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Random composite: stirred or shaken?

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A James Bond's (JB) catchphrase "shaken, not stirred" is explored for the problem of effective conductivity of composites. The superconductivity critical index s for the conductivity of random non-overlapping disks turns out to be distinctly different for shaking and stirring protocols. In the case of stirring modeled by random walks the formula $s(\tau) = 0.5 + 0.8\sqrt[3]{\tau}$ is deduced for evolution of the critical index with the normalized time $0 \le \tau \le 1$, which is proportional to the number of random walks and serving as the disorder measure. Strikingly, the coefficient 0.8 is very close to the critical index for shaking protocol and 0.5 is the critical index for regular lattices. The obtained formula for s is based on the analytical solution to the 2D conductivity problem of randomly distributed disks up to $O(x^{19})$, where x denotes the concentration of inclusions and its extension to special 3D composites.

Key words: effective conductivity of composites, superconductivity critical index, shaking and stirring protocols, random composites.

1. Introduction

Estimation of the effective properties began with Maxwell's approach (1873) and Clausius-Mossotti (1850) formula for dilute random composites. The modern constructive homogenization theory is still of great interest in connection with vital fields of nanocomposites [1–5], interdependent networks [6], and in research of an invisibility cloak (see [7] and the works cited therein).

First, we discuss 2D problems. We consider the most prolific case of ideally conducting circular discs with the concentration x, embedded regularly or *randomly* in an otherwise uniform locally isotropic host. The conductivity of the host is normalized to unity. The considered problem is called the superconductivity problem. The term *random* can be described from the probabilistic point of view. To this end, we introduce a parameter τ, corresponding to a probabilistic distribution of disks. In particular, regular and, generally, deterministic locations of disks correspond to degenerate distributions with a single probabilistic event. The effective conductivity is presented then in the form of series in the volume fraction x (concentration) of the disks on the plane $\sigma^{(\tau)}(x) = 1 + 2x + 2x^2 + a_3^{(\tau)}x^3 + a_4^{(\tau)}x^4 + \cdots$. The effective properties of dilute composites are described by famous Clausius–Mossotti formula obtained from the series by cutting the terms $O(x^3)$. Rayleigh (1893), and McPhedran *et al.* [12, 13], calculated a number terms for the series of regular composites.

Bruggeman (see, e.g., [19]), suggested a self-consistent formalism leading to the family of effective medium approximations (EMAs) [1, 2, 3, 16, 20]. EMAs could be applied to various mixtures [21], including even quantum tunneling in the most intriguing case of nanocomposites [4, 5]. For macroscopically isotropic composites, the second-order term in x does not depend on the location of inclusions, while the third-order term does [8, 9, 10, 11]. This implies that any EMA is valid only up to the third-order term. For macroscopically anisotropic composites, the EMA can be applied with confidence only within the first-order approximation, that is, it is impossible to write a universal formula independent of τ. We focus our attention on estimation of the coefficients $a_m^{(\tau)}$ for randomly located inclusions and refer to [16, 17, 18] for the theory of corresponding bounds. Every deterministic or random composite has its series representation $\sigma^{(\tau)}$ and the same statement holds for the dependence of critical index on τ.

Recently, a novel approach in the theory of $2D$ composites was suggested in [8]. The line of thought dedicated to derivation of an arbitrary long series in x, and allowing to take into account the particles spatial arrangements (e.g., distribution of particles in a matrix) was developed. The latter task accomplished through a direct computer simulations for lattice, or by means of some Monte Carlo (MC) algorithm (protocol depending on τ) for off-lattice continuum percolation models [22].

In the current work, we compare two different protocols: rather intuitive shaking and stirring. In the context of 2D problems, shaking and stirring mean locations of the infinitely long unidirectional cylinders (fibers) in which random perturbations of fibers take place in the section plane perpendicular to fibers. We find that the critical index for superconductivity (or conductivity) is protocol-dependent as well. In the framework of random walks, the interpolation formula is deduced for evolution of the critical index s with the measure of disorder τ:

$$s(\tau) = 0.5 + 0.8\sqrt[3]{\tau}, \qquad 0 \le \tau \le 1. \tag{1.1}$$

The method of section can be applied to random protocols for 3D composites similar to [25], and it is performed in Section 6 for the special type of random composites.

2. Elements of theory

The deterministic boundary value problem governed by Laplace's equation, in the case of non-overlapping disks can be solved exactly for arbitrary locations of inclusions [8, 9, 10, 11]. The effective conductivity is written as an expression which contains the geometrical and physical parameters, such as radius of the disk and material constants, explicitly in a symbolic form. The exact formula does not contain any free parameters. It is written in the form presented above, with all coefficients $a_m^{(\tau)}$ being expressed in exact closed form. In practice one should truncate the series, hence an approximate formula arises.

In the random case, the local conductivity tensor can be considered as a random function of spatial variables. First, deterministic boundary value problem should be solved for arbitrary locations of inclusions, i.e., for all events in the considered probabilistic space by the method of functional equations [8, 9, 10, 11]. When an approximate formula for the deterministic case is deduced, the random case is treated through the ensemble averaging performed through direct MC computations. More precisely, the mathematical expectation $\langle a_m^{(\tau)} \rangle$ is calculated in the framework of the fixed probabilistic distribution of disks. Thus we avoid computation of the correlation functions [18], and compute $\langle a_m^{(\tau)} \rangle$ through their weighted moments, conveniently expressed through the sums of products of Eisenstein functions [8, 9, 10, 11]. These moments yield a direct method for computation of the effective properties, which does not involve knowledge of the correlation functions. It follows from simulations for the uniform distribution of a non-overlapping disks, that in order to reach high accuracy in the effective conductivity one needs to solve the corresponding boundary value problems for at least 81 inclusions per cell, repeated at least 1500 times.

The effective properties are obtained after averaging explicit analytical expressions for the deterministic composites over the probabilistic space. The structure of the space reflects the actual physical means to create randomness, such as shaking and stirring. The resulting truncated long series for square, hexagonal and fully random structures were explicitly presented in [26, 27, 28]. The method to extract the critical index s from the polynomials was also described.

In the present paper, we construct a set of truncated series considering τ as a non-negative disorder parameter, $\tau = 0$ corresponds to regular arrays and $\tau = \infty$ to the theoretically disordered location of disks obeying the uniform non-overlapping distribution. This sets of polynomial yields the dependence $s = s(\tau)$

of the critical index on the degree of disorder τ. The effective conductivity is expected to tend to infinity as a power-law, as the concentration x tends to the maximal value x_c for the hexagonal array $\sigma(x) \simeq A(x_c - x)^{-s}$. The critical superconductivity index (exponent) s for the lattice percolation problem is believed to be close to 1.3 [29]. The critical amplitude A is a non-universal parameter. From the phase interchange theorem [33] it follows that in two-dimensions, the superconductivity index is equal to the conductivity index t. Various experiments confirm the value of $t \approx 1.3$ [34, 35, 36, 37]. For regular arrays of cylinders, the index is much smaller, $s = \frac{1}{2}$ [1, 2, 3, 12, 13, 14, 15] and the critical amplitude is also known with good precision for square and hexagonal lattices [12, 13, 14, 15, 26].

In the general case of continuum percolation, the value of t can be much larger than 1.3 [38, 39, 40, 41, 42, 43], which is in agreement with the experiments in [1]. It is believed that continuum percolation problems can be mapped into the lattice problems with conducting bonds whose conductivity is drawn from a probability density law [38, 39, 40, 41, 42, 43].

In this paper, the representative hexagonal cell serves as the domain where random composite is generated as a probabilistic distribution of non-overlapping disks, by means of some Monte-Carlo algorithm (protocol). The number of inclusions per cell can be taken arbitrarily large and still the shape of the cell does somewhat influences the final result.

In [26, 27, 28], the random sequential addition (RSA) protocol was utilized. This RSA-protocol produced the following series in concentration [26]: $\sigma^{\text{RSA}} = 1 + 2x + 2x^2 + 5.00392x^3 + 6.3495x^4$. The coefficients on x^k $(k = 5, 6, 7, 8)$ vanish with very high precision. A good estimate for the critical index s was obtained by applying the $D-$Log Padé method [44]. The result is $s = 1.28522$ for the critical index. The corrected regular lattice approximation [26, 27, 28], gives a higher value of $s = 1.31561$.

3. Random walks

Random walks (RWs) [8, 9], can be considered as a model of mechanical stirring (!) of the host particles with matrix [45]. Initially, random points (100 points per periodicity cell is taken in simulation) are generated, at first being put onto the nodes of the hexagonal array. Let each point move in a randomly chosen direction with some step. Thus, each center obtains a new coordinate. This move is repeated many times, without particles overlap. If particle does overlap with some previously generated particle, it remains an overlap of particles blocked at this step. After a large number of walks (steps), the obtained locations of the centers can be considered as a sought statistical realization, defining random composite. The number of steps N is proportional to the real time scale.

The coefficients $a_i^{(\tau)}$ for disordered locations do not differ much after 30 steps. Hence, we introduce the time scale $\tau = N/30$ which is equal to unity when the full disorder is achieved in our simulations. All the sought properties should be considered as functions of τ. In particular, approximation polynomials for the effective conductivity (truncated power series) acquire the “time" dependence, as well as the critical index $s(\tau)$ and the amplitude $A(\tau)$ extrapolated from them. The RW protocol can be applied for arbitrary concentrations including those very close to x_c, where $x_c = \frac{\pi}{\sqrt{12}}$ stands also for the maximum volume fraction of $2D$ composites achieved for the regular hexagonal array of disks.

Below, the series in the truncated numerical form are presented for $\tau = 1$

$$\sigma^{\mathrm{RW}}(x) = 1 + 2x + 2x^2 + 5.13057x^3 + 5.9969x^4 + \Delta(x). \tag{3.1}$$

The coefficients on x^k ($k = 5, 6, 7, 8$) are small. The remainder $\Delta(x)$ has a highly irregular form (see Supplementary Material[1]), and does not contribute to the critical properties or general expressions for the conductivity. The approximation polynomials σ^{RW} and σ^{RSA} appear numerically as almost the same. The polynomial σ^{RSA} was constructed by fitting at $x = 0.1, 0.2, \ldots, 0.9$, while σ^{RW} by fitting at $x = 0.3, 0.35, \ldots, 0.9$. Always present starting terms, $1 + 2x + 2x^2$, ensure that the region of small concentrations is approximated properly. When the set of points $x = 0.1, 0.2, \ldots, 0.9$ is used to construct σ^{RW}, the results for critical properties appear worse [26, 27, 28].

Let us employ a standard approach to the critical index calculation [44]. To this end, let us apply the following transformation:

$$z = \frac{x}{x_c - x} \tag{3.2}$$

to the original series, to make calculations with different approximants more convenient. To such transformed series we apply the $D - \mathrm{Log}$ transformation. By applying the Padé approximants $P_{n,n+1}(z)$ to the transformed series one can readily obtain the sequence of approximations $s_n = \lim_{z\to\infty}(zP_{n,n+1}(z))$. The result $s_2 = 1.24078$ is reasonably good, but can be further improved.

The corrected regular lattice approximation rests on the idea of corrected approximants [26, 27, 28, 46, 47, 48]. To start such approximation one has to select the initial approximation to be corrected, in order to describe regular hexagonal array of inclusions, namely

$$f^*_{0,r}(x) = \frac{(0.419645x + 1)^{3.45214}}{\sqrt{1 - 1.10266x}}. \tag{3.3}$$

This formula incorporates the critical index $\frac{1}{2}$ of the regular hexagonal lattice, the threshold for the hexagonal lattice and the two starting, effective medium

[1]See: http://am.ippt.pan.pl/supplementary/am-v68p229-suppl.pdf

terms from the series. Let us divide the original series by $f^*_{0,r}$, extracting the part corresponding to the random effects only. Then, let us express the new series in terms of z, apply $D-$Log transformation and call the transformed series $K_r(z)$. The transformed series can be processed with different approximants, e.g., iterated roots [46, 47, 48]. The following sequence of corrected approximations the critical index arises:

$$s_n = s^{(0)} + \lim_{z\to\infty}(z\, r_n(z)), \tag{3.4}$$

where $r_n(z)$ stands for the iterated root of n-th order[26], constructed for the series $K_r(z)$ with such a power at infinity that defines constant correction to the initial approximation $s^{(0)} = 1/2$. The second-order iterated root has a simple form

$$r_2(z) = \frac{v_0 z^2}{(v_2 z^2 + (v_1 z + 1)^2)^{3/2}}.$$

The parameters v_i are computed from the series $K_r(z)$. The effective conductivity can be expressed in a closed form [26]. For a very large $N = 300$, the results $s_2 = 1.3367$ and $A_2 = 1.6651$ are close to RSA and they tend to depend very weakly on Δ. If we simply set $\Delta = 0$, then they change to $s = 1.3275$ and $A = 1.6837$, which supports our view that only the starting four terms are relevant when the critical region is concerned.

4. Random shaking

Randomness can be also by introduced to the MC simulation by gentle shaking, through the random locations of the centers of the disks [49]. According to JB, stirring should be avoided and shaking is preferred.

Following [49], let us consider the unit cell with identical inclusions whose centers are random variables. Each center is uniformly distributed in a disk of the radius d called the shaking parameter. Centers of these disks form a hexagonal array on the plane whereas the disks by themselves do not form the periodic array. Hence, we investigate a random shaking of the disks about the periodic hexagonal array. The shaking parameter d does not have to be small and therefore our results are not perturbative. The parameter d is chosen so that the disks cannot touch each other. The original approach to random shaking (RS) in [49] did not include the threshold for continuum percolation.

Heuristically, the shaking geometries provide a reasonable approximation for random mixtures at not very high concentrations. There is not much room for the inclusions to move around, when their density is relatively high. Therefore, the inclusions can naturally form some kind of a random shaking pattern. For the shaking model, the cubic term depends on the random locations of inclusions.

The expansion is presented in the truncated numerical form as follows:

$$\sigma^{RS}(x) = 1 + 2x + 2x^2 + 2.50496x^3 + 1.34794x^4 + 2.28669x^5 + \cdots , \tag{4.1}$$

up to to the terms of 18-th order inclusively (see Supplementary Material). Applying the $D-$ Log Padé technique we find: $s_1 = 0.944643$, $s_2 = 0.803755$, $s_3 = 0.79748$, $s_4 = 0.808613$, $s_5 = 0.440396$, $s_6 = 0.329953$, $s_7 = 0.812099$, $s_8 = 0.816668$, and $s_9 = 0.812114$. The critical amplitude for $n = 9$ equals 1.53383. The result $s = 0.812114$ is weakly sensitive to the value of shaking parameter d used for computations. Thus, JB prefers the shaken composite ("martini") with the critical index of $0.81 - 0.82$, which is very much different from the stirred result 1.3.

5. Temporal crossover for RW

In this section we intend to obtain a dependence of the critical index and critical amplitude for the RW protocol, dependent on the degree of disorder quantified by the time τ. The formula is going to be constructed in such a manner that for "zero"-randomness ($\tau = 0$) it is going to behave as the regular hexagonal lattice. For "maximum"-randomness $\tau \to \infty$, we expect to have a random composite. All the cases with intermediate degrees of randomness for finite τ, are expected to fall in between the two cases of τ described above. The critical behavior of regular composites occurs for x very close to x_c, due to the direct contact of particles in the whole area of composite, with $s = 1/2$ without an explicit non-linearity or randomness. On the other hand, randomness dominates in the case of continuum percolation.

We concede that the two limiting cases of small τ (quasi-regular composite), and the large τ (random composite), should be considered separately. Final formula will be obtained by matching the two behaviors, as a "regular-to-random" crossover. In the case of large τ, we literally apply the technique of corrected regular lattice approximation described above, since it is fairly accurate for the random case for the number of steps $N = 30$. For very small τ, we have a regular hexagonal array of disks, and we consider a peculiar critical phenomena using different initial approximation, $f^*_{0,h}(\tau = 0)$, in place of $f^*_{0,r}$. General form:

$$f^*_{0,h} = b_0 + b_1 \frac{1}{\sqrt{x_c} - x} + b_2\sqrt{x_c - x} + b_3(x_c - x)$$

is the same as in [26, 27, 28], where the coefficients of the series were obtained from the exact formulae. In the present work, we derive approximating polynomial for the regular case following exactly the same procedure as for the random case. The following values for the coefficients in $f^*_{0,h}(\tau = 0)$ are as follows:

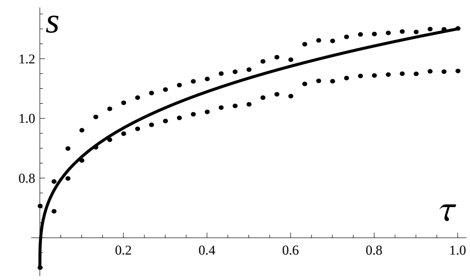

FIG. 1. Interpolation curve for the critical index $s(\tau) = 0.5 + 0.8\sqrt[3]{\tau}$ (shown as solid line), between the two sets of calculations. The upper set corresponds to the initial approximation $f^*_{0,r}$, while the lower set of points to the initial approximation $F^*_{0,h}$.

$b_0 = -6.94364$, $b_1 = 5.18603$, $b_2 = 3.33683$ and $b_3 = -0.749575$. The calculations of s and A for arbitrary τ, are identical to the random case described above. Complete expressions for the approximating polynomials for all τ can be found in Supplementary Material. Interpolation between the two sets of calculations for the critical index $s(\tau)$ and the curve (1.1) are shown and explained in Fig. 1. Good saturation of the results is achieved already at $N = 30$. Thus, for small and moderately large times the value of index is bounded by its regular and random values, $0.5 \leq t \leq 1.3$. It is believed that physics of 2D regular and irregular composites is related to the so-called “necks" certain areas between closely spaced disks. Randomness (stirring) adds to the regularly formed necks an additional random component. For a very strong disorder this additional contribution is, the above part estimated with a good precision for the shaking protocol. Similar statement holds for the critical amplitude. The initial approximation $f^*_{0,r}$ underestimates critical amplitude at small τ, while the initial approximation $f^*_{0,h}(\tau = 0)$ overestimates the amplitude at large τ. The amplitude is bounded by its random and regular values, $1.71 \leq A \leq 5.19$. In addition, not only regular or random cases but the whole spectrum of composites can be studied.

6. 3D extensions

The homogenization theory of random media [50] demonstrates that the effective properties of composites can be precisely determined as the mathematical expectation of the effective constants of the statistically representative cells. Consider a 3D cuboid cell $Q_{000} = \{(x, y, z) \in \mathbb{R}^3 : 0 < x < a,\ 0 < y < a, 0 < z < h\}$ of the size $a \times a \times h$ displayed in Fig. 2. It is assumed that unidirectional cylinders have the height h and their axis is parallel to the axis OZ. We have a disks distribution in each section of the cell perpendicular to the axis OZ. It is assumed that this distribution is isotropic in the plane XY. Let the cell

Q_{000} with fixed inclusions represents 3D random composites. Let all the cells $Q_{klm} = \{(x, y, z) \in \mathbb{R}^3 : k < x < a + k, l < y < a + l, m < z < h + m\}$ (k, l, m run over integers) to be obtained by the same random distribution of cylinders but with different statistical realizations. Examples of such composites for regular, shaken and RW distributions are displayed in Fig. 3. According to the homogenization theory [50], the 3D composite consisting of the cells Q_{klm} has the effective conductivity tensor

$$S = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}. \tag{6.1}$$

The component σ coincides with the effective conductivity of 2D composites discussed above. The component σ_z is equal to infinity because of percolation in the regular and shaken cases displayed in Fig. 3. The random case displayed in the third picture of Fig. 3 requires a separate investigation.

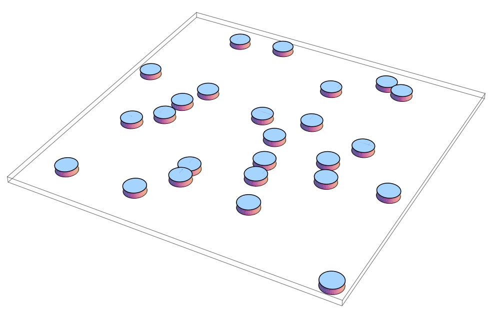

Fig. 2. Unidirectional cylinders in the 3D cuboid cell of the size $a \times a \times h$.

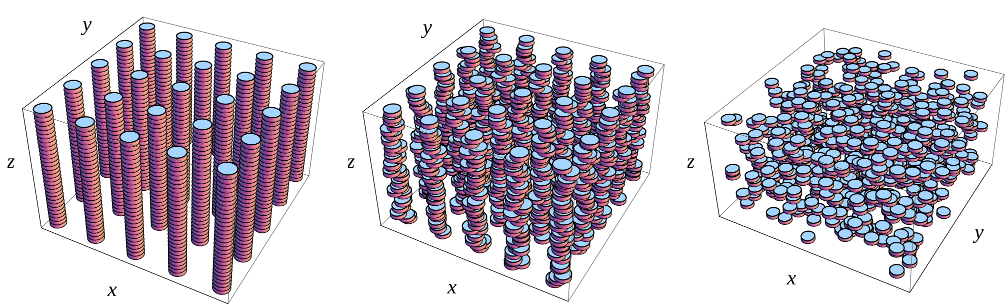

Fig. 3. 3D composites corresponding to regular, shaken and RW 2D structures.

7. Conclusion

The main result of the present paper is the interpolation formula (1.1) deduced for evolution of the critical index. We conclude that both shaking and stirring can, in principle, produce the same effective conductivity. However, shaking (RS protocol) is preferable to stirring (RW protocol) because in practice it is easier to control the shaking parameter d, then the stirring parameter τ. On the other hand, stirring is much better than shaking when one needs to create a random composite.

Interpolation is suggested for evolution of the critical index with numbers of steps (time), with a small-time limit dominated by regular composite critical behavior, and large-time limit dominated by random composite. Two different methods had to be applied to capture the different behaviors. Modern 3D printing techniques might be able eventually to produce actual composite from any pattern generated by any protocol.

Note that in the mechanics of continuum, the anti-plane elasticity can be solved by analogy to 2D conductivity problem. Hence, formula (1.1) is valid for the anti-plane elasticity problem for fibrous composites.

References

1. D.S. McLachlan, G. Sauti, *The AC and DC Conductivity of Nanocomposites*, Journal of Nanomaterials, **2007**, 30389, 9 pages, 2007.

2. J. Wu, D.S. McLachlan, *Scaling behaviour of the complex conductivity of graphite-boron nitride systems*, Phys. Rev. B **58**, 14880–14887, 1998.

3. D.S. McLachlan, G. Sauti, C. Chiteme, *Static dielectric function and scaling of the ac conductivity for universal and nonuniversal percolation systems*, Phys. Rev. B, **76**, 014201, 1–13, 2007.

4. G. Ambrosetti, C. Grimaldi, I. Balberg, T. Maeder, A. Danani, P. Ryser, *Solution of the tunneling-percolation problem in the nanocomposite regime*, Phys. Rev B, **81**, 155434, 1–12, 2010.

5. G. Ambrosetti, I. Balberg, C. Grimaldi, *Percolation-to-hopping crossover in conductor-insulator composites*, Phys. Rev. B, **82**, 134201, 2010.

6. M. Danziger, A. Bashan, S. Havlin, *Interdependent resistor networks with process-based dependency*, New J. Phys., **17**, 043046, 1–8, 2015.

7. J. O'Neill, Ö. Selsil, R.C. McPhedran, A.B. Movchan, N.V. Movchan, *Active cloaking of inclusions for flexural waves in thin elastic plates*, Q. J. Mechanics Appl. Math., **68**, 3, 263–288, 2015.

8. V. Mityushev, N. Rylko, *Maxwell's approach to effective conductivity and its limitations*, Q. J. Mechanics Appl. Math., **66**, 241–251, 2013.

9. R. Czapla, W. Nawalaniec, V. Mityushev, *Effective conductivity of random two-dimensional composites with circular non-overlapping inclusions*, Comput. Mat. Sci., **63**, 118–126, 2012.

10. V. Mityushev, *Steady heat conduction of a material with an array of cylindrical holes in the nonlinear case*, IMA J. Appl. Math., **61**, 91–102, 1998.

11. V. Mityushev, *Exact solution of the $\mathbb{R}$-linear problem for a disk in a class of doubly periodic functions*, JAFA **2**, 115–127, 2007.

12. J.B. Keller, *Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders*, J. Appl. Phys. **34**, 991–993, 1963.

13. W.T. Perrins, D.R. McKenzie, R.C. McPhedran, *Transport properties of regular array of cylinders*, Proc. R. Soc. A, **369**, 207–225, 1979.

14. R.C. McPhedran, *Transport properties of cylinder pairs and of the square array of cylinders*, Proc. R. Soc. Lond. A, **408**, 31–43, 1986.

15. R.C. McPhedran, L. Poladian, G.W. Milton, *Asymptotic studies of closely spaced, highly conducting cylinders*. Proc. R. Soc. A, **415**, 185–196, 1988.

16. D.J. Bergman, *The dielectric constant of a composite material-a problem in classical physics*, Phys. Rep., **43**, 377–407, 1978.

17. G.W. Milton, *The Theory of Composites*, Cambridge Univ. Press, 2002.

18. S. Torquato, *Random Heterogeneous Materials: Microstructure and Macroscopic Properties*, Springer-Verlag, New York, 2002.

19. A.V. Goncharenko, *Generalizations of the Bruggeman equation and a concept of shape-distributed particle composites*, Phys. Rev. E, **68**, 041108, 1–13, 2003.

20. M. Sahimi, B.D. Hughes, L. E. Scriven, H.T. Davis, *Real-space renormalization and effective-medium approximation to the percolation conduction problem*, Phys. Rev. B, **28**, 307–311, 1983.

21. S. Gluzman, A. Kornyshev, A. Neimark, *Electrophysical Properties of metal-solid electrolyte composite*, Phys. Rev. B, **52**, 927–938, 1995.

22. J.W. Eischen, S. Torquato, *Determining elastic behavior of composites by the boundary element method*, J. Appl. Phys., **74**, 159–170, 1993.

23. S. Torquato, T. M. Truskett, P.G. Debenedetti, *Is random close packing of spheres well defined?* Phys. Rev. Lett., **84**, 2064–2067, 2000.

24. S. Torquato, F.H. Stillinger, *Jammed hard-particle packings: From Kepler to Bernal and beyond*, Reviews of Modern Physica, **82**, 2634–2672, 2010.

25. N. Rylko, *Representative volume element in 2D for disks and in 3D for balls*, Journal of Mechanics of Materials and Structures, **9**, 427–439, 2014.

26. S. Gluzman, V. Mityushev, W. Nawalaniec, G. Starushenko, *Effective Conductivity and Critical Properties of a Hexagonal Array of Superconducting Cylinders, Contributions in Mathematics and Engineering. In Honor of Constantin Caratheodory*, P.M. Pardalos and T.M. Rassias [Eds.], Springer, 2016; arxiv.org/abs/1508.05068.

27. S. Gluzman, V. Mityushev, *Series, index and threshold for random 2D composite*, Arch. Mech., **67**, 75–93, 2015.

28. S. Gluzman, V. Mityushev, W. Nawalaniec, *Cross-properties of the effective conductivity of the regular array of ideal conductors*, Arch. Mech., **66**, 287–301, 2014.

29. D.C. Hong, S. Havlin, H.J. Herrmann, H.E. Stanley, *Breakdown of Alexander–Orbach conjecture for percolation: Exact enumeration of random walks on percolation backbones*, Phys. Rev. B, **30**, 4083–4086, 1984.

30. J.-M. Normand, H.J. Herrmann, M. Hajjar, *Precise calculation of the dynamical exponent of two-dimensional percolation*, J. Stat. Phys., **52**, 441–446, 1988.

31. D.J. Frank, C.J. Lobb, *Highly efficient algorithm for percolative transport studies in two dimensions*, Phys. Rev. B, **37**, 302–307, 1988.

32. P. Grassberger, *Conductivity exponent and backbone dimension in 2d percolation*, Physica A, **262**, 251–263, 1999.

33. J.B. Keller, *A theorem on the conductivity of a composite medium*, J. Math. Phys., **5**, 548–549, 1964.

34. L.N. Smith, C.J. Lobb, *Percolation in two-dimensional conductor-insulator networks with controllable anisotropy*, Phys. Rev. B, **20**, 3653–3658, 1979.

35. L. Benguigui, *Experimental study of the elastic properties of a percolating system*, Phys. Rev. Lett. **53**, 2028–2030, 1984.

36. Y. Chen, S. Schuh, *Effective properties of random composites: Continuum calculations versus mapping to a network*, Phys. Rev. E, **80**, 040103, 1–4, 2009.

37. N. Lebovka, M. Lisunova, Ye.P. Mamunya, N. Vygornitskii, *Scaling in percolation behaviour in conductive-insulating composites with particles of different size*, J. Phys. D., **39**, 2264–2272, 2006.

38. T.C. Lubensky, A.-M.S. Tremblay, *ϵ-expansion for transport exponents of continuum percolating systems*, Phys. Rev. B, **34**, 3408–3417, 1986.

39. S. Feng, B. Halperin, P.N. Sen, *Transport properties of continuum systems near the percolation threshold*, Phys. Rev. B, **35**, 197–214, 1987.

40. K.M. Golden, *Critical behavior of transport in lattice and continuum percolation models*, Phys. Rev. Lett., **78**, 3935–3938, 1997.

41. D.R. Baker, G. Paul, S. Sreenivasan, H.E. Stanley, *Continuum percolation threshold for interpenetrating squares and cubes*, Phys.Rev. E, **66**, 046136, 1–5, 2002.

42. M. Murat, S. Marianer, D.J. Bergman, *A transfer matrix study of conductivity and permeability exponents in continuum percolation*, J. Phys. A, **19**, L275–L279, 1998.

43. I. Balberg, *Limits on the continuum-percolation transport exponents*, Phys. Rev. B, **57**, 13351–13354, 1998.

44. G.A. Baker, P. Graves-Moris, *Padé Approximants*, Cambridge University, Cambridge, 1996.

45. V. Mityushev, W. Nawalaniec, *Basic sums and their random dynamic changes in description of microstructure of 2D composites*, Comput. Mater. Sci., **97**, 64–74, 2015.

46. S. Gluzman, V.I. Yukalov, *Extrapolation of perturbation-theory expansions by self-similar approximants*, European Journal of Applied Mathematics, **25**, 595–628, 2014.

47. S. GLUZMAN, V.I. YUKALOV, *Self-similar extrapolation from weak to strong coupling*, J. Math. Chem., **48**, 883–913, 2010.

48. V. YUKALOV, S. GLUZMAN, *Self similar crossover in statistical physics*, Physica A, **273**, 401–415, 1999.

49. L. BERLYAND, V. MITYUSHEV, *Generalized Clauisius-Mossotti Formula for Random Composite with Circular Fibers*, J. Stat. Phys., **102**, 115–145, 2001.

50. J.J. TELEGA, *Stochastic homogenization: convexity and nonconvexity*, [in:] P.P. Castañeda, J.J. Telega, B. Gambin [Eds.], Nonlinear Homogenization and its Applications to Composites, Polycrystals and Smart Materials, NATO Science Series, Kluwer Academic Publishers, Dordrecht, 305–346, 2004.

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