicable, when the governing equations are coupled, to a wide range of problems in thermodynamics and fluid dynamics [78].

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