

Asymptotic solutions for generalized thermoelasticity with variable thermal material properties

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IN THIS PAPER, A UNIFIED GENERALIZED THERMOELASTIC SOLUTION WITH VARIABLE THERMAL MATERIAL PROPERTIES IS PROPOSED in the context of different generalized models of thermoelasticity, including thermoelasticity with one thermal relaxation time (LS theory), thermoelasticity with two thermal relaxation times (GL theory) and thermoelasticity without energy dissipation (GN theory). The unified form of governing equations is presented by introducing unifier parameters. The unified formulations are derived and given for isotropic homogenous materials with variable thermal material properties. The Laplace transform techniques and the Kirchoff's transformation are used to obtain general solutions for any set of boundary conditions in the physical domain. Asymptotic solutions for a specific problem of an elastic half-space with variable thermal conductivity and a specific heat, whose boundary is subjected to a thermal shock, are derived by means of the limit theorem of Laplace transform. In the context of these asymptotic solutions, some generalized thermoelastic phenomena are observed. Especially, the jumps at the wavefronts induced by the propagation of finite signal speed for the heat are clearly noticed. In addition, the effect of variable characteristics of material properties on thermoelastic behaviors is revealed by a comparison with the results obtained in the case of constant material properties.

Key words: generalized thermoelasticity, variable material properties, asymptotic solutions, thermal shock.

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1. Introduction

Prediction of thermoelastic behaviors in solids involving rapid transient heat conduction is of considerable practical importance in engineering sciences due to the widespread use of some new processing techniques such as pulse laser irradiation and rapid solidification. Some experiments [1] have proved that a heat signal propagates in an elastic medium with a finite speed when heat conduction takes

place at low temperature or short time interval. This means that the conventional coupled theory of thermoelasticity [2], which predicts the infinite thermal propagation speed, cannot accurately predict these specific heat phenomena. In order to overcome this shortcoming inherent in the conventional coupled theory of thermoelasticity, some efforts were made from a different perspective, and a wave-type equation disclosing a finite propagation speed of the heat wave was proposed by different researchers [3–6]. In accordance with these modified models, and the known generalized theories of thermoelasticity, the thermoelastic behavior involving finite propagation speed of heat signal was investigated in [7–9]. In these investigations, the “second sound” effects in solids, proposed by LANDAU and LIFSHITZ [10], were revealed by the solutions of problems involving short time interval, such as transient thermal shock.

Since these generalized theories are not used in the variation of material properties at the time of their establishment, most investigations have been conducted with an assumption of constant material properties, which limits the applicability of the results obtained from these generalized theories to certain ranges of temperature [11]. At high temperature, material properties are no longer constant but change with temperature, which has some effects on thermoelastic behavior of materials. It is necessary to take into account the actual behaviors of material properties. EZZAT *et al.* [11], YOUSSEF [12], AOUADI [13], OTHMAN and KUMAR [14], ALLAM *et al.* [15] and ABBAS [16] considered variable material properties and studied various thermoelastic problems in the context of different generalized theories, respectively, where the assumption that material parameters are the linear functions of the reference temperature was used to simplify the solutions of governing equations. XIONG and TIAN [17] and HE *et al.* [18] analyzed the thermoelastic response based on a linear function of each material parameter with respect to real temperature, and pointed out that the effect of variable material properties would be enhanced to a certain extent. The same linear relations between material properties and real temperature were also used by SHERIEF and EL-LATIEF [19] to solve a thermoelastic interaction in a half-space formed from a material with variable thermal conductivity.

When taking into account the complexity of the solutions for these generalized models, especially for some generalized thermoelastic problems involving variable material properties, the method to obtain exact solutions is very important in revealing the generalized thermoelastic behavior. The integral transform combined with numerical inversion was mostly used in previous investigations, in which the Laplace transform technique was first used to derive the general solution in the physical domain; then, the corresponding numerical solutions in the time domain were obtained by the numerical inversion [20]. Since the truncation error and the discretization error generated from the numerical inversion would decrease the precision of numerical solutions, the wave-like be-

haviors of heat transfer, especially the jumps located at each wavefront could not be revealed accurately and the “second sound” effects in solids induced by the propagation of heat signal with a finite speed were also weakened. The other method used to solve these generalized models is the direct solution in the time domain by means of numerical techniques such as the finite element method [21]. The advantage of this method is that it avoids the complicated integral transform and inverse transform, while some specific problems with complicated conditions can be solved. However, the shortcomings of numerical techniques such as the dependence on difference schemes, the reliability of the meshing of grids and the calculation errors would restrain the application of these generalized theories. Recently, an asymptotic analysis method has been introduced to solve some thermoelastic problems by means of different generalized theories [22, 23], in which the limit theorems of Laplace transform were used to simplify the general solutions in the physical domain, and the corresponding asymptotic solutions were derived in the time domain by the inverse Laplace transform. In the studies of thermal shock problem involving different generalized theories [22–24], it is assumed that a generalized thermoelastic phenomenon cannot be accurately analyzed (predicted) by using the above-mentioned two methods. However, the explicit expressions describing the propagation of heat signal can be obtained, which is important in showing the effect of each characteristic factor of the generalized theories of thermoelastic behaviors.

In this paper, the thermoelastic behaviors of an elastic medium with variable thermal material properties are investigated in the context of different generalized theories of thermoelasticity. An asymptotic analysis method is introduced to provide unified solutions for different generalized models. The propagation of the thermal wave and thermoelastic wave induced by external thermal shock are obtained by the solution of a specific problem of an elastic half-space formed from an isotropic homogeneous material with the variable thermal conductivity and specific heat, and its boundary subjected to a thermal shock. Additionally, the distributions of the displacement, temperature and each stress component are also illustrated. The comparison with the predictions obtained from the case of constant material properties is conducted to evaluate the effects of variable material properties on thermoelastic behaviors.

2. Unified formulation of generalized thermoelasticity

Due to the LS, GL and GN theories of generalized thermoelasticity [3-5], the fundamental equations for the isotropic homogeneous material in a unified form are presented by introducing the terms η_1 and η_2 as the unifier parameters. These equations, in general form, can be expressed as

– equation of motion:

$$(2.1) \quad \rho \ddot{u}_i = \rho f_i + \sigma_{ij,j},$$

– constitutive equation:

$$(2.2) \quad \sigma_{ij} = \lambda \gamma_{kk} \delta_{ij} + 2\mu \gamma_{ij} - \beta(\theta + \tau_1 \dot{\theta}) \delta_{ij},$$

– linear strain-displacement relation:

$$(2.3) \quad \gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

– energy equation:

$$(2.4) \quad q_{i,i} = \rho r - \rho c_p(\dot{\theta} + \tau_2 \ddot{\theta}) - T_0 \beta \dot{\gamma}_{kk} + c_i \dot{\theta}_{,i},$$

– heat conduction equation:

$$(2.5) \quad \eta_1 q_i + \tau_0 \dot{q}_i + \eta_2 \ddot{q}_i = -\eta_1 k \theta_{,i} - \eta_2 k^* \dot{\theta}_{,i} - c_i \dot{\theta}.$$

In the above-mentioned equations, u_i are the components of the displacement vector, q_i are the components of the heat flux vector, f_i are the components of the body force per unit mass, σ_{ij} are the components of the stress tensor, γ_{ij} are the components of the strain tensor, $\theta = T - T_0$ is the temperature increment, T is the absolute temperature, T_0 is the reference temperature, ρ is the mass density, k is the thermal conductivity, c_p is the specific heat at constant strain, r is the internal heat source, $\beta = (3\lambda + 2\mu)\alpha_T$ is the thermal-mechanical coefficient, α_T is the coefficient of linear thermal expansion, λ and μ are the Lamé's constants, τ_0 and τ_1, τ_2 are the relaxation time constant for LS and GL models, respectively, c_i are the components of new material constant proposed in GL model, and for isotropic material $c_i = 0$, and k^* is the new material constant associated with the GN model. Also, η_1 and η_2 are the terms introduced to consolidate all three theories into a unified system of equations. Meanwhile, the superscript dot ($\dot{\cdot}$) and the subscript comma ($,$) denote the derivatives of time t and coordinates x_i ($i = 1, 2, 3$), respectively.

Equations (2.1)–(2.5) can be reduced to the governing equations of LS, GL and GN theories by different values of unifier parameters and the corresponding material constants:

- 1) LS model: $\eta_1 = 1, \eta_2 = 0, c_i = 0, \tau_1 = \tau_2 = 0$.
- 2) GL model: $\eta_1 = 1, \eta_2 = 0, \tau_0 = 0$.
- 3) GN model: $\eta_1 = 0, \eta_2 = 1, c_i = 0, \tau_0 = \tau_1 = \tau_2 = 0$.

Also, if $\eta_1 = 1, \eta_2 = 0, c_i = 0$ and $\tau_0 = \tau_1 = \tau_2 = 0$, Eqs. (2.1)–(2.5) can be reduced to the governing equations of the classical coupled theory of thermoelasticity [2].

3. Governing equations with variable thermal material properties

We consider a half-space ($x \geq 0$) formed from an isotropic homogeneous material with variable thermal conductivity and variable specific heat. The boundary plane of the half-space is assumed to be traction free and is subjected to a sudden temperature rise. We assume that there is no body force or internal heat source affecting the half-space.

Due to the physics of this problem, it is clear that all the considered functions will depend on t and x only. Thus, the components of the displacement vector have the following forms:

$$(3.1) \quad u_x = u(x, t), \quad u_y = u_z = 0.$$

Substituting Eq. (3.1) into the linear strain-displacement relations (2.3), we obtain

$$(3.2) \quad \gamma_{xx} = u_{,x}, \quad \gamma_{yy} = \gamma_{zz} = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0.$$

Due to the constitutive equation (2.2), the non-zero stress components can be expressed as

$$(3.3) \quad \sigma_{xx} = (\lambda + 2\mu)u_{,x} - \beta(\theta + \tau_1\dot{\theta}),$$

$$(3.4) \quad \sigma_{yy} = \sigma_{zz} = \lambda u_{,x} - \beta(\theta + \tau_1\dot{\theta}).$$

The equation of motion without a body force can be rewritten as

$$(3.5) \quad \rho\ddot{u} = (\lambda + 2\mu)u_{,xx} - \beta(\theta_{,x} + \tau_1\dot{\theta}_{,x}).$$

Combining the energy equation (2.4) with heat conduction equation (2.5), and considering the variable characteristics of thermal conductivity k , k^* and specific heat c_p , the temperature equation without an internal heat source can be written as

$$(3.6) \quad (\eta_1 k + \eta_2 k^*)\theta_{,xx} + (\eta_1 k_{,x} + \eta_2 k^*_{,x})\theta_{,x} \\ = \rho c_p [\eta_1(\dot{\theta} + \tau_2\ddot{\theta}) + \tau_0\ddot{\theta} + \eta_2\ddot{\theta}] + T_0\beta(\dot{\gamma}_{xx} + \tau_0\ddot{\gamma}_{xx} + \eta_2\ddot{\gamma}_{xx}) + \rho\dot{c}_p(\tau_0\dot{\theta} + \eta_2\dot{\theta}).$$

For most materials, the thermal material properties change with the temperature increment θ and these temperature increment-dependent relations are linear in some range of the temperature T [18, 19]. Thus, the following linear relations of the thermal material parameters k , k^* and c_p dependent on temperature increment θ are used:

$$(3.7) \quad k = k(\theta) = k_0(1 + \chi_1\theta),$$

$$(3.8) \quad c_p = \frac{k}{\rho\kappa} = c_{p0}(1 + \chi_1\theta),$$

$$(3.9) \quad k^* = k^*(\theta) = k_0^*(1 + \chi_2\theta),$$

where k_0 , c_{p0} and k_0^* are the corresponding thermal conductivity and specific heat at reference temperature T_0 , and they are constant, κ is the thermal diffusivity at reference temperature T_0 and it is also constant, and χ_1 and χ_2 are the small quantities which indicate the influence of the temperature deviation.

Substituting the linear relations (3.7)–(3.9) into Eq. (3.6) results in

$$(3.10) \quad (\eta_1 k + \eta_2 k^*)\theta_{,xx} + (\eta_1 \chi_1 k_0 + \eta_2 \chi_2 k_0^*)\theta_{,x}\theta_{,x} \\ = \rho c_p [\eta_1 (\dot{\theta} + \tau_2 \ddot{\theta}) + \tau_0 \ddot{\theta} + \eta_2 \ddot{\theta}] + T_0 \beta (\dot{\gamma}_{xx} + \tau_0 \ddot{\gamma}_{xx} + \eta_2 \ddot{\gamma}_{xx}) + \chi_1 \rho c_{p0} \dot{\theta} (\tau_0 \dot{\theta} + \eta_2 \dot{\theta}).$$

For convenience, the Kirchhoff's transformation is introduced as follows:

$$(3.11) \quad \phi = \frac{1}{k_0} \int_0^\theta k(\theta_1) d\theta_1.$$

Applying the above transformation to Eq. (3.10), we obtain

$$(3.12) \quad (\eta_1 k_0 + \eta_2 \chi_2^* k_0^*)\phi_{,xx} \\ = \rho c_{p0} [\eta_1 (\dot{\phi} + \tau_2 \ddot{\phi}) + \tau_0 \ddot{\phi} + \eta_2 \ddot{\phi}] + T_0 \beta (\dot{\gamma}_{xx} + \tau_0 \ddot{\gamma}_{xx} + \eta_2 \ddot{\gamma}_{xx}),$$

where $\chi_2^* = \chi_2/\chi_1$.

Similarly, applying transformation (3.11) to Eqs. (3.7)–(3.9) and neglecting some small quantities generated by introducing ϕ to replace θ , the equation of motion and the non-zero stress components can be approximately expressed as

$$(3.13) \quad \rho \ddot{u} = (\lambda + 2\mu)u_{,xx} - \beta(\phi_{,x} + \tau_1 \dot{\phi}_{,x}),$$

$$(3.14) \quad \sigma_{xx} = (\lambda + 2\mu)u_{,x} - \beta(\phi + \tau_1 \dot{\phi}),$$

$$(3.15) \quad \sigma_{yy} = \sigma_{zz} = \lambda u_{,x} - \beta(\phi + \tau_1 \dot{\phi}).$$

4. Asymptotic solutions of the problems

4.1. General solutions in the transform domain

For simplicity, the following non-dimensional variables are introduced:

$$x^* = av_e x, \quad t^* = av_e^2 t, \quad \tau_i^* = av_e^2 \tau_i \quad (i = 0, 1, 2), \\ u^* = av_e \frac{\lambda + 2\mu}{\beta T_0} u, \quad \phi^* = \frac{\phi}{T_0}, \quad \theta^* = \frac{\theta}{T_0}, \quad \sigma_{ii}^* = \frac{1}{\beta T_0} \sigma_{ii}.$$

Substituting these non-dimensional variables into equations (3.12)–(3.15) and dropping, for convenience, the asterisks, we obtain

$$(4.1) \quad (\eta_1 + \eta_2 \chi_2^* v_t^2) \frac{\partial^2 \phi}{\partial x^2} = \left[\eta_1 \frac{\partial \phi}{\partial t} + (\tau_0 + \tau_2 + \eta_2) \frac{\partial^2 \phi}{\partial t^2} \right] \\ + \vartheta \left[\eta_1 \frac{\partial^2 u}{\partial x \partial t} + (\tau_0 + \eta_2) \frac{\partial^3 u}{\partial x \partial t^2} \right],$$

$$(4.2) \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial \phi}{\partial x} + \tau_1 \frac{\partial^2 \phi}{\partial x \partial t} \right),$$

$$(4.3) \quad \sigma_{xx} = \frac{\partial u}{\partial x} - \left(\phi + \tau_1 \frac{\partial \phi}{\partial x} \right),$$

$$(4.4) \quad \sigma_{yy} = \sigma_{zz} = k_v \frac{\partial u}{\partial x} - \left(\phi + \tau_1 \frac{\partial \phi}{\partial x} \right),$$

where $a = \rho c_{p0}/k_0$ is the thermal viscosity constant at reference temperature, $v_e = \sqrt{(\lambda + 2\mu)/\rho}$ is the speed of thermal elastic wave, $v_t = \sqrt{k_0^*/(\rho c_{p0} v_e^2)}$ is the non-dimensional speed of thermal wave, $\vartheta = T_0 \beta^2 / (\rho c_{p0} (\lambda + 2\mu))$ is the thermal coupling constant, and $k_v = \lambda / (\lambda + 2\mu)$ is the non-dimensional constant.

Applying to both sides of Eqs. (4.1)–(4.4), the Laplace transform defined as

$$L\{t\} = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

we obtain

$$(4.5) \quad (\eta_1 + \eta_2 \chi_2^* v_t^2) \frac{d^2 \bar{\phi}}{dx^2} = [\eta_1 s \bar{\phi} + (\tau_0 + \tau_2 + \eta_2) s^2 \bar{\phi}] \\ + \vartheta \left[\eta_1 s \frac{d\bar{u}}{dx} + (\tau_0 + \eta_2) s^2 \frac{d\bar{u}}{dx} \right],$$

$$(4.6) \quad s^2 \bar{u} = \frac{d^2 \bar{u}}{dx^2} - \left(\frac{d\bar{\phi}}{dx} + \tau_1 s \frac{d\bar{\phi}}{dx} \right),$$

$$(4.7) \quad \bar{\sigma}_{xx} = \frac{d\bar{u}}{dx} - (\bar{\phi} + \tau_1 s \bar{\phi}),$$

$$(4.8) \quad \bar{\sigma}_{yy} = \bar{\sigma}_{zz} = k_v \frac{d\bar{u}}{dx} - (\bar{\phi} + \tau_1 s \bar{\phi}).$$

Eliminating terms \bar{u} and $\bar{\phi}$ separately by combining Eq. (4.5) with Eq. (4.6) results in

$$(4.9) \quad \frac{d^4 \bar{\varphi}_i}{dx^4} - (s^2 + \omega_1 + \omega_2) \frac{d^2 \bar{\varphi}_i}{dx^2} + \omega_1 s^2 \bar{\varphi}_i = 0,$$

where $\bar{\varphi}_i$ ($i = 1, 2$) indicate term \bar{u} or $\bar{\phi}$, respectively, and

$$\omega_1 = \frac{[\eta_1 + (\tau_0 + \tau_2 + \eta_2)s]s}{\eta_1 + \eta_2 \chi_2^* v_t^2}, \quad \omega_2 = \frac{\vartheta[\eta_1 + (\tau_0 + \eta_2)s](1 + \tau_1 s)s}{\eta_1 + \eta_2 \chi_2^* v_t^2}.$$

The general solution of Eq. (4.9) can be expressed as

$$(4.10) \quad \bar{\varphi}_i = A_{1i}(s) \exp(-R_1x) + B_{1i}(s) \exp(-R_2x) \\ + C_{1i}(s) \exp(R_1x) + D_{1i}(s) \exp(R_2x),$$

where $R_{1,2}^2$ are the roots of the following characteristic equation:

$$R^4 - [s^2 + \omega_1 + \omega_2]R^2 + \omega_1s^2 = 0,$$

$A_{1i}(s)$, $B_{1i}(s)$, $C_{1i}(s)$ and $D_{1i}(s)$ are coefficients depending on parameters and are determined by the given boundary conditions.

Considering the bounded solutions with large x for the elastic half-space problem, the positive exponential part of expression (4.10) should be omitted, then we get

$$(4.11) \quad \bar{u} = A_{11}(s) \exp(-R_1x) + B_{11}(s) \exp(-R_2x),$$

$$(4.12) \quad \bar{\phi} = A_{12}(s) \exp(-R_1x) + B_{12}(s) \exp(-R_2x),$$

Substituting these general solutions (4.11) and (4.12) into Eqs. (4.5) or (4.6) results in

$$(4.13) \quad A_{12}(s) = -\frac{R_1^2 - s^2}{(1 + \tau_1s)R_1} A_{11}(s), \quad B_{12}(s) = -\frac{R_2^2 - s^2}{(1 + \tau_1s)R_2} B_{11}(s).$$

For the above relation, Eq. (4.12) can be rewritten as

$$(4.14) \quad \bar{\phi} = -\frac{R_1^2 - s^2}{(1 + \tau_1s)R_1} A_{11}(s) \exp(-R_1x) - \frac{R_2^2 - s^2}{(1 + \tau_1s)R_2} B_{11}(s) \exp(-R_2x).$$

Here, the following non-dimensional boundary conditions on the boundary plane $x = 0$ are introduced:

$$(4.15) \quad \theta(0, t) = \theta_0 H(t), \quad \sigma_{xx}(0, t) = 0,$$

where $H(t)$ is the Heaviside unit function and θ_0 is a non-dimensional constant.

Applying the Laplace transform to the above boundary condition (4.15) and considering Eq. (3.11), we get

$$(4.16) \quad \bar{\phi}(0, s) = \frac{\phi_0}{s}, \quad \bar{\sigma}_{xx}(0, s) = 0,$$

where $\phi_0 = (1 + \chi_1\theta_0)\theta_0/2$.

Substituting these boundary conditions into general solutions (4.11) and (4.14) and non-zero components of stress (4.7) and (4.8), the general solutions for u , ϕ and σ_{ii} ($i = 1, 2, 3$) in the transform domain can be derived as

$$(4.17) \quad \bar{u} = -\frac{(1 + \tau_1 s)\phi_0}{(R_1^2 - R_2^2)s} [R_1 \exp(-R_1 x) - R_2 \exp(-R_2 x)],$$

$$(4.18) \quad \bar{\phi} = \frac{\phi_0}{(R_1^2 - R_2^2)s} [(R_1^2 - s^2) \exp(-R_1 x) - (R_2^2 - s^2) \exp(-R_2 x)],$$

$$(4.19) \quad \bar{\sigma}_{xx} = \frac{(1 + \tau_1 s)s\phi_0}{R_1^2 - R_2^2} [\exp(-R_1 x) - \exp(-R_2 x)],$$

$$(4.20) \quad \bar{\sigma}_{yy} = \bar{\sigma}_{zz} \\ = \frac{(1 + \tau_1 s)\phi_0}{(R_1^2 - R_2^2)s} [(k_w R_1^2 + s^2) \exp(-R_1 x) - (k_w R_2^2 + s^2) \exp(-R_2 x)],$$

where $k_w = k_v - 1$.

4.2. Asymptotic solutions in the time domain

Solutions of u , ϕ and σ_{ii} ($i = 1, 2, 3$) in the time domain can also be obtained from the above transform solutions by means of inverse Laplace transform. Since the complicated expressions of roots R_i ($i = 1, 2$) are contained in these transform solutions, it is practically impossible to construct the exact solutions in a closed form in the time domain by inverse Laplace transform. Hence, the numerical inverse method was used to give some numerical predictions in previous investigations [11–19]. Unfortunately, the truncation error and the discretization error generated in the numerical inversion would lead to a poor prediction and the ‘second sound’ effect induced by the propagation of heat signal with a finite speed could not be described accurately [25]. In order to overcome this shortcoming inherent in numerical inversion, an asymptotic analysis method [22, 23] is introduced to solve this problem.

Due to the characteristics equation, the roots R_i ($i = 1, 2$) can be expressed as

$$(4.21) \quad R_{1,2} = \sqrt{\left(\frac{s^2 + \omega_1 + \omega_2 \pm \sqrt{(s^2 + \omega_1 + \omega_2)^2 - 4\omega_1 s^2}}{2}\right)}.$$

In accordance with the limit theorem of the Laplace transform, the transform parameter s would be a high value when the thermal duration is very short (t is a low value). Then, some approximations for the roots R_i ($i = 1, 2$) can be derived as [22–24]

$$(4.22) \quad R_{1,2} \approx k_{1,2}s + m_{1,2},$$

where

$$k_{1,2} = \left[\frac{1 + \tau_0 + \tau_2 + \eta_2/(\chi_2^* v_t^2) + \vartheta(\tau_0 + \tau_1 + \eta_2/(\chi_2^* v_t^2)) \pm \sqrt{a_1}}{2} \right]^{1/2},$$

$$m_{1,2} = \eta_1 \frac{1 + \vartheta \pm b_1/\sqrt{a_1}}{4k_{1,2}},$$

$$a_1 = [1 + \tau_0 + \tau_2 + \eta_2/(\chi_2^* v_t^2) + \vartheta(\tau_0 + \tau_1 + \eta_2/(\chi_2^* v_t^2))]^2 - 4(\tau_0 + \tau_2 + \eta_2/(\chi_2^* v_t^2)),$$

$$b_1 = \eta_1[(1 + \vartheta)(\tau_0 + \tau_2 + \vartheta(\tau_0 + \tau_1)) + \vartheta - 1].$$

Substituting Eq. (4.22) into the general solutions (4.17)-(4.20) derived in the transform domain, the forms convenient to inverse the Laplace transform can be obtained. By using the standard results of the Laplace transform theory, the asymptotic solutions of u , ϕ and σ_{ii} ($i = 1, 2, 3$) in time domain can be derived as

$$(4.23) \quad u = -\frac{\phi_0}{\sqrt{a_1}} \exp(-m_1 x) \left[k_1 \tau_1 + \left(k_1 + m_1 \tau_1 - \frac{b_1}{a_1} k_1 \tau_1 \right) (t - k_1 x) \right] H(t - k_1 x) \\ + \frac{\phi_0}{\sqrt{a_1}} \exp(-m_2 x) \left[k_2 \tau_1 + \left(k_2 + m_2 \tau_1 - \frac{b_1}{a_1} k_2 \tau_1 \right) (t - k_2 x) \right] H(t - k_2 x),$$

$$(4.24) \quad \phi = \frac{\phi_0}{\sqrt{a_1}} \exp(-m_1 x) \left[k_3 + \left(m_3 - \frac{b_1}{a_1} k_3 \right) (t - k_1 x) \right] H(t - k_1 x) \\ - \frac{\phi_0}{\sqrt{a_1}} \exp(-m_2 x) \left[k_4 + \left(m_4 - \frac{b_1}{a_1} k_4 \right) (t - k_2 x) \right] H(t - k_2 x),$$

$$(4.25) \quad \sigma_{xx} = \frac{\phi_0}{\sqrt{a_1}} \exp(-m_1 x) \\ \times \left\{ \left(1 - \frac{b_1}{a_1} \tau_1 \right) \left[1 - \frac{b_1}{a_1} (t - k_1 x) \right] H(t - k_1 x) + \tau_1 \delta(t - k_1 x) \right\} \\ - \frac{\phi_0}{\sqrt{a_1}} \exp(-m_2 x) \left\{ \left(1 - \frac{b_1}{a_1} \tau_1 \right) \left[1 - \frac{b_1}{a_1} (t - k_2 x) \right] H(t - k_2 x) + \tau_1 \delta(t - k_2 x) \right\},$$

$$(4.26) \quad \sigma_{yy} = \sigma_{zz} \\ = \frac{\phi_0}{\sqrt{a_1}} \exp(-m_1 x) \left\{ (1 + k_w k_5 + \tau_1 k_w m_3) \left[1 - \frac{b_1}{a_1} (t - k_1 x) \right] H(t - k_1 x) \right. \\ \left. + k_w m_3 (t - k_1 x) H(t - k_1 x) + (1 + k_w k_5) \tau_1 \delta(t - k_1 x) \right\} \\ - \frac{\phi_0}{\sqrt{a_1}} \exp(-m_2 x) \left\{ (1 + k_w k_6 + \tau_1 k_w m_4) \left[1 - \frac{b_1}{a_1} (t - k_2 x) \right] H(t - k_2 x) \right. \\ \left. + k_w m_4 (t - k_2 x) H(t - k_2 x) + (1 + k_w k_6) \tau_1 \delta(t - k_2 x) \right\},$$

where

$$\begin{aligned}
 k_{3,4} &= \frac{-1 + \tau_0 + \tau_2 + \eta_2/(\chi_2^* v_t^2) + \vartheta(\tau_0 + \tau_1 + \eta_2/(\chi_2^* v_t^2)) \pm \sqrt{a_1}}{2}, \\
 m_{3,4} &= \eta_1 \frac{1 + \vartheta \pm b_1/\sqrt{a_1}}{2}, \\
 k_{5,6} &= \frac{1 + \tau_0 + \tau_2 + \eta_2/(\chi_2^* v_t^2) + \vartheta(\tau_0 + \tau_1 + \eta_2/(\chi_2^* v_t^2)) \pm \sqrt{a_1}}{2},
 \end{aligned}$$

$\delta(x)$ is the Dirac delta function.

For the expression of ϕ , the temperature increment θ can be obtained by solving Eq. (3.11) to give

$$(4.27) \quad \theta = \frac{-1 + \sqrt{1 + 2\chi_1\phi}}{\chi_1}.$$

5. Numerical results and discussions

5.1. Wave propagation analysis

The Heaviside unit function in the above asymptotic solutions (4.23)–(4.26) predicts the occurrence of waves. It is obvious that each of u , ϕ and σ_{ii} ($i = 1, 2, 3$) is made of two parts and each part corresponds to a wave propagation with a finite speed, one is the thermal wave (T-wave), and the other is the thermal elastic wave (E-wave). Due to the property of the Heaviside unit function, the non-dimensional wave propagation velocities and positions of each wavefront can be obtained as

$$(5.1) \quad v_{1,2} = 1/k_{1,2}, \quad \xi_{1,2} = t/k_{1,2}.$$

In combination with the expressions of parameters $k_{1,2}$, we can observe that the velocities and positions of wavefronts are dependent on relaxation time constants τ_0 , τ_1 , τ_2 and thermal coupling constant ϑ for the LS and GL models, respectively. Furthermore, when the relaxation time constants are $\tau_0 \rightarrow 0$ and $\tau_2 \rightarrow 0$, which corresponds to the Fourier heat conduction, we have $v_1 \rightarrow \{1, 1/\sqrt{1 + \vartheta\tau_1}\}$ and $v_2 \rightarrow \infty$. This means that the velocity with subscript 1 is the propagation velocity of E-wave, and the velocity with subscript 2 is the propagation velocity of T-wave. For GN model, the velocities and positions of wavefronts only depend on the thermal coupling constants ϑ , and for $\vartheta \rightarrow 0$, we have $v_1 \rightarrow 1$ and $v_2 \rightarrow v_t$. Considering the non-dimensional velocity v_t indicates the thermal propagation velocity without the coupling effect, we have the same conclusion in the LS and GL models for the meaning of velocity subscripts 1 and 2.

Meanwhile, we also observe that k_1 is a monotonously increasing function of thermal coupling constant ϑ , while k_2 is a monotonously decreasing function of ϑ for the LS, GL and GN models, which means that the coupling effect has different impact on the propagation of T-wave and E-wave. The velocity of T-wave is increasing with an increase of the value of ϑ , and the corresponding position of wavefront is closer to the thermal boundary. For E-wave, it is the opposite with an increase of the value of ϑ .

5.2. Thermoelastic response analysis

Now for the illustration of thermoelastic response involving variable thermal material properties in the context of different generalized theories, the following non-dimensional constants are introduced to numerical calculations, as in [21, 23]:

$$\tau_0 = 0.5, \quad \tau_1 = 0.5, \quad \tau_2 = 0.25, \quad \vartheta = 0.02, \quad \chi_1 = \chi_2 = -0.25, \quad v_t^2 = 4, \quad \theta_0 = 1.$$

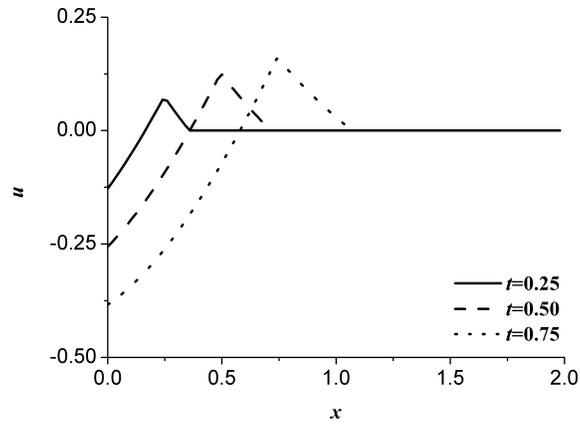
Figures 1–4 show the distributions of displacement u , temperature θ and each component of stress σ_{ii} for different generalized theories of thermoelasticity (LS, GL and GN models) at a wide range of x ($0 \leq x \leq 2$) and at different time. As for these distributions, a phenomenon important for generalized thermoelastic problem, in which all of u , θ and σ_{ii} vanish identically at all points beyond the faster wavefront, is observed. The displacement is continuous at all the positions including the locations of the wavefronts for LS and GN model, but it is discontinuous at both the wavefronts for GL model, which means that one portion of matter penetrates into another, and this phenomenon violates the continuum hypothesis. These results can also be obtained by substituting the wavefronts $t = k_1x$ and $t = k_2x$ into Eq. (4.23), then the discontinuities for the displacement are as follows:

$$\begin{aligned} [u]_{k_{1,2}x} &= 0 && \text{(LS and GN models),} \\ [u]_{k_{1,2}x} &= \mp \frac{\phi_0}{\sqrt{a_1}} \exp(-m_{1,2}x) k_{1,2} \tau_1 && \text{(GL model).} \end{aligned}$$

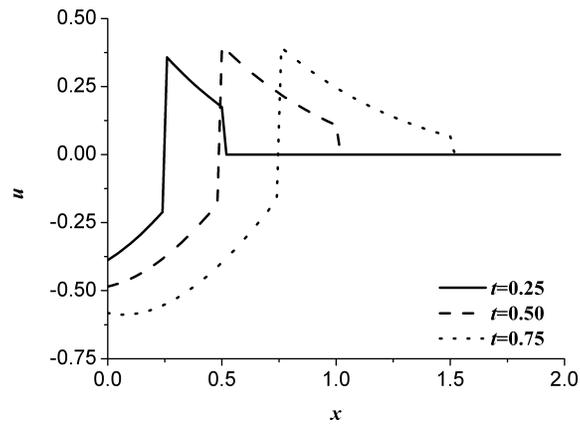
The temperature is discontinuous at both the wavefronts for LS, GL and GN models, and these discontinuities are also obtained by substituting the wavefronts $t = k_1x$ and $t = k_2x$ into Eqs. (4.24) and (4.27), which can be expressed as

$$[\theta]_{k_{1,2}x} = \frac{1}{\chi_1} \left(-1 + \sqrt{1 \pm \frac{2\chi_1\phi_0}{\sqrt{a_1}} k_{3,4} \exp(-m_{1,2}x)} \right) \text{ (LS, GL and GN models).}$$

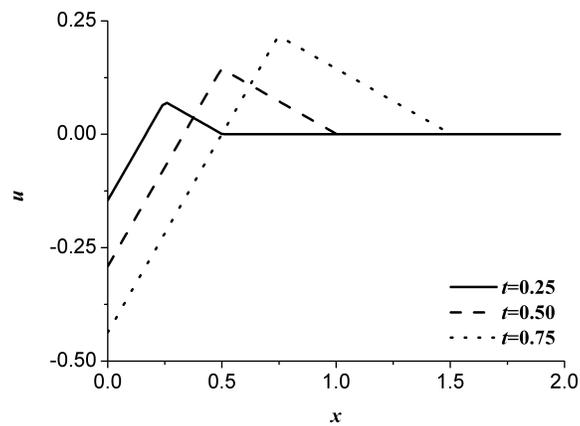
It is notable that the effect of the magnitude of jump at the first wavefront on temperature is very small, which leads to the fact that the first discontinuity



(a) LS model

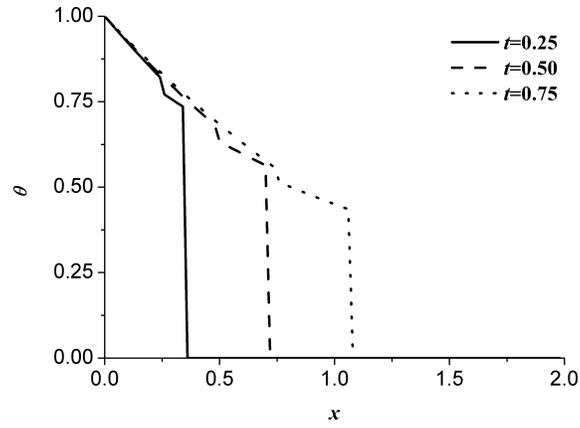


(b) GL model

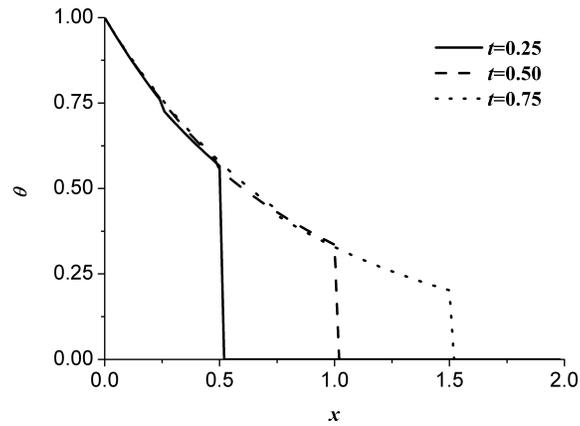


(c) GN model

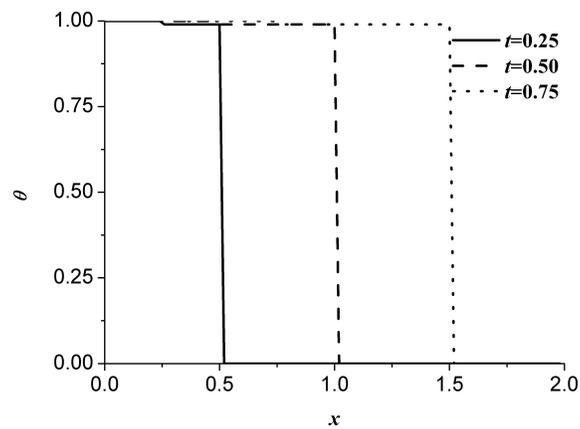
FIG. 1. Distribution of the displacement u with x at different time t and $\chi_1 = \chi_2 = -0.25$ for three generalized models (LS, GN and GS).



(a) LS model



(b) GL model



(c) GN model

FIG. 2. Distribution of the temperature θ with x at different time t and $\chi_1 = \chi_2 = -0.25$ for three generalized models (LS, GL and GN).

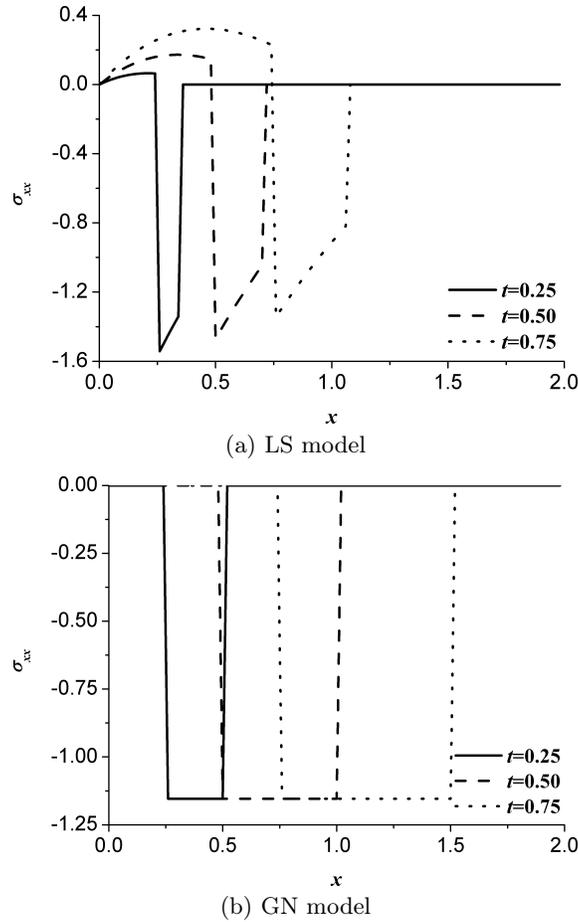


FIG. 3. Distribution of the stress component σ_{xx} with x at different time t and $\chi_1 = \chi_2 = -0.25$ for generalized models (LS and GN).

cannot be clearly shown in Fig. 2. This means that E-wave has only a small effect on temperature. Furthermore, LS and GL models give similar predictions for temperature distribution, whose magnitudes of jumps exponentially decrease with the propagation of E-wave and T-wave, while for GN model, the magnitudes of jumps remain unchanged at all the positions, which is consistent with the assumption that there is no dissipation of thermal energy in GN theory.

In accordance with Eqs. (4.25) and (4.26), the expression of each stress component exhibits the Dirac delta function for GL model, which means the magnitudes of the stress components are finite at both the wavefronts. Therefore, the stress distributions for LS and GN models are only illustrated in Figs. 3 and 4. The values of stress still are constant in each of the intervals for GN model,

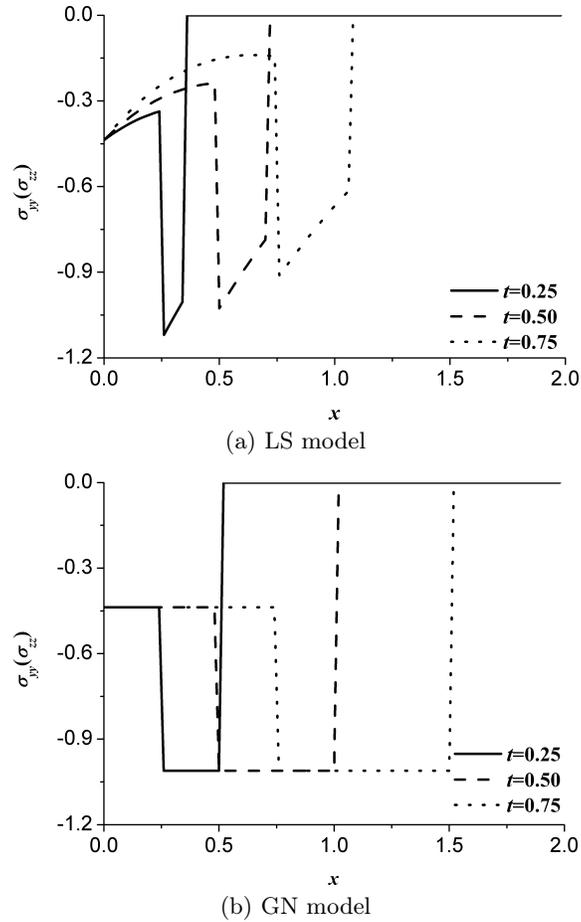
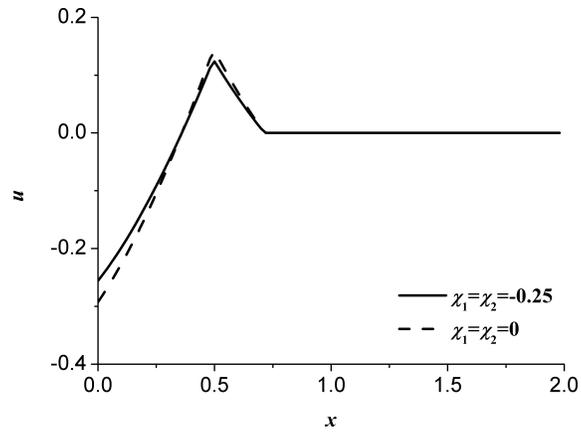


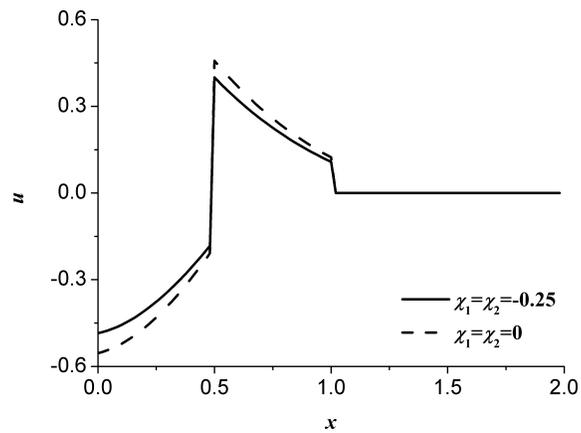
FIG. 4. Distribution of the stress component $\sigma_{yy}(\sigma_{zz})$ with x at different time t and $\chi_1 = \chi_2 = -0.25$ for generalized models (LS and GN).

but the stress increases at first and then decreases steadily in the first interval ($0 \leq x < t/k_1$), and then increases quickly again in the second interval ($t/k_1 < x < t/k_2$), and finally vanishes in the last interval ($t/k_2 < x < \infty$) for LS model.

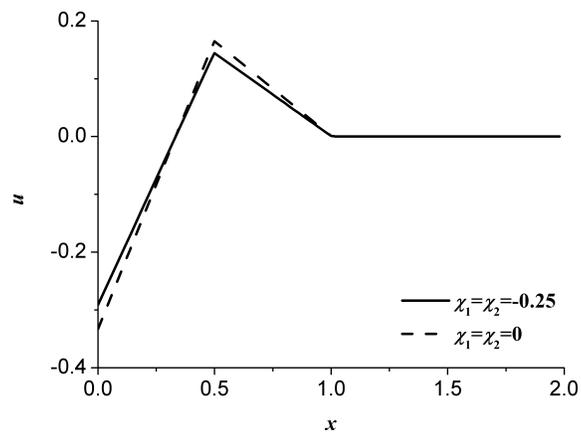
For Eqs. (3.7)–(3.9), if the parameters χ_1 and χ_2 are taken to be zero, the thermal material parameters k , c_p and k^* are constants, which corresponds to the case of constant material properties. Figures 5–7 display the distributions of displacement u , temperature θ and stress component σ_{xx} for different generalized theories at the given time and different values of parameters χ_1 and χ_2 . Some differences can be observed in the comparisons between the cases of variable and constant thermal material properties. First, both peak values of the displacement, temperature and stress are decreased for three models when the



(a) LS model

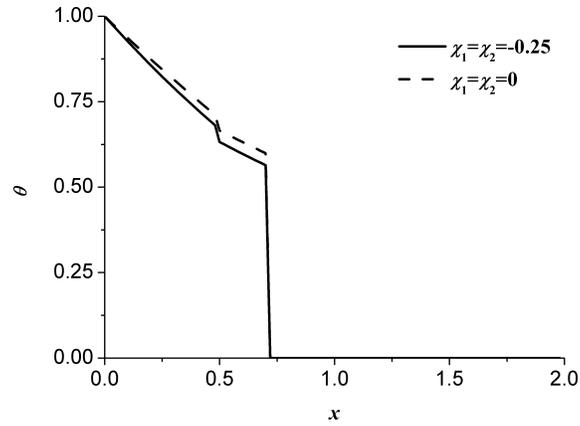


(b) GL model

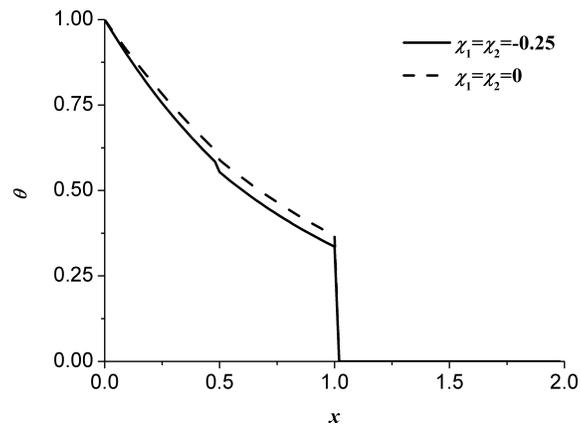


(c) GN model

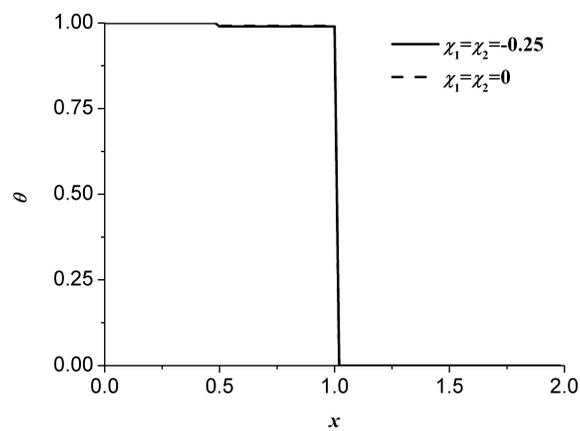
FIG. 5. Effects of parameters χ_1 and χ_2 on the distribution of displacement u at $t = 0.5$ for three models (LS, GL and GN).



(a) LS model



(b) GL model



(c) GN model

FIG. 6. Effects of parameters χ_1 and χ_2 on the distribution of temperature θ at $t = 0.5$ for three models (LS, GL and GN).

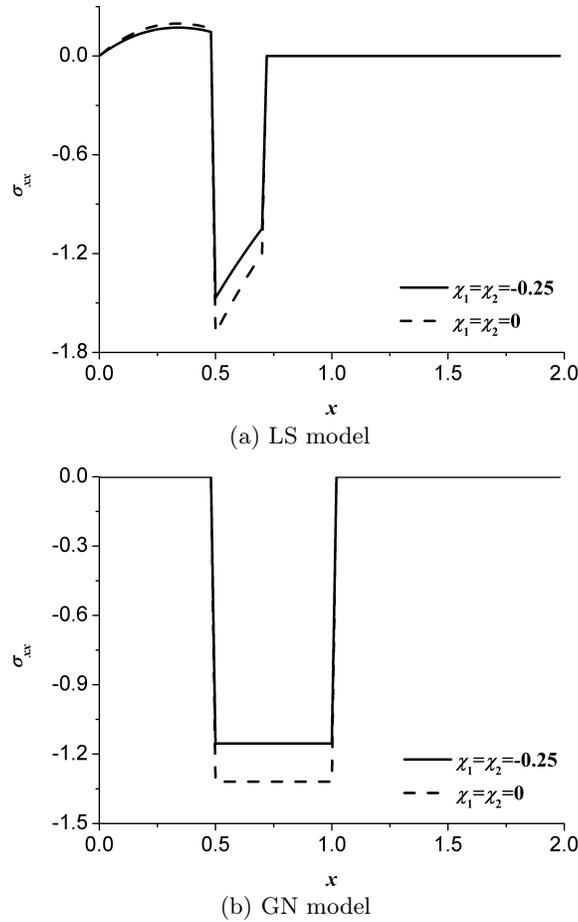


FIG. 7. Effects of parameters χ_1 and χ_2 on the distribution of stress component σ_{xx} at $t = 0.5$ for LS and GN models.

variable characteristics of thermal material properties are considered, but it can be noticed that the first peak value for temperature shows only a small change compared with the other change. Second, the locations of jumps for temperature and stress remain unchanged for two cases, which means that the effects of variable thermal material properties on thermoelastic behaviors are only reflected in the distributions of each field.

6. Conclusions

The methods, presented in the above described generalized theories, were used to effectively solve the governing equations. These methods are very impor-

tant in the studies of thermoelastic behavior involving finite propagation speed of heat signal and lacking necessary experiments. In this paper, the asymptotic approach, based on the Laplace transform and the Kirchhoff's transformation, was proposed to solve these complicated governing equations with variable material properties. The analytical solutions of each physical field as well as the explicit expressions of two waves' velocities and their wavefronts were obtained by this asymptotic approach, which clearly showed the thermoelastic behavior involving finite propagation speed of heat signal and variable material properties. Using the asymptotic solutions in an elastic half-space problem with the boundary subjected to a thermal shock, we obtained some conclusions as follows:

1) All the distributions of the displacement u , temperature θ and each stress component σ_{ii} ($i = 1, 2, 3$) are staged by the influence of the propagation of heat signal with a finite speed, which vanish identically at all points beyond the faster wavefront. The regions of each distribution are enlarging with propagation of E-wave and T-wave. The peak values of temperature and each stress component are decreasing for LS and GL models, while, for GN model, they remain unchanged assuming that there is no thermal energy dissipation.

2) The variable thermal material properties have a significant effect on thermoelastic behaviors and these effects mainly focus on the change of the magnitudes of thermoelastic response. Although different predictions for the distributions of each field are obtained by different generalized theories, for the effect of variable thermal material properties on thermoelastic behaviors, these generalized theories give the same predictions, except for the numerical differences in solutions,

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References

1. K. MITRA, S. KUMAR, A. VDDAVARZ, M.K. MOALLEMI, *Experimental evidence of hyperbolic heat conduction in processed meat*, ASME J. Heat Trans., **117**, 568–573, 1995.
2. M.A. BIOT, *Thermoelasticity and irreversible thermodynamic*, J. Appl. Phys., **27**, 240–253, 1956.
3. H.W. LORD, Y. SHULMAN, *A generalized dynamic theory of thermoelasticity*, J. Mech. Phys. Solids, **15**, 299–309, 1967.
4. A.E. GREEN, K.A. LINDSAY, *Thermoelasticity*, J. Elast., **2**, 1–7, 1972.

5. A.E. GREEN, P.M. NAGHDI, *Thermoelasticity without energy dissipation*, J. Elast., **31**, 189–208, 1993.
6. Y.Z. POVSTENKO, *Fractional Cattaneo-type equations and generalized thermoelasticity*, J. Therm. Stresses, **34**, 97–114, 2011.
7. D.S. CHANDRASEKHARAIHAH, *Hyperbolic thermoelasticity: a review of recent literature*, Appl. Mech. Rev., **51**, 795–729, 1998.
8. R.B. HETNARSKI, J. IGNACZAK, *Generalized thermoelasticity*, J. Therm. Stresses, **22**, 451–476, 1999.
9. X.G. TIAN, Y.P. SHEN, *Research progress in generalized thermoelastic problems*, Advances in Mechanics, **42**, 1–11, 2012.
10. L.D. LANDAU, E.M. LIFSHITZ, *Fluid Mechanics*, Pergamon Press, Oxford, UK, 2nd ed., 1987.
11. M.A. EZZAT, A.S. EL-KARAMANY, A.A. SAMAAAN, *The dependence of the modulus of elasticity on reference temperature in generalized thermoelasticity with thermal relaxation*, Appl. Math. Comput., **147**, 169–189, 2004.
12. H.M. YOUSSEF, *Dependence of modulus of elasticity and thermal conductivity on reference temperature in generalized thermoelasticity for an infinite material with a spherical cavity*, Appl. Math. Mech., **26**, 470–475, 2005.
13. M. AOUADI, *Generalized thermo-piezoelectric problems with temperature-dependent properties*, Int. J. Solids Struct., **43**, 6347–6358, 2006.
14. M.I.A. OTHMAN, R. KUMAR, *Reflection of magneto-thermoelasticity waves with temperature dependent properties in generalized thermoelasticity*, Int. Comm. Heat Mass Transfer, **36**, 513–520, 2009.
15. M.N. ALLAM, K.A. ELSIBAI, A.E. ABOUELREGAL, *Magneto-thermoelasticity for an infinite body with a spherical cavity and variable material properties without energy dissipation*, Int. J. Solids Struct., **47**, 2631–2638, 2010.
16. I.A. ABBAS, *Eigenvalue approach in a three-dimensional generalized thermoelastic interaction with temperature-dependent material properties*, Comput. Math. Appl., **68**, 2036–2056, 2014.
17. Q.L. XIONG, X.G. TIAN, *Transient magneto-thermoelastic response for a semi-infinite body with voids and variable material properties during thermal shock*, Int. J. Appl. Mech., **3**, 161–185, 2011.
18. T.H. HE, S.H. SHI, *Effect of temperature-dependent problems on thermoelastic problem with thermal relaxations*, Acta Mech. Solida Sin., **27**, 412–419, 2014.
19. H.H. SHERIEF, A.M. ABD EL-LATIEF, *Effect of variable thermal conductivity on a half-space under the fractional order theory of thermoelasticity*, Int. J. Mech. Sci., **74**, 185–189, 2013.
20. N.M. EL-MAGHRABY, H.M. YOUSSEF, *State space approach to generalized thermoelastic problem with thermomechanical shock*, Appl. Math. Comput., **156**, 577–586, 2004.
21. X.G. TIAN, Y.P. SHEN, C.Q. CHEN, T.H. HE, *A direct finite element method study of generalized thermoelastic problems*, Int. J. Solids Struct., **43**, 2050–2063, 2006.

22. M. BALLA, *Analytical study of the thermal shock problem of a half-space with various thermoelastic models*, Acta Mech., **89**, 73–92, 1991.
23. Y.Z. WANG, X.B. ZHANG, X.N. SONG, *A unified generalized thermoelasticity solution for the transient thermal shock problem*, Acta Mech., **223**, 735–743, 2012.
24. Y.Z. WANG, X.B. ZHANG, D. LIU, *Asymptotic analysis of generalized thermoelasticity for axisymmetric plane strain problem with temperature-dependent material properties*, Int. J. Appl. Mech., **5**, 1350023-20, 2013.
25. A.S. EL-KARAMANY, M.A. EZZAT, *Thermal shock problem in generalized thermoviscoelasticity under four theories*, Int. J. Eng. Sci., **42**, 649–671, 2004.

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