Arch. Mech., **67**, 1, pp. 75–93, Warszawa 2015

Series, index and threshold for random 2D composite

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Effective conductivity of a 2D random composite is expressed in the form of long series in the volume fraction of ideally conducting disks. The problem of a *direct* reconstruction of the critical index for superconductivity from the series is solved with good accuracy, for the first time. General analytical expressions for conductivity in the whole range of concentrations are derived and compared with the regular composite and existing models.

Key words: two-dimensional random composite, effective conductivity, critical index, crossover formula.

1. Introduction

It is frequently that only lower order formulae can be obtained analytically for the effective conductivity problem. On the other hand, they cannot be analytically solved in general case because of the complicated random geometrical structure of the composite. Therefore, hard numerical computations are applied. Of course, advanced computational approaches can be useful for practical purposes important for mechanical engineering. But an exact or an accurate approximate formula has its own merits, adding substantially to the problem's general understanding. In particular, some sophisticated expressions/approximants can capture an asymptotic behavior near singular points when numerical methods usually fail.

In the present paper, we deduce such a formula for a 2D, two-component composite made from a collection of non-overlapping, identical, ideally conducting circular disks, embedded randomly in an otherwise uniform locally isotropic host (see Fig. 1). The effective conductivity problem for an insulating or ideally conducting inclusions is called the conductivity and superconductivity problem, respectively [1]. The problem and its approximate solution go back to Maxwell, see e.g. [2].

There are two important unresolved problems in the theory of random composites:

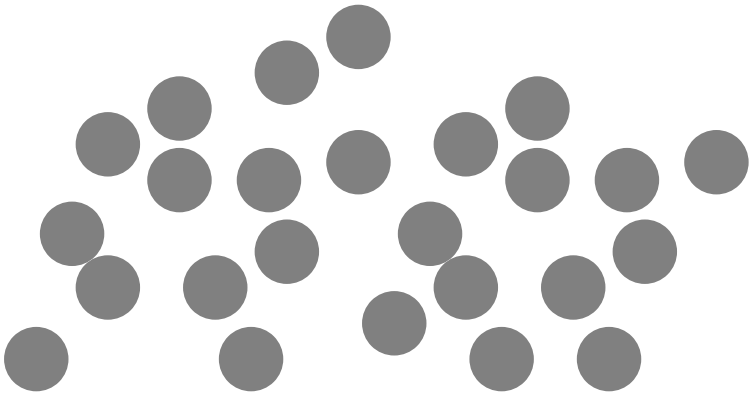

FIG. 1. Randomly distributed disks.

1. what quantity should stand for the maximum volume fraction x_c of random composites [3], and

2. theoretical explanation of the values of critical indices for conductivity and superconductivity denoted by t and s, respectively [1].

Recently, a novel technique for deriving expansions in concentration was suggested [4]. It combines analytic and numeric methods for solving the conductivity problem directly in the 2D case. It is applicable both for regular [5] and random cases. Thus, we proceed to the case of a 2D random composite, where the series in concentration for the effective conductivity by itself, will be presented and analyzed systematically, following generally to [4, 5]. The series will be used to estimate the index and the threshold in 2D random case.

The problem of defining the threshold is highly non-trivial, since the random closest packing of hard spheres turned out to be ill-defined, and cannot stand for the maximum volume fraction. It depends on the protocol employed to produce the random packing as well as other system characteristics [3].

The problem seems less acute in two dimensions, where various protocols seems to agree on what quantity should stand for the maximum volume fraction of random composites [6, 7, 8, 4, 9]. Namely it is the concentration of $\frac{\pi}{\sqrt{12}} \approx 0.9069$, attained only for the regular hexagonal array of disks. The sought value for a long time was thought to be close to 0.82, and considered as random close packing value [10]. It was recognized recently, that it does not correspond to the maximally random jammed state [3]. For volume fractions above 0.82 some local order is present and irregular packing is polycrystalline, forming rather large triangular coordination domains-grains. In present paper, a protocol leading to $x_c = \frac{\pi}{\sqrt{12}}$ is used, although our method can be applied with another protocol with unknown x_c.

All attempts to explain the value of critical indices through geometrical quantities of percolation problem, i.e. universally [1, 11], had failed so far and the indices are considered independent. From the phase interchange theorem [12] it follows that in two-dimensions, the superconductivity index is equal to the conductivity index [1, 13, 14].

While it is clear that using expansions in concentration for the conductivity, one should be able to address the two problems, in practice there are no more than two terms available for random systems [11], because of the serious technical difficulties. No method even such powerful as renormalization, or resummation approaches can draw reliable conclusions systemically, based on such short series [11]. "In fact, the age-old method of series expansions is also blocked by the same difficulties..." [11].

This concerns also the whole family of self consistent methods (SCMs) which include Maxwell's approach, effective medium approximations, differential schemes etc. SCMs are valid only for a dilute composites when interactions between inclusions do not matter [2]. The idea to correct a self consistent method (SCM) result $t = s = 1$ (in all dimensions) using the series in concentration remained, therefore, practically unattainable (see nevertheless, [15]).

We should also mention an indirect approach to estimating t for resistor networks from resistive susceptibility via scaling relations [16]. This approach also is based on a resummation techniques.

2. Series for conductivity, random 2D

In order to correctly define the effective conductivity tensor $\boldsymbol{\sigma}$ of random composites, the probabilistic distribution of disks of radius r must be introduced, since already the second order term of $\boldsymbol{\sigma}$ in concentration depends on the distribution [2]. For macroscopically isotropic composites, the third order term begins to depend on the distribution [1, 2]. In the present paper, we consider a uniform non-overlapping distribution when a set of independent and identically distributed (i.i.d.) points $\mathbf{a}_i$ are located in the plane in such a way that $|\mathbf{a}_i - \mathbf{a}_j| \geq 2r$.

For $r = 0$ we return to the Poisson distribution and for the maximally possible concentration $x_c = \frac{\pi}{\sqrt{12}}$, the distribution degenerates to the unique location, the hexagonal array. The tensor $\boldsymbol{\sigma}$ is expressed through the scalar effective conductivity σ as follows $\boldsymbol{\sigma} = \sigma\mathbf{I}$, where $\mathbf{I}$ is the unit tensor.

In the present paper, the numerical computations are performed only for the hexagonal representative cell. This assumption does not restrict our investigation since the number of inclusions per cell can be taken arbitrary large, hence, the shape of the cell does not impact on the final result.

Consider sufficiently large number of non-overlapping circular disks of radius r with the centers $\mathbf{a}_k$. The formal definition of the random variable has to be

statistically realized to get numerical results. The protocol for the data generation is based on the Monte Carlo simulations [4] and can be shortly described as follows. At the beginning, the centers $\mathbf{a}_k$ are located at the nodes of the regular hexagonal lattice and further randomly moved without overlapping. After sufficiently long random walks the centers form a statistical event satisfying the considered distribution. Using these locations of disks we compute coefficients of σ in x many times and take the average. Detailed description of the computational method and all relevant parameters for simulations can be found in [4, 9].

Consider a set of centers $\mathbf{a}_k = (a_{k1}, a_{k2}) \in \mathbb{R}^2$ expressed by complex numbers $a_k = a_{k1} + ia_{k2} \in \mathbb{C}$ $(k = 1, 2, \ldots, N)$ in the single representative cell $Q_{(0,0)}$ (see Fig. 1). The cell $Q_{(0,0)}$ is taken as the rhombus of the unit square with an angle 60^0. A rigorous theory of the representative volume elements is described in [17]. In accordance with simulations performed in [4, 9] the uniform non-overlapping distribution of disks can be realized by the cell $Q_{(0,0)}$ containing 64–100 disks. Moreover, the effective conductivity of 1500 configurations (events) have to be averaged to get results practically not depending on the shape of the cell. Let q be a natural number; $k_0, k_1, \ldots, k_q$ run over 1 to N, $m_j \geq 2$. Let $\mathbf{C}$ be the operator of complex conjugation. The sum over all k_s

$$e_{m_1 \ldots m_q} := \frac{1}{N^{1+\frac{m_1+\cdots+m_q}{2}}} \times \sum_{k_0 k_1 \ldots k_q} E_{m_1}(a_{k_0} - a_{k_1})\overline{E_{m_2}(a_{k_1} - a_{k_2})} \cdots \mathbf{C}^{q+1} E_{m_q}(a_{k_{q-1}} - a_{k_q}) \tag{2.1}$$

is called the basic sum of the multi-order $\mathbf{m} = (m_1, \ldots, m_q)$.

The effective conductivity tensor of 2D composites with non-overlapping identical disks can be expanded in the concentration [4]

$$\sigma(x) = 1 + 2\rho x(1 + A_1 x + A_2 x^2 + \ldots), \tag{2.2}$$

where

$$\begin{aligned}
A_1 &= \frac{1}{\pi} e_2, \qquad A_2 = \frac{1}{\pi^2} e_{22}, \qquad A_3 = \frac{1}{\pi^3}\left[-2^2 e_{33} + e_{222}\right], \\
A_4 &= \frac{1}{\pi^4}\left[3e_{44} - 2(e_{332} + e_{233}) + e_{2222}\right], \\
A_5 &= \frac{1}{\pi^5}\left[-4e_{55} + (3e_{442} + 6e_{343} + 3e_{244})\right. \\
&\quad \left. - 2(e_{3322} + e_{2332} + e_{2233}) + e_{22222}\right], \qquad (2.3) \\
A_6 &= \frac{1}{\pi^6}\left[5e_{66} - 4(e_{255} + 3e_{354} + 3e_{453} + e_{552})\right. \\
&\quad + (3e_{2244} + 6e_{2343} + 4e_{3333} + 3e_{2442} + 6e_{3432} + 3e_{4422}) \\
&\quad \left. - 2(e_{22233} + e_{22332} + e_{23322} + e_{33222}) + e_{222222})\right].
\end{aligned}$$

The higher order coefficients A_n can also be written in a closed form.

In the general case of a two-phase composite the so-called contrast parameter ρ [4] should be included into consideration explicitly. We are interested here in the case of a high contrast regular composites, when the conductivity of the inclusions is much larger than the conductivity of the host. I.e., the highly conducting inclusions are replaced by the ideally conducting inclusions with infinite conductivity. In this case the contrast parameter is equal to unity and remains implicit. The conductivity of the matrix is normalized by unity as well. The case of a high contrast is considered to be the most difficult for the expansion techniques [1], to the point of becoming meaningless for $\rho = 1$ [4].

Note, that in regular case the coefficients of the power series expansion of $\sigma(x)$ are expressed through elliptic functions by exact formulae and are exact [18, 19]. Using the same formulae as in the regular case but for the random structure of the composite, yields an approximate power-series. Below, this expansion is presented in the truncated numerical form following [4, 9]

$$\sigma(x) = 1 + 2x + 2x^2 + 5.00392x^3 + 6.3495x^4 + O(x^9). \tag{2.4}$$

The higher order terms, or remainder $\Delta(x)$ in (2.4), can be specified as well

$$\begin{aligned}\Delta(x) = {} & 0.0000186711x^9 + 9.57157 \times 10^{-10}x^{10} + 0.0570669x^{14} \\ & + 27.2148x^{15} + 7.06377x^{16} + 1.63666 \times 10^{-6}x^{17}.\end{aligned} \tag{2.5}$$

It has highly irregular form and does not contribute to the critical properties or general expressions for the conductivity discussed in the paper. The coefficients on x^k ($k = 5, 6, 7, 8$) vanish in (2.4) with the precision 10^{-10}. Since we are dealing with the limiting case of a perfectly conducting inclusions when the conductivity of inclusions tends to infinity, the effective conductivity is also expected to tend to infinity as a power-law, as the concentration x tends to the maximal value x_c for the hexagonal array,

$$\sigma(x) \simeq A(x_c - x)^{-s}. \tag{2.6}$$

The critical superconductivity index (exponent) s believed to be close to $\frac{4}{3} \approx 1.3$ [1, 20]. This value is known from numerical simulations, while rigorously it can be anywhere between one and two [21]. The critical amplitude A is an unknown non-universal parameter.

For regular arrays of cylinders the index is much smaller, $s = \frac{1}{2}$ [22, 23] and the critical amplitude is also known with good precision. The critical behavior of regular composites is practically never mentioned together with other critical phenomena [24]. It is remarkable that a relatively "simple" Laplace equation for the potential, when complemented with a non-trivial boundary conditions in the regular domain of inclusions, behaves critically even without explicit nonlinearity or randomness, typical to the phase transitions and percolation phenomena [24].

Overall effective conductivity of random systems is expected to be higher by order(s) of magnitude as the threshold is approached [25]. By punching holes in a sheet of metal, it was determined that $t = s = 1.2 \pm 0.1$ [26].

Expression (2.4) is different from the regular case of superconducting inclusions forming a hexagonal lattice with the same value of threshold x_c [23, 1].

Also there is no explicit expression presented in [23], which can be compared directly to (2.4), one can use as a proxy their cumulative expression (2.7), written below for the particular case of the superconducting disks,

$$\sigma = 1 - \frac{2x}{\frac{0.075422x^6}{1-1.06028x^{12}} + x - 1}. \tag{2.7}$$

The expansion of (2.7) is characterized by rather regular behavior of the coefficients,

$$\sigma_{\rm reg}(x) = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + 2.15084x^7 + 2.30169x^8 + O(x^9). \tag{2.8}$$

Comparing (2.4) and (2.8) one can see how and where the random effects captured by our procedure contribute most.

Randomness does seem to make the third and fourth order coefficients in the series much larger, signifying a stronger coupling between the inclusions already at small concentrations. The starting two (non-trivial) terms are common for the series (2.4) and (2.8). They are exact and typical for all effective medium type approximations [36]. One can also anticipate that some different methods will have to be suggested for the random composite, compared with regular case [5].

The problem of interest can be formulated mathematically as follows. Given the polynomial approximation (2.4) of the function $\sigma(x)$, to estimate the convergence radius x_c of the Taylor series of $\sigma(x)$, and to determine parameters of the asymptotically equivalent approximation(2.6) near $x = x_c$. When such extrapolation problem is solved, one can derive the formula for all concentrations., i.e, solve an interpolation problem. Alternatively in the latter problem, one may assume that the critical behavior is known in advance and proceed with interpolation from the start.

3. Critical point

3.1. Padé approximants

Probably the simplest (technically) way to estimate the position of a critical point, is to apply the Padé approximants $P_{n,m}(x)$, which is nothing else but a ratio of the two polynomials $P_n(x)$ and $P_m(x)$ of the order n and m, respectively. The coefficients are derived directly from the coefficients of the given power series [27, 28] from the requirement of asymptotic equivalence.

Padé approximants are the best locally, rational approximations of power series. Their poles determine singular points of the approximated functions [28]. Calculations with Padé approximants are particularly straightforward and can be performed with *Mathematica*®. Their application does not require any preliminary knowledge of the critical index and identifies the critical point with the position of a simple pole.

But in the case of series (2.4), the direct application of Padé approximants leads to a poorly convergent results in the diagonal sequence $P_{n,n}$, with the best estimate for the threshold 0.85269, obtained form the approximant $P_{3,3}$. The results do not improve significantly if other sequences are considered. We attribute the problem to the relative shortage of the coefficients in (2.4), and to the triviality of the lower-order terms.

In order to compensate for the unchanging values of the coefficients in the starting orders, we considered also transformed in a various ways sequences of approximants $P_{n,n}$. The most significant improvement is achieved when we multiply the original series (2.4) by the Clausius-Mossotti function $p(x) = \frac{1+x}{1-x}$, applying then a diagonal Padé approximants. There is now a better estimate for the critical point, $x_4 = 0.914241$. The percentage error obtained from the approximant $P_{4,4}$, equals to 0.8095%. Note, that in order to obtain higher-order approximants, the Padé technique will routinely use trivial, zero values for the missing higher-order coefficients in the expansion (2.4). Only a minimal number of such trivial conditions allowing for a reasonable estimates are utilized. The same will apply to calculations with different approximants below.

The estimates does not improve if different types of approximants, other than Padé, are applied. We conclude that some special efforts are in order to reconstruct the threshold from the series (2.4) more accurately.

3.2. Corrected threshold

Assuming that x_c is known approximately from the Padé-estimates above, let us estimate an improved, or corrected value of threshold, employing general idea of corrected approximants [29].

The technique of corrected approximants allows to treat different groups of coefficients in expansion differently and separately if needed. The solution can be patched together from several different sequences of approximants, not necessarily of the same type. Such approach is potentially more flexible and powerful then "classical" technique employing just one type of approximants. This program will be realized below both for the threshold and critical index.

For the series (2.4), it seems natural to treat the two starting terms separately and the rest of the series consider as a correction. Assume also that the initial threshold value is available from previous Padé-estimates, and is equal to

$x_4 = 0.914241$. Of course, formulation of the initial approximation is a crucial step. Note that using the value of 0.85269} as initial guess will lead to the wrong results.

We will attempt now to correct the value of x_4. Also, the value of the critical index $s = 4/3$ will be incorporated into the initial approximation. Let us observe that the factorized approximations to the effective conductivity σ can be always represented as a product of two factors: critical part $C(x) = (1 - \frac{x}{x_c})^{-s}$ and of the rest, i.e. regular part $R(x)$. As shown by Fuchs, such factorization also holds for non-analytic solutions to the homogeneous linear differential equations [30]. So one can most generally express the threshold

$$x_c = \frac{xC^{1/s}(x)}{C^{1/s}(x) - 1}. \tag{3.1}$$

The subsequent steps are described below for the particular case, but without any loss of generality.. At first, we should obtain the solution explicitly as a factor approximant [31, 32]. The simplest factor approximant

$$\sigma = \frac{(2.94539x + 1)^{0.183879}}{(1 - 1.0938x)^{4/3}}. \tag{3.2}$$

Such approximant satisfy the two non-trivial starting terms from the series (2.4), and incorporates the accepted value of 4/3 for the critical index and the trial value of threshold x_4. Let us look for another solution, supposedly in the same form, but with an exact, yet unknown threshold x_c,

$$\sigma' = (2.94539x + 1)^{0.183879} \left(1 - \frac{x}{x_c}\right)^{-4/3}. \tag{3.3}$$

Such solution retains the regular part $R(x) = (2.94539x + 1)^{0.183879}$ from the factor approximant (3.2). One can express the new threshold formally,

$$x_c = \frac{x\left(\frac{\sigma'}{R(x)}\right)^{-1/s}}{\left(\frac{\sigma'}{R(x)}\right)^{-1/s} - 1}, \tag{3.4}$$

since $\sigma'(x)$ is also unknown. To make it practical we are obliged to use for σ' a concrete expression, namely the series (2.4), so that instead of a true threshold which is a number, we have an effective threshold $X_c(x)$, given as an expansion around an approximate threshold $x_0 = x_4$,

$$X_c(x) = 0.914241 - 0.954606x^2 + 0.213995x^3 + \cdots \tag{3.5}$$

which should become a number corresponding to the true threshold x_c, as $x \to x_c$. Moreover, let us apply re-summation procedure to the expansion

(3.5) using again factor approximants $F^*(x)$, and define the sought approximate threshold X_c^* self-consistently,

$$X_c^* = x_0 - 0.954606x^3F^*(X_c^*). \tag{3.6}$$

As we approach the threshold, the RHS of (3.6) should become the threshold. Since factor approximants are defined as F_k^* for arbitrary number of terms k, we will also have a sequence of $X_{c,k}^*$. Reasonably solution arises in the six order,

$$\begin{aligned} F_6^*(x) = (4.75828x+1)^{0.601565}&(1-(0.548713-0.738153i)x)^{-2.31965+3.81508i} \\ &\times(1-(0.548713+0.738153i)x)^{-2.31965-3.81508i}. \end{aligned} \tag{3.7}$$

Expression (3.7) matches (3.5) up to the 8-order terms included. Solving (3.6), we obtain $X_{c,6}^* = 0.910181$. In the next even order there is no improvement for $X_{c,8}$.

The percentage error of such estimate is 0.361822%. We indeed managed to extract a significant part of the threshold value at the first step, producing reasonable estimate x_0 and then, corrected it a bit using higher orders terms from the series. Mind that the threshold is a purely geometrical quantity, not very sensitive therefore to the value of critical index employed in the calculations.

The technique of corrected threshold can be successfully applied in the case of a regular composite [5], with different way to define the initial approximation to be corrected. It was also tested with good results for the high-temperature series for the 2D Ising model [33]. In those cases the corresponding series coefficients behave (increase) regularly and standard methods also work well.

4. Critical index s

Conventionally, one would first apply the following transformation,

$$z = \frac{x}{x_c - x} \;\Leftrightarrow\; x = \frac{zx_c}{z+1} \tag{4.1}$$

to the original series, to make calculations with different approximants more convenient. The transformation maps the segment to a half-line.

4.1. Direct application of factor approximants

The most straightforward way to estimate index s is to apply factor approximants [31, 32] (in terms of the variable z), so that possible corrections to the "mean-field" value unity, appear additively, by definition. Following the standard procedure, the simplest factor approximant is written as follows, $\sigma_3^* =$

$1 + b_1 z\,(b_2 z + 1)^{c_2}$, where $c_2 = -0.01357$, $b_1 = 1.8138$, $b_2 = 26.6303$, and the critical index is simply $1 + c_2 = 0.996504$.

In the next order the value of critical index improves significantly,

$$\sigma_5^* = 1 + \frac{b_1 x\big(\frac{b_3 x}{x_c - x} + 1\big)^{c_3}\big(\frac{b_2 x}{x_c - x} + 1\big)^{c_2}}{x_c - x}, \tag{4.2}$$

where $b_1 = 1.8138$, $b_2 = 1.33199 + 2.39138i$, $b_3 = 1.33199 - 2.39138i$, $c_2 = 0.148893 + 0.102399i$, $c_3 = 0.148893 - 0.102399i$. The critical index is equal to $1 + c_2 + c_3 = 1.297786$. Such estimate and corresponding expression (4.2) are already good. To confirm them we resort to different methods applied below.

4.2. D-Log Padé

Let us see the results of a standard approach to the critical index calculation [27]. To this end let us again express the original series in terms of z, and to such transformed series $M_1(z)$ apply the $D - \mathrm{Log}$ transformation [27, 34] (differentiate Log of $M_1(z)$), and call the transformed series $M(z)$. Applying the Padé approximants $P_{n,n+1}(z)$ to $M(z)$. one can readily obtain the sequence of approximations s_n for the critical index s,

$$s_n = \lim_{z\to\infty}(zP_{n,n+1}(z)). \tag{4.3}$$

In the case of (2.4), this method turns to be quite accurate. Namely, the best (and only) result is $s_2 = 1.28522$. Explicitly we obtain

$$P_{2,3}(z) = \frac{15.7323z^2 + 3.89669z + 1.8137993642342}{12.241z^3 + 9.442z^2 + 4.14836z + 1}. \tag{4.4}$$

The effective conductivity can be reconstructed [35], from an effective critical index (or β-function) approximated in our case by the approximant $P_{2,3}(z)$,

$$\sigma^*(x) = \exp\Bigg(\int_0^{\frac{x}{x_c - x}} P_{2,3}(z)\,dz\Bigg). \tag{4.5}$$

Calculating the integral we obtain

$$\sigma^*(x) = e^{1.25174 - 0.436689\tan^{-1}(\frac{2.05578x + 0.389134}{0.9069 - x})} \times \left(\frac{0.574015x + 0.386326}{0.9069 - x}\right)^{1.08699}\left(\frac{x^2 - 0.0409192x + 0.186347}{(0.9069 - x)^2}\right)^{0.0991144}. \tag{4.6}$$

Also, the critical amplitude evaluates as 1.57888. Eq. (4.6) will be compared below with other formula for the effective conductivity valid everywhere.

4.3. Corrected mean-field approximation

Let us look for the solution in the form of a simple pole, also satisfying the two starting non-trivial terms from the series (2.4),

$$f_0^*(x) = \frac{(1.35495x+1)^{0.662269}}{1-1.10266x}, \tag{4.7}$$

so that our zero approximation with $s^{(0)} = 1$ for the critical index, is typical for various SCMs.

Let us divide then the original series (2.4) by f_0^*, express the newly found series in term of variable z, then apply $D - Log$ transformation and call the transformed series $K(z)$. Applying now the Padé approximants $P_{n,n+1}(z)$, one can obtain the following sequence of corrected SCM approximations for the critical index,

$$s_n = s^{(0)} + \lim_{z\to\infty}(zP_{n,n+1}(z)). \tag{4.8}$$

The "corrected" values for the critical index can be calculated readily and we have now a unique good estimates, $s_2 = 1.37959$.

Effective conductivity can be reconstructed using the complete expression for the effective critical index, employing the approximant $P_{2,3}(z)$),

$$P_{2,3}(z) = \frac{5.98481z^2}{15.7666z^3 + 11.1117z^2 + 4.66455z + 1}, \tag{4.9}$$

and

$$\sigma^*(x) = f_0^*(x)\exp\left(\int_0^{\frac{x}{x_c-x}} P_{2,3}(z))\,dz\right). \tag{4.10}$$

The integral can be found analytically, so that $\sigma^*(x) = f_0^*(x)F(x)$, with

$$F(x) = 1.68244e^{0.388802\tan^{-1}(2.19461+\frac{2.39204}{x-0.9069})} \times \left(\frac{0.9069}{0.9069-x} + 0.631139\right)^{0.280524}\left(\frac{x(x-0.00867304)+0.169156}{(0.9069-x)^2}\right)^{0.049532}. \tag{4.11}$$

4.4. Corrected regular lattice approximation

In a separate paper, we intend to present a generalization of (2.4), i.e. the transition formula from the regular hexagonal array to the random array (2.4). We expect to obtain a dependence of the critical index on the degree of randomness. For "zero"-randomness we expect to have a regular hexagonal lattice. For

"maximum"-randomness we expect to have a random composite of the present paper. All cases with intermediate degrees of randomness are expected to fall in between the two cases. Accordingly, one should be able to describe the regular and random composites within a single formalism.

To this end let us select the initial approximation to be corrected, as describing a regular hexagonal array of inclusions, namely

$$f^*_{0,r}(x) = \frac{(0.419645x+1)^{3.45214}}{\sqrt{1-1.10266x}}. \tag{4.12}$$

This formula incorporates the critical index 1/2 of the regular hexagonal lattice, the threshold for the hexagonal lattice and the two starting, effective medium terms from the series (2.8, 2.4).

Let us divide the original series (2.4) by $f^*_{0,r}$, thus extracting the part corresponding to the random effects only.

Then express the newly found series in term of variable z, then apply $D-P\,\mathrm{Log}$ transformation and call the transformed series $K_r(z)$. Let us also process the transformed series

$$K_r(z) = 7.52332z^2 - 35.008z^3 + 86.1167z^4 - 141.937z^5 + \cdots \tag{4.13}$$

with different approximants, such as iterated roots [29]. One can obtain the following sequence of corrected approximations to the critical index,

$$s_n = 1/2 + \lim_{z\to\infty}(z\, r_n(z)), \tag{4.14}$$

where $r_n(z)$ stands for the iterated root of nth order [29], constructed for the series $K_r(z)$ with such a power at infinity that defines constant correction to $s^{(0)}$. Calculations with iterated roots are really easy since at each step we need to compute only one new coefficient, while keeping all preceding from previous steps. The power at infinity is selected in order to compensate for the factor z and extract the correction to regular lattice value. Namely,

$$r_1(z) = \frac{7.52332z^2}{(1.55109z+1)^3}, \quad r_2(z) = \frac{7.52332z^2}{(1.99241z^2+(1.55109z+1)^2)^{3/2}}. \tag{4.15}$$

Correspondingly,

$$\sigma^*(x) = f^*_{0,r}(x)\exp\Bigg(\int_0^{\frac{x}{x_c-x}} r_2(z)\,dz\Bigg). \tag{4.16}$$

The following result for the critical index s follows, approximated by $s_2 = 1.31561$.

$$\sigma^*(x) = 0.121708 f^*_{0,r}(x) \times \exp\Bigg(\frac{(0.64454x - 1.38151)x + 0.72278}{(x-0.9069)^2\sqrt{\frac{x(1x+0.435329)+0.3582}{(x-0.9069)^2}}} - 0.815613\sinh^{-1}\left(\frac{2.0171(x+0.494058)}{x-0.9069}\right)\Bigg). \tag{4.17}$$

Similar techniques were applied also to the regular case [5] with the goal of calculating independently the critical index. In the random case we proceed by extrapolating from the side of a diluted regime and to the high-concentration regime close to x_c; while in the regular case we first derived an approximation to the high-concentration regime and then extrapolated to the less concentrated regime. There are indications that physics of a 2D regular and irregular composites is related to the so-called "necks", certain areas between closely spaced disks [22, 25]. Randomness eases the necks formation.

5. Final formula for all concentrations

We proceed to derivation of the formula valid for all concentrations assuming now that the critical index and threshold are both known. Such program is less ambitious but still entails calculation of the critical amplitude not known in advance. Let us discuss briefly some formulae for the effective conductivity from [4, 36] valid for all concentrations. The first formula (Eq. (22), [4]) is nothing else but an improved Padé conditioned by appearance of a simple pole at x_c, [4],

$$\sigma_M(x) = \frac{0.014x + 0.001}{x^2 + 0.261x + 0.076} + \frac{3.223}{x - 1.247} - \frac{3.237}{x - 0.9069}. \tag{5.1}$$

We also employ Eq. (5) from [36], adjusting it with regard to the threshold and critical index values. It exemplifies a crossover from the diluted regime where SCM is valid, to the percolation regime with typical critical behavior. as described generally in [39, 40]. The expression below represents a quasi-fractional extension of the two-point Padé approximants, according to [37],

$$\sigma_A = \frac{\frac{x^2\left(\frac{x}{0.9069-x}+1\right)^{4/3}}{(0.9069-x)^2} + \frac{1.97425x}{0.9069-x} + 1}{\frac{0.877834x^2}{(0.9069-x)^2} + \frac{0.160454x}{0.9069-x} + 1}. \tag{5.2}$$

5.1. Modified Padé formula

Our suggestion for the conductivity valid for all concentrations in the random case is based on [38, 5]. To derive an explicit formula, let us first apply to the series $M_1(z)$ another transformation, $T(z) = M_1(z)^{-1/s}$, with $s = 4/3$, in order to get rid of the power-law behavior at infinity. Applying Padé approximants to $T(z)$ one can readily obtain the sequence of approximations A_n for the critical amplitude A,

$$A_n = x_c^s \lim_{z\to\infty} (zP_{n,n+1}(z)))^{-s}, \tag{5.3}$$

There are only few reasonable estimates for the amplitude, $A_0 = 1.32316$, $A_1 = 1.20082$ and $A_2 = 1.54817$.

Following the prescription above, we obtain can reconstruct the effective conductivity explicitly (for $n = 2$),

$$\sigma_p^*(x) = \left(\frac{x(x(0.651542\ -1.16957x) - 0.0407068) + 0.373419}{x(x(x + 1.01058) + 0.519422) + 0.373419}\right)^{-4/3}. \tag{5.4}$$

5.2. Factors

Let us also apply factor approximants to the series $M_1(z)$ directly. There is a convergence now with increasing number of terms. The best factor approximant appears to be given as follows,

$$F_5^*(z) = (B_1z + 1)^{c_1}(B_2z + 1)^{c_2}(B_3z + 1)^{s-c_1-c_2}, \tag{5.5}$$

where $B_1 = 1.00734 - 2.05598i$, $B_2 = 2.2771$, $B_3 = 1.00734 + 2.05598i$, $c_1 = 0.0705011 - 0.253723i$, $c_2 = 1.19233$. The critical amplitude is simply, the limiting value of the approximant $F_5^*(z)$ calculated as $z \to \infty$,

$$A = x_c^s B_1^{c_1} B_2^{c_2} B_3^{s-c_1-c_2} = 1.49445. \tag{5.6}$$

To finalize, one should apply to the formula (5.5) an inverse transformation (4.1). Expression (5.5) and (5.4) appear to be very close.

Various expressions are shown in Fig.2. Note, that significant deviations of the Padé formula (5.1) (with typical value of the critical index $s = 1$) compared to our results, start around $x = 0.82$. The two formulae, (4.17) and (5.4), happen to be very close to each other everywhere, although the former is the result of extrapolation from the low-concentration region, and the latter is an interpolation between the two limiting behaviors. Another numerical formula (4.10) is slightly higher than (5.4), while formula (4.6) gives result lower than (5.4).

Closed-form expression for the effective conductivity of the regular hexagonal array of disks is presented in [23], see (2.7). Since it is supposed to be defined

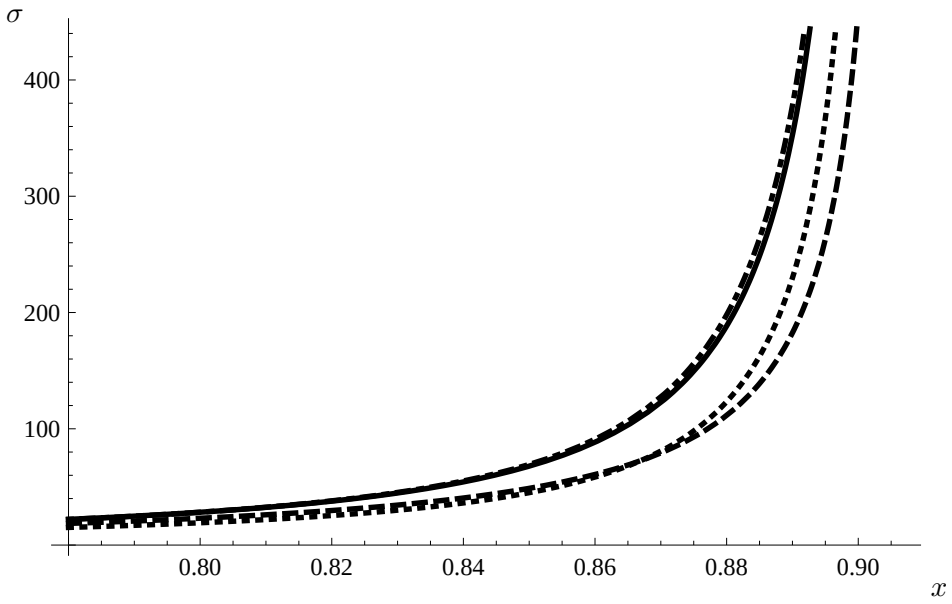

FIG. 2. Our formulae (4.10) (dot-dashed), (5.4) (solid) are compared with improved Padé approximant (5.1) from [4] (dashed) and quasi-fractional approximant (5.2) from [36] (dotted).

in the same domain of concentrations as in the random case, a comparison can explicitly quantify the role of a randomness (irregularity) of the composite. But in the most interesting region of large x, the relevant formula (2.7) fails. In order to estimate an enhancement factor due to randomness we can still use the numerical results tabulated in [23]. In particular, an enhancement factor at $x = 0.9$, is about 20, compared with (5.5) and (5.4).

6. Discussion

In this paper, we developed a direct approach to the effective conductivity of the random 2D arrangements of an ideally conducting cylinders, based on the series (2.4). Despite its relatively short length, such series turned out to be very informative. Due to such favorable concatenation we are able to confine our study to closed form expressions fro the effective conductivity for all volume fractions, possibly suggesting that the problem of 2D high-contrast random composite is tractable.

We confirm the position of a threshold for the effective conductivity and calculate accurately the value of a superconductivity critical index by four different methods. Finally we obtain a crossover expression (5.4), valid for arbitrary concentrations. Original resummation techniques had to get involved and ex-

tended in order to solve the three above mentioned problems. Most notably it includes the method of corrected approximants extensions for the critical point and critical index calculation [29, 5]. The methods of corrected critical point and corrected critical index are in the same mold as the traditional renormalization group [41, 42, 43, 35, 34].

Our main achievement is a direct (independent on other indices) calculation of the critical index for superconductivity $s = \frac{4}{3} \approx 1.3$ from the series in concentration, starting from the equations describing the composite. Randomness does tend to make the series coefficients (2.4) to become larger compared with regular case. Our method of corrected critical index allows thus to correct effectively the value of the critical index given by the large family of self consistent methods, the most popular among them being ever useful effective medium approximation [11, 2]. We cannot yet completely exclude the possibility that s may depend on the protocol. Further studies are needed with different protocols.

Interestingly, the effective viscosity of two-dimensional suspensions also diverges near the threshold x_c with the critical exponent $\frac{4}{3}$ [45]. Although a rigorous analogy does not hold for viscosity and conductivity, it seems to hold in practice, including three-dimensional case [46]. Such quantities as a 2D fluid permeability [1], and the effective elastic modulus for the 2D system of antiplane cracks [47], are expected to be characterized by the same value of critical exponent $\frac{4}{3}$. It would be both extremely interesting and challenging to explain such universality based on the series in concentration and applying the techniques described in the paper. On the other hand, an elastic properties characterized by the effective Young modulus [1], are characterized by more than 3 times larger value of the index analogous to t, while the so-called superelasticity index analogous to s, seems to have close value of 1.24. Explanation of such differences directly from the series, presents another challenge.

We believe that construction of series analogous to (2.4) by the method [48] will yield accurate results also in 3D .

Acknowledgement

We are grateful to Wojciech Nawalaniec for computer derivation of the formula (2.4).

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Received May 29, 2014; revised version January 1, 2015.