

Effective boundary condition method and Rayleigh waves in orthotropic half-spaces coated by a thin layer with sliding contact

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THIS PAPER STUDIES THE PROPAGATION of Rayleigh waves in an orthotropic elastic half-space coated by a thin orthotropic elastic layer. The half-space and the layer are assumed to be either compressible or incompressible and they are in sliding contact with each other. The main aim of the paper is to establish approximate secular equations of the wave for all (four) possibilities of a compressible or incompressible half-space covered with a compressible or incompressible thin layer, except the case of a compressible half-space coated by a compressible layer that has been considered [19]. In order to do that, the effective boundary condition method is employed and the approximate third-order secular equations regarding the dimensionless thickness of the layer are derived. It is shown that these approximate secular equations have a high accuracy. Based on the obtained secular equations, the effect of incompressibility on the Raleigh wave propagation is considered through some numerical examples. It is shown that incompressibility strongly affects the Raleigh wave velocity and the effect becomes stronger when the coating is incompressible.

Key words: Rayleigh waves, orthotropic elastic half-space, thin orthotropic elastic layer, sliding contact, effective boundary conditions, effective boundary condition method, approximate secular equation.

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1. Introduction

THE STRUCTURES MADE OF A THIN FILM ATTACHED TO SOLIDS, modeled as half-spaces coated by a thin layer, are widely applied in modern technology. The measurement of mechanical properties of thin films deposited on half-spaces before and during loading plays an important role in health monitoring of these structures in applications, see for example MAKAROV *et al.* [1], EVERY [2] and references therein. Among various measurement methods, the surface/guided wave method is most widely used [2], because it is non-destructive and brings reduced cost, short inspection time and greater coverage (HESS *et al.* [3]), and for which the guided Rayleigh wave is a versatile and convenient tool (HESS *et al.* [3], KUCHLER and RICHTER [4]).

For the Rayleigh-wave approach, the explicit dispersion relations of Rayleigh waves supported by thin-film/substrate interactions are employed as theoretical bases for extracting the mechanical properties of thin films from experimental data. They are therefore the main purpose of any investigation of Rayleigh waves propagating in half-spaces covered by a thin layer.

TIERSTEN [5] and BOVIK [6] assumed that the layer and the half-space are both isotropic, and the authors derived approximate secular second-order equations. For this case, VINH and ANH [7] obtained a fourth-order approximate secular equation with a very high accuracy. STEIGMANN and OGDEN [8] considered a transversely isotropic layer with residual stress overlying an isotropic half-space and the authors obtained an approximate second-order dispersion relation. WANG *et al.* [9] considered an isotropic half-space covered by a thin electrode layer and they obtained an approximate secular equation of first-order. In VINH and LINH [10] and VINH *et al.* [11] the layer and the half-space are both assumed to be orthotropic, and approximate secular equations of third-order were obtained. In VINH and LINH [12] the layer and the half-space are both subjected to homogeneous pre-strains and an approximate secular equation of third-order was established which is valid for any pre-strain and for a general strain energy function.

In the above mentioned investigations, the contact between the layer and the half-space is assumed to be perfectly bonded. For the case of sliding contact, ACHENBACH and KESHAVA [13] derived an approximate secular equation of third-order by replacing the thin layer by a plate modeled by Mindlin's plate theory [14]. The layer and the half-space are both isotropic and compressible. However, this approximate secular equation includes the shear coefficient, originating from Mindlin's plate theory [14], whose usage should be avoided as noted by *Touratier* [15], MULLER and TOURATIER [16] and STEPHEN [17]. Recently, VINH *et al.* [18] derived a fourth-order approximate secular equation with a very high accuracy.

For the case when the layer and the half-space are both orthotropic, an approximate secular equation of third-order was established recently by VINH and ANH [19] for a compressible half-space coated by a thin compressible layer. There are four possibilities of a (compressible or incompressible) half-space covered with a (compressible or incompressible) thin layer. Therefore, the derivation of approximate secular equations for the remaining three combinations is needed.

By using the effective boundary condition method, the approximate secular equations of third-order are derived in terms of the dimensionless thickness of the layer. It is shown numerically that these approximate secular equations have a high accuracy. Based on the obtained secular equations, the effect of incompressibility on the Raleigh wave propagation is considered carrying out some

numerical examples. It is shown that the incompressibility (of half-spaces and coating layers) strongly affects the Rayleigh wave velocity and the effect becomes stronger when the coating is incompressible.

The paper is organized as follows. In Section 2, the effective boundary condition method is recalled. In Section 3, the pre-effective boundary conditions for a compressible or incompressible orthotropic elastic layer are presented. In Sections 4 and 6, the propagation of Rayleigh waves in an incompressible orthotropic elastic half-space coated by a thin incompressible (compressible) orthotropic elastic layer is considered. Section 5 deals with the propagation of Rayleigh waves in a compressible orthotropic elastic half-space coated by a thin incompressible orthotropic elastic layer. The approximate secular equations of third-order are established using the effective boundary condition method. In Section 7, as an application of the obtained results, the influence of incompressibility of half-spaces and layers on the Rayleigh wave velocity is investigated numerically using the derived approximate secular equations.

2. The effective boundary condition method

Taking into consideration the thin-layer assumption, approximate explicit secular equations of Rayleigh waves can be derived by replacing approximately the entire effect of the thin layer on the half-space by *the so-called effective boundary conditions* that relate linearly the displacements and the stresses of the half-space at its surface. Then, by ignoring the coating, the Rayleigh wave can be considered as a Rayleigh wave propagating in the half-space, without the layer whose surface is subjected to *the effective boundary conditions*. This approach is called *the effective boundary condition method*. It is worth noting that this method is applicable not only to the Rayleigh wave propagation but also to other problems.

To derive the effective boundary conditions, first we need the so-called *pre-effective boundary conditions* that relate linearly the displacements and the stresses of the layer at its flat bottom, where the layer is in contact with the half-space. The effective boundary conditions are then derived by using the contact conditions of the half-space and the layer.

For obtaining the pre-effective boundary conditions, ACHENBACH and KESHAVA [13], and TIERSTEN [5] replaced the thin layer with a plate modeled by different theories. Mindlin's plate theory and the plate theory of low-frequency extension and flexure, while BOVIK [6] expanded the stresses at the top surface of the layer into Taylor series in its thickness. The Taylor expansion technique was then developed by NIKLASSON *et al.* [20], ROKHLIN and HUANG [21, 22], BENVENISTE [23], STEIGMANN and OGDEN [8], VINH and LINH [10, 12], VINH *et al.* [11, 18] and VINH and ANH [7, 19].

While the effective boundary conditions are immediately obtained from the pre-effective boundary condition for the welded contact because of the continuity of displacements and stresses through the interface between the half-space and the layer, the situation does not occur for the sliding contact, due to the discontinuity of the horizontal displacement component. To derive these conditions, the displacement and the stresses are restricted to the class of plane waves.

3. Pre-effective boundary conditions for thin orthotropic layers

Consider a thin orthotropic elastic layer that occupies the domain $-h \leq x_2 \leq 0$. The layer is in sliding contact with an orthotropic elastic half-space $x_2 \geq 0$ through the plane $x_2 = 0$. Note that some quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer.

3.1. A compressible layer

Suppose that the layer is compressible. According to VINH and LINH [10], VINH and ANH [19], the approximate pre-effective third-order boundary conditions have the following form:

$$(3.1) \quad \bar{\sigma}_{12} + h(r_1 \bar{\sigma}_{22,1} - r_3 \bar{u}_{1,11} - \bar{\rho} \ddot{u}_1) + \frac{h^2}{2} \left[r_2 \bar{\sigma}_{12,11} + \frac{\bar{\rho}}{\bar{c}_{66}} \ddot{\sigma}_{12} - r_3 \bar{u}_{2,111} - \bar{\rho}(1+r_1) \ddot{u}_{2,1} \right] + \frac{h^3}{6} \left(r_4 \bar{\sigma}_{22,111} + \bar{\rho} r_5 \ddot{\sigma}_{22,1} - r_6 \bar{u}_{1,1111} - \bar{\rho} r_7 \ddot{u}_{1,11} - \frac{\bar{\rho}^2}{\bar{c}_{66}} \ddot{u}_{1,tt} \right) = 0, \quad \text{at } x_2 = 0,$$

$$(3.2) \quad \bar{\sigma}_{22} + h(\bar{\sigma}_{12,1} - \bar{\rho} \ddot{u}_2) + \frac{h^2}{2} \left[r_1 \bar{\sigma}_{22,11} + \frac{\bar{\rho}}{\bar{c}_{22}} \ddot{\sigma}_{22} - r_3 \bar{u}_{1,111} - \bar{\rho}(1+r_1) \ddot{u}_{1,1} \right] + \frac{h^3}{6} \left[r_2 \bar{\sigma}_{12,111} + \bar{\rho} r_8 \ddot{\sigma}_{12,1} - r_3 \bar{u}_{2,1111} - \bar{\rho}(1+2r_1) \ddot{u}_{2,11} - \frac{\bar{\rho}^2}{\bar{c}_{22}} \ddot{u}_{2,tt} \right] = 0, \quad \text{at } x_2 = 0,$$

where

$$(3.3) \quad \begin{aligned} r_1 &= \frac{\bar{c}_{12}}{\bar{c}_{22}}, & r_2 &= r_1 + \frac{r_3}{\bar{c}_{66}}, & r_3 &= \frac{\bar{c}_{12}^2 - \bar{c}_{11}\bar{c}_{22}}{\bar{c}_{22}}, \\ r_4 &= r_1 r_2 + \frac{r_3}{\bar{c}_{22}}, & r_5 &= \frac{1+r_1}{\bar{c}_{22}} + \frac{r_1}{\bar{c}_{66}}, & r_6 &= (r_1+r_2)r_3, \\ r_7 &= r_1^2 + 2r_2, & r_8 &= \frac{1+r_1}{\bar{c}_{66}} + \frac{1}{\bar{c}_{22}}. \end{aligned}$$

3.2. An incompressible layer

Let the layer be incompressible. Then, the approximate pre-effective third-order boundary conditions are given by (see VINH *et al.* [11])

$$(3.4) \quad \bar{\sigma}_{12} + h \left(\bar{\sigma}_{22,1} + \bar{\delta} \bar{u}_{1,11} - \bar{\rho} \ddot{u}_1 \right) + \frac{h^2}{2} \left(r_9 \bar{\sigma}_{12,11} + \frac{\bar{\rho}}{\bar{c}_{66}} \ddot{\sigma}_{12} + \bar{\delta} \bar{u}_{2,111} - 2\bar{\rho} \ddot{u}_{2,1} \right) \\ + \frac{h^3}{6} \left(r_9 \bar{\sigma}_{22,111} + \frac{\bar{\rho}}{\bar{c}_{66}} \ddot{\sigma}_{22,1} - r_{10} \bar{u}_{1,1111} - \bar{\rho} r_{11} \ddot{u}_{1,11} - \frac{\bar{\rho}^2}{\bar{c}_{66}} \ddot{u}_{1,tt} \right) = 0, \quad \text{at } x_2 = 0,$$

$$(3.5) \quad \bar{\sigma}_{22} + h \left(\bar{\sigma}_{12,1} - \bar{\rho} \ddot{u}_2 \right) + \frac{h^2}{2} \left(\bar{\sigma}_{22,11} + \bar{\delta} \bar{u}_{1,111} - 2\bar{\rho} \ddot{u}_{1,1} \right) \\ + \frac{h^3}{6} \left(r_9 \bar{\sigma}_{12,111} + \frac{2\bar{\rho}}{\bar{c}_{66}} \ddot{\sigma}_{12,1} + \bar{\delta} \bar{u}_{2,1111} - 3\bar{\rho} \ddot{u}_{2,11} \right) = 0, \quad \text{at } x_2 = 0,$$

where

$$(3.6) \quad r_9 = 1 - \frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_{10} = \bar{\delta} \left(\frac{\bar{\delta}}{\bar{c}_{66}} - 2 \right), \quad r_{11} = 2r_9 + 1, \quad \bar{\delta} = \bar{c}_{11} + \bar{c}_{22} - 2\bar{c}_{12}.$$

REMARK 1.

(i) If the contact between the layer and the half-space is welded, i.e., the displacements and the stresses are continuous through the interface of the layer and the half-space, we immediately obtain the effective boundary conditions from the pre-effective boundary conditions by replacing \bar{u}_1 , \bar{u}_2 , $\bar{\sigma}_{12}$ and $\bar{\sigma}_{22}$ with u_1 , u_2 , σ_{12} and σ_{22} , respectively.

(ii) For the sliding contact, the situation is rather different. The horizontal displacement is not required to be continuous through the interface; the effective boundary conditions are not immediately obtained from the pre-effective boundary conditions. As shown below, for obtaining these conditions we have to restrict the displacement and the stress to those with the plane wave motion.

4. Rayleigh waves in an incompressible half-space coated by a thin incompressible layer

4.1. Approximate effective third-order boundary conditions

Consider the propagation of a Rayleigh wave, traveling (in coated half-space) with velocity c (> 0) and wave number k (> 0) in the x_1 -direction and decaying in the x_2 -direction. The displacements and the stresses of the wave are sought in the form

$$(4.1) \quad \bar{u}_1 = \bar{U}_1(y) e^{ik(x_1-ct)}, \quad \bar{u}_2 = \bar{U}_2(y) e^{ik(x_1-ct)}, \\ \bar{\sigma}_{12} = ik\bar{T}_1(y) e^{ik(x_1-ct)}, \quad \bar{\sigma}_{22} = ik\bar{T}_2(y) e^{ik(x_1-ct)}$$

for the layer, and

$$(4.2) \quad \begin{aligned} u_1 &= U_1(y)e^{ik(x_1-ct)}, & u_2 &= U_2(y)e^{ik(x_1-ct)}, \\ \sigma_{12} &= ikT_1(y)e^{ik(x_1-ct)}, & \sigma_{22} &= ikT_2(y)e^{ik(x_1-ct)} \end{aligned}$$

into the half-space, where $y = kx_2$. Substituting (4.1) for (3.4) and (3.5) yields

$$(4.3) \quad \begin{aligned} & i\bar{T}_1(0) \left[1 + \frac{\varepsilon^2}{2} \left(-r_9 - \frac{\bar{\rho}c^2}{\bar{c}_{66}} \right) \right] + \bar{T}_2(0) \left[-\varepsilon + \frac{\varepsilon^3}{6} \left(r_9 + \frac{\bar{\rho}c^2}{\bar{c}_{66}} \right) \right] \\ & + \bar{U}_1(0) \left[\varepsilon \left(-\bar{\delta} + \bar{\rho}c^2 \right) + \frac{\varepsilon^3}{6} \left(-r_{10} - \bar{\rho}c^2 r_{11} - \frac{\bar{\rho}^2 c^4}{\bar{c}_{66}} \right) \right] \\ & + i\bar{U}_2(0) \left[\varepsilon^2 \left(-\frac{\bar{\delta}}{2} + \bar{\rho}c^2 \right) \right] = 0, \\ & \bar{T}_1(0) \left[-\varepsilon + \frac{\varepsilon^3}{6} \left(r_9 + \frac{2\bar{\rho}c^2}{\bar{c}_{66}} \right) \right] + i\bar{T}_2(0) \left(1 - \frac{\varepsilon^2}{2} \right) \\ & + i\bar{U}_1(0) \left[\varepsilon^2 \left(-\frac{\bar{\delta}}{2} + \bar{\rho}c^2 \right) \right] + \bar{U}_2(0) \left[\varepsilon\bar{\rho}c^2 + \frac{\varepsilon^3}{6} (\bar{\delta} - 3\bar{\rho}c^2) \right] = 0, \end{aligned}$$

where $\varepsilon = kh$ is the dimensionless thickness of the layer. Let the contact between the layer and the half-space be sliding, then we have

$$(4.4) \quad \sigma_{12} = 0, \quad \bar{\sigma}_{12} = 0, \quad u_2 = \bar{u}_2, \quad \sigma_{22} = \bar{\sigma}_{22} \quad \text{at } x_2 = 0,$$

or, in view of Eqs. (4.1) and (4.2), equivalently

$$(4.5) \quad T_1(0) = 0, \quad \bar{T}_1(0) = 0, \quad U_2(0) = \bar{U}_2(0), \quad T_2(0) = \bar{T}_2(0).$$

Using the second equation of (4.5) in (4.3) and eliminating \bar{U}_1 we have

$$(4.6) \quad i\bar{T}_2(0)(a_1 + a_2\varepsilon^2) = -(a_3\varepsilon + a_4\varepsilon^3)\bar{U}_2(0),$$

where

$$(4.7) \quad \begin{aligned} a_1 &= r_v^2 x - \bar{e}_\delta, & a_2 &= \frac{1}{6} \left[\bar{e}_\delta(2 - \bar{e}_\delta) + 2\bar{e}_\delta r_v^2 x - r_v^4 x^2 \right], \\ a_3 &= \bar{c}_{66} r_v^2 x (r_v^2 x - \bar{e}_\delta), & a_4 &= \frac{\bar{c}_{66}}{12} \left[\bar{e}_\delta^2 - 2r_v^2 x (\bar{e}_\delta^2 - 2\bar{e}_\delta r_v^2 x + r_v^4 x^2) \right], \end{aligned}$$

in which

$$(4.8) \quad x = \frac{c^2}{c_2^2}, \quad \bar{e}_\delta = \frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_v = \frac{c_2}{c_2}, \quad c_2 = \sqrt{\frac{c_{66}}{\rho}}, \quad \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}.$$

From the last two equations of (4.5) and Eq. (4.6) it follows that

$$(4.9) \quad T_2(0)(a_1 + a_2\varepsilon^2) = i(a_3\varepsilon + a_4\varepsilon^3)U_2(0).$$

This is the desired approximate third-order effective boundary condition with which the total effect of the layer on the half-space is approximately replaced. From the first equations of (4.5) and (4.9), the surface $x_2 = 0$ of the half-space is subjected to the following conditions:

$$(4.10) \quad T_1(0) = 0, \quad T_2(0)(a_1 + a_2\varepsilon^2) = iU_2(0)(a_3\varepsilon + a_4\varepsilon^3).$$

4.2. An approximate secular third-order equation

Now we can ignore the layer and consider the propagation of Rayleigh waves in the incompressible elastic half-space whose surface $x_2 = 0$ is subjected to the boundary conditions (4.10). According to OGDEN and VINH [24], the displacement components of a Rayleigh wave travelling with velocity c and wave number k in the x_1 -direction and decaying in the x_2 -direction are determined by (4.2)_{1,2} in which $U_1(y)$ and $U_2(y)$ are given by

$$(4.11) \quad \begin{aligned} U_1(y) &= -k(b_1B_1e^{-b_1y} + b_2B_2e^{-b_2y}), \\ U_2(y) &= -ik(B_1e^{-b_1y} + B_2e^{-b_2y}), \end{aligned}$$

where B_1 and B_2 are the constants to be determined and b_1 and b_2 are the roots of the equation

$$(4.12) \quad \hat{\gamma}b^4 - (2\hat{\beta} - X)b^2 + (\hat{\gamma} - X) = 0,$$

with positive real parts (for ensuring the decay conditions), $X = \rho c^2$, and

$$(4.13) \quad \hat{\gamma} = c_{66}, \quad \hat{\beta} = (\delta - 2\hat{\gamma})/2, \quad \delta = c_{11} + c_{22} - 2c_{12}$$

From Eq. (4.12) it follows that

$$(4.14) \quad b_1^2 + b_2^2 = \frac{2\hat{\beta} - X}{\hat{\gamma}} := S, \quad b_1^2 b_2^2 = \frac{\hat{\gamma} - X}{\hat{\gamma}} := P.$$

It is not difficult to verify that if the Rayleigh wave exists ($\rightarrow b_1$ and b_2 have positive real parts), then

$$(4.15) \quad 0 < X < c_{66}$$

and

$$(4.16) \quad b_1 b_2 = \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}.$$

Introducing (4.11) and (4.2)_{1,2} into the stress-strain relations (Eq. (6) in [24]) and using the equations of motion (Eq. (7) in [24]) yield

$$(4.17) \quad \begin{aligned} \sigma_{12} &= k^2(\beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}) e^{ik(x_1 - ct)}, \\ \sigma_{22,1} &= k^3(\gamma_1 B_1 e^{-b_1 y} + \gamma_2 B_2 e^{-b_2 y}) e^{ik(x_1 - ct)}, \end{aligned}$$

in which

$$(4.18) \quad \beta_n = c_{66}(b_n^2 + 1), \quad \gamma_n = (X - \delta + \beta_n)b_n, \quad n = 1, 2.$$

From (4.2)_{3,4} and (4.17) it follows that

$$(4.19) \quad \begin{aligned} T_1(y) &= -ik(\beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}), \\ T_2(y) &= -k(\gamma_1 B_1 e^{-b_1 y} + \gamma_2 B_2 e^{-b_2 y}). \end{aligned}$$

Substituting into Eqs. (4.10) Eqs. (4.11) and (4.19) provides a homogeneous system of linear equations for B_1 and B_2

$$(4.20) \quad \begin{cases} f(b_1)B_1 + f(b_2)B_2 = 0, \\ F(b_1)B_1 + F(b_2)B_2 = 0, \end{cases}$$

where

$$(4.21) \quad F(b_n) = \gamma_n(a_1 + a_2\varepsilon^2) + (a_3\varepsilon + a_4\varepsilon^3), \quad f(b_n) = \beta_n, \quad n = 1, 2.$$

Due to $B_1^2 + B_2^2 \neq 0$, the determinant of coefficients of the system (4.20) must vanish. This gives

$$(4.22) \quad f(b_1)F(b_2) - f(b_2)F(b_1) = 0.$$

Introducing (4.21) into (4.22) leads to the secular equation of the wave

$$(4.23) \quad D_0 + D_1\varepsilon + D_2\varepsilon^2 + D_3\varepsilon^3 = 0,$$

where

$$(4.24) \quad \begin{aligned} D_0 &= a_1[(X - \delta)\sqrt{P} + X], & D_1 &= a_3\sqrt{S + 2\sqrt{P}}, \\ D_2 &= a_2[(X - \delta)\sqrt{P} + X], & D_3 &= a_4\sqrt{S + 2\sqrt{P}}, \end{aligned}$$

in which S and P are given by (4.14). Equation (4.23), in which D_k are given by (4.24), is the desired approximate secular third-order equation. It is fully explicit. In the dimensionless form this equation is written as

$$(4.25) \quad E_0 + E_1\varepsilon + E_2\varepsilon^2 + E_3\varepsilon^3 = 0,$$

where

$$\begin{aligned}
 E_0 &= (r_v^2 x - \bar{e}_\delta)[(x - e_\delta)\sqrt{P} + x], \quad E_1 = r_\mu r_v^2 x (r_v^2 x - \bar{e}_\delta)\sqrt{S + 2\sqrt{P}}, \\
 E_2 &= \frac{1}{6}[\bar{e}_\delta(2 - \bar{e}_\delta) + 2\bar{e}_\delta r_v^2 x - r_v^4 x^2][(x - e_\delta)\sqrt{P} + x], \\
 E_3 &= \frac{1}{12}r_\mu[\bar{e}_\delta^2 - 2r_v^2 x(\bar{e}_\delta^2 - 2\bar{e}_\delta r_v^2 x + r_v^4 x^2)]\sqrt{S + 2\sqrt{P}}, \\
 S &= e_\delta - 2 - x, \quad P = 1 - x
 \end{aligned}
 \tag{4.26}$$

and

$$e_\delta = \frac{\delta}{c_{66}}, \quad r_\mu = \frac{\bar{c}_{66}}{c_{66}}.
 \tag{4.27}$$

It is clear from Eqs. (4.25) and (4.26) that the squared dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ depends on five dimensionless parameters $e_\delta, \bar{e}_\delta, r_\mu, r_v$ and ε . Note that $e_\delta > 0, \bar{e}_\delta > 0$ (see [24]). When $\varepsilon = 0$, from Eq. (4.25) and the first of Eq. (4.26) it follows that

$$(x - e_\delta)\sqrt{1 - x} + x = 0.
 \tag{4.28}$$

This is the secular equation of Rayleigh waves in an incompressible orthotropic elastic half-space (see [24]).

Figure 1 presents the dependence on $\varepsilon = k.h \in [0, 1.5]$ of the dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ that is calculated by the exact secular equation

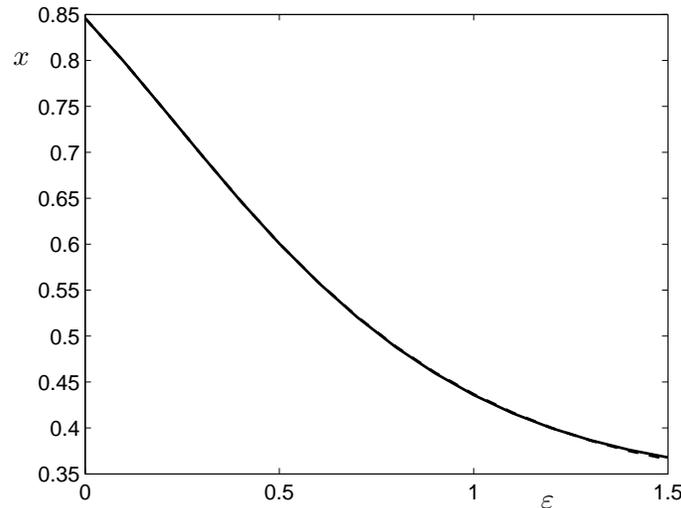


FIG. 1. Plots of the dimensionless Rayleigh wave velocity $x(\varepsilon)$ in the interval $[0, 1.5]$ that is calculated by the exact secular equation (solid line) and by the approximate secular equation (4.25) (dashed line). Here we take $r_\mu = 1, r_v = 1.5$ and $e_\delta = 3, \bar{e}_\delta = 3.5$.

(having the form of a 6×6 determinant) and marked (solid line) and by the approximate secular equation (4.25) (dashed line). Here we take $r_\mu = 1$, $r_v = 1.5$, $e_\delta = 3$, $\bar{e}_\delta = 3.5$. It is seen in Fig. 1 that the exact velocity curve and the third-order approximate one almost totally coincide with each other for the values of $\varepsilon \in [0, 1.5]$. The maximum absolute error in the interval $[0, 1.5]$ is 0.002 (at $\varepsilon = 1.5$). This shows that the approximate secular equation (4.25) is a very good approximation.

4.3. Isotropic case

When the layer and the half-space are both transversely isotropic (with the isotropic axis being the x_3 -axis)

$$(4.29) \quad c_{11} = c_{22}, \quad \bar{c}_{11} = \bar{c}_{22}, \quad c_{11} - c_{12} = 2c_{66}, \quad \bar{c}_{11} - \bar{c}_{12} = 2\bar{c}_{66},$$

we can see that

$$(4.30) \quad e_\delta = \bar{e}_\delta = 4, \quad S = 2 - x.$$

With the use of Eqs. (4.29) and (4.30), Eq. (4.26) is simplified to

$$(4.31) \quad \begin{aligned} E_0 &= (r_v^2 x - 4) \left[(x - 4) \sqrt{1 - x} + x \right], \\ E_1 &= r_\mu r_v^2 x (r_v^2 x - 4) (1 + \sqrt{1 - x}), \\ E_2 &= -\frac{1}{6} (8 - 8r_v^2 x + r_v^4 x^2) \left[(x - 4) \sqrt{1 - x} + x \right], \\ E_3 &= \frac{1}{6} r_\mu [8 - r_v^2 x (16 - 8r_v^2 x + r_v^4 x^2)] (1 + \sqrt{1 - x}). \end{aligned}$$

When the layer and the half-space are both isotropic, E_0 , E_1 , E_2 and E_3 are also given by Eq. (4.31), but in this case $x = \rho c^2 / \mu$, $r_\mu = \bar{\mu} / \mu$, μ and $\bar{\mu}$ are the shear moduli.

5. Rayleigh waves in an incompressible half-space coated by a thin compressible layer

5.1. Effective third-order boundary conditions

Using the same technique as the one in Subsection 4.1, from the pre-effective boundary conditions (3.1) and (3.2) we derive the approximate third-order effective boundary conditions for this case (see also [19])

$$(5.1) \quad \begin{aligned} T_1(0) &= 0, \\ T_2(0)(a_5 + a_6 \varepsilon^2) &= iU_2(0)(a_7 \varepsilon + a_8 \varepsilon^3), \end{aligned}$$

where

$$\begin{aligned}
 a_5 &= xr_v^2 - \bar{e}_d, \\
 a_6 &= -\frac{1}{6} \left\{ \bar{e}_d (\bar{e}_d - 2\bar{e}_2\bar{e}_3) + r_v^2 x (\bar{e}_2^2\bar{e}_3^2 + 2\bar{e}_2\bar{e}_3 - 3\bar{e}_1\bar{e}_2 - 2\bar{e}_d) + r_v^4 x^2 (1 + 3\bar{e}_2) \right\}, \\
 (5.2) \quad a_7 &= \bar{c}_{66} r_v^2 x (r_v^2 x - \bar{e}_d), \\
 a_8 &= \frac{\bar{c}_{66}}{12} \left\{ \bar{e}_d^2 + r_v^2 x \left[2\bar{e}_d (\bar{e}_2\bar{e}_3 - \bar{e}_d - 1) \right. \right. \\
 &\quad \left. \left. + r_v^2 x (1 - 2\bar{e}_2\bar{e}_3 - \bar{e}_2^2\bar{e}_3^2 + 4\bar{e}_d + 2\bar{e}_1\bar{e}_2) - 2r_v^4 x^2 (1 + \bar{e}_2) \right] \right\},
 \end{aligned}$$

in which

$$\begin{aligned}
 (5.3) \quad x &= \frac{c^2}{c_2^2}, \quad \bar{e}_1 = \frac{\bar{c}_{11}}{\bar{c}_{66}}, \quad \bar{e}_2 = \frac{\bar{c}_{66}}{\bar{c}_{22}}, \quad \bar{e}_3 = \frac{\bar{c}_{12}}{\bar{c}_{66}}, \quad \bar{e}_d = \bar{e}_1 - \bar{e}_2\bar{e}_3^2, \\
 r_\mu &= \frac{\bar{c}_{66}}{c_{66}}, \quad r_v = \frac{c_2}{\bar{c}_2}, \quad c_2 = \sqrt{\frac{c_{66}}{\rho}}, \quad \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}.
 \end{aligned}$$

Note that the total effect of the layer on the half-space is replaced approximately by the second equation of (5.1).

5.2. An approximate secular third-order equation

Now we can omit the layer and consider the Rayleigh wave propagating in the incompressible orthotropic elastic half-space (without the coating) whose surface $x_2 = 0$ is subjected to the boundary conditions (5.1).

By substituting into Eqs. (5.1) Eqs.(4.11) and (4.19) we obtain homogeneously linear equations for B_1 and B_2 . Making the determinant of coefficients of this system leads to the third-order approximate dispersion equation of the wave:

$$(5.4) \quad D_0 + D_1\varepsilon + D_2\varepsilon^2 + D_3\varepsilon^3 = 0,$$

where

$$\begin{aligned}
 (5.5) \quad D_0 &= a_5 [(X - \delta)\sqrt{P} + X], \quad D_1 = a_7 \sqrt{S + 2\sqrt{P}}, \\
 D_2 &= a_6 [(X - \delta)\sqrt{P} + X], \quad D_3 = a_8 \sqrt{S + 2\sqrt{P}},
 \end{aligned}$$

in which S and P are given by (4.14). In the dimensionless form, Eq. (5.4) becomes

$$(5.6) \quad E_0 + E_1\varepsilon + E_2\varepsilon^2 + E_3\varepsilon^3 = 0,$$

where

$$\begin{aligned}
 E_0 &= (r_v^2 x - \bar{e}_d) [(x - e_\delta) \sqrt{P} + x], & E_1 &= r_\mu r_v^2 x (r_v^2 x - \bar{e}_d) \sqrt{S + 2\sqrt{P}}, \\
 E_2 &= -\frac{1}{6} \left\{ r_v^4 x^2 (1 + 3\bar{e}_2) + r_v^2 x \left[\bar{e}_2^2 \bar{e}_3^2 + 2\bar{e}_2 \bar{e}_3 - 2\bar{e}_d - 3\bar{e}_1 \bar{e}_2 \right] \right. \\
 &\quad \left. - \bar{e}_d (2\bar{e}_2 \bar{e}_3 - \bar{e}_d) \right\} [(x - e_\delta) \sqrt{P} + x], \\
 E_3 &= \frac{1}{12} r_\mu \left\{ \bar{e}_d^2 + r_v^2 x \left[2\bar{e}_d (\bar{e}_2 \bar{e}_3 - \bar{e}_d - 1) \right. \right. \\
 &\quad \left. \left. + r_v^2 x (1 - 2\bar{e}_2 \bar{e}_3 - \bar{e}_2^2 \bar{e}_3^2 + 4\bar{e}_d + 2\bar{e}_1 \bar{e}_2) - 2r_v^4 x^2 (1 + \bar{e}_2) \right] \right\} \sqrt{S + 2\sqrt{P}}, \\
 S &= e_\delta - 2 - x, & P &= 1 - x.
 \end{aligned}
 \tag{5.7}$$

It is clear from Eqs. (5.6) and (5.7) that the squared dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ depends on seven dimensionless parameters e_δ , \bar{e}_1 , \bar{e}_2 , \bar{e}_3 , r_μ , r_v and ε . Note that $e_\delta > 0$, $\bar{e}_1 > 0$, $\bar{e}_2 > 0$ and $\bar{e}_1 - \bar{e}_2 \bar{e}_3^2 > 0$ (see [25]). When $\varepsilon = 0$, from Eq. (5.6) and the first equation of (5.7) it implies

$$(x - e_\delta) \sqrt{1 - x} + x = 0.
 \tag{5.8}$$

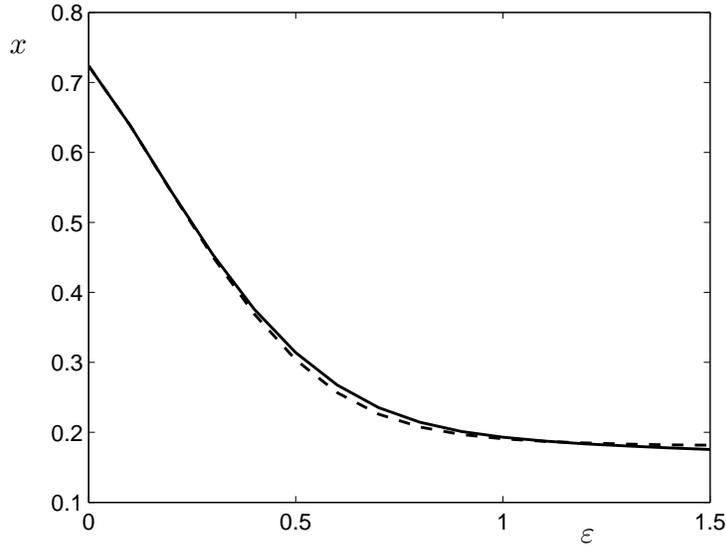


FIG. 2. Plots of the dimensionless Rayleigh wave velocity $x(\varepsilon)$ in the interval $[0, 1.5]$ that is calculated by the exact secular equation (solid line) and by the approximate secular equation (5.6) (dashed line). Here we take $e_\delta = 2.1$, $\bar{e}_1 = 1.8$, $\bar{e}_2 = 1.2$, $\bar{e}_3 = 0.6$, $r_v = 3$, $r_\mu = 0.5$.

This is again the secular equation of Rayleigh waves in an incompressible orthotropic elastic half-space (see [24]).

Figure 2 presents the dependence on $\varepsilon = k.h \in [0, 1.5]$ of the dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ that is calculated by the exact secular equation (having the form of a 6×6 determinant) (solid line) and by the approximate secular equation (5.6) (dashed line). The dimensionless parameters are taken as $e_\delta = 2.1$, $\bar{e}_1 = 1.8$, $\bar{e}_2 = 1.2$, $\bar{e}_3 = 0.6$, $r_v = 3$, $r_\mu = 0.5$. It is shown in Fig. 2 that the exact velocity curve and the third-order approximate one are very close to each other for the values of $\varepsilon \in [0, 1.5]$. The maximum absolute error in the interval $[0, 1.5]$ is 0.0109 (at $\varepsilon = 0.6$). This means that the approximate secular equation (5.6) has a high accuracy.

5.3. Isotropic case

When the layer is isotropic and the half-space is transversely isotropic (with the isotropic axis being the x_3 axis), we have

$$(5.9) \quad \bar{c}_{11} = \bar{c}_{22} = \bar{\lambda} + 2\bar{\mu}, \quad \bar{c}_{12} = \bar{\lambda}, \quad \bar{c}_{66} = \bar{\mu}, \quad c_{11} = c_{22}, \quad c_{11} - c_{12} = 2c_{66}$$

and consequently

$$(5.10) \quad \bar{e}_1 = 1/\bar{\gamma}, \quad \bar{e}_2 = \bar{\gamma}, \quad \bar{e}_3 = 1/\bar{\gamma} - 2, \quad \bar{e}_d = 4(1 - \bar{\gamma}), \quad e_\delta = 4, \quad S = 2 - x,$$

where $\bar{\gamma} = \bar{\mu}/(\bar{\lambda} + 2\bar{\mu})$. Taking into account (5.10), the expressions (5.7) of E_k are simplified to

$$(5.11) \quad \begin{aligned} E_0 &= [r_v^2 x - 4(1 - \bar{\gamma})] [(x - 4)\sqrt{1 - x} + x], \\ E_1 &= r_\mu r_v^2 x [r_v^2 x - 4(1 - \bar{\gamma})] (1 + \sqrt{1 - x}), \\ E_2 &= -\frac{1}{6} [8(1 - \bar{\gamma}) + 4r_v^2 x (\bar{\gamma}^2 - 2) + r_v^4 x^2 (1 + 3\bar{\gamma})] [(x - 4)\sqrt{1 - x} + x], \\ E_3 &= \frac{1}{6} r_\mu \left\{ 8(1 - \bar{\gamma})^2 + r_v^2 x [8(-2 + 3\bar{\gamma} - \bar{\gamma}^2) + 2r_v^2 x (4 - 2\bar{\gamma} - \bar{\gamma}^2) \right. \\ &\quad \left. - r_v^4 x^2 (1 + \bar{\gamma}) \right\} (1 + \sqrt{1 - x}). \end{aligned}$$

When the layer and the half-space are both isotropic, the expressions (5.11) are unchanged, but in this case: $x = \rho c^2/\mu$, $r_\mu = \bar{\mu}/\mu$, $\bar{\mu}$ and μ are the shear moduli.

6. Rayleigh waves in a compressible half-space coated by a thin incompressible layer

According to Subsection 4.1, the approximate effective third-order boundary condition for this case is given by (4.10). The Rayleigh wave can now be considered as a Rayleigh wave propagating in the compressible orthotropic elastic

half-space whose surface $x_2 = 0$ is subjected to the boundary conditions (4.10). According to [25], the displacement components of a Rayleigh wave travelling with velocity c and wave number k in the x_1 -direction and decaying in the x_2 -direction are determined by (4.2)_{1,2} in which $U_1(y)$ and $U_2(y)$ are given by

$$(6.1) \quad \begin{aligned} U_1(y) &= b_1 B_1 e^{-b_1 y} + b_2 B_2 e^{-b_2 y}, \\ U_2(y) &= \alpha_1 B_1 e^{-b_1 y} + \alpha_2 B_2 e^{-b_2 y}, \end{aligned}$$

where B_1 and B_2 are the constants, b_1 and b_2 are roots of the characteristic equation

$$(6.2) \quad c_{22}c_{66}b^4 + [(c_{12} + c_{66})^2 + c_{22}(X - c_{11}) + c_{66}(X - c_{66})]b^2 + (c_{11} - X)(c_{66} - X) = 0$$

whose real parts are positive to ensure the decay condition, $X = \rho c^2$, and

$$(6.3) \quad \begin{aligned} \alpha_k &= i\beta_k, \\ \beta_k &= \frac{b_k(c_{12} + c_{66})}{c_{22}b_k^2 - c_{66} + \rho c^2} = \frac{c_{11} - \rho c^2 - c_{66}b_k^2}{(c_{12} + c_{66})b_k}, \end{aligned}$$

where $k = 1, 2$ and $i = \sqrt{-1}$. From (6.2) we have

$$(6.4) \quad \begin{aligned} b_1^2 + b_2^2 &= -\frac{(c_{12} + c_{66})^2 + c_{22}(X - c_{11}) + c_{66}(X - c_{66})}{c_{22}c_{66}} := S, \\ b_1^2 b_2^2 &= \frac{(c_{11} - X)(c_{66} - X)}{c_{22}c_{66}} := P. \end{aligned}$$

It is not difficult to verify that if the Rayleigh wave exists ($\rightarrow b_1$ and b_2 having positive real parts), then

$$(6.5) \quad 0 < X < \min\{c_{11}, c_{66}\}$$

and

$$(6.6) \quad b_1 b_2 = \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}.$$

Using (4.2)_{1,2} and (6.1) in the stress-strain relations (Eq. (2.2) in [25]) we obtain the expressions of σ_{12} and σ_{22} that are given by (4.2)_{3,4}, in which

$$(6.7) \quad \begin{aligned} T_1(y) &= ic_{66}[(b_1 + \beta_1)B_1 e^{-b_1 y} + (b_2 + \beta_2)B_2 e^{-b_2 y}], \\ T_2(y) &= [(c_{12} - c_{22}b_1\beta_1)B_1 e^{-b_1 y} + (c_{12} - c_{22}b_2\beta_2)B_2 e^{-b_2 y}]. \end{aligned}$$

Introducing Eqs. (6.1) and (6.7) into Eqs. (4.10) provides a homogeneous system of two linear equations for B_1 and B_2 , and making its determinant of coefficients equal to 0 yields the third-order approximate dispersion equation of the wave:

$$(6.8) \quad D_0 + D_1\varepsilon + D_2\varepsilon^2 + D_3\varepsilon^3 = 0$$

where

$$(6.9) \quad \begin{aligned} D_0 &= a_1 \left[(c_{12}^2 - c_{11}c_{22} + c_{22}X)b_1b_2 + (c_{11} - X)X \right], \\ D_1 &= a_3(c_{11} - X)(b_1 + b_2), \\ D_2 &= a_2 \left[(c_{12}^2 - c_{11}c_{22} + c_{22}X)b_1b_2 + (c_{11} - X)X \right], \\ D_3 &= a_4(c_{11} - X)(b_1 + b_2), \end{aligned}$$

in which b_1b_2 and $b_1 + b_2$ are given by Eq. (6.6). Equation (6.8) is the desired approximate third-order secular equation. It is totally explicit. In the dimensionless form this equation is written as

$$(6.10) \quad E_0 + E_1\varepsilon + E_2\varepsilon^2 + E_3\varepsilon^3 = 0,$$

where

$$(6.11) \quad \begin{aligned} E_0 &= (r_v^2x - \bar{e}_\delta) \left[(e_2x - e_d)b_1b_2 + (e_1 - x)x \right], \\ E_1 &= r_\mu r_v^2x (r_v^2x - \bar{e}_\delta) (e_1 - x) (b_1 + b_2), \\ E_2 &= \frac{1}{6} \left[\bar{e}_\delta(2 - \bar{e}_\delta) + 2\bar{e}_\delta r_v^2x - r_v^4x^2 \right] \left[(e_2x - e_d)b_1b_2 + (e_1 - x)x \right], \\ E_3 &= \frac{1}{12} r_\mu \left[\bar{e}_\delta^2 - 2r_v^2x(\bar{e}_\delta^2 - 2\bar{e}_\delta r_v^2x + r_v^4x^2) \right] (e_1 - x) (b_1 + b_2), \\ b_1b_2 &= \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}, \\ P &= \frac{(1-x)(e_1-x)}{e_2}, \\ S &= \frac{e_2(e_1-x) + 1-x - (1+e_3)^2}{e_2} \end{aligned}$$

and

$$(6.12) \quad e_1 = \frac{c_{11}}{c_{66}}, \quad e_2 = \frac{c_{22}}{c_{66}}, \quad e_3 = \frac{c_{12}}{c_{66}}, \quad e_d = e_1e_2 - e_3^2.$$

It is clear from Eqs. (6.10) and (6.11) that the squared dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ depends on seven dimensionless parameters $e_1, e_2, e_3,$

\bar{e}_δ , r_μ , r_ν and ε . Note that $e_1 > 0$, $e_2 > 0$, $e_1 e_2 - e_3^2 > 0$, $\bar{e}_\delta > 0$ (see [25]). If $\varepsilon = 0$, from Eq. (6.10) and the first equation of (6.11) it follows

$$(6.13) \quad (e_2 x - e_d) \sqrt{P} + (e_1 - x)x = 0.$$

Equation (6.13) is the secular equation of Rayleigh waves propagating in an orthotropic elastic half-space (see [26] and [25]).

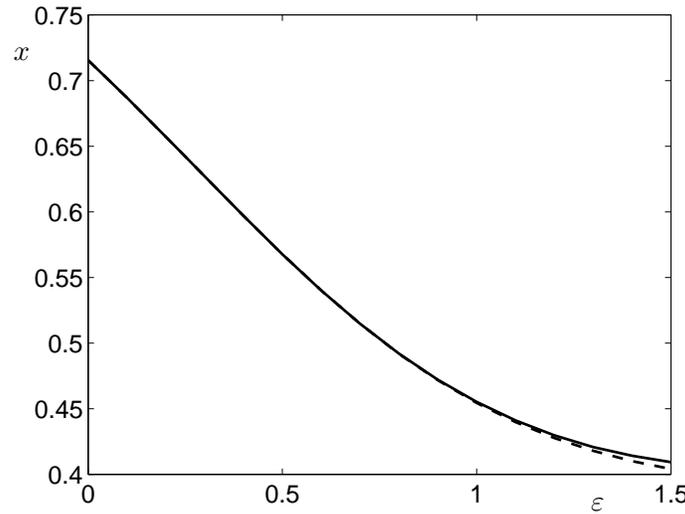


FIG. 3. The Rayleigh wave velocity curves drawn by solving the exact dispersion (solid line) and by the approximate secular equation (6.10) (dashed line) with $r_\mu = 0.8$, $r_\nu = 1.3$, $e_1 = 2.5$, $e_2 = 3$, $e_3 = 1.5$ and $\bar{e}_\delta = 3.2$.

Figure 3 presents the dependence on $\varepsilon = k.h \in [0, 1.5]$ of the dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ that is calculated by the exact secular equation (in the form of a 6×6 determinant) (solid line) and by the approximate secular equation (6.10) (dashed line) with $r_\mu = 0.8$, $r_\nu = 1.3$, $e_1 = 2.5$, $e_2 = 3$, $e_3 = 1.5$ and $\bar{e}_\delta = 3.2$. Figure 3 shows that the exact velocity curve and the third-order approximate one almost totally coincide with each other for the values of $\varepsilon \in [0, 1.5]$. The maximum absolute error in the interval $[0, 1.5]$ is 0.0052 (at $\varepsilon = 1.5$). This shows that the approximate secular equation (6.10) is a good approximation.

6.1. Isotropic case

When the layer is transversely isotropic (with the isotropic axis being the x_3 -axis) and the half-space is isotropic, i.e., $c_{11} = c_{22} = \lambda + 2\mu$, $c_{12} = \lambda$, $c_{66} = \mu$, $\bar{c}_{11} = \bar{c}_{22}$ and $\bar{c}_{11} - \bar{c}_{12} = 2\bar{c}_{66}$, then from Eqs. (3.6), (4.8), (6.3) and (6.4), it is

easy to verify that

$$(6.14) \quad \bar{e}_\delta = 4, \quad b_1 = \sqrt{1 - \gamma x}, \quad b_2 = \sqrt{1 - x}, \quad \beta_1 = b_1, \quad \beta_2 = \frac{1}{b_2},$$

where $\gamma = \mu/(\lambda + 2\mu)$. With the help of Eq. (6.14), the secular equation (6.10) is simplified to

$$(6.15) \quad \bar{E}_0 + \bar{E}_1 \varepsilon + \bar{E}_2 \varepsilon^2 + \bar{E}_3 \varepsilon^3 = 0,$$

where

$$(6.16) \quad \begin{aligned} \bar{E}_0 &= (r_v^2 x - 4) [(x - 2)^2 - 4b_1 b_2], \quad \bar{E}_1 = r_\mu r_v^2 x^2 b_1 (r_v^2 x - 4), \\ \bar{E}_2 &= -\frac{1}{6} [8 - 8r_v^2 x + r_v^4 x^2] [(x - 2)^2 - 4b_1 b_2], \\ \bar{E}_3 &= \frac{1}{6} r_\mu x b_1 [8 - r_v^2 x (16 - 8r_v^2 x + r_v^4 x^2)], \end{aligned}$$

here $r_\mu = \bar{c}_{66}/\mu$. When the layer and the half-space are both isotropic, \bar{E}_k , ($k = 0, 1, 2, 3$) are also calculated by Eq. (6.16), but in this case $x = \rho c^2/\mu$, $r_\mu = \bar{\mu}/\mu$, $\bar{\mu}$ and μ are the shear moduli.

7. Numerical examples

In this section, as an example of application of the obtained approximate secular equations, we consider numerically the influence of incompressibility on the Rayleigh wave velocity. For this purpose we consider four examples. In the first example, a compressible half-space is coated either by a compressible layer or by an incompressible layer. These two layers have the same elastic constants. In the second example, the compressible half-space is replaced by an incompressible half-space. In the third and fourth examples, two different (compressible and incompressible) half-spaces with the same elastic constants are covered with the same compressible or incompressible layer.

In particular, in the first example we take $e_1 = 2.5$, $e_2 = 3$ and $e_3 = 1.5$ for the half-space and $\bar{e}_1 = 2.2$, $\bar{e}_2 = 1.8$ and $\bar{e}_3 = 0.5$ for the layers and $r_\mu = 1$ and $r_v = 1.2$.

In the second example, we choose $e_\delta = 2.5$ for the half-space and $\bar{e}_1 = 4.6$, $\bar{e}_2 = 1$ and $\bar{e}_3 = 1$ for the layers and $r_\mu = 0.8$ and $r_v = 2$.

In the third example, the dimensionless parameters $\bar{e}_1 = 1.8$, $\bar{e}_2 = 1$ and $\bar{e}_3 = 0.6$ are used for the layer and $e_1 = 2.5$, $e_2 = 3$ and $e_3 = 1.5$ for the half-spaces and $r_\mu = 0.5$ and $r_v = 3$.

In the last case, they are $\bar{e}_\delta = 3.5$ for the layer and $e_1 = 2.8$, $e_2 = 3.2$ and $e_3 = 1.5$ for the half-spaces and $r_\mu = 1$ and $r_v = 1.5$.

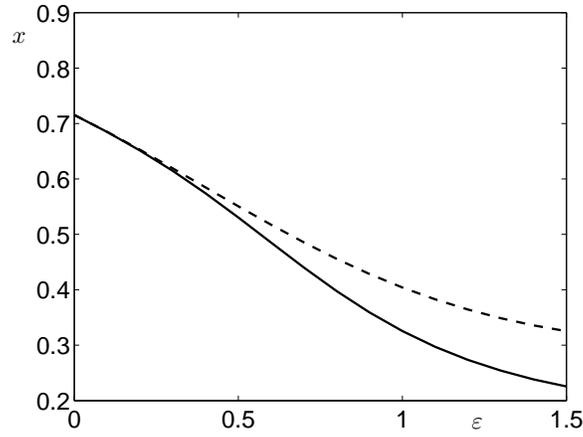


FIG. 4. A compressible half-space coated by a compressible layer (solid line *drawn by solving* Eq. (39) in [19]), by an incompressible layer (dashed line *drawn by solving* (6.10)). Here we take $e_1 = 2.5$, $e_2 = 3$ and $e_3 = 1.5$ for the half-space and $\bar{e}_1 = 2.2$, $\bar{e}_2 = 1.8$ and $\bar{e}_3 = 0.5$ for the layers and $r_\mu = 1$ and $r_v = 1.2$.

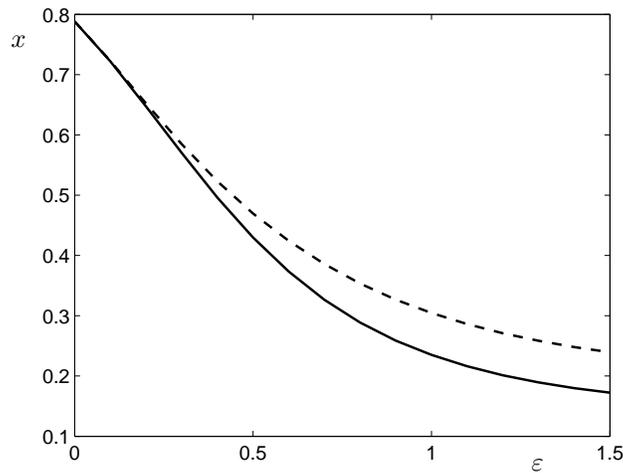


FIG. 5. An incompressible half-space coated by a compressible layer (solid line *drawn by solving* (5.6)), by an incompressible layer (dashed line *drawn by solving* (4.25)). Here we take $e_\delta = 2.5$ for the half-space and $\bar{e}_1 = 4.6$ and $\bar{e}_2 = 1$, $\bar{e}_3 = 1$ for the layers and $r_\mu = 0.8$ and $r_v = 2$.

The numerical results of the first, second, third and fourth examples are presented in Figs. 4, 5, 6 and 7, respectively. To establish the wave velocity curves, the approximate secular equations (4.25), (5.6), (6.10) obtained in this paper and Eq. (39) in [19] are employed.

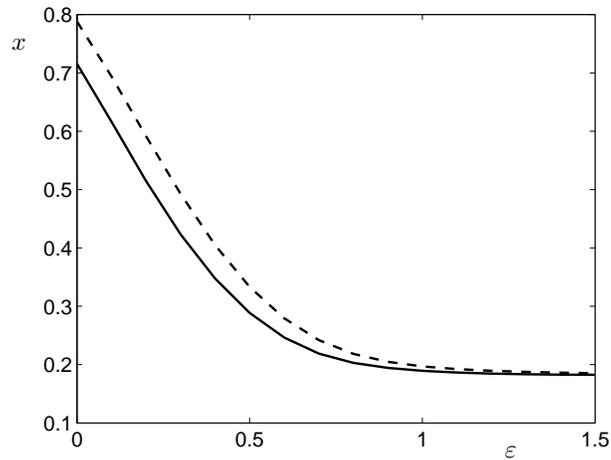


FIG. 6. A compressible layer coats a compressible half-space (solid line *drawn by solving* Eq. (39) in [19]), and coats an incompressible half-space (dashed line *drawn by solving* (5.6)). Here we take $\bar{e}_1 = 1.8$, $\bar{e}_2 = 1$ and $\bar{e}_3 = 0.6$ for the layer and $e_1 = 2.5$, $e_2 = 3$ and $e_3 = 1.5$ for the half-spaces and $r_\mu = 0.5$ and $r_\nu = 3$.

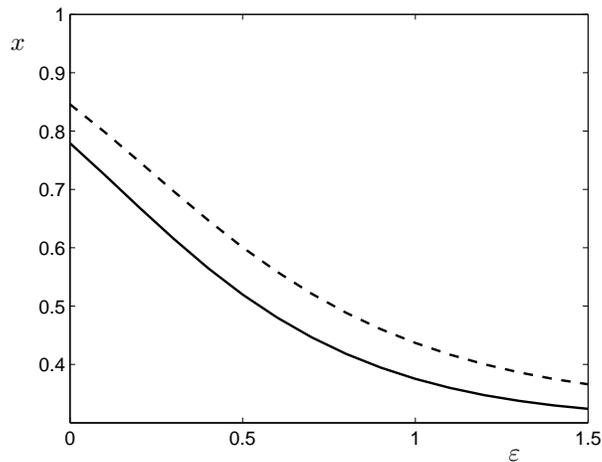


FIG. 7. An incompressible layer coats a compressible half-space (solid line *drawn by solving* (5.6)), and an incompressible half-space (dashed line *drawn by solving* (4.25)). Here we take $\bar{e}_\delta = 3.5$ for the layer and $e_1 = 2.8$, $e_2 = 3.2$ and $e_3 = 1.5$ for the half-spaces and $r_\mu = 1$ and $r_\nu = 1.5$.

It is shown in Figs. 4–7:

- (i) Incompressibility strongly affects the Rayleigh wave velocity.
- (ii) The effect of incompressible coating layers is considerably stronger than the one of incompressible half-spaces.
- (iii) Incompressibility increases the Rayleigh wave velocity.

8. Conclusions

In this paper, the propagation of Rayleigh waves in an orthotropic elastic half-space coated by a thin orthotropic elastic layer is considered. The half-space and the layer are in sliding contact with each other and they are compressible or incompressible. By using the effective boundary condition method, the approximate third-order secular equations regarding of the dimensionless thickness of the layer are derived for three combinations: incompressible/incompressible, compressible/incompressible and incompressible/compressible. It is shown that these approximate secular equations are good approximations. Based on the obtained secular equations, the effect of incompressibility on the Rayleigh wave velocity is examined. It is shown that the Rayleigh wave velocity is strongly affected by the incompressibility of half-spaces and coating layers, and the incompressibility of coating layers has a stronger effect than the one of half-spaces. The incompressibility increases the Rayleigh wave velocity.

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