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A parametric study on the elastic-plastic deformation of a centrally heated two-layered composite cylinder with free ends

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In this paper, an elastic-plastic deformation of a centrally heated two-layered composite cylinder with free ends subjected to uniformly distributed internal energy generation within an inner cylinder is studied using Tresca's yield condition and its associated flow rule. Stress, strain and displacement distributions in the composite cylinder made of elastic-perfectly plastic material are derived considering the influence of geometric parameters as well as material properties such as yield strength, modulus of elasticity, Poisson's ratio, coefficient of thermal conduction and coefficient of thermal expansion. Yielding starts at the outer boundary or at the axis corresponding to an 'edge regime' of Tresca's prism in both cases. Propagations of the plastic regions are studied due to an increase of a heat generation.

Key words: elastic-plastic stress analysis, thermal stress, composite cylinder, Tresca's criterion, yielding.

1. Introduction

In many engineering applications, it is necessary to conduct an elastic-plastic stress analysis. One of the reasons for this analysis is a thermal stress generated due to heat generation in bodies such as electrical conductors and nuclear reactors. The determination of the elastic-plastic and thermal stresses in cylindrical or spherical bodies has been achieved in a wide range of applications [1–4]. Gulgec studied the influence of the temperature dependence of the yield stress on the stress distribution in a heat generating elastic-plastic cylinder with fixed ends [5]. Different studies on the similar topic were also conducted by various authors [6–9]. In a similar problem, Eraslan [10] subjected concentric tubes to free and radially constrained boundaries . In his study, the heat was assumed to be generated at a constant rate either within the inner or the outer tube. Orcan investigated the thermal stresses for both elastic and plastic deformations in a heat generating elastic-plastic homogeneous cylinder with free ends [11]. More recently, Ozturk and Gulgec studied first an onset of yield due to material properties, and next they conducted an elastic-plastic stress analysis of two-layered heat generating composite cylinder with fixed ends, in addition to the effect of the material properties on the onset of yield due to thermal stresses, respectively [12, 13].

The purpose of this paper is to obtain the complete solution and present a parametric analysis of elastic-plastic behaviour of the composite cylinder with free ends, which consists of two materials concentric to each other.

2. Problem statement and temperature distribution

The geometry of the considered problem consists of an infinitely long solid cylinder of a radius "a" placed inside a tube with the inner radius "a" and the outer radius "b" (Fig. 1). This composite system is assumed to have free ends and the problem can be treated as a generalized plane strain problem. Since the end effects are ignored in the present analysis, the solution will be valid for sections located sufficiently away from the ends. It is assumed that heat is generated only within the inner cylinder at a uniform rate and transferred outside through the surrounding tube to an ambient kept at constant temperature. The material parameters are supposed to be independent of the temperature, because the equations derived in this paper are general equations that are not aimed for a specific material. It is not possible to derive a unique correlation for all materials, related with the dependence on the temperature. So, anyone who wants to use these equations for a specific material can insert the correlation for temperature dependence into the general equations and then follow the procedure given in this paper.

The involved geometric and the thermo-mechanical parameters can be listed as follows:

a, b: inner and outer radii of the tube,
E_1, E_2: elasticity moduli,
α_1, α_2: coefficients of thermal expansion,
ν_1, ν_2: Poisson's ratios,
λ_1, λ_2: coefficients of thermal conduction,
$\sigma_0^I, \sigma_0^{II}$: yield strength of the solid cylinder and yield strength of the tube, respectively.

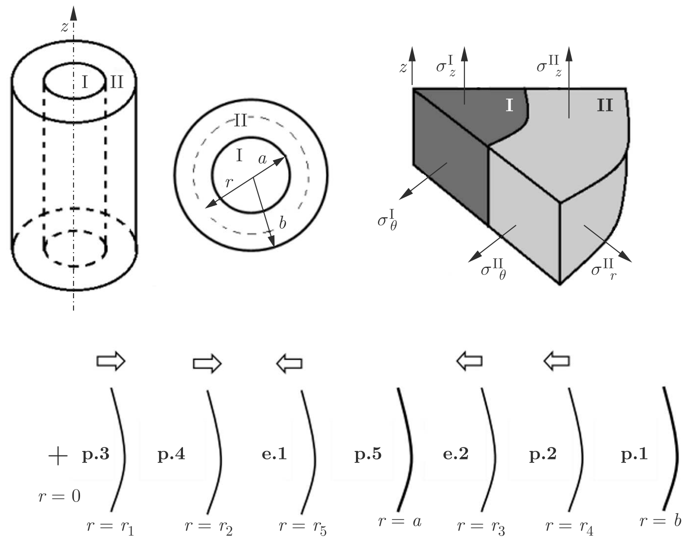


FIG. 1. Composite cylinder geometry and plastic regions with their propagation directions ("e" stands for elastic region, "p" for plastic region).

The temperature distributions of the inner solid cylinder and the outer tube are denoted by T_1 and T_2 respectively. The reference temperature at the outer surface is T_0. The unsteady, coupled heat equation in polar coordinates, in its general form for the solid cylinder can be written as [14]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda_1 r\frac{\partial T_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\lambda_1\frac{\partial T_1}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\lambda_1\frac{\partial T_1}{\partial z}\right)+q''' = \rho_1 C_{p1}\frac{\partial T_1}{\partial t}+(3\lambda_1+2\mu_1)\alpha_1 T_1\frac{\partial D_1}{\partial t}, \tag{2.1}$$

where r, ϕ, z are the cylindrical coordinates, t is the time, T_1 is temperature distribution, q''' is internal energy generation per unit volume per unit time, ρ_1 is the density, C_{p1} is the specific heat, λ_1 and μ_1 are Lame constants, and D_1 is the dilatation. If the internal energy generation is slowly increased, the governing equation for the radial temperature distribution is reduced to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda_1 r\frac{\partial T_1}{\partial r}\right)+q'''=0. \tag{2.2}$$

The temperature distribution for the solid (inner) cylinder becomes

$$T_1=-\frac{q'''r^2}{4\lambda_1}+A_1\ln r+A_2. \tag{2.3}$$

At $r = 0$, T_1 has to be finite, hence $A_1 = 0$ and

$$T_1 = -\frac{q''' r^2}{4\lambda_1} + A_2. \tag{2.4}$$

Solving Eq. (2.2) for $q''' = 0$, since there is no heat generation in the tube (outer hollow cylinder), we find that

$$T_2 = A_3 \ln r + A_4. \tag{2.5}$$

Using the conditions of continuity of the temperature and heat flux at the interface, the temperature is finite at $r = 0$ and equal to T_0 at $r = b$. The temperature distributions in the solid cylinder and in the tube take the following forms respectively:

$$T_1 = -\frac{q'''}{4}\left(\frac{r^2 - a^2}{\lambda_1} + \frac{2a^2}{\lambda_2}\ln\frac{a}{b}\right) + T_0, \tag{2.6}$$

$$T_2 = -\frac{q''' a^2}{2\lambda_2}\ln\frac{r}{b} + T_0. \tag{2.7}$$

3. Elastic deformation

By means of the basic equations of linear elasticity theory, the stress and the radial displacement distributions take the following forms [13].

3.1. Solid cylinder

$$\sigma_r^I = -\frac{E_1\alpha_1}{(1-\nu_1)}\theta^I(0,r) + \frac{C_1}{2} + \frac{C_2}{r^2}, \tag{3.1}$$

$$\sigma_\theta^I = \frac{E_1\alpha_1}{(1-\nu_1)}[\theta^I(0,r) - T_1] + \frac{C_1}{2} - \frac{C_2}{r^2}, \tag{3.2}$$

$$\sigma_z^I = -\frac{E_1\alpha_1}{(1-\nu_1)}T_1 + \nu_1 C_1 + E_1\varepsilon_z, \tag{3.3}$$

$$u^I = -\frac{1+\nu_1}{E_1}\frac{C_2}{r} + \left[\alpha_1\frac{1+\nu_1}{1-\nu_1}\theta^{\mathrm{I}}(0,r) + \frac{C_1}{2E_1}(1+\nu_1)(1-2\nu_1) - \nu_1\varepsilon_z\right]r. \tag{3.4}$$

3.2. Tube

$$\sigma_r^{II} = -\frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,r) + \frac{C_3}{2} + \frac{C_4}{r^2}, \tag{3.5}$$

$$\sigma_\theta^{II} = \frac{E_2\alpha_2}{(1-\nu_2)}[\theta^{II}(a,r) - T_2] + \frac{C_3}{2} - \frac{C_4}{r^2}, \tag{3.6}$$

$$\sigma_z^{II} = -\frac{E_2\alpha_2}{(1-\nu_2)}T_2 + \nu_2 C_3 + E_2\varepsilon_z, \tag{3.7}$$

$$u^{II} = -\frac{1+\nu_2}{E_2}\frac{C_4}{r} + \left[\alpha_2\frac{1+\nu_2}{1-\nu_2}\theta^{II}(a,r) + \frac{C_3}{2E_2}(1+\nu_2)(1-2\nu_2) - \nu_2\varepsilon_z\right]r. \tag{3.8}$$

In the above expressions,

$$\theta^{\rm I}(0,r) = \frac{1}{r^2}\int\limits_0^r T_1 r\,dr \quad \text{and} \quad \theta^{\rm II}(a,r) = \frac{1}{r^2}\int\limits_a^r T_2 r\,dr.$$

The stresses and displacement are continuous at the interface and are bounded at the axis, the surface of the tube is free of any traction, and for an infinitely long cylinder with free ends

$$\int\limits_0^a \sigma_z^I 2\pi r dr + \int\limits_a^b \sigma_z^{II} 2\pi r\,dr = 0.$$

When these conditions are enforced, it is found that $C_2 = 0$ and the other unknowns are determined as follows:

$$C_1 = 2\left[\frac{E_1\alpha_1}{(1-\nu_1)}\theta^I(0,a) + \frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,b) + C_4\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right], \tag{3.9}$$

$$C_3 = 2\left[\frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,b) - \frac{C_4}{b^2}\right], \tag{3.10}$$

$$C_4 = \frac{\theta^{II}(a,b)D_1 - \theta^I(0,a)\frac{2E_1\alpha_1}{(1-2\nu_2)} - \frac{E_1\varepsilon_z(\nu_2-\nu_1)}{(1+\nu_1)(1-2\nu_1)}}{\frac{1}{a^2} - \frac{1}{b^2} + \frac{E_1}{E_2}\frac{(1+\nu_2)}{(1+\nu_1)(1-2\nu_1)}\left[\frac{(1-2\nu_2)}{b^2} + \frac{(1+\nu_2)}{a^2}\right]}, \tag{3.11}$$

where $D_1 = -\frac{E_2\sigma_0^{II}}{(1-\nu_2)} + E_1\alpha_2\frac{((1+\nu_2)(1-2\nu_2)}{(1+\nu_1)(1-2\nu_1)(1-\nu_2))}$,

$$\varepsilon_z = \frac{\frac{E_1\alpha_1}{(1-\nu_1)}\left[\frac{q'''}{4}a^2\left(-\frac{1}{2\lambda_1} + \frac{2}{\lambda_2}\ln\frac{a}{b}\right) + 2\nu_1\theta^I(0,a)\right] + \frac{E_2\alpha_2}{(1-\nu_2)}D_2D_3D_4}{\left[-E_1 - E_2\left(\frac{b^2}{a^2} - 1\right) + \frac{(\nu_1-\nu_2)}{(1+\nu_1)(1-2\nu_1)}E_1D_4\right]}, \tag{3.12}$$

where

$$D_2 = \frac{q'''}{2}\frac{a^2}{\lambda_2}\left[-\frac{b^2}{2a^2} - \left(\ln\frac{a}{b} - \frac{1}{2}\right)\right] + 2\theta^{II}(a,b)\left[\nu_1 + \nu_2\left(\frac{b^2}{a^2} - 1\right)\right],$$

$$D_3 = \theta^{II}(a,b)\left[-\frac{E_2\sigma_0^{II}}{(1-\nu_2)} + E_1\alpha_2\frac{(1+\nu_2)(1-2\nu_2)}{(1+\nu_1)(1-2\nu_1)(1-\nu_2)}\right] - 2\theta^I(0,a)\frac{E_1\alpha_1}{(1-2\nu_1)},$$

$$D_4 = \frac{2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)(\nu_1 - \nu_2)}{\left\{\frac{1}{a^2} - \frac{1}{b^2} + \frac{E_1}{E_2}\frac{(1+\nu_2)}{(1+\nu_1)(1-2\nu_1)}\left[\frac{(1-2\nu_2)}{b^2} + \frac{(1+\nu_2)}{a^2}\right]\right\}}.$$

3.3. Dimensionless parameters

In order to generalize the obtained results, the parameters are made dimensionless as follows: $\overline{r} = r/b$ – dimensionless radius, $Q = a/b$ – geometric parameter, $\overline{E} = E_1/E_2$ – ratio of Young's moduli, $\overline{\nu} = \nu_1/\nu_2$ – ratio of Poisson's ratios, $\overline{\alpha} = \alpha_1/\alpha_2$ – ratio of thermal expansion coefficients, $\overline{\lambda} = \lambda_1/\lambda_2$ – ratio of thermal conduction coefficients, $\overline{\sigma}_0 = \sigma_0^{II}/\sigma_0^{I}$ – ratio of yield limits, and $\overline{q}''' = (q'''\alpha_1 E_1 b^2)/(\sigma_0^{\mathrm{I}}\lambda_1)$ – dimensionless thermal load parameter. The stress, displacement and plastic strain components are made dimensionless in the following forms:

$$\overline{\sigma}_r = \frac{\sigma_r}{\sigma_0^{\mathrm{I}}}, \qquad \overline{\sigma}_\theta = \frac{\sigma_\theta}{\sigma_0^{\mathrm{I}}}, \qquad \overline{\sigma}_z = \frac{\sigma_z}{\sigma_0^{\mathrm{I}}}, \qquad \overline{u} = \frac{E_1 u}{\sigma_0^{\mathrm{I}} b},$$
$$\overline{\varepsilon}_r^p = \frac{E_1\varepsilon_r^p}{\sigma_0^{\mathrm{I}}}, \qquad \overline{\varepsilon}_\theta^p = \frac{E_1\varepsilon_\theta^p}{\sigma_0^{\mathrm{I}}}, \qquad \overline{\varepsilon}_z^p = \frac{E_1\varepsilon_z^p}{\sigma_0^{\mathrm{I}}},$$

and the temperature distributions in the two layers are written in dimensionless forms as $\overline{T}_1 = (T_1 - T_0)E_1\alpha_1/\sigma_0^I$ and $\overline{T}_2 = (T_2 - T_0)E_1\alpha_1/\sigma_0^I$ (Fig. 2).

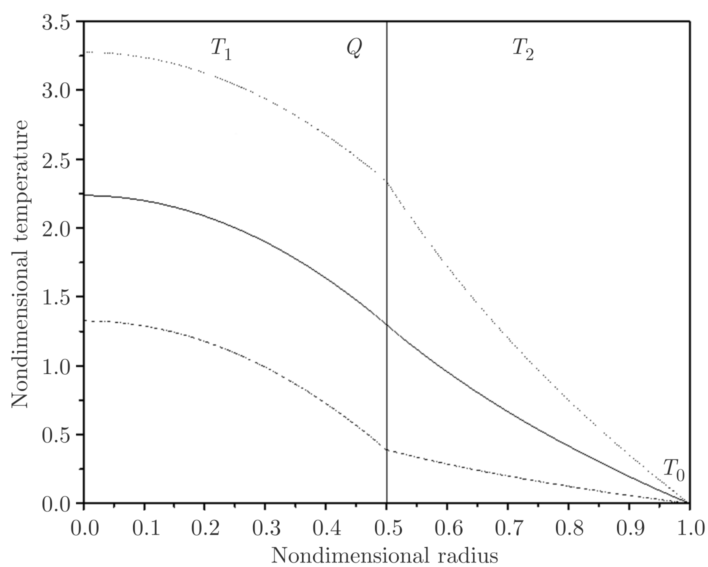


Fig. 2. Dimensionless (nondimensional) temperature distribution for $Q = 0.5$, $\bar{q}''' = 15$, $\overline{\lambda} = 0.3$, $\overline{\lambda} = 1.0$ and $\overline{\lambda} = 1.8$.

3.4. Elastic limit analysis

The elastic-plastic constitutive equations for the generalized plane strain problem in cylindrical coordinates are given by [15]:

$$\varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] + \alpha T + \varepsilon_r^p, \tag{3.13}$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] + \alpha T + \varepsilon_\theta^p, \tag{3.14}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)] + \alpha T - (\varepsilon_r^p + \varepsilon_\theta^p), \tag{3.15}$$

where ε_z = constant for generalized plane strain. Assuming that the two layers forming the composite tube have the same thermo-mechanical properties $\overline{E} = 1$, $\overline{\alpha} = 1$, $\overline{\lambda} = 1$, $\overline{\nu} = 1$, $\overline{\sigma}_0 = 1$, the dependence of onset of yield at the axis and at the surface on the geometric parameter $Q = a/b$ is investigated. Yielding starts both at the surface and at the axis of the composite cylinder for $Q \geq 0.4199$ (Fig. 3). It is noted that the thermal load required for the onset of yield increases exponentially as the geometric parameter Q decreases.

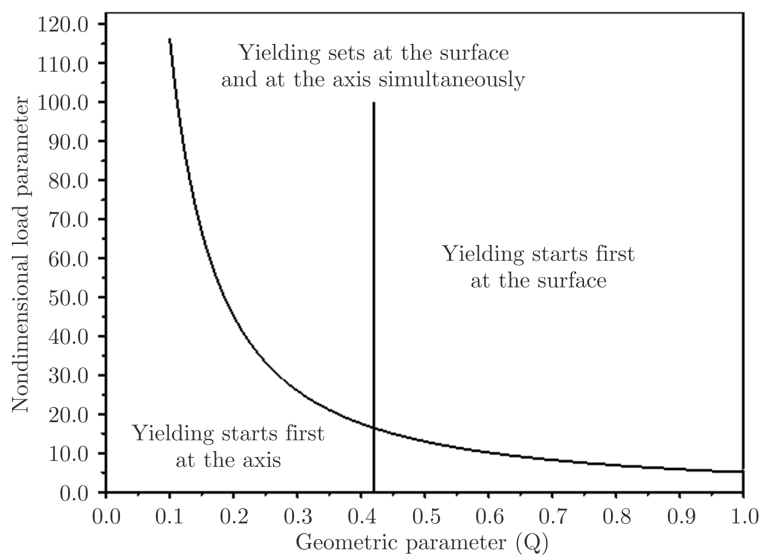


FIG. 3. Dependence of onset of yield on the geometric parameter Q for $\bar{E} = 1$, $\bar{\alpha} = 1$, $\bar{\lambda} = 1$, $\bar{\nu} = 1$ and $\bar{\sigma}_0 = 1$.

4. Onset of yield at the surface of the composite cylinder

Yielding starts at the surface of the composite cylinder in which the stress state corresponds to an edge regime of Tresca's yield surface with the principal stresses $\sigma_\theta = \sigma_z > \sigma_r$. Two plastic regions develop simultaneously and expand inwards through the tube, while the solid cylinder stays purely elastic (Fig. 1). In this section, the general equations for determining the plastic stress-strain relations for Tresca's yield criterion are derived based on a unified approach by Drucker [15].

4.1. Plastic region 1 ($r_4 < r < b$)

In this edge regime, Tresca's yield functions

$$f_1 = (\sigma_\theta)_1^{II} - (\sigma_r)_1^{II} - \sigma_0^{II}, \tag{4.1}$$

$$f_2 = (\sigma_z)_1^{II} - (\sigma_r)_1^{II} - \sigma_0^{II} \tag{4.2}$$

and the associated flow rule [16]

$$d\varepsilon_{ij}^p = \sum_k \frac{\partial f_k}{\partial \sigma_{ij}} d\lambda_k \tag{4.3}$$

lead to

$$d\varepsilon_{ij}^p = \sum_{k=1}^{2} \frac{\partial f_k}{\partial \sigma_{ij}} \, d\lambda_k = \frac{\partial f_1}{\partial \sigma_{ij}} d\lambda_1 + \frac{\partial f_2}{\partial \sigma_{ij}} d\lambda_2 \tag{4.4}$$

which, when substituting non-zero stress components, results in

$$d\varepsilon_\theta^p = \sum_{k=1}^{2} \frac{\partial f_k}{\partial \sigma_\theta} d\lambda_k = \frac{\partial f_1}{\partial \sigma_\theta} d\lambda_1 + \frac{\partial f_2}{\partial \sigma_\theta} d\lambda_2 = d\lambda_1, \tag{4.5}$$

$$d\varepsilon_z^p = \sum_{k=1}^{2} \frac{\partial f_k}{\partial \sigma_z} \, d\lambda_k = \frac{\partial f_1}{\partial \sigma_z} d\lambda_1 + \frac{\partial f_2}{\partial \sigma_z} d\lambda_2 = d\lambda_2, \tag{4.6}$$

$$d\varepsilon_r^p = \sum_{k=1}^{2} \frac{\partial f_k}{\partial \sigma_r} \, d\lambda_k = \frac{\partial f_1}{\partial \sigma_r} d\lambda_1 + \frac{\partial f_2}{\partial \sigma_r} d\lambda_2 = -(d\lambda_1 + d\lambda_2). \tag{4.7}$$

Applying these relations to the plastic region 1, we obtain: $d(\varepsilon_\theta^p)_1^{II} = d\lambda_1$, $d(\varepsilon_z^p)_1^{II} = d\lambda_2$ and $d(\varepsilon_r^p)_1^{II} = -(d\lambda_1 + d\lambda_2)$, where the subscripts 1 and 2 of $d\lambda$ are only index numbers, irrelevant to the plastic region numbers. Hence,

$$d(\varepsilon_r^p)_1^{II} = -[d(\varepsilon_\theta^p)_1^{II} + d(\varepsilon_z^p)_1^{II}], \tag{4.8}$$

where the superscript II denotes the tube, and the subscript 1 the plastic region 1. According to Eqs. (4.1) and (4.2), the equation of equilibrium leads to

$$(\sigma_r)_1^{II} = \sigma_0^{II} \ln r + C_5, \tag{4.9}$$

$$(\sigma_z)_1^{II} = (\sigma_\theta)_1^{II} = \sigma_0^{II}(1 + \ln r) + C_5. \tag{4.10}$$

Using plastic incompressibility, the elastic formula for dilatation leads to a differential equation for the radial displacement with the general solution

$$u_1^{II} = \frac{1-2\nu_2}{E_2} \frac{r}{2} \left[\sigma_0^{II} \left(\frac{1}{2} + 3 \ln r \right) + 3C_5 \right] + 3\alpha_2 r \theta^{II}(r_4, r) - \frac{1}{2}\varepsilon_z r + \frac{C_6}{r}, \tag{4.11}$$

where

$$\theta^{II}(r_4, r) = \frac{1}{r^2} \int_{r_4}^{r} T_2(r) r dr.$$

The plastic strains are then obtained as

$$(4.12) \qquad (\varepsilon_r^p)_1^{II} = \frac{1-2\nu_2}{E_2} \frac{1}{2} [\sigma_0^{II} \ln r + C_5] + \frac{7-6\nu_2}{4E_2} \sigma_0^{II} - \alpha_2 [3\theta^{II}(r_4, r) - 2T_2(r)] - \frac{1}{2}\varepsilon_z - \frac{C_6}{r^2},$$

$$(4.13) \qquad (\varepsilon_z^p)_1^{II} = \varepsilon_z - \frac{1}{E_2}[(1-\nu_2)\sigma_0^{II} + (1-2\nu_2)(\sigma_0^{II} \ln r + C_5)] - \alpha_2 T_2(r).$$

4.2. Plastic region 2 ($r_3 < r < r_4$)

In this plastic region, due to the inequality $(\sigma_\theta)_2^{II} > (\sigma_z)_2^{II} > (\sigma_r)_2^{II}$, we obtain

$$(4.14) \qquad f = (\sigma_\theta)_2^{II} - (\sigma_r)_2^{II} - \sigma_0^{II},$$

and thus the following can be written: $d(\varepsilon_r^p)_2^{II} = -(d\varepsilon_\theta^p)_2^{II}$, $d(\varepsilon_z^p)_2^{II} = 0$. The stresses, displacement and plastic strains are derived as

$$(4.15) \qquad (\sigma_r)_2^{II} = \sigma_0^{II} \ln r + C_7,$$

$$(4.16) \qquad (\sigma_\theta)_2^{II} = \sigma_0^{II}(1 + \ln r) + C_7,$$

$$(4.17) \qquad u_2^{II} = \frac{(1+\nu_2)(1-2\nu_2)}{E_2} r[\sigma_0^{II} \ln r + C_7] + 2(1+\nu_2)\alpha_2 r \theta^{II}(r_3, r) - \nu_2 \varepsilon_z r + \frac{C_8}{r},$$

$$(4.18) \qquad (\varepsilon_\theta^p)_2^{II} = -(1-\nu_2^2)\frac{\sigma_0^{II}}{E_2} + (1+\nu_2)\alpha_2[2\theta^{II}(r_3, r) - T_2(r)] + \frac{C_8}{r^2},$$

$$(4.19) \qquad (\varepsilon_r^p)_2^{II} = -(\varepsilon_\theta^p)_2^{II},$$

where

$$\theta^{II}(r_3, r) = \frac{1}{r^2} \int_{r_3}^{r} T_2(r) r dr.$$

4.3. Conditions and determination of the unknowns

Two elastic regions surrounded by two plastic regions contain 11 unknowns: C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, the axial strain ε_z, and the border radii r_3 and r_4.

For the determination of these unknowns, the following boundary and continuity conditions are enforced:

$$\text{at } r=0 \quad u_e^I \text{ is finite,} \tag{4.20}$$

$$\text{at } r=a \quad (\sigma_r)_e^{II} = (\sigma_r)_e^I, \tag{4.21}$$

$$\text{at } r=a \quad (u)_e^{II} = (u)_e^I, \tag{4.22}$$

$$\text{at } r=r_3 \quad (\sigma_r)_e^{II} = (\sigma_r)_2^{II}, \tag{4.23}$$

$$\text{at } r=r_3 \quad (\varepsilon_\theta^p)_2^{II} = 0, \tag{4.24}$$

$$\text{at } r=r_3 \quad (\sigma_\theta)_e^{II} - (\sigma_r)_e^{II} = \sigma_0^{II}, \tag{4.25}$$

$$\text{at } r=r_4 \quad (\sigma_r)_1^{II} = (\sigma_r)_2^{II}, \tag{4.26}$$

$$\text{at } r=r_4 \quad (\sigma_\theta)_2^{II} = (\sigma_z)_2^{II}, \tag{4.27}$$

$$\text{at } r=r_4 \quad (\varepsilon_\theta^p)_2^{II} = (\varepsilon_\theta^p)_2^I, \tag{4.28}$$

$$\text{at } r=b \quad (\sigma_r)_1^{II} = 0, \tag{4.29}$$

$$\int_0^a (\sigma_z)_e^I r\,dr + \int_a^{r3} (\sigma_z)_e^{II} r\,dr + \int_{r3}^{r4} (\sigma_z)_2^{II} r\,dr + \int_{r4}^{b} (\sigma_z)_1^{II} r\,dr = 0, \tag{4.30}$$

where the last one is the free end condition. They result in $C_2 = 0$ and

$$C_1 = 2\left[\frac{E_1\alpha_1}{(1-\nu_1)}\theta^I(0,a) + \frac{C_3}{2} + \frac{C_4}{a^2}\right], \tag{4.31}$$

$$C_3 = 2\left[\frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,r_3) + \sigma_0^{II}\ln\frac{r_3}{b} - \frac{C_4}{r_3^2}\right], \tag{4.32}$$

$$C_4 = \frac{r_3^2}{2}\left[\frac{E_2\alpha_2}{(1-\nu_2)}[2\theta^{II}(a,r_3) - T_2(r_3)] - \sigma_0^{II}\right], \tag{4.33}$$

$$C_5 = -\sigma_0^{II}\ln b, \tag{4.34}$$

$$C_7 = C_5, \tag{4.35}$$

$$E_2\varepsilon_z = \sigma_0^{II}\left[1-\nu_2+(1-2\nu_2)\ln\frac{r_4}{b}\right] + E_2\alpha_2 T_2(r_4), \tag{4.36}$$

$$\begin{aligned} C_6 = C_8 - \frac{\sigma_0^{II}}{2E_2}r_4^2&\left[\frac{1+2\nu_2-4\nu_2^2}{2} + (1-2\nu_2)\ln\frac{r_4}{b}\right] \\ &- \alpha_2 r_4^2[\nu_2 T_2(r_4) - 2(1+\nu_2)\theta^{II}(r_3,r_4)] + \frac{1}{2}\varepsilon_z r_4^2, \end{aligned} \tag{4.37}$$

$$C_8 = r_3^2\left[(1-\nu_2^2)\frac{\sigma_0^{II}}{E_2} + (1+\nu_2)\alpha_2 T_2(r_3)\right], \tag{4.38}$$

$$(1+\nu_1)(1-2\nu_1)\frac{C_1}{2E_1} - (\nu_1-\nu_2)\varepsilon_z = (1+\nu_2)(1-2\nu_2)\frac{C_3}{2E_2} - \alpha_1\frac{1+\nu_1}{1-\nu_1}\theta^{\mathrm{I}}(0,a) - \frac{1+\nu_2}{E_2}\frac{C_4}{a^2}. \tag{4.39}$$

The integrals in Eq. (4.30) are evaluated as

$$\int_0^a (\sigma_z)_e^I r\,dr = \frac{E_1\alpha_1}{16(1-\nu_1)}\left[a^4+2a^4\left(-1+2\frac{\lambda_1}{\lambda_2}\ln\frac{a}{b}\right)\right] + \frac{1}{2}(\nu_1 C_1+\varepsilon_z E_1)a^2, \tag{4.40}$$

$$\int_a^{r3} (\sigma_z)_e^{II} r\,dr = \frac{E_2\alpha_2}{\lambda_2}\frac{q'''}{4(1-\nu_2)}a^2\left[r_3^2\left(\ln\frac{r_3}{b}-\frac{1}{2}\right)-a^2\left(\ln\frac{a}{b}-\frac{1}{2}\right)\right] + \frac{1}{2}(\nu_2 C_3+E_2\varepsilon_z)(r_3^2-a^2), \tag{4.41}$$

$$\int_{r3}^{r4} (\sigma_z)_2^{II} r\,dr = \left[\frac{E_2\alpha_2}{\lambda_2}\frac{q'''}{4}a^2+\nu_2\sigma_0^{II}\right]\left[r_4^2\left(\ln\frac{r_4}{b}-\frac{1}{2}\right)-r_3^2\left(\ln\frac{r_3}{b}-\frac{1}{2}\right)\right] + \frac{r_4^2-r_3^2}{2}(\nu_2\sigma_0^{II}+E_2\varepsilon_z), \tag{4.42}$$

$$\int_{r4}^{b} (\sigma_z)_1^{II} r\,dr = \left[\frac{b^2}{4}-\frac{r_4^2}{2}\left(\ln\frac{r_4}{b}+\frac{1}{2}\right)\right]. \tag{4.43}$$

The border radii r_3 and r_4 are evaluated from the simultaneous solutions of the free surface condition and Eq. (4.30).

5. Onset of yield at the axis of the solid cylinder

For smaller values of the geometric parameter $Q = a/b$ yielding starts first at the center of the composite cylinder and the two plastic regions, plastic regions 3 and 4 expand outward simultaneously with different rates of propagation (Fig. 1). Stress states in the principal stress space corresponds to $\sigma_\theta = \sigma_r > \sigma_z$ and $\sigma_\theta > \sigma_r > \sigma_z$ respectively.

5.1. Plastic region 3 $(0 < r < r_1)$

In this edge regime, the two yield functions

$$f_1 = (\sigma_\theta)_3^I - (\sigma_z)_3^I - \sigma_0^I, \tag{5.1}$$

$$f_2 = (\sigma_r)_3^I - (\sigma_z)_3^I - \sigma_0^I \tag{5.2}$$

lead to

$$(\sigma_r)_3^I = (\sigma_\theta)_3^I = C_9, \tag{5.3}$$

$$(\sigma_z)_3^I = C_9 - \sigma_0^I. \tag{5.4}$$

Using plastic incompressibility, we obtain the differential equation

$$\frac{du_3^I}{dr} + \frac{u_3^I}{r} = \frac{(1-2\nu_1)}{E_1}[(\sigma_r)_3^I + (\sigma_\theta)_3^I + (\sigma_z)_3^I] + 3\alpha_1 T_1(r) - \varepsilon_z \tag{5.5}$$

with the general solution

$$u_3^I = \frac{1-2\nu_1}{E_1}\frac{r}{2}[3C_9 - \sigma_0^I] + 3\alpha_1 r\theta^I(0,r) - \frac{1}{2}\varepsilon_z r + \frac{C_{10}}{r}. \tag{5.6}$$

The plastic strains are then as follows:

$$(\varepsilon_\theta^p)_3^I = \frac{1}{2E_1}[(1-2\nu_1)C_9 - \sigma_0^I] + \alpha_1[3\theta^I(0,r) - T_1(r)] - \frac{1}{2}\varepsilon_z + \frac{C_{10}}{r^2}, \tag{5.7}$$

$$(\varepsilon_r^p)_3^I = \frac{1}{2E_1}[(1-2\nu_1)C_9 - \sigma_0^I] - \alpha_1[3\theta^I(0,r) - 2T_1(r)] - \frac{1}{2}\varepsilon_z - \frac{C_{10}}{r^2}, \tag{5.8}$$

$$(\varepsilon_z^p)_3^I = \varepsilon_z - \frac{1}{E_1}[(1-2\nu_1)C_9 - \sigma_0^I] - \alpha_1 T_1(r). \tag{5.9}$$

5.2. Plastic region 4 ($r_1 < r < r_2$)

By using the yield function

$$f = (\sigma_\theta)_4^I - (\sigma_z)_4^I - \sigma_0^I, \tag{5.10}$$

the plastic strain increments $d(\varepsilon_z^p)_4^I = -d(\varepsilon_\theta^p)_4^I$, $d(\varepsilon_r^p)_4^I = 0$, the equilibrium and the compatibility equations, we derive the following differential equation for the tangential stress:

$$r^2\frac{d^2(\sigma_\theta)_4^I}{dr^2} + 3r\frac{d(\sigma_\theta)_4^I}{dr} + \frac{(1-2\nu_1)}{2(1-\nu_1)}(\sigma_\theta)_4^I = \frac{1}{2(1-\nu_1)}\left[\sigma_0^I - E_1\alpha_1\left(T_1 + 5r\frac{dT_1}{dr} + 2r^2\frac{d^2T}{dr^2}\right) + E_1\varepsilon_z\right], \tag{5.11}$$

with the general solution

$$(\sigma_\theta)_4^I = C_{11}r^{-(1-M)} + C_{12}r^{-(1+M)} + \frac{(\sigma_0^I + E_1\varepsilon_z)}{1-2v_1} + \frac{E_1\alpha_1}{4(1-v_1)}[(1+2M)\theta_1 + (1+2M)\theta_2 - 4T_1], \tag{5.12}$$

$$(5.13)\qquad (\sigma_r)_4^I = \frac{1}{M}\Big\{C_{11}r^{-(1-M)} - C_{12}r^{-(1+M)} + \frac{E_1\alpha_1}{4(1-v_1)}[(1-2M)\theta_1 - (1+2M)\theta_2]\Big\} + \frac{1}{(1-2v_1)}(\sigma_0^I + E_1\varepsilon_z),$$

where

$$\theta_1 = r^{-(1-M)}\int_{r_1}^{r} T_1\, r^{-M}\, dr,$$

$$\theta_2 = r^{-(1+M)}\int_{r_1}^{r} T_1\, r^{M} dr,$$

$$M = \sqrt{\frac{1}{2(1-\nu_1)}}.$$

Therefrom, the radial displacement and the plastic strain components are defined as

$$(5.14)\qquad E_1u_4^I = 2\Big\{\Big(1 - v_1 - \frac{v_1}{M}\Big)\Big[C_{11}r^M + \frac{1-2M}{4(1-v_1)}E_1\alpha_1\theta_1 r\Big] + \Big(1 - v_1 + \frac{v_1}{M}\Big)\Big[C_{12}r^{-M} + \frac{1+2M}{4(1-v_1)}E_1\alpha_1\theta_2 r\Big]\Big\} + (1+v_1)\sigma_0^I r + E_1\varepsilon_z r,$$

$$(5.15)\qquad E_1(\varepsilon_\theta^p)_4^I = \Big(1 - v_1 - \frac{v_1}{M}\Big)\Big[C_{11}r^{-(1-M)} + \frac{(1-2M)}{4(1-v_1)}E_1\alpha_1\theta_1\Big] + \Big(1 - v_1 + \frac{v_1}{M}\Big)\Big[C_{12}r^{-(1+M)} + \frac{1+2M}{4(1-v_1)}E_1\alpha_1\theta_2\Big].$$

5.3. Conditions and determination of the unknowns

For the determination of the 11 unknowns: C_1, C_2, C_3, C_4, C_9, C_{10}, C_{11}, C_{12} the axial strain ε_z, and the border radii r_1 and r_2, the following boundary and continuity conditions are used:

$$(5.16)\qquad \text{at } r = 0 \quad u_3^I \text{ is finite},$$

$$(5.17)\qquad \text{at } r = r_1 \quad (\sigma_r)_3^I = (\sigma_r)_4^I,$$

$$(5.18)\qquad \text{at } r = r_1 \quad (\sigma_\theta)_3^I = (\sigma_\theta)_4^I,$$

$$(5.19)\qquad \text{at } r = r_1 \quad (\varepsilon_r^p)_3^I = 0,$$

(5.20) at $r = r_2 \quad (\varepsilon_\theta^p)_4^I = 0,$

(5.21) at $r = r_2 \quad (\sigma_r)_4^I = (\sigma_r)_e^I,$

(5.22) at $r = r_2 \quad (\sigma_\theta)_e^I = (\sigma_\theta)_4^I,$

(5.23) at $r = a \quad (\sigma_r)_e^{II} = (\sigma_r)_e^I,$

(5.24) at $r = a \quad (u)_e^{II} = (u)_e^I,$

(5.25) at $r = b \quad (\sigma_r)_e^{II} = 0,$

$$\int_0^{r1} (\sigma_z)_3^I \, r\,dr + \int_{r1}^{r2} (\sigma_z)_4^I \, r\,dr + \int_{r2}^{a} (\sigma_z)_e^I \, r\,dr + \int_a^b (\sigma_z)_e^{II} \, r\,dr = 0. \tag{5.26}$$

The above conditions lead to $C_{10} = 0$ and

$$\begin{aligned} C_1 = {} & \left(1+\frac{1}{M}\right) C_{11} r_2^{-(1-M)} + \left(1-\frac{1}{M}\right) C_{12} r_2^{-(1+M)} + \frac{E_1\alpha_1}{4(1-\nu_1)} \\ & \times \left[\left(1+\frac{1}{M}\right)(1-2M)\theta_1(r_1,r_2) + \left(1-\frac{1}{M}\right)(1+2M)\theta_2(r_1,r_2)\right] \\ & + \frac{2}{(1-2\nu_1)}(\sigma_0^I + E_1\varepsilon_z), \end{aligned} \tag{5.27}$$

$$\begin{aligned} C_2 = {} & -\frac{1}{2}\left[\left(1-\frac{1}{M}\right) C_{11} r_2^{1+M} + \left(1+\frac{1}{M}\right) C_{12} r_2^{1-M}\right] - \frac{E_1\alpha_1}{8(1-\nu_1)} \\ & \left[\left(1-\frac{1}{M}\right)(1-2M)\theta_1(r_1,r_2) + \left(1+\frac{1}{M}\right)(1+2M)\theta_2(r_1,r_2)\right] r_2^2, \end{aligned} \tag{5.28}$$

$$C_3 = 2\left[\frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,b) - \frac{C_4}{b^2}\right], \tag{5.29}$$

$$C_4 = \frac{a^2b^2}{(b^2-a^2)}\left[-\frac{E_1\alpha_1}{(1-\nu_1)}\theta^I(r_2,a) - \frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,b) + \frac{C_1}{2} + \frac{C_2}{a^2}\right], \tag{5.30}$$

$$C_9 = \frac{1}{(1-2\nu_1)}[2E_1\alpha_1[3\theta^I(0,r_1) - 2T_1(r_1)] + E_1\varepsilon_z + \sigma_0^I], \tag{5.31}$$

$$\begin{aligned} C_{11} = {} & \frac{1}{2}E_1\alpha_1 r_1^{(1-M)}\left\{\frac{6(1+M)}{(1-2\nu_1)}\theta^I(0,r_1) \right. \\ & \left. + \left[\frac{1}{(1-\nu_1)} - \frac{4(1+M)}{(1-2\nu_1)}\right] T_1(r_1)\right\}, \end{aligned} \tag{5.32}$$

$$\begin{aligned} C_{12} = {} & \frac{1}{2}E_1\alpha_1 r_1^{(1+M)}\left\{-\frac{6(1-M)}{(1-2\nu_1)}\theta^I(0,r_1) \right. \\ & \left. + \left[\frac{1}{(1-\nu_1)} - \frac{4(1-M)}{(1-2\nu_1)}\right] T_1(r_1)\right\}, \end{aligned} \tag{5.33}$$

(5.34)

$$\varepsilon_z = \frac{\begin{bmatrix} \frac{C_2}{a^2}\Big\{-\frac{(1+\nu_1)}{E_1}-\frac{b^2}{(b^2-a^2)}\Big[-\frac{(1+\nu_2)}{E_2}-\frac{(1+\nu_2)(1-2\nu_2)}{E_2 b^2}a^2\Big]\Big\} \\ +\theta^I(r_2,a)\Big\{\alpha_1\Big(\frac{1+\nu_1}{1-\nu_1}\Big)+\frac{a^2b^2}{(b^2-a^2)}\frac{E_1\alpha_1}{(1-\nu_1)}\Big[-\frac{(1+\nu_2)}{E_2a^2}-\frac{(1+\nu_2)(1-2\nu_2)}{E_2b^2}\Big]\Big\} \\ \Big\{\Big(1+\frac{1}{M}\Big)C_{11}r_2^{-(1-M)}+\Big(1-\frac{1}{M}\Big)C_{12}r_2^{-(1+M)}+\frac{2}{(1-2\nu_1)}\sigma_0^I \\ +\frac{E_1\alpha_1}{4(1-\nu_1)}\Big[\Big(1+\frac{1}{M}\Big)\theta_1(r_1,r_2)(1-2M)+\Big(1-\frac{1}{M}\Big)(1+2M)\theta_2(r_1,r_2)\Big]\Big\} \\ \Big\{\frac{1}{2E_1}(1+\nu_1)(1-2\nu_1)-\frac{a^2b^2}{(b^2-a^2)}\frac{1}{2}\Big[-\frac{(1+\nu_2)}{E_2a^2}-\frac{(1+\nu_2)(1-2\nu_2)}{E_2b^2}\Big]\Big\}+(\nu_2\nu_1)\varepsilon_z \\ +\frac{E_2\alpha_2}{(1-\nu_2)}\theta^{II}(a,b)\Big\{\frac{a^2b^2}{(b^2a^2)}\Big[-\frac{(1+\nu_2)}{E_2a^2}-\frac{(1+\nu_2)(1-2\nu_2)}{E_2b^2}\Big]-\frac{(1+\nu_2)(1-2\nu_2)}{E_2}\Big\} \end{bmatrix}}{-(1+\nu_1)+\frac{1}{(1+Q^2)}\frac{1}{(1-2\nu_1)}\{-E(1+\nu_2)[1+Q^2(1-2\nu_2)]\}+(\nu_1-\nu_2)}.$$

From Eq. (5.20) we have the following relationship:

$$\text{(5.35)}\qquad \left(1-\nu_1-\frac{\nu_1}{M}\right)\left[C_{11}r_2^{-(1-M)}+\frac{(1-2M)}{4(1-\nu_1)}E_1\alpha_1\theta_1(r_1,r_2)\right] + \left(1-\nu_1+\frac{\nu_1}{M}\right)\left[C_{12}r_2^{-(1+M)}+\frac{(1+2M)}{4(1-\nu_1)}E_1\alpha_1\theta_2(r_1,r_2)\right]=0,$$

and the integrals in the free end condition take the following forms:

$$\text{(5.36)}\qquad \int_0^{r1}(\sigma_z)_3^I r\,dr=(C_9-\sigma_0^I)\frac{r_1^2}{2},$$

$$\text{(5.37)}\qquad \int_{r1}^{r2}(\sigma_z)_4^I r\,dr=\frac{C_{11}(r_2^{1+M}-r_1^{1+M})}{1+M} + \frac{C_{12}(r_2^{1-M}-r_1^{1-M})}{1-M}+\frac{E_1\varepsilon_z+2\nu_1\sigma_0^I}{(1-2V_1)}\frac{(r_2^2-r_1^2)}{2} + \frac{E_1}{4(1-V_1)}\left[(1-2M)\int_{r_1}^{r_2}\theta_1(r_1,r)r\,dr + (1+2M)\int_{r_1}^{r_2}\theta_2(r_1,r)r\,dr-4\int_{r_1}^{r_2}T_1(r)r\,dr\right],$$

$$\text{(5.38)}\qquad \int_{r_1}^{r_2}\theta_1(r_1,r)r\,dr = -\frac{q'''}{4}\left[\frac{r_2^4-r_1^4}{4\lambda_1(3-M)}+\frac{a^2}{1-M}\left(-\frac{1}{\lambda_1}+\frac{2}{\lambda_2}\ln\frac{a}{b}\right)\right.$$

$$\times \left\{ \frac{r_2^2 - r_1^2}{2} - \frac{1}{1+M} \left[r_2^2 \left(\frac{r_1}{r_2} \right)^{(1-M)} - r_1^2 \right] \right\}$$

$$- \frac{1}{(1+M)} \left[r_2^2 \left(\frac{r_1}{r_2} \right)^{(1-M)} - r_1^2 \right] \frac{r_1^2}{\lambda_1 (3-M)} \Bigg],$$

$$\int_{r_1}^{r_2} \theta_2(r_1, r) r\, dr = -\frac{q'''}{4} \left[\frac{r_2^4 - r_1^4}{4\lambda_1(3+M)} + \frac{a^2}{1+M} \left(-\frac{1}{\lambda_1} + \frac{2}{\lambda_2} \ln \frac{a}{b} \right) \right. \tag{5.39}$$

$$\left\{ \frac{r_2^2 - r_1^2}{2} - \frac{1}{1-M} \left[r_2^2 \left(\frac{r_1}{r_2} \right)^{(1+M)} - r_1^2 \right] \right\}$$

$$- \frac{1}{(1-M)} \left[r_2^2 \left(\frac{r_1}{r_2} \right)^{(1+M)} - r_1^2 \right] \frac{r_1^2}{\lambda_1 (3-M)} \Bigg],$$

$$\int_{r_1}^{r_2} T_1(r) r\, dr = -\frac{q'''}{4} \left[\frac{r_2^4 - r_1^4}{4\lambda_1} + \frac{(r_2^2 - r_1^2)}{2} a^2 \left\{ -\frac{1}{\lambda_1} + \frac{2}{\lambda_2} \ln \frac{a}{b} \right\} \right], \tag{5.40}$$

$$\int_{r_2}^{a} (\sigma_z)_{e1}^{I} r\, dr \tag{5.41}$$

$$= \frac{E_1 \alpha_1}{(1-\nu_1)} \frac{q'''}{4} \left[\frac{a^4 - r_2^4}{4\lambda_1} + \frac{(a^2 - r_2^2)}{2} a^2 \left(-\frac{1}{\lambda_1} + \frac{2}{\lambda_2} \ln \frac{a}{b} \right) \right]$$

$$+ \frac{(\nu_1 C_1 + E_1 \varepsilon_z)}{2} (a^2 - r_2^2),$$

$$\int_{a}^{b} (\sigma_z)_{e1}^{II} r\, dr = \frac{1}{2} (\nu_2 C_3 + E_2 \varepsilon_z)(b^2 - a^2) \tag{5.42}$$

$$+ \frac{E_2 \alpha_2}{(1-\nu_2)} \frac{q'''}{2} \frac{a^2}{\lambda_2} \left[\frac{b^2}{2} \left(\ln \frac{b}{b} - \frac{1}{2} \right) - \frac{a^2}{2} \left(\ln \frac{a}{b} - \frac{1}{2} \right) \right].$$

In the first stage of elastic-plastic deformation, the border radii r_1 and r_2 are evaluated as a function of the thermal load from the simultaneous solution of Eqs. (5.26) and (5.35).

6. Onset of yield at the interface of the composite cylinder

Depending on the choice of the material parameters, beyond a certain critical load parameter, another plastic region, plastic region 5 can be generated at the outer surface of solid cylinder (the interface of the composite cylinder). This region propagates inwards as plastic regions 3 and 4 expand outwards until they meet when the elastic region vanishes and the solid cylinder reaches the fully plastic state (Fig. 1).

6.1. Plastic region 5 $(r_5 < r < a)$

In this region,$(\sigma_\theta)_5^I > (\sigma_z)_5^I > (\sigma_r)_5^I$ and the yield function

$$f = (\sigma_\theta)_5^I - (\sigma_r)_5^I - \sigma_0^I \tag{6.1}$$

requires that $d(\varepsilon_r^p)_5^I = -(d\varepsilon_\theta^p)_5^I$ and $d(\varepsilon_z^p)_5^I = 0$. Hence,

$$(\sigma_r)_5^I = \sigma_0^I \ln r + C_{13}, \tag{6.2}$$

$$(\sigma_\theta)_5^I = \sigma_0^I(1 + \ln r) + C_{13}, \tag{6.3}$$

$$(\sigma_z)_5^I = \nu_1[\sigma_0^I(2\ln r + 1) + 2C_{13}] + E_1\varepsilon_z - E_1\alpha_1 T_1(r), \tag{6.4}$$

$$u_5^I = \frac{(1+\nu_1)(1-2\nu_1)}{E_1} r(\sigma_0^I \ln r + C_{13}) + 2(1+\nu_1)\alpha_1 r\theta^I(r_5, r) - \nu_1\varepsilon_z r + \frac{C_{14}}{r}, \tag{6.5}$$

$$(\varepsilon_\theta^p)_5^I = -(1-\nu_1^2)\frac{\sigma_0^I}{E_1} + (1+\nu_1)\alpha_1[2\theta^{II}(r_5, r) - T_1(r)] + \frac{C_{14}}{r^2}, \tag{6.6}$$

where

$$\theta^I(r_5, r) = \frac{1}{r^2}\int_{r_5}^{r} T_1(r) r\, dr.$$

6.2. Conditions and results

For this stage of plastic deformation, the total number of unknowns is 17 and they are C_1, C_2, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, ε_z, r_1, r_2, r_4 and r_5. In Section 5.3, the unknowns C_1, C_2, C_9, C_{10}, C_{11}, C_{12} were determined, and in Section 4.3, the unknowns C_5, C_6, C_7, ε_z were defined. Using the conditions: $(\varepsilon_\theta^p)_5^I = 0$ at $r = r_5$, $(\sigma_r)_2^{II} = (\sigma_r)_5^I$ and $(u)_2^{II} = (u)_5^I$ at $r = a = r_3$, it follows that

$$C_8 = a^2\left\{\sigma_0^{II}\ln\frac{a}{b}\left[\frac{(1+\nu_1)(1-2\nu_1)}{E_1} - \frac{(1+\nu_2)(1-2\nu_2)}{E_2}\right] + 2(1+\nu_1)\alpha_1\theta^I(r_5, a) + (\nu_2 - \nu_1)\varepsilon_Z + \frac{C_{14}}{a^2}\right\}, \tag{6.7}$$

$$C_{13} = \sigma_0^{II}\ln\frac{a}{b} - \sigma_0^I \ln a, \tag{6.8}$$

$$C_{14} = r_5^2\left\{(1-\nu_1^2)\frac{\sigma_0^I}{E_1} + (1+\nu_1)\alpha_1 T^I(r_5)\right\}. \tag{6.9}$$

The unknown border radii will be determined from the simultaneous solution of the following nonlinear equations:

$$\left(1-\nu_1-\frac{\nu_1}{M}\right)\left[C_{11}r_2^{-(1-M)}+\frac{(1-2M)}{4(1-\nu_1)}E_1\alpha_1\theta_1(r_1,r_2)\right] + \left(1-\nu_1+\frac{\nu_1}{M}\right)\left[C_{12}r_2^{-(1+M)}+\frac{(1+2M)}{4(1-\nu_1)}E_1\alpha_1\theta_2(r_1,r_2)\right]=0, \tag{6.10}$$

$$\sigma_0^I\ln r_5+C_{13}+\frac{E_1\alpha_1}{(1-\nu_1)}\theta^I(r_2,r_5)-\frac{C_1}{2}-\frac{C_2}{r_5^2}=0, \tag{6.11}$$

$$\frac{2}{(1-\nu_1)}E_1\alpha_1\theta^I(r_2,r_5)-\frac{E_1\alpha_1}{(1-\nu_1)}T_1(r_5)-2\frac{C_2}{r_5^2}-\sigma_0^{II}=0, \tag{6.12}$$

and the free end condition is

$$\int_0^{r1}(\sigma_z)_3^I r\,dr+\int_{r1}^{r2}(\sigma_z)_4^I r\,dr+\int_{r2}^{r5}(\sigma_z)_e^I r\,dr + \int_{r5}^{a}(\sigma_z)_5^I r\,dr+\int_a^{r4}(\sigma_z)_2^{II} r\,dr+\int_{r4}^{b}(\sigma_z)_1^{II} r\,dr=0, \tag{6.13}$$

where

$$\int_{r_5}^{a}(\sigma_z)_5^I r\,dr=\nu_1\sigma_0^I\left[(a^2-r_5^2)\frac{\sigma_0^{II}}{\sigma_0^I}\ln\frac{a}{b}-r_5^2\ln\frac{r_5}{a}\right] + E_1\alpha_1\frac{q'''}{4}\left\{\frac{a^4-r_5^4}{4\lambda_1}+\frac{a^2-r_5^2}{2}\left[a^2\left(-\frac{1}{\lambda_1}+\frac{2}{\lambda_2}\ln\frac{a}{b}\right)\right]\right\}+E_1\varepsilon_z\frac{a^2-r_5^2}{2}. \tag{6.14}$$

7. Numerical results

In this section, the conditions for the onset of yield and the effects of all the parameter ratios on the distribution of stresses and plastic strains in different stages of elastic-plastic deformations are investigated. In the presentation of the numerical results, the ratios of the material parameters are taken as 1.0 unless they are specified differently.

7.1. Effect of the parameter Q

First, the case in which all the dimensionless parameters are taken as 1.0 is considered. Yielding starts simultaneously both at the axis and at the surface of the composite cylinder for the dimensionless load parameter $\overline{q}'''$ =17.541 and for the geometric parameter Q=0.4199. For the values less than the critical value of 0.4199, yielding starts at the axis and for the values greater than the critical value of 0.4199 yielding starts at the outer surface. For Q=0.4 < 0.4199 as shown in Fig. 4, yielding starts first at the axis at $\overline{q}'''$ =18.431 for $\overline{\sigma}_0$= 1, and plastic regions 3 and 4 expand outward with increasing thermal loads. At $\overline{q}'''$ =19.156 the elastic-plastic behavior of the composite cylinder enters the second stage in which plastic flow starts at the outer surface, and plastic regions 1 and 2 propagate inward. The plastic-elastic border r_2 reaches the interface at r=a and $\overline{q}'''$ =23.591, while the inner portion of the outer cylinder is still elastic. The stresses and deformations corresponding to $\overline{q}'''$ =20.944 are plotted in Fig. 5. The composite cylinder reaches the fully plastic state with $r_2 = a = r_3$ at $\overline{q}''' = 26.052$.

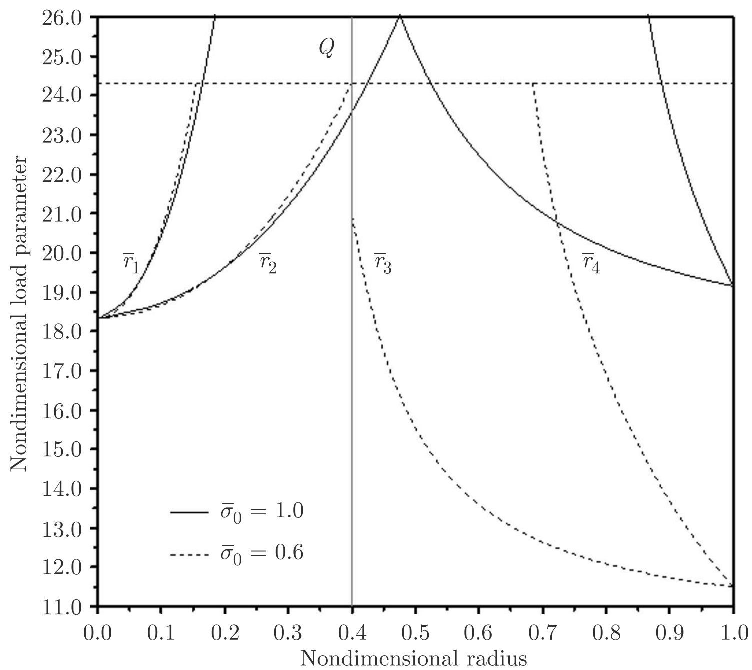
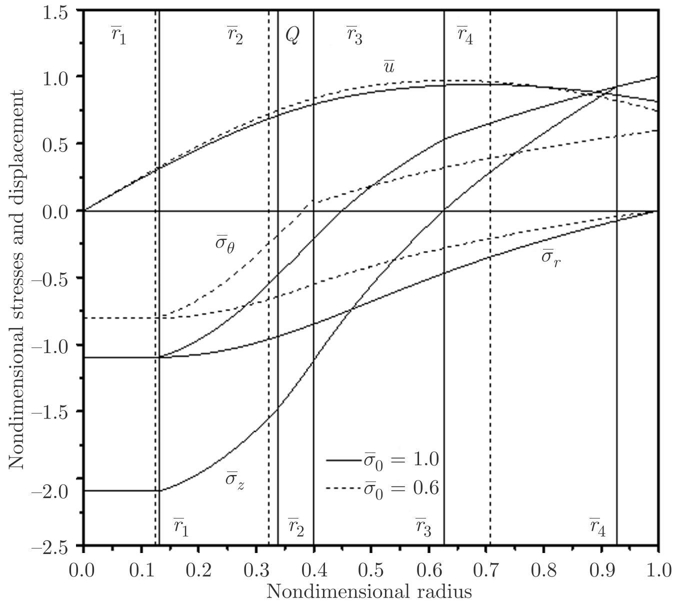


FIG. 4. Expansion of plastic regions as a function of the load parameter for $Q = 0.4$, $\bar{\sigma}_0 = 1.0$ and $\bar{\sigma}_0 = 0.6$.

7.2. Effect of the parameter $\overline{\sigma}_0$

It was shown in Fig. 4 that for $\overline{\sigma}_0 = 1$ yielding starts at the axis when $Q = 0.4$. On the other hand, for small values of the yield limit ratios yielding starts first at the outer surface of the tube. For $\overline{\sigma}_0 = 0.6$, the corresponding load parameter of the onset of yield at the surface of the tube is $\overline{q}''' = 11.5$. In Fig. 5, the stresses and the deformations are compared for the chosen yield limit ratios. If the yield limit ratio is decreased further, yielding starts again at the



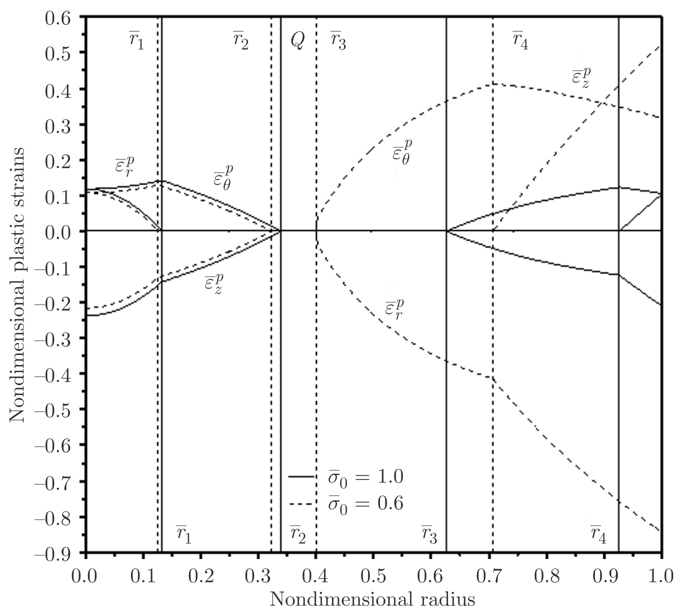


Fig. 5. a) Stresses and displacement, b) plastic strains for $Q = 0.4$, $\bar{q}''' = 20.944$, $\bar{\sigma}_0 = 1.0$ and $\bar{\sigma}_0 = 0.6$.

outer surface, but this time at smaller load parameters. When $Q = 0.8$ for both $\overline{\sigma}_0 = 1.0$ and $\overline{\sigma}_0 = 1.4$ yielding starts at the surface for the load parameters of $\overline{q}''' = 6.4798$ and $\overline{q}''' = 9.0720$ respectively, while the tube has a relatively higher resistance to yielding.

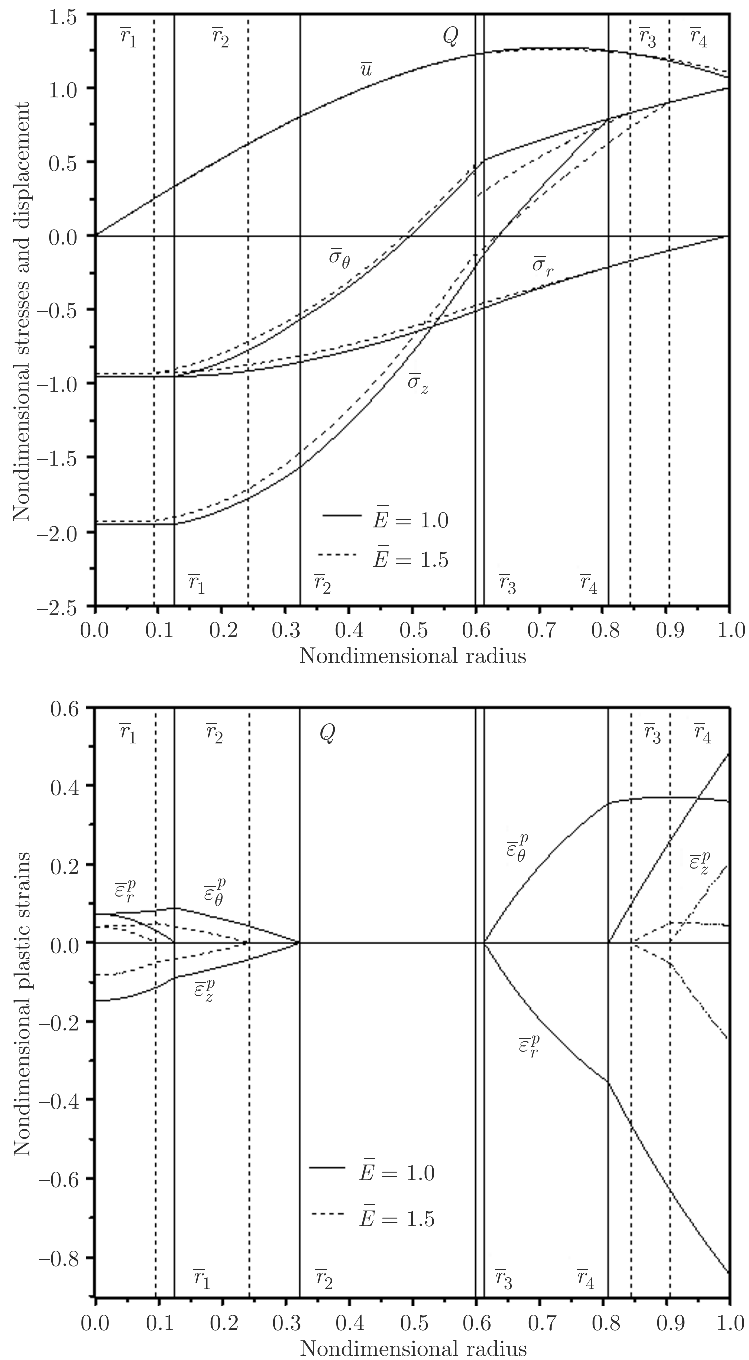


FIG. 6. a) Stresses and displacement; b) plastic strains for $Q = 0.6$, $\bar{q}''' = 15$, $\overline{E} = 1.0$ and $\overline{E} = 1.5$.

7.3. Effect of the parameter $\overline{E}$

The effect of the parameter $\overline{E}$ on the stress and deformation distribution in the second stage of elastic-plastic behavior is shown in Fig. 6. For $Q = 0.6$ and $\overline{q}''' = 15$ for $\overline{E} = 1$, the border radii are evaluated as $\overline{r}_1 = 0.12546$, $\overline{r}_2 = 0.32300$, $\overline{r}_3 = 0.61268$, $\overline{r}_4 = 0.80856$; whereas for $\overline{E} = 1.5$ the border radii are evaluated as $\overline{r}_1 = 0.09410$, $\overline{r}_2 = 0.24226$, $\overline{r}_3 = 0.84334$ and $\overline{r}_4 = 0.90494$. It is shown that, as the Young modulus of the inner component becomes greater, i.e., for $\overline{E} > 1$, the inner and outer plastic regions stay in a narrow range. However, the inner and outer plastic regions propagate faster for $\overline{E} < 1$. For the same geometric parameters $Q = 0.5$, $\overline{q}''' = 15$ and $\overline{E} = 0.5$, the border radii become $\overline{r}_1 = 0.04401$, $\overline{r}_2 = 0.11331$, $\overline{r}_3 = 0.51084$ and $\overline{r}_4 = 0.76464$, which justifies the case presented above.

7.4. Effect of the parameter $\overline{\nu}$

For $Q = 0.6$, $\overline{q}''' = 15$ and $\overline{\nu} = 1$, the radii were given in the previous section. For $\overline{\nu} = 0.7$, we have $\overline{r}_1 = 0.06529$, $\overline{r}_2 = 0.17805$, $\overline{r}_3 = 0.67017$ and $\overline{r}_4 = 0.82228$. This shows that as $\overline{\nu}$ gets smaller all of the plastic regions cannot expand at the same rate. For $Q = 0.5$, $\overline{q}''' = 15$ and $\overline{\nu} = 1$, the border radii are $\overline{r}_1 = 0.02298$, $\overline{r}_2 = 0.07679$, $\overline{r}_3 = 0.70994$, $\overline{r}_4 = 0.91793$, whereas for $\overline{\nu} = 1.38$, the border radii become $\overline{r}_1 = 0.81678$, $\overline{r}_2 = 0.32794$, $\overline{r}_3 = 0.51285$, $\overline{r}_4 = 0.89412$. Compared with $\overline{\nu} = 1$, it is found that when $\overline{\nu} > 1$ the plastic regions expand at a faster rate. In Fig. 7, it can be also noted that the plastic strains for $\overline{\nu} > 1$ are considerably greater than those for $\overline{\nu} = 1$ at the same load parameter.

7.5. Effect of the parameter $\overline{\lambda}$

The effect of the parameter $\overline{\lambda}$ on the temperature distribution was shown in Fig. 2. As $\overline{\lambda}$ increases the temperature distribution inside the composite cylinder increases. The effect of $\overline{\lambda}$ on elastic-plastic stress distribution is evident from its contribution on the temperature distribution. For $Q = 0.6$, $\overline{q}''' = 15$ and $\overline{\lambda} = 0.6$ the border radii are $\overline{r}_1 = 0.03211$, $\overline{r}_2 = 0.08267$, $\overline{r}_3 = 0.68031$ and $\overline{r}_4 = 0.95227$; whereas for $\overline{\lambda} = 1$ they become $\overline{r}_1 = 0.12546$, $\overline{r}_2 = 0.32300$, $\overline{r}_3 = 0.61268$ and $\overline{r}_4 = 0.80856$.

7.6. Effect of the parameter $\overline{\alpha}$

The effect of the coefficient of thermal expansion ratio $\overline{\alpha}$ is shown in Fig. 8. For $Q = 0.5$ and $\overline{q}''' = 15$ when $\overline{\alpha} = 1.2$, the border radii easily propagate (except for the outermost plastic region) and become $\overline{r}_1 = 0.08353$, $\overline{r}_2 = 0.21505$, $\overline{r}_3 = 0.55869$ and $\overline{r}_4 = 0.95456$. On the other hand, for $\overline{\alpha} = 0.8$ the border radii

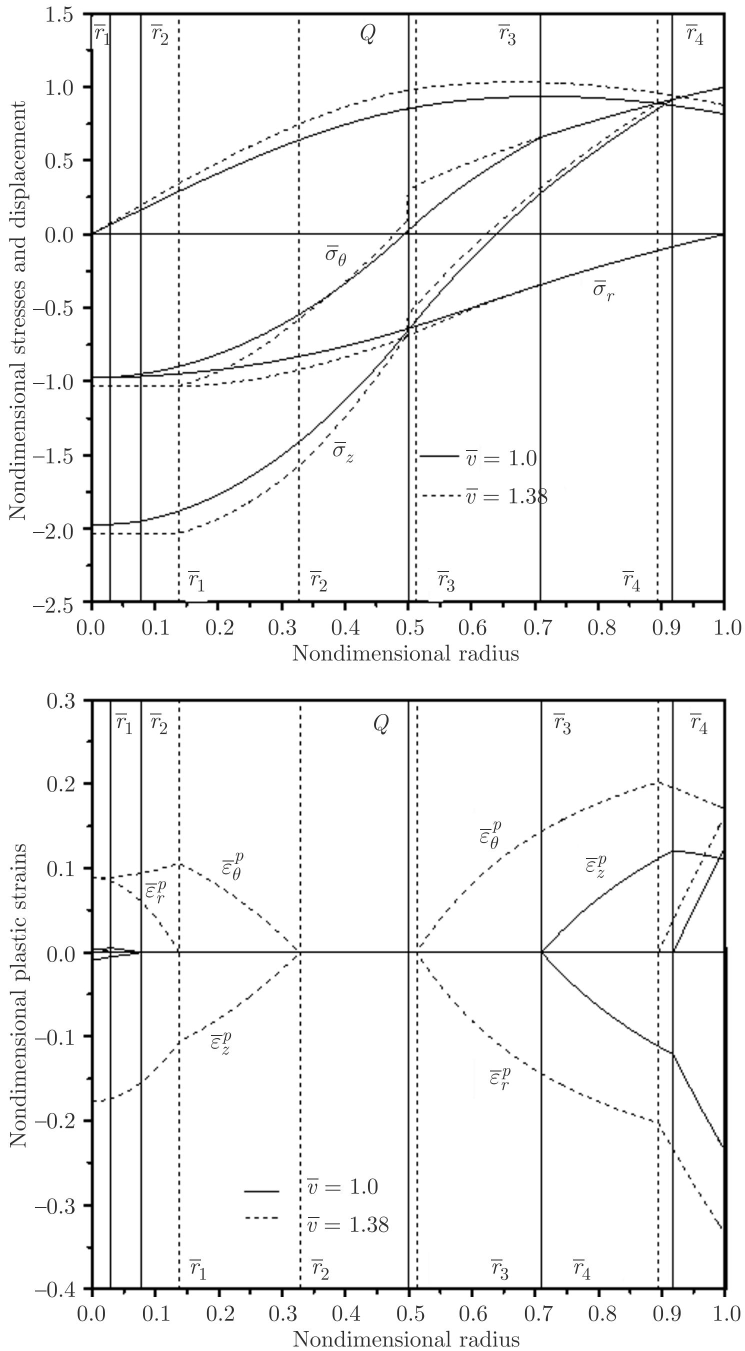


FIG. 7. a) Stresses and displacement, b) plastic strains for $Q = 0.5$, $\bar{q}''' = 15$, $\bar{\nu} = 1.0$ and $\bar{\nu} = 1.38$.

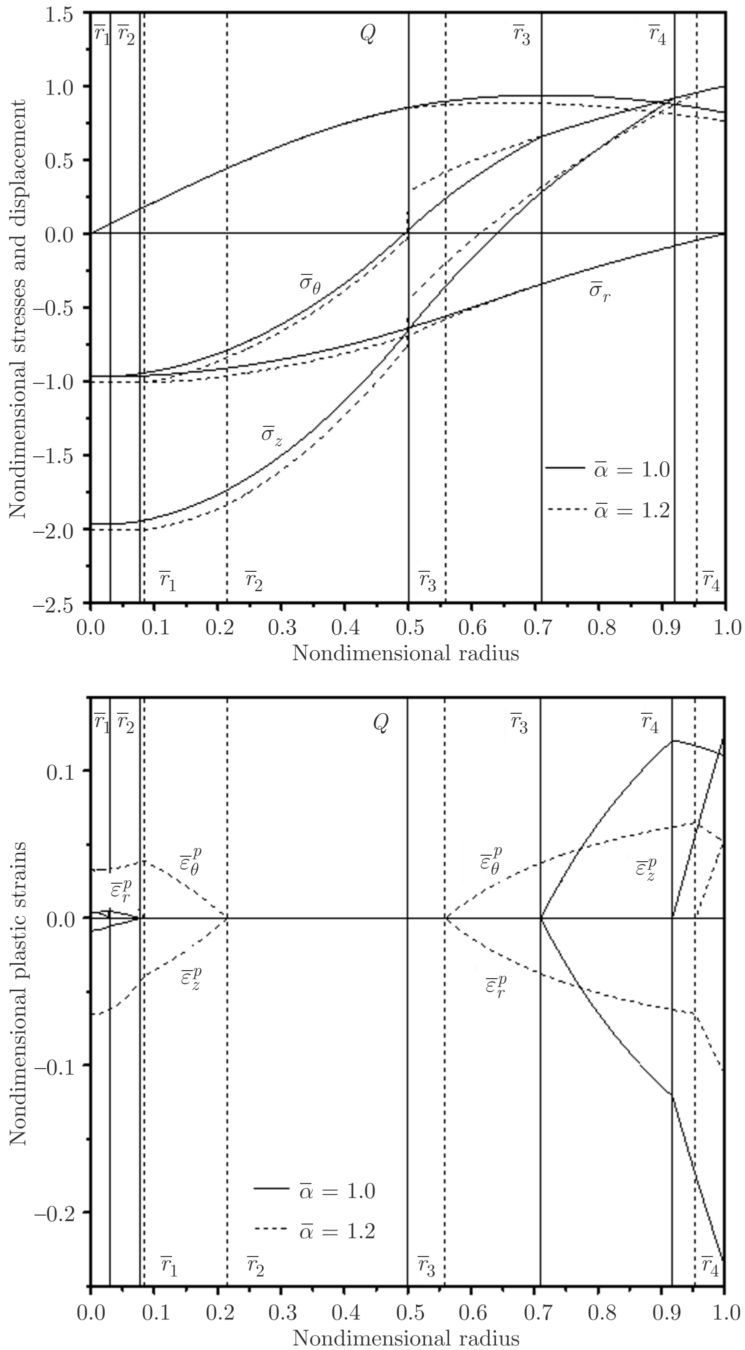


Fig. 8. a) Stresses and displacement, b) plastic strains for $Q = 0.5$, $\bar{q}''' = 15$, $\bar{\alpha} = 1.0$ and $\bar{\alpha} = 1.2$.

are $\overline{r}_1 = 0.06455$, $\overline{r}_2 = 0.16618$, $\overline{r}_3 = 0.70517$ and $\overline{r}_4 = 0.81678$, which shows that the expansion of plastic regions is retarded for $\overline{\alpha} < 1$.

8. Conclusions

The advantage of using a two-layered composite cylinder, due to its higher thermal load carrying capacity, becomes apparent when the results presented in this study are compared to those given by ORCAN for a homogeneous solid cylinder in [11]. As the thickness of the tubular shell is increased, onset of yield at the outer surface and its propagation become more difficult, and for sufficiently small values of the geometric parameter Q, yielding occurs at the axis if higher thermal load is supplied. The choice of the greater Young modulus for the solid cylinder, i.e., as $\overline{E}$ increases, causes the propagation of plastic regions to become more difficult, leading to wider elastic regions. On the other hand, choosing smaller values for each of the parameters $\overline{\nu}$, $\overline{\alpha}$ and $\overline{\lambda}$ results in a retardation of the expansion of plastic regions. It is concluded that thermally induced elastic-plastic deformations of the composite cylinder can be decreased significantly by a proper selection of the ratios of thermo-mechanical properties.

References

1. Z. Mróz, K. Kowalczyk–Gajewska, J. Maciejewski, R.B. Pęcherski, *Tensile or compressive plastic deformation of cylinders assisted by cyclic torsion*, Arch. Mech., **58**, 6, 497–527, 2006.
2. I.-D. Ghiba, *On the deformation of transversely isotropic porous elastic, circular cylinder*, Arch. Mech., **61**, 5, 407–421, 2009.
3. L. H. You, Z. Y. Zheng, *Elastic–plastic thermo-mechanical response of composite co-axial cylinders*, J. Mater. Sci., **41**, 15, 4901–4913, 2006.
4. Y. Orcan, U. Gamer, *Elastic-plastic deformation of a centrally heated cylinder*, Acta Mech., **90**, 1–4, 61–80, 1991.
5. M. Gulgec, *Influence of the temperature dependence of the yield stress on the stress distribution in a heat generating elastic-plastic cylinder*, ZAMM, **79**, 3, 210–216, 1999.
6. M. Gulgec, Y. Orcan, *Elastic-plastic deformation of a heat generating tube with temperature dependent yield stress*, Int. J. Eng. Sci., **38**, 1, 89–106, 2000.
7. Y. Orcan, M. Gulgec, *Influence of the temperature dependence of the yield stress on the stress distribution in a heat-generating tube with free ends*, J. Therm. Stress., **23**, 6, 529–547, 2000.
8. C.L. Tan, K.H. Lee, *Elastic-plastic stress analysis of a cracked thick-walled cylinder*, J. Strain Analysis, **18**, 4, 253–260, 1983.
9. M.A. Irfan, W. Chapman, *Thermal stresses in radiant tubes due to axial, circumferential and radial temperature distributions*, Appl. Therm. Eng., **29**, 10, 1913–1920, 2009.

10. A.N. Eraslan, *Stress distribution in energy generating two-layer tubes subjected to free and radially constrained boundary conditions*, Int. J. Mech. Sci., **45**, 3, 469–496, 2003.

11. Y. Orcan, *Thermal stresses in a heat generating elastic-plastic cylinder with free ends*, Int. J. Eng. Sci., **32**, 8, 883–898, 1994.

12. A. Ozturk, M. Gulgec, *Determination of onset of yield due to material properties in a heat generating two-layered compound cylinder*, Applied Mechanics and Materials, **325-326**, 22–27, 2013.

13. A. Ozturk, M. Gulgec, *Influence of the material properties on the elastic-plastic deformation in a heat generating composite solid cylinder*, Journal of the Faculty of Engineering and Architecture of Gazi University, **28**, 2, 283–292, 2013.

14. F.P. Incropera, D.P. De Witt, *Fundamentals of Heat and Mass Transfer*, p. 45, 2nd Ed., John Wiley & Sons, New York, 1985.

15. A. Mendelson, *Plasticity: Theory and Application*, p. 137, pp. 111–116, Macmillan, New York, 1968.

16. W.T. Koiter, *General theorems for elastic-plastic solids*, [in:] Progress in Solid Mechanics, I.N. Sneddon and R. Hill [Eds.], North-Holland Publishing Company, Amsterdam, pp. 165–221, 1960.

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