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Some considerations of fundamental solution in micropolar thermoelastic materials with double porosity

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This paper is concerned with micropolar thermoelastic materials which have a double porosity structure. The system of the equations of the assumed model is based on the equations of motion, equilibrated stress equations of motion and heat conduction equation for material with double porosity. The explicit expressions for the fundamental solution of the system of equations in the case of steady vibrations are presented. The desired solutions are obtained by the use of elementary functions. Some basic properties are also established.

Key words: micropolar thermoelasticity, double porosity, fundamental solution, steady vibrations.

1. Introduction

Porous media theories play an important role in many branches of engineering including materials science, petroleum industry, chemical engineering, biomechanics and other such fields of engineering. The development and the intensive investigation of the theories of continua with microstructures arise due to the wide use of porous materials in engineering and technology. Representation of a fluid-saturated porous medium as a single-phase material has been virtually abandoned. The material with the pore spaces, such as concrete, can be treated easily because all concrete ingredients have the same motion when the concrete body is deformed. However, this situation becomes more complicated if the pores are filled with liquid and, in that case, the solid and liquid phases have different motions. Due to these different motions, the different material properties and the complicated geometry of pore structures, the mechanical behavior of a fluid-saturated porous thermoelastic medium becomes very difficult. Therefore, the researchers have tried to overcome this difficulty in the past and we see many studies of porous media in the literature. A brief historical background of these theories is given by de Boer [1, 2].

As far as we are concerned with the studies of porous media in the modern era, Biot [3] proposed the general theory of three-dimensional deformation of fluid-saturated porous salts. The Biot theory is based on the assumption of compressible constituents, and till recently, some of his results have been used as standard references and basis for subsequent analysis in acoustics, geophysics and other such fields. Another interesting theory is given by Bowen [4], and de Boer and Ehlers [5], in which all the constituents of a porous medium are assumed to be incompressible. The fluid-saturated porous material is modeled as a two-phase system consisting of an incompressible solid phase and incompressible fluid phase, thus meeting the many problems in engineering practice, e.g., in soil mechanics. One important generalization of the Biot's theory of poroelasticity that has been studied extensively started with the research by Barenblatt *et al.* [6], where the double porosity model was first proposed to express the fluid flow in hydrocarbon reservoirs and aquifers.

The double porosity model represents a new possibility for the study of important problems concerning the civil engineering. It is well known that, under super-saturation conditions due to water of other fluid effects, the so-called neutral pressures generate unbearable stress states on the solid matrix and on the fracture faces, with severe (sometimes disastrous) instability effects like landslides, rock fall or soil fluidization (typical phenomenon connected with propagation of seismic waves). In such a context, it seems possible, acting suitably on the boundary pressure state, to regulate the internal pressures in order to deactivate the noxious effects related to neutral pressures; finally, a further but connected positive effect could be the lightening of the solid matrix/fluid system.

Wilson and Aifantis [7] presented the theory of consolidation with double porosity. Khaled, Beskos and Aifantis [8] employed the finite element method to consider the numerical solutions of the differential equation of the theory of consolidation with double porosity developed by Aifantis [7]. Wilson and Aifantis [9] discussed the propagation of acoustic waves in a fluid-saturated porous medium. In particular, the propagation of acoustic waves in a fluid-saturated porous medium containing a continuously distributed system of fractures was discussed. The porous medium was assumed to consist of two degrees of porosity and the resulting model thus yielded three types of longitudinal waves, one associated with the elastic properties of the matrix material and one for each of the fluids in the pore space and the fracture space.

Beskos and Aifantis [10] presented the theory of consolidation with double porosity and obtained the analytical solutions to two boundary value problems. Khalili and Valliappan [11] studied the unified theory of flow and deformation in double porous media. Aifantis [12–15] introduced a multi-porous system and studied the mechanics of diffusion in solids. Moutsopoulos *et al.* [16] developed the numerical simulation of transport phenomena by using the double porosity/diffusivity continuum model. Khalili and Selvadurai [17] presented a fully coupled constitutive model for thermo-hydro-mechanical analysis in an elastic media with double porosity structure. Pride and Berryman [18] studied the linear dynamics of double-porosity dual-permeability materials. Straughan [19] studied the stability and uniqueness of double porous elastic media.

Svanadze [20–24] investigated some problems on elastic solids, viscoelastic solids and thermoelastic solids with double porosity. Scarpetta *et al.* [25, 26] proved the uniqueness theorems in the theory of thermoelasticity for solids with double porosity, and also obtained the fundamental solutions in the theory of thermoelasticity for solids with double porosity.

Nunziato and Cowin [27] developed a nonlinear theory of elastic material with voids. Later, Cowin and Nunziato [28] developed a theory of linear elastic materials with voids for the mathematical study of the mechanical behavior of porous solids. They also considered several applications of the linear theory by investigating the response of the materials to homogeneous deformations, pure bending of beams and small amplitudes of acoustic waves. Nunziato and Cowin have established a theory for the behavior of porous solids, in which the skeletal or matrix materials are elastic and the interstices are voids of material.

Iesan and Quintanilla [29] used the Nunziato–Cowin theory of materials with voids to derive a theory of thermoelastic solids which have a double porosity structure. This theory is not based on Darcy's law. In contrast with the classical theory of elastic materials with double porosity, the double porosity structure in the case of equilibrium is influenced by the displacement field. Marin *et al.* [56] presented a new model for micropolar bodies with double porosity.

The mechanical behavior of solids with voids andsolids containing microscopic components cannot be described by means of the classical theory of elasticity. In reality, almost all materials possess microstructure and in such materials, microstructural motions cannot be ignored. Eringen [30] introduced the theory of micropolar elasticity which had attracted much interest in recent years because of its possible usefulness in investigating the deformation properties of solids for which the classical theory is inadequate. The micropolar theory has been useful in investigating the material consisting of bar-like molecules, which exhibit the microrotational effects and can support body and surface couples. A micropolar continuum is a collection of interconnected particles in the form of small rigid bodies undergoing both translational and rotational motions. The force at a point of the surface element of bodies is solely characterized by stress vector and couple stress vector at that point.

The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua thermal effect. The comprehensive review of this theory was given by Eringen [31] and Nowacki [32]. Touchert *et al.* [33] derived the basic equations of the linear theory of micropolar coulpled thermoelasticity. Chandrasekharaiah [34] developed a heat flux dependent micropolar thermoelasticity. Boschi and Iesan [35] extended a generalized theory of micropolar thermoelasticity to permit for the transmission of heat as thermal waves at finite speed.

The construction of fundamental solutions has a great importance in many mathematical, physical and engineering problems. To investigate the boundary value problems of the theory of elasticity and thermoelasticity by potential method, it is necessary to construct a fundamental solution of systems of partial differential equations and to establish their basic properties, respectively. Hetnarski [36, 37] studied the fundamental solutions in the classical theory of coupled thermoelasticity. The information related to fundamental solutions of differential equations is presented in the books by Hörmander [38, 39]. Various authors [41–54] have derived the fundamental solutions in different theories of continuum mechanics.

In this paper, the fundamental solution of system of equations in the case of steady vibrations in terms of elementary functions is constructed and the basic properties of the fundamental solution are established. The aspects of the particular cases of Scarpetta *et al.* [25], Scarpetta [42], Ciarlette *et al.* [45] and Svanadze [51] are also deduced in the present investigation.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space R^3,

$$|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \qquad \mathbf{D}_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

and let t denote the time variable.

Following Marin *et al.* [56], the basic equations for isotropic, homogeneous micropolarthermoelastic material with double porosity structure, in the absence of body forces, body couples, extrinsic equilibrated body forces and heat sources, are:

$$
\begin{aligned}
&(\mu+\kappa)\Delta\bar{\mathbf{u}}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\bar{\mathbf{u}} \\
&\qquad +\kappa\operatorname{curl}\bar{\boldsymbol{\Phi}}+b\operatorname{grad}\bar{\varphi}+d\operatorname{grad}\bar{\psi}-\beta\operatorname{grad}\bar{T}=\rho\ddot{\bar{\mathbf{u}}}, \\
&(\gamma\Delta-2\kappa)\bar{\boldsymbol{\Phi}}+(\alpha+\beta)\operatorname{grad}\operatorname{div}\bar{\boldsymbol{\Phi}} \\
&\qquad +\kappa\operatorname{curl}\bar{\mathbf{u}}+c_0\operatorname{grad}\bar{\varphi}+d_0\operatorname{grad}\bar{\psi}=\rho_j\ddot{\bar{\boldsymbol{\Phi}}}, \\
&\alpha\Delta\bar{\varphi}+b_1\Delta\bar{\psi}-b\operatorname{div}\bar{\mathbf{u}}-\alpha_1\bar{\varphi}-\alpha_3\bar{\psi}+\gamma_1\bar{T}-c_0\operatorname{div}\bar{\boldsymbol{\Phi}}=\kappa_1\ddot{\bar{\varphi}}, \\
&b_1\Delta\bar{\varphi}+\gamma_0\Delta\bar{\psi}-d\operatorname{div}\bar{\mathbf{u}}-\alpha_3\bar{\varphi}-\alpha_2\bar{\psi}+\gamma_2\bar{T}-d_0\operatorname{div}\bar{\boldsymbol{\Phi}}=\kappa_2\ddot{\bar{\psi}}, \\
&K^*\Delta\bar{T}-\beta T_0\operatorname{div}\dot{\bar{\mathbf{u}}}-\gamma_1T_0\dot{\bar{\varphi}}-\gamma_2T_0\dot{\bar{\psi}}=\rho C^*\dot{\bar{T}},
\end{aligned}
\tag{2.1}
$$

where $\bar{\mathbf{u}}=(\bar{u}_1,\bar{u}_2,\bar{u}_3)$ is the displacement vector; $\bar{\boldsymbol{\Phi}}=(\bar{\Phi}_1,\bar{\Phi}_2,\bar{\Phi}_3)$ is the microrotation vector, λ and μ are the Lame's constants, ρ is the mass density, ρ_j is coefficient of inertia, $\beta=(3\lambda+2\mu+\kappa)\alpha_t$, α_t is the linear thermal expansion, C^* is the specific heat at constant strain, $\bar{u}_i$ is the displacement components, κ_1 and κ_2 are coefficients of equilibrated inertia, $\bar{\varphi}$ and $\bar{\psi}$ are the volume fraction fields corresponding to pores and fissures respectively, K^* is the coefficient of thermal conductivity and b, d, b_1, γ_0, γ_1, γ_2, c_0, d_0, κ are constitutive coefficients, δ_{ij} is the Kronecker's delta, and $\bar{T}$ is the temperature change measured from the absolute temperature T_0 ($T_0\neq 0$); a superposed dot represents differentiation with respect to time variable t and Δ is the Laplacian operator.

If the displacement vector $\bar{\mathbf{u}}$, microrotation vector $\bar{\boldsymbol{\Phi}}$, volume fractions fields $\bar{\varphi}$, $\bar{\psi}$ and temperature distribution $\bar{T}$ have a harmonic time variation as

$$
\{\bar{\mathbf{u}},\bar{\boldsymbol{\Phi}},\bar{\varphi},\bar{\psi},\bar{T}\}(x,t)=\operatorname{Re}[\{\mathbf{u},\boldsymbol{\Phi},\varphi,\psi,T\}(\mathbf{x})e^{-i\omega t}],
\tag{2.2}
$$

using (2.2) in (2.1) yields the system of steady vibrations as

$$
\begin{aligned}
&[(\mu+\kappa)\Delta+\rho\omega^2]\mathbf{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} \\
&\qquad +\kappa\operatorname{curl}\boldsymbol{\Phi}+b\operatorname{grad}\varphi+d\operatorname{grad}\psi-\beta\operatorname{grad}T=0, \\
&(\gamma\Delta+\mu_1)\boldsymbol{\Phi}+(\alpha+\beta)\operatorname{grad}\operatorname{div}\boldsymbol{\Phi}+\kappa\operatorname{curl}\mathbf{u}+c_0\operatorname{grad}\varphi+d_0\operatorname{grad}\psi=0, \\
&(\alpha\Delta+\mu_2)\varphi+(b_1\Delta-\alpha_3)\psi-b\operatorname{div}\mathbf{u}+\gamma_1T-c_0\operatorname{div}\boldsymbol{\Phi}=0, \\
&(b_1\Delta-\alpha_3)\varphi+(\gamma_0\Delta+\mu_3)\psi-d\operatorname{div}\mathbf{u}+\gamma_2T-d_0\operatorname{div}\boldsymbol{\Phi}=0, \\
&(k_3\Delta-\rho C^*)T-\beta T_0\operatorname{div}\mathbf{u}-\gamma_1T_0\varphi-\gamma_2T_0\psi=0,
\end{aligned}
\tag{2.3}
$$

where ω is the oscillation frequency ($\omega>0$), and

$$
\mu_1=\rho_j\omega^2-2\kappa,\quad \mu_2=\kappa_1\omega^2-\alpha_1,\quad \mu_3=\kappa_2\omega^2-\alpha_2,\quad k_3=-\frac{K^*}{i\omega}.
$$

Introducing the matrix differential operator

$$
\mathbf{E}(\mathbf{D}_x)=\|E_{gh}(\mathbf{D}_x)\|_{9\times 9},
$$

where

$$E(\mathbf{D}_x) = [(\mu + k)\Delta + \rho\omega^2]\delta_{mn} + (\lambda + \mu)\frac{\partial^2}{\partial x_m \partial x_n},$$

$$E_{m,n+3}(\mathbf{D}_x) = E_{m+3,n}(\mathbf{D}_x) = \kappa \sum_{r=1}^{3} \varepsilon_{mrn} \frac{\partial}{\partial x_r},$$

$$E_{m7}(\mathbf{D}_x) = -E_{7m}(\mathbf{D}_x) = b\frac{\partial}{\partial x_m},$$

$$E_{m8}(\mathbf{D}_x) = -E_{8m}(\mathbf{D}_x) = d\frac{\partial}{\partial x_m},$$

$$E_{m9}(\mathbf{D}_x) = -\beta\frac{\partial}{\partial x_m},$$

$$E_{m+3,n+3}(\mathbf{D}_x) = (\gamma\Delta + \mu_1)\delta_{mn} + (\alpha + \beta)\frac{\partial^2}{\partial x_m \partial x_n},$$

$$E_{m+3,7}(\mathbf{D}_x) = -E_{7,n+3}(\mathbf{D}_x) = c_0\frac{\partial}{\partial x_m},$$

$$E_{m+3,8}(\mathbf{D}_x) = -E_{8,n+3}(\mathbf{D}_x) = d_0\frac{\partial}{\partial x_m},$$

$$E_{77}(\mathbf{D}_x) = \alpha\Delta + \mu_2, \qquad E_{78}(\mathbf{D}_x) = E_{87}(\mathbf{D}_x) = b_1\Delta - \alpha_3,$$

$$E_{79}(\mathbf{D}_x) = \gamma_1, \qquad E_{89}(\mathbf{D}_x) = \gamma_2, \qquad E_{97}(\mathbf{D}_x) = -\gamma_1 T_0,$$

$$E_{98}(\mathbf{D}_x) = -\gamma_2 T_0, \qquad E_{88}(\mathbf{D}_x) = \gamma_0\Delta + \mu_3,$$

$$E_{9m}(\mathbf{D}_x) = -\beta T_0\frac{\partial}{\partial x_m}, \qquad E_{m+3,9}(\mathbf{D}_x) = 0 = E_{9,n+3}(\mathbf{D}_x),$$

$$E_{99}(\mathbf{D}_x) = k_3\Delta - \rho C^*, \qquad m, n = 1, 2, 3,$$

δ_{mn} is the Kronecker's delta and ε_{mrn} is the alternating symbol.

The system (2.3) can be written as

$$\mathbf{E}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{\Phi}, \varphi, \psi, T)$ is a nine-component vector function on R^3.

We assume that

$$\alpha_4\alpha_5\alpha_6 k_3 \gamma(\mu + \kappa) \neq 0, \tag{2.4}$$

where $\alpha_4 = \lambda + 2\mu + \kappa$, $\alpha_5 = \alpha + \beta + \gamma$, $\alpha_6 = \alpha\gamma_0 - b_1^2$. Evidently, if conditions in (2.4) are satisfied, then E is the elliptic differential operator [38].

Definition. The fundamental solution of the system (2.3) (the fundamental matrix of operator $\mathbf{E}$) is the matrix $\mathbf{\Lambda}(\mathbf{x}) = \|\mathbf{\Lambda}_{gh}(x)\|_{9\times 9}$ satisfying condition [38]

$$\mathbf{E}(\mathbf{D}_x)\mathbf{\Lambda}(\mathbf{x}) = \delta(x)\mathbf{I}(\mathbf{x}), \tag{2.5}$$

where δ is the Dirac delta, $\mathbf{I} = \|\delta_{gh}\|_{9\times 9}$ is the unit matrix, and $\mathbf{x} \in \mathbf{R}^3$.

Now, we construct the matrix $\mathbf{\Lambda}(x)$ in terms of elementary functions and we also establish some basic properties.

3. Fundamental solution of the system of equations of steady vibrations

We consider the system of equations

$$\begin{gathered}[(\mu+\kappa)\Delta+\rho\omega^2]\mathbf{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}\\ +\kappa\operatorname{curl}\mathbf{\Phi}-b\operatorname{grad}\varphi-d\operatorname{grad}\psi-\beta T_0\operatorname{grad}T=\mathbf{F}',\\ (\gamma\Delta+\mu_1)\mathbf{\Phi}+(\alpha+\beta)\operatorname{grad}\operatorname{div}\mathbf{\Phi}+\kappa\operatorname{curl}\mathbf{u}\\ -c_0\operatorname{grad}\varphi-d_0\operatorname{grad}\psi=\mathbf{F}'',\\ (\alpha\Delta+\mu_2)\varphi+(b_1\Delta-\alpha_3)\psi+b\operatorname{div}u-\gamma_1T_0T+c_0\operatorname{div}\mathbf{\Phi}=f',\\ (b_1\Delta-\alpha_3)\varphi+(\gamma_0\Delta+\mu_3)\psi+d\operatorname{div}\mathbf{u}-\gamma_2T_0T+d_0\operatorname{div}\mathbf{\Phi}=f'',\\ (k_3\Delta-\rho C^*)T-\beta\operatorname{div}\mathbf{u}+\gamma_1\varphi+\gamma_2\psi=f''',\end{gathered}\tag{3.1}$$

where $\mathbf{F}'$ and $\mathbf{F}''$ are three-component vector functions on $\mathbf{R}^3$; f', f'' and f''' are scalar functions on $\mathbf{R}^3$.

The system (3.1) may be written in the form

$$\mathbf{E}^{tr}(\mathbf{D}_x)\mathbf{U}(x)=\mathbf{Q}(x),\tag{3.2}$$

where $\mathbf{E}^{tr}$ is the transpose of matrix $\mathbf{E}$, $\mathbf{Q}=(\mathbf{F}',\mathbf{F}'',f',f'',f''')$ is the nine-component vector function on $\mathbf{R}^3$, and $\mathbf{x}\in\mathbf{R}^3$.

Applying the operator div to first and second equation of system (3.1), we obtain

$$\begin{gathered}[\alpha_4\Delta+\rho\omega^2]\operatorname{div}\mathbf{u}-b\Delta\varphi-d\Delta\psi-\beta T_0\Delta T=\operatorname{div}\mathbf{F}',\\ (\alpha_5\Delta+\mu_1)\operatorname{div}\mathbf{\Phi}-c_0\Delta\varphi-d_0\Delta\psi=\operatorname{div}\mathbf{F}'',\\ (\alpha\Delta+\mu_2)\varphi+(b_1\Delta-\alpha_3)\psi+b\operatorname{div}\mathbf{u}-\gamma_1T_0T+c_0\operatorname{div}\mathbf{\Phi}=f',\\ (b_1\Delta-\alpha_3)\varphi+(\gamma_0\Delta+\mu_3)\psi+d\operatorname{div}\mathbf{u}-\gamma_2T_0T+d_0\operatorname{div}\mathbf{\Phi}=f'',\\ (k_3\Delta-\rho C^*)T-\beta\operatorname{div}\mathbf{u}+\gamma_1\varphi+\gamma_2\psi=f'''.\end{gathered}\tag{3.3}$$

The system (3.3) can be written as

$$\mathbf{H}(\Delta)\mathbf{S}=\tilde{\mathbf{Q}},\tag{3.4}$$

where

$$\begin{aligned}\mathbf{S}&=(\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{\Phi},\varphi,\psi,T),\\ \tilde{\mathbf{Q}}&=(\operatorname{div}\mathbf{F}',\operatorname{div}\mathbf{F}'',f',f'',f''')=(f_1,f_2,f_3,f_4,f_5)\end{aligned}$$

and

$$\mathbf{H}(\Delta) = \|H_{mn}(\Delta)\|_{5\times 5} = \begin{Vmatrix} \alpha_4\Delta + \rho\omega^2 & 0 & -b\Delta & -d\Delta & -\beta T_0\Delta \\ 0 & \alpha_5\Delta + \mu_1 & -c_0\Delta & -d_0\Delta & 0 \\ b & c_0 & \alpha\Delta + \mu_2 & b_1\Delta - \alpha_3 & -\gamma_1 T_0 \\ d & d_0 & b_1\Delta - \alpha_3 & \gamma_0\Delta + \mu_3 & -\gamma_2 T_0 \\ -\beta & 0 & \gamma_1 & \gamma_2 & k_3\Delta - \rho C^* \end{Vmatrix}_{5\times 5}. \tag{3.5}$$

The system (3.3) may be written as

$$\Gamma_1(\Delta)\mathbf{S} = \mathbf{\Psi}, \tag{3.6}$$

where

$$\begin{gathered} \mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5), \qquad \Psi_n = g_1 \sum_{m=1}^{5} H^*_{mn} f_m, \\ \Gamma_1(\Delta) = g_1 \det \mathbf{H}(\Delta), \qquad g_1 = \frac{1}{\alpha_4\alpha_5\alpha_6 k_3}, \qquad n = 1,2,3,4,5, \end{gathered} \tag{3.7}$$

where H^*_{mn} is the cofactor of the element H_{mn} of the matrix $\mathbf{H}$.

From (3.5) and (3.6), we see that

$$\Gamma_1(\Delta) = \prod_{m=1}^{5} (\Delta + \xi_m^2),$$

where ξ_m^2, $m = 1,2,3,4,5$ are the roots of the equation $\Gamma_1(-\chi) = 0$ (with respect to χ).

Applying the operator $(\gamma\Delta + \mu_1)$ and $\kappa\,\mathrm{curl}$ to Eqs. $(3.1)_1$ and $(3.1)_2$, respectively, we obtain

$$\begin{aligned} (\gamma\Delta + \mu_1)[(\mu+\kappa)\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \rho\omega^2\mathbf{u}] + \kappa(\gamma\Delta + \mu_1)\,\mathrm{curl}\,\mathbf{\Phi} \\ = (\gamma\Delta + \mu_1)[F' + b\,\mathrm{grad}\,\varphi + d\,\mathrm{grad}\,\psi + \beta T_0\,\mathrm{grad}\,T], \end{aligned} \tag{3.8}$$

$$\kappa(\gamma\Delta + \mu_1)\,\mathrm{curl}\,\mathbf{\Phi} = -\kappa^2\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} + \kappa\,\mathrm{curl}\,\mathbf{F}''. \tag{3.9}$$

When using (3.9) and equality

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \Delta\mathbf{u} \tag{3.10}$$

in Eq. (3.8), we obtain

$$\begin{aligned} [\{(\gamma\Delta + \mu_1)(\mu+\kappa) + \kappa^2\}\Delta + \rho\omega^2(\gamma\Delta + \mu_1)]\mathbf{u} \\ + [(\lambda+\mu)(\gamma\Delta + \mu_1) - \kappa^2]\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} \\ = (\gamma\Delta + \mu_1)[F' + b\,\mathrm{grad}\,\varphi + d\,\mathrm{grad}\,\psi + \beta T_0\,\mathrm{grad}\,T] - \kappa\,\mathrm{curl}\,\mathbf{F}''. \end{aligned} \tag{3.11}$$

Applying the operator $\Gamma_1(\Delta)$ to Eq. (3.11) and after using Eq. (3.6), we obtain

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{u} = \mathbf{\Psi}', \tag{3.12}$$

where

$$\Gamma_2(\Delta) = g_2[\gamma(\mu+\kappa)\Delta^2 + (\mu\mu_1 + \mu_1\kappa + \kappa^2 + \rho\omega^2\gamma)\Delta + \rho\omega^2\mu_1],$$
$$g_2 = \frac{1}{\gamma(\mu+\kappa)}$$

and

$$\mathbf{\Psi}' = g_2[(\gamma\Delta + \mu_1)(\mathit{\Lambda}_1\mathbf{F}' + b\,\mathrm{grad}\,\Psi_3 + d\,\mathrm{grad}\,\Psi_4 + \beta T_0\,\mathrm{grad}\,\Psi_5) - \kappa\mathit{\Lambda}_1\,\mathrm{curl}\,\mathbf{F}'' - \{(\lambda+\mu)(\gamma\Delta+\mu_1) - \kappa^2)\}\kappa\,\mathrm{grad}\,\Psi_1]. \tag{3.13}$$

We see that

$$\Gamma_2(\Delta) = (\Delta + \xi_6^2)\,(\Delta + \xi_7^2), \tag{3.14}$$

where ξ_6^2 and ξ_7^2 are the roots of the equation (with respect to χ).

$$\gamma(\mu+\kappa)\chi^2 + (\mu\mu_1 + \mu_1\kappa + \kappa^2 + \rho\omega^2\gamma)\chi + \rho\omega^2\mu_1 = 0.$$

Similarly, from Eqs. $(3.1)_1$,$(3.1)_2$ and (3.6) we obtain

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{\Phi} = \mathbf{\Psi}'', \tag{3.15}$$

where

$$\begin{aligned}\mathbf{\Psi}'' = {} & g_2\mathit{\Lambda}_1(\Delta)[-\kappa\,\mathrm{curl}\,\mathbf{F}' + \{(\mu+\kappa)\Delta + \rho\omega^2\}\mathbf{F}'' \\ & - g_2[(\alpha+\beta)\{(\mu+\kappa)\Delta + \rho\omega^2\} - \kappa^2]\,\mathrm{grad}\,\mathit{\Phi}_2 \\ & + c_0 g_2[(\mu+\kappa)\Delta + \rho\omega^2]\,\mathrm{grad}\,\mathit{\Phi}_3 + d_0 g_2[(\mu+\kappa)\Delta + \rho\omega^2]\,\mathrm{grad}\,\mathit{\Phi}_4.\end{aligned} \tag{3.16}$$

From Eqs. (3.12), (3.15) and (3.6), we obtain

$$\mathbf{\Gamma}(\Delta)\mathbf{U}(x) = \tilde{\mathbf{\Psi}}(x), \tag{3.17}$$

where $\mathbf{\Psi} = (\mathbf{\Psi}', \mathbf{\Psi}'', \Psi_3, \Psi_4, \Psi_5)$ and

$$\Gamma(\Delta) = \|\Gamma_{ij}(\Delta)\|_{9\times 9},$$
$$\Gamma_{pp}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{m=1}^{7}(\Delta + \xi_m^2),$$
$$\Gamma_{nn}(\Delta) = \Gamma_1(\Delta), \qquad \Gamma_{ij}(\Delta) = 0,$$
$$p = 1,\dots,6, \qquad n = 7,8,9, \qquad i,j = 1,\dots,9, \qquad i \neq j.$$

In what follows we have

$$
\begin{aligned}
r_{l1}(\Delta) &= g_1 g_2[(\gamma\Delta+\mu_1)(bH^*_{l3}+dH^*_{l4}+\beta T_0 H^*_{l5}) \\
&\qquad - \{(\lambda+\mu)(\gamma\Delta+\mu_1)-\kappa^2\}H^*_{l1}], \\
r_{l2}(\Delta) &= c_0 g_1 g_2[(\mu+\kappa)\Delta+\rho\omega^2]H^*_{l3}+d_0 g_1 g_2[(\mu+\kappa)\Delta+\rho\omega^2]H^*_{l4} \\
&\qquad - g_1 g_2[(\alpha+\beta)\{(\mu+\kappa)\Delta+\rho\omega^2\}-\kappa^2]H^*_{l2}, \\
r_{lj}(\Delta) &= g_1 H^*_{lj}, \qquad l=1,2,3,4,5, \qquad j=3,4,5.
\end{aligned} \tag{3.18}
$$

It is evident that $r_{12}(\Delta) = r_{21}(\Delta)$. From Eqs. (3.13) and (3.15) ,by virtue of Eqs.(3.7) and (3.18), we have

$$
\begin{aligned}
\boldsymbol{\Psi}' &= [g_2(\gamma\Delta+\mu_1)\Gamma_1\mathbf{I}+r_{11}\,\mathrm{grad}\,\mathrm{div}]\mathbf{F}'+[-\kappa g_2\Gamma_1\,\mathrm{curl}+r_{21}\,\mathrm{grad}\,\mathrm{div}]\mathbf{F}'' \\
&\quad + r_{31}\,\mathrm{grad}\, f' + r_{41}\,\mathrm{grad}\, f'' + r_{51}\,\mathrm{grad}\, f''', \\
\boldsymbol{\Psi}'' &= [-\kappa b_2\Gamma_1\,\mathrm{curl}+r_{12}\,\mathrm{grad}\,\mathrm{div}]\mathbf{F}' \\
&\quad + [g_2\{(\mu+\kappa)\Delta+\rho\omega^2\}\Gamma_1\mathbf{I}+r_{22}\,\mathrm{grad}\,\mathrm{div}]\mathbf{F}'' \\
&\quad + r_{32}\,\mathrm{grad}\, f' + r_{42}\,\mathrm{grad}\, f'' + r_{52}\,\mathrm{grad}\, f''', \\
\Psi_j &= r_{1j}\,\mathrm{div}\,\mathbf{F}' + r_{2j}\,\mathrm{div}\,\mathbf{F}'' + r_{3j} f' + r_{4j} f'' + r_{5j} f'', \qquad j=3,4,5,
\end{aligned} \tag{3.19}
$$

where $\mathbf{I} = \|\delta_{ej}\|_{3\times 3}$ is the unit matrix.

Therefore, from Eq. (3.18), we have

$$
\tilde{\Psi}(x) = \mathbf{N}^{tr}(\mathbf{D}_x)\mathbf{Q}(x), \tag{3.20}
$$

where

$$
\begin{aligned}
\mathbf{N}(\mathbf{D}_x) &= \|N_{gh}(\mathbf{D}_x)\|_{9\times 9}, \\
N_{mn}(\mathbf{D}_x) &= g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{mn}+r_{11}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}, \\
N_{m,n+3}(\mathbf{D}_x) &= N_{m+3,n}(\mathbf{D}_x) \\
&= -\kappa g_2\Gamma_1(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r}+r_{12}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}, \\
N_{mp}(\mathbf{D}_x) &= r_{1,p-4}(\Delta)\frac{\partial}{\partial x_m}, \\
N_{m+3,n+3}(\mathbf{D}_x) &= g_2\{(\mu+\kappa)\Delta+\rho\omega^2\}\Gamma_1(\Delta)\delta_{mn}+r_{22}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}, \\
N_{m+3,p}(\mathbf{D}_x) &= r_{2,p-4}(\Delta)\frac{\partial}{\partial x_m}, \qquad N_{pm}(\mathbf{D}_x) = r_{p-4,1}\frac{\partial}{\partial x_m}, \\
N_{p,m+3}(\mathbf{D}_x) &= r_{p-4,2}(\Delta)\frac{\partial}{\partial x_m}, \\
N_{pq}(\mathbf{D}_x) &= r_{p-4,q-4}(\Delta), \qquad m,n=1,2,3, \qquad p,q=7,8,9.
\end{aligned} \tag{3.21}
$$

In view of Eqs. (3.2) and (3.20), from Eq. (3.17) it is found that $\mathbf{\Gamma U} = \mathbf{N}^{tr}\mathbf{E}^{tr}\mathbf{U}$. It is evident that $\mathbf{N}^{tr}\mathbf{E}^{tr} = \mathbf{\Gamma}$, and hence

$$\mathbf{E}(\mathbf{D}_x)\mathbf{N}(\mathbf{D}_x) = \mathbf{\Gamma}(\Delta). \tag{3.22}$$

We assume that

$$\xi_m^2 \neq \xi_n^2 \neq 0, \qquad m, n = 1, 2, \ldots, 7 \text{ and } m \neq n.$$

Let

$$\mathbf{Z}(x) = \|Z_{ej}(\mathbf{D}_x)\|_{9\times 9}, \qquad Z_{mm}(x) = \sum_{n=1}^{6} s_{1n}\varsigma_n(x),$$

$$Z_{m+3,m+3}(x) = \sum_{n=5}^{7} s_{2n}\varsigma_n(x),$$

$$Z_{77}(x) = Z_{88}(x) = Z_{99}(x) = \sum_{n=1}^{4} s_{3n}\varsigma_n(x),$$

$$Z_{ej}(x) = 0, \qquad m = 1, 2, 3, \qquad e, j = 1, 2, \ldots, 9, \qquad e \neq j,$$

where

$$\begin{aligned}
&\varsigma_n(x) = -\frac{1}{4\pi|x|}e^{i\xi_n|x|},\\
&s_{1l} = \prod_{\substack{m=1\\ m\neq l}}^{6} (\xi_m^2 - \xi_l^2)^{-1}, \qquad l = 1, 2, 3, 4, 5, 6,\\
&s_{2e} = \prod_{\substack{m=5\\ m\neq e}}^{7} (\xi_m^2 - \xi_e^2)^{-1}, \qquad e = 5, 6, 7,\\
&s_{3j} = \prod_{\substack{m=1\\ m\neq j}}^{4} (\xi_m^2 - \xi_j^2)^{-1}, \qquad j = 1, 2, 3, 4.
\end{aligned} \tag{3.23}$$

Therefore, the matrix $\mathbf{Z}$ is the fundamental matrix of operator $\mathbf{\Gamma}(\Delta)$, that is,

$$\mathbf{\Gamma}(\Delta)\mathbf{Z}(x) = \delta(x)\mathbf{I}(x). \tag{3.24}$$

Introducing the matrix

$$\mathbf{\Lambda}(x) = \mathbf{N}(\mathbf{D}_x)\mathbf{Z}(x) \tag{3.25}$$

and using (3.24), in Eqs. (3.22) and (3.25), we obtain

$$\mathbf{E}(\mathbf{D}_x)\mathbf{\Lambda}(x) = \mathbf{E}(\mathbf{D}_x)\mathbf{N}(\mathbf{D}_x)\mathbf{Z}(x) = \mathbf{\Gamma}(\Delta)\mathbf{Z}(x) = \delta(x)\mathbf{I}(x). \tag{3.26}$$

Hence, $\mathbf{\Lambda}(x)$ is the solution of Eq. (2.5).

Now, we will prove the following theorem.

Theorem 1. *The matrix* $\mathbf{\Lambda}(x)$ *defined by Eq.* (3.26) *is the fundamental solution of system* (2.3).

Remark. The fundamental solution $\mathbf{\Lambda}(x)$ of system (2.3) is constructed for $\xi_m \neq \xi_n \neq 0$ $(m, n = 1, 2, \dots, 7$ and $m \neq n)$. Evidently, using the above method, it is possible to construct the fundamental solution of system (2.3) for the cases where $\xi_m = 0$ and $\xi_m = \xi_n$.

4. Basic properties of the matrix $\mathbf{\Lambda}(x)$

Corollary 1. *Each column of the matrix* $\mathbf{\Lambda}(x)$ *is the solution of the system* (2.3) *at every point* $x \in \mathbf{R}^3$ *except the origin.*

Corollary 2. *If conditions in* (2.4) *are satisfied, then the fundamental solution of the system*

$$\begin{aligned} (\mu+\kappa)\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} &= 0,\\ \gamma\Delta\mathbf{\Phi} + (\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{\Phi} &= 0,\\ \alpha\Delta\varphi + b_1\Delta\psi &= 0,\\ b_1\Delta\varphi + \gamma_0\Delta\psi &= 0,\\ k_3\Delta T &= 0 \end{aligned} \tag{4.1}$$

is the matrix

$$\mathbf{\Omega}(x) = \|\Omega_{mn}(x)\|_{9\times 9},$$

where

$$\begin{aligned} \mathbf{\Omega}_{lj}(x) &= \left(\frac{1}{\alpha_4}\,\mathrm{grad}\,\mathrm{div} - \frac{1}{\mu+\kappa}\,\mathrm{curl}\,\mathrm{curl}\right)\lambda_1(x),\\ \mathbf{\Omega}_{lm}(x) &= \mathbf{\Omega}_{ml}(x) = \mathbf{\Omega}_{l,m+3}(x) = \mathbf{\Omega}_{m+3,l}(x) = 0,\\ \mathbf{\Omega}_{77}(x) &= \frac{\alpha}{\alpha_6}\lambda_2(x), \qquad \mathbf{\Omega}_{78}(x) = \mathbf{\Omega}_{87}(x) = \frac{-b_1}{\alpha_6}\lambda_2(x),\\ \mathbf{\Omega}_{88}(x) &= \frac{\gamma_0}{\alpha_6}\lambda_2(x), \qquad \mathbf{\Omega}_{99}(x) = \frac{1}{k_3}\lambda_2(x),\\ \mathbf{\Omega}_{l+3,j+3}(x) &= \left(\frac{1}{\alpha_5}\,\mathrm{grad}\,\mathrm{div} - \frac{1}{\gamma}\,\mathrm{curl}\,\mathrm{curl}\right)\lambda_1(x), \end{aligned}$$

$$\mathbf{\Omega}_{79}(x) = \mathbf{\Omega}_{97}(x) = 0, \qquad \mathbf{\Omega}_{98}(x) = \mathbf{\Omega}_{89}(x) = 0,$$
$$\lambda_1(x) = -\frac{|x|}{8\pi}, \qquad \lambda_2(x) = -\frac{1}{4\pi|x|}, \qquad l,j = 1,2,3, \qquad m = 4,5,6.$$

Lemma 1. *If conditions* (2.4) *are satisfied, then*

$$\begin{aligned} \Delta r_{l1}(\Delta) &= g_1\Gamma_2(\Delta)H^*_{l1}(\Delta) - g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{l1}, \\ \Delta r_{l2}(\Delta) &= g_1\Gamma_2(\Delta)H^*_{l2}(\Delta) - g_2[(\mu+\kappa)\Delta+\rho\omega^2]\Gamma_1(\Delta)\delta_{l2}, \quad l = 1,2,3,4,5. \end{aligned} \tag{4.2}$$

Proof. Using the equality

$$(\alpha_4\Delta + \rho\omega^2)H^*_{l1} - \Delta(bH^*_{l3} + dH^*_{l4} + \beta T_0 H^*_{l5}) = \frac{1}{g_1}\delta_{l1}\Gamma_1(\Delta), \qquad l = 1,2,3,4,5,$$

Eq. $(3.18)_1$ implies that

$$\begin{aligned} \Delta r_{l1}(\Delta) = g_1 g_2[(\gamma\Delta + \mu_1)\{(\alpha_4\Delta + \rho\omega^2)H^*_{l1} - \frac{1}{g_1}\delta_{l1}\Gamma_1(\Delta)\} \\ - \{(\lambda+\mu)(\gamma\Delta+\mu_1) - \kappa^2\}\Delta H^*_{l1}] \\ = g_1 g_2[(\gamma\Delta+\mu_1)\{(\mu+\kappa)\Delta+\rho\omega^2\} + \kappa^2\Delta]H^*_{l1} \\ - g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{l1} \\ = g_1\Gamma_2(\Delta)H^*_{l1}(\Delta) - g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{l1}. \end{aligned}$$

Similarly, from Eqs. $(3.18)_2$ and

$$(\alpha_5\Delta + \mu_1)H^*_{l2} - c_0\Delta H^*_{l3} - d_0\Delta H^*_{l4} = \frac{1}{g_1}\delta_{l2}\Gamma_1(\Delta), \qquad l = 1,2,3,4,5$$

we obtain

$$\begin{aligned} \Delta r_{l2}(\Delta) &= g_1 g_2[\{(\mu+\kappa)\Delta+\rho\omega^2)\}\{c_0\Delta H^*_{l3} + d_0\Delta H^*_{l4} + (\alpha_5-\gamma)\Delta H^*_{l2}\} + \kappa^2\Delta H^*_{l2}] \\ &= g_1 g_2\left[\left\{(\gamma\Delta+\mu_1)H^*_{l2} - \frac{1}{g_1}\delta_{l2}\Gamma_1(\Delta)\right\}\{(\mu+\kappa)\Delta+\rho\omega^2\} + \kappa^2\Delta H^*_{l2}\right] \\ &= g_1\Gamma_2(\Delta)H^*_{l2}(\Delta) - g_2[(\mu+\kappa)\Delta+\rho\omega^2)]\Gamma_1(\Delta)\delta_{l2}. \end{aligned}$$

Lemma 2. *If conditions* (2.4) *are satisfied and* $\mathbf{x} \in \mathbf{R}^3 \setminus \{\mathbf{0}\}$, *then*

$$\begin{aligned} &\left[r_{l1}(-\xi_m^2) - \frac{g_2}{\xi_m^2}(-\gamma\xi_m^2+\mu_1)\Gamma_1(-\xi_m^2)\delta_{l1}\right]\zeta_j(x) \\ &\qquad = -\frac{g_1}{\xi_m^2}\Gamma_2(-\xi_m^2)H^*_{l1}(-\xi_m^2)\zeta_j(x), \\ &\left[r_{l2}(-\xi_m^2) - \frac{g_2}{\xi_m^2}\{-(\mu+\kappa)\xi_m^2+\rho\omega^2\}\Gamma_1(-\xi_m^2)\delta_{l2}\right]\zeta_j(x) \\ &\qquad = -\frac{g_1}{\xi_m^2}\Gamma_2(-\xi_m^2)H^*_{l2}(-\xi_m^2)\zeta_j(x), \qquad l = 1,2,3,4,5. \end{aligned} \tag{4.3}$$

Proof. We obtain Eqs. (4.3) when we use the following equality

$$\Delta\zeta_j(x) = -\xi_m^2\zeta_j(x)$$

in the system of Eqs. (4.1).

THEOREM 2. *If conditions* (2.4) *are satisfied and* $\mathbf{x} \in \mathbf{R}^3 \setminus \{\mathbf{0}\}$, *then*

$$\mathbf{\Theta}^{(1)}(x) = \operatorname{grad}\operatorname{div}\sum_{m=1}^{5} v_{1m}\zeta_m(x) - \operatorname{curl}\operatorname{curl}\sum_{e=6}^{7} v_{1e}\zeta_e(x),$$

$$\mathbf{\Theta}^{(2)}(x) = \mathbf{\Theta}^{(3)}(x) = v_{20}\operatorname{curl}[\zeta_6(x) - \zeta_7(x)] + \operatorname{grad}\operatorname{div}\sum_{m=1}^{5} v_{2m}\zeta_m(x),$$

$$\mathbf{\Theta}^{(4)}(x) = \operatorname{grad}\operatorname{div}\sum_{m=1}^{5} v_{1m}\zeta_m(x) - \operatorname{curl}\operatorname{curl}\sum_{e=6}^{7} v_{4e}\zeta_m(x),$$

$$\mathbf{\Theta}^{(n)}_{er}(x) = \frac{\partial}{\partial x_e}\sum_{m=1}^{5} v_{nrj}\zeta_m(x),$$

$$\mathbf{\Theta}^{(n+2)}_{re}(x) = \frac{\partial}{\partial x_e}\sum_{m=1}^{5} v_{n+2,rj}\zeta_m(x),$$

$$\mathbf{\Theta}^{(9)}_{re}(x) = \sum_{m=1}^{5} v_{9qrj}\zeta_m(x), \qquad e = 1,2,3, \quad r,q = 1,2, \quad n = 5,6,$$

where

$$\mathbf{\Theta} = \|\mathbf{\Theta}_{ej}\|_{8\times 8} = \begin{Vmatrix} \mathbf{\Theta}^{(1)} & \mathbf{\Theta}^{(2)} & \mathbf{\Theta}^{(5)} \\ \mathbf{\Theta}^{(3)} & \mathbf{\Theta}^{(4)} & \mathbf{\Theta}^{(6)} \\ \mathbf{\Theta}^{(7)} & \mathbf{\Theta}^{(8)} & \mathbf{\Theta}^{(9)} \end{Vmatrix}_{9\times 9}, \qquad \mathbf{\Theta}^{(n)} = \|\mathbf{\Theta}^{(n)}_{ej}\|_{3\times 3},$$

$$\mathbf{\Theta}^{(q)} = \|\mathbf{\Theta}^{(q)}_{ej}\|_{3\times 2}, \qquad \mathbf{\Theta}^{(r)} = \|\mathbf{\Theta}^{(r)}_{ej}\|_{2\times 3}, \qquad \mathbf{\Theta}^{(9)} = \|\mathbf{\Theta}^{(9)}_{ej}\|_{2\times 2},$$

$n = 1,2,3,4$, $q = 5,6$, $r = 7,8$ *and*

$$
\begin{aligned}
v_{1m} &= -\frac{g_1}{\xi_m^2}s_{2j}H^*_{l1}(-\xi_m^2), \qquad v_{1e} = \frac{(-1)^e g_2}{\xi_e^2(\xi_6^2-\xi_7^2)}(\zeta\xi_e^2 - \mu_1),\\
v_{2m} &= c_0 g_1(-bk_3\xi_m^2 + b\alpha_3 + d\alpha_4 - \beta\alpha_5), \qquad v_{20} = \frac{\kappa g_2}{\xi_6^2 - \xi_7^2},\\
v_{4m} &= -\frac{g_1}{\xi_m^2}s_{2m}H^*_{22}(-\xi_m^2), \qquad v_{4e} = \frac{(-1)^e g_2}{k_e^2(k_6^2-k_7^2)}[(\mu+\kappa)\xi_m^2 + \rho\omega^2],\\
v_{erm} &= g_1 s_{2m}H^*_{e-4,r+2}(-\xi_m^2),\\
v_{e+2,rm} &= g_1 s_{2m}H^*_{r+2,e-4}(-\xi_m^2),\\
v_{9qrm} &= g_1 s_{2m}H^*_{q+2,r+2}(-\xi_m^2), \quad q,r = 1,2,\ m = 1,2,3,4,5,\ e = 6,7.
\end{aligned}
\tag{4.4}
$$

Proof. When using

$$\mathbf{I}_{\zeta_m}(\mathbf{x}) = -\frac{1}{\xi_m^2}(\mathrm{grad}\,\mathrm{div} - \mathrm{curl}]curl)\zeta_m(\mathbf{x}), \qquad \mathbf{x} \neq \mathbf{0}$$

and Eqs. (3.10), (3.21), (3.23), and (3.25), we obtain

$$\begin{aligned}\mathbf{\Theta}^{(1)}(x) &= [g_2(\gamma\Delta + \mu_1)\Gamma_1(\Delta)\mathbf{I} + r_{11}(\Delta)\,\mathrm{grad}\,\mathrm{div}]\sum_{m=1}^{7} s_{1m}\zeta_m(x) \\ &= \sum_{m=1}^{7} s_{1m}[\{r_{l1}(-\xi_m^2) - \frac{g_2}{\xi_m^2}(-\gamma\xi_m^2 + \mu_1)\Gamma_1(-\xi_m^2)\}\,\mathrm{grad}\,\mathrm{div} \\ &\quad + \frac{g_2}{\xi_m^2}(-\gamma\xi_m^2 + \mu_1)\Gamma_1(-\xi_m^2)\,\mathrm{curl}\,\mathrm{curl}]\zeta_m(x).\end{aligned} \tag{4.5}$$

Using $(4.3)_1$ in (4.5), we obtain

$$\begin{aligned}\mathbf{\Theta}^{(1)}(x) = \sum_{m=1}^{7} s_{1m}\Bigg[\Bigg\{-\frac{g_1}{\xi_m^2}\Gamma_2(-\xi_m^2)H_{l1}^{*}(-\xi_m^2)\Bigg\}\,\mathrm{grad}\,\mathrm{div} \\ + \frac{g_2}{\xi_m^2}(-\gamma\xi_m^2 + \mu_1)\Gamma_1(-\xi_m^2)\,\mathrm{curl}\,\mathrm{curl}\Bigg]\zeta_m(x).\end{aligned}$$

By virtue of Eq. (4.4) and the equalities

$$\begin{aligned}\Gamma_1(-\xi_m^2)s_{1m} &= \begin{cases} 0 & m = 1,2,3,4,5, \\ (-1)^m(\xi_6^2 - \xi_7^2)^{-1} & m = 6,7, \end{cases} \\ \Gamma_2(-\xi_m^2)s_{1m} &= \begin{cases} s_{2m} & m = 1,2,3,4,5, \\ 0 & m = 6,7. \end{cases}\end{aligned} \tag{4.6}$$

From Eq. (4.5), we obtain

$$\begin{aligned}\mathbf{\Theta}^{(1)}(x) =&\ \mathrm{grad}\,\mathrm{div}\sum_{m=1}^{5}\left[-\frac{g_1}{\xi_m^2}s_{2m}H_{l1}^{*}(-\xi_m^2)\right]\zeta_m(x) \\ &- \frac{g_2}{\xi_m^2}\,\mathrm{curl}\,\mathrm{curl}\sum_{e=6}^{7}\frac{(-1)^e g_2(\gamma\xi_e^2 - \mu_1)}{\xi_m^2(\xi_6^2 - \xi_7^2)}\zeta_e(x) \\ =&\ \mathrm{grad}\,\mathrm{div}\sum_{m=1}^{5} v_{1m}\zeta_m(x) - \mathrm{curl}\,\mathrm{curl}\sum_{e=6}^{7} v_{1e}\zeta_e(x).\end{aligned}$$

Other formulae of Theorem 2 can be proved in the similar manner.

Theorem 3. *The relationships*

$$\begin{aligned}
&\mathbf{\Lambda}_{gh}(x) - \mathbf{\Omega}_{gh}(x) = \text{const} + O(|x|),\\
&\frac{\partial^p}{\partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3}}[\mathbf{\Lambda}_{gh}(x) - \mathbf{\Omega}_{gh}(x)] = O(|x|^{1-p}),
\end{aligned} \tag{4.7}$$

and

$$|\mathbf{\Lambda}_{ej}(\mathbf{x})| < \text{const}|\mathbf{x}|^{-1},\ |\mathbf{\Lambda}_{e+3,j+3}(\mathbf{x})| < \text{const}|\mathbf{x}|^{-1},\ |\mathbf{\Lambda}_{nn}(\mathbf{x})| < \text{const}|\mathbf{x}|^{-1} \tag{4.8}$$

hold in the neighborhood of the origin, where $p = p_1 + p_2 + p_3$, $p \geq 1$, $p_j \geq 0$, $e, j = 1, 2, 3$, $g, h = 1, 2, \ldots, 9$, $n = 7, 8, 9$.

Proof. It is evident from Theorem 2 and Corollary 2 that

$$\mathbf{\Lambda}^{(1)}(\mathbf{x}) - \mathbf{\Omega}^{(1)}(\mathbf{x}) = \mathbf{G}(\mathbf{x}), \tag{4.9}$$

where

$$\begin{aligned}
&\mathbf{G}(\mathbf{x}) = \|G_{em}(x)\|_{3\times 3} = \operatorname{grad}\operatorname{div}\zeta^{(1)}(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\zeta^{(2)}(\mathbf{x}),\\
&\zeta^{(1)}(\mathbf{x}) = \sum_{m=1}^{5} v_{1m}\zeta_m(\mathbf{x}) - \frac{1}{\alpha_4}\lambda_1(\mathbf{x}),\\
&\zeta^{(2)}(\mathbf{x}) = \sum_{e=6}^{7} v_{1e}\zeta_e(\mathbf{x}) - \frac{1}{\mu+\kappa}\lambda_1(\mathbf{x}).
\end{aligned} \tag{4.10}$$

From Eq. (4.10), in the neighborhood of the origin, we have

$$\begin{aligned}
\zeta^{(1)}(\mathbf{x}) &= -\frac{1}{8\pi}\left[2\sum_{m=1}^{5} v_{1m}\sum_{n=0}^{\infty}\frac{i^n\xi_m^n}{n!}|\mathbf{x}|^{n-1} - \frac{1}{\alpha_4}|\mathbf{x}|\right]\\
&= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{m=1}^{5} v_{1m} - |\mathbf{x}|\left(\sum_{m=1}^{5} v_{1m}\xi_m^2 + \frac{1}{\alpha_4}\right)\right]\\
&\quad - \frac{i}{4\pi}\sum_{m=1}^{5} v_{1m}\xi_m + \zeta^{(3)}(\mathbf{x}),\\
\zeta^{(2)}(\mathbf{x}) &= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{e=6}^{7} v_{1e} - |\mathbf{x}|\left(\sum_{e=6}^{7} v_{1e}\xi_e^2 + \frac{1}{\mu+\kappa}\right)\right]\\
&\quad - \frac{i}{4\pi}\sum_{e=6}^{7} v_{1e}\xi_e + \zeta^{(4)}(\mathbf{x}),
\end{aligned} \tag{4.11}$$

where

$$\zeta^{(3)}(x) = -\frac{1}{4\pi}\sum_{m=1}^{5} v_{1m}\sum_{n=3}^{\infty}\frac{i^n\xi_m^n}{n!}|\mathbf{x}|^{n-1},$$
$$\zeta^{(4)}(x) = -\frac{1}{4\pi}\sum_{e=6}^{7} v_{1e}\sum_{n=3}^{\infty}\frac{i^n\xi_e^n}{n!}|\mathbf{x}|^{n-1}. \tag{4.12}$$

Therefore, from Eq. (4.12), in the neighborhood of the origin, we obtain

$$\zeta^{(p)}(\mathbf{x}) = O(|\mathbf{x}|^2), \qquad \frac{\partial}{\partial x_e}\zeta^{(p)}(\mathbf{x}) = O(|\mathbf{x}|),$$
$$\frac{\partial^2}{\partial x_e \partial x_m}\zeta^{(p)}(\mathbf{x}) = \mathrm{const} + O(|\mathbf{x}|), \qquad e,m = 1,2,3, \quad p = 3,4. \tag{4.13}$$

By virtue of Eq. (4.11) and the inequalities

$$\sum_{m=1}^{5} v_{1m} = \sum_{e=6}^{7} v_{1e} = -\frac{1}{\rho\omega^2},$$
$$\sum_{m=1}^{5} v_{1m}\xi_m^2 + \frac{1}{\alpha_4} = 0,$$
$$\sum_{e=6}^{7} v_{1e}\xi_e^2 + \frac{1}{\mu+\kappa} = 0,$$
$$(\mathrm{grad\,div} - \mathrm{curl\,curl})\frac{1}{|\mathbf{x}|} = \Delta\frac{1}{|\mathbf{x}|} = 0, \qquad \mathbf{x} \neq \mathbf{0}. \tag{4.14}$$

From Eq. (4.10), we obtain

$$\mathbf{G}(\mathbf{x}) = \mathrm{grad\,div}\,\zeta^{(1)}(x) - \mathrm{curl\,curl}\,\zeta^{(2)}(x). \tag{4.15}$$

When using Eqs. (4.13) and (4.14) in Eq. (4.9), we obtain the relationship $(4.7)_1$ for $g,h = 1,2,3$. Similarly, other formulae of Eq. (4.7) can be proved.

We can obtain inequalities (4.8) from Eqs. (4.7) as

$$|\mathbf{\Omega}_{mn}(\mathbf{x})| < \mathrm{const}|\mathbf{x}|^{-1}, \qquad |\mathbf{\Omega}_{m+3,n+3}(\mathbf{x})| < \mathrm{const}|\mathbf{x}|^{-1},$$
$$|\mathbf{\Omega}_{hh}(\mathbf{x})| < \mathrm{const}|\mathbf{x}|^{-1}, \qquad m,n = 1,2,3, \quad h = 7,8,9.$$

Hence, the matrix $\mathbf{\Omega}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{\Lambda}(\mathbf{x})$ in the neighborhood of the origin.

5. Special cases

(i) Neglecting the thermal and micropolarity effect in system of equations (2.3) yields the system of steady vibrations for homogeneous isotropic elastic material with double porosity as follows:

$$
\begin{aligned}
&[\mu\Delta + \rho\omega^2]\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + b\,\mathrm{grad}\,\varphi + d\,\mathrm{grad}\,\psi = 0,\\
&(\alpha\Delta + \mu_2)\varphi + (b_1\Delta - \alpha_3)\psi - b\,\mathrm{div}\,\mathbf{u} = 0,\\
&(b_1\Delta - \alpha_3)\varphi + (\gamma_0\Delta + \mu_3)\psi - d\,\mathrm{div}\,\mathbf{u} = 0.
\end{aligned}
\tag{5.1}
$$

The derived fundamental solution for the system of equations (5.1) is similar to the solution obtained by Svanadze [51].

(ii) In the absence of single porosity parameter in the system of equations (2.3), we obtain the system of steady vibrations for homogeneous isotropic micropolar thermoelastic material with voids as follows:

$$
\begin{aligned}
&[(\mu+\kappa)\Delta + \rho\omega^2]\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}\\
&\qquad +\kappa\,\mathrm{curl}\,\mathbf{\Phi} + b\,\mathrm{grad}\,\varphi - \beta\,\mathrm{grad}\,T = 0,\\
&(\gamma\Delta + \mu_1)\mathbf{\Phi} + (\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{\Phi} + \kappa\,\mathrm{curl}\,\mathbf{u} + c_0\,\mathrm{grad}\,\varphi = 0,\\
&(\alpha\Delta + \mu_2)\varphi - b\,\mathrm{div}\,\mathbf{u} + \gamma_1 T - c_0\,\mathrm{div}\,\mathbf{\Phi} = 0,\\
&(k_3\Delta - \rho C^*)T - \beta T_0\,\mathrm{div}\,\mathbf{u} - \gamma_1 T_0\varphi = 0.
\end{aligned}
\tag{5.2}
$$

Obtaining the fundamental solution of the system of equations (5.2) is the same as the one given by Ciarlette *et al.* [45].

(iii) In the absence of single porosity parameter and thermal effect in the system of equations (2.3), the system of steady vibrations for homogeneous isotropic micropolar elastic material with voids is

$$
\begin{aligned}
&[(\mu+\kappa)\Delta + \rho\omega^2]\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \kappa\,\mathrm{curl}\,\mathbf{\Phi} + b\,\mathrm{grad}\,\varphi = 0,\\
&(\gamma\Delta + \mu_1)\mathbf{\Phi} + (\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{\Phi} + \kappa\,\mathrm{curl}\,\mathbf{u} + c_0\,\mathrm{grad}\,\varphi = 0,\\
&(\alpha\Delta + \mu_2)\varphi - b\,\mathrm{div}\,\mathbf{u} - c_0\,\mathrm{div}\,\mathbf{\Phi} = 0.
\end{aligned}
\tag{5.3}
$$

The resulting fundamental solution obtained from the system of equations (5.3) is in agreement with those obtained by Scarpetta [42].

(iv) In the absence of micropolarity effect, we obtain the system of steady vibrations for homogeneous isotropic thermoelastic material with double porosity as

$$
\begin{aligned}
&(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + b\,\mathrm{grad}\,\varphi + d\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,T = 0,\\
&(\alpha\Delta + \mu_2)\varphi + (b_1\Delta - \alpha_3)\psi - b\,\mathrm{div}\,\mathbf{u} + \gamma_1 T = 0,\\
&(b_1\Delta - \alpha_3)\varphi + (\gamma_0\Delta + \mu_3)\psi - d\,\mathrm{div}\,\mathbf{u} + \gamma_2 T = 0,\\
&(k_3\Delta - \rho C^*)T - \beta T_0\,\mathrm{div}\,\mathbf{u} - \gamma_1 T_0\varphi - \gamma_2 T_0\psi = 0.
\end{aligned}
\tag{5.4}
$$

The derived fundamental solution from the system of equations (5.4) is similar to the solution obtained by Scarpetta *et al.* [25], with some modification.

6. Concluding remarks

1. The constructed fundamental solution $\mathbf{\Lambda}(\mathbf{x})$ of the system (2.3) can be used

(i) To solve the boundary value problems by using boundary element method.

(ii) For constructing the surface and volume potentials and establishing their basic properties [40].

(iii) For investigating three-dimensional boundary value problems in micropolar thermoelastic materials with double porosity by potential method [40].

2. It is possible to represent the fundamental solutions of the systems of equations in different theories of continuum mechanics by using the method applied in this paper.

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