Arch. Mech., **67**, 6, pp. 457–475, Warszawa 2015

Effects of phase-lags in a thermoviscoelastic orthotropic continuum with a cylindrical hole and variable thermal conductivity

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This article presents an analytical solution for the effect of phase-lags on a generalized plane strain thermoviscoelastic orthotropic medium with a cylindrical cavity subjected to a thermal shock from varying heat. It is assumed that the cylindrical cavity is made of Kelvin–Vogt type material. The general solutions for field quantities are obtained using the method of Laplace transforms. The results are graphically presented to illustrate the effect of phase-lags, viscoelasticity and variability of thermal conductivity on the studied fields. Comparisons are also presented with those in the absence of viscosity and variability of thermal conductivity.

Key words: thermoviscelasticity, orthotropic medium, cylindrical hole, variable thermal conductivity, dual-phase-lags.

1. Introduction

The theory of coupled thermoelasticity was formulated by Biot [1] to eliminate the paradox inherent in the classical uncoupled theory which states that elastic changes have no effect on temperature. However, the heat equations for both of the two mentioned theories, though different, are of the diffusion type that predicts infinite speeds of propagation for heat waves, which is contrary to physical observations. The theory of coupled thermoelasticity was extended by Lord and Shulman [2] and Green and Lindsay [3] by including thermal relaxation time in constitutive relations. Additional discussions of the generalized thermoelasticity theory are given in the literature (see, for example, Chandarasekharaia [4], Aouadi [5], Hetnarski and Eslami [6] and Zenkour [7]).

The dual-phase-lag (DPL) model describes the interactions between photons and electrons on a microscopic level as retarding sources causing a delayed response on a macroscopic scale. The DPL model was first proposed by Tzou [8, 9]. The proposed model was a modification of the classical thermoelastic model in which the Fourier law is replaced by an approximation to a modified Fourier law with two different time translations: a phase lag of the heat flux τ_q and a phase lag of the temperature gradient τ_θ (see Tzou [10]). For macroscopic formulation, it is convenient to use the DPL model for investigation of the micro-structural effect on the behavior of heat transfer. The physical meanings and the applicability of the DPL model have been supported by experimental results in [10]. A Taylor series approximation of the modified Fourier law, together with the remaining field equations, leads to a complete system of equations describing DPL thermoelastic model. The model transmits thermoelastic disturbance in a wave-like manner if the approximation is linear with respect to τ_q and τ_θ $(0 \leq \tau_\theta < \tau_q)$ or quadratic in τ_q and linear in τ_θ, with $\tau_q > 0$ and $\tau_\theta > 0$. This theory is developed in a rational way to produce a fully consistent theory which incorporates thermal pulse transmission in a very logical manner. Roychoudhuri [11] has studied thermoelastic wave propagation in an elastic half-space in the context of a generalized DPL model. Another generalization, known as three-phase-lag thermoelasticity model, was extended by Roychoudhuri [12]. Akbarzadeh and Pasini [13] have presented a theoretical framework including three-phase-lag, dual-phase-lag and hyperbolic heat conduction to study the thermal responses of one-dimensional multilayered systems, functionally graded solid media and porous materials.

Viscoelasticity is of interest in different applications. It is linked to a variety of microphysical processes and can be therefore used as an experimental probe in those processes. The linear viscoelastic governing equations are constructed by means of the Boltzmann superposition principle. The linear theory of viscoelasticity may be extended to the corresponding thermo-viscoelasticity theory at finite strains. To do this, several requirements should be taken under consideration. The first requirement is that compatibility with the second law of thermodynamics should be satisfied. The second is that the constitutive theory of finite thermoelasticity can be reduced during a sufficiently fast (or slow) deformation process. The third is that it can be interpreted by microscopic deformation mechanisms under reasonable physical assumptions. The last is that it can adequately represent as many experimental data as possible with limited number of material parameters.

The linear theory of viscoelasticity has been formulated and applied to situations in which environmental factors such as temperature are assumed to be constant. However, the mechanical response of a viscoelastic material is sensitive to variations in such environmental factors as temperature, humidity and presence of diffusion. Viscoelasticity is one of the principal types of inelastic behavior. Pervasive categories of materials such as polymers, rubber, glass, concrete, asphalt, ice, rock salt, sound dampers, as well as biological and geological substances and elastic materials at high temperature, behave viscoelastically and are characterized by time dependent energy dissipation due to creep and/or relaxation. Viscoelasticity and related phenomena are of great importance in the study of biological materials. Just as strain can be measured in more than one way, so the related rate of strain can be measured in a number of different ways.

Viscoelastic materials play an important role in many branches of engineering, technology and, in recent years, biomechanics [14]. Viscoelastic materials such as amorphous polymers, semi-crystalline polymers and biopolymers can be modeled to determine their stress or strain interactions as well as their temporal dependencies. The study of viscoelastic behavior in bone is of interest in several contexts. Bone is a hierarchical solid that contains structure at multiple length scales. The study of bone viscoelasticity is best placed in the context of strain levels and frequency components associated with normal activities and with applications of diagnostic tools [15]. The investigations of the solutions of viscoelastic wave equations, velocities of seismic wave propagating and the attenuation of seismic waves in viscoelastic media are very important for geophysical prospecting technology. The linear theories of the viscoelasticity and thermosviscoelasticity of binary mixtures where the individual components are modeled as Kelvin–Voight viscoelastic materials were developed in [15, 16]. Svanadze [17] has considered basic boundary value problems (BVPs) of steady vibrations in the linear theories of the viscoelasticity and thermo-viscoelasticity for Kelvin–Voigt materials. He has also generalized some basic results of the classical theories of elasticity and thermoelasticity, and established the uniqueness theorems of the basic internal and external BVPs [17].

The Kelvin–Voigt model is one of the macroscopic mechanical models often used to describe the viscoelastic behavior of a material. This model represents the delayed elastic response subjected to stress when the deformation is time dependent but recoverable. The dynamic interaction of thermal and mechanical fields in solids has great practical applications in modern aeronautics, astronautics, nuclear reactors and high-energy particle accelerators. Several problems of wave propagation in a linear viscoelastic solid have been discussed by many researchers. Additionally, with the rapid development of polymer science and the plastic industry, as well as the wide use of materials under high temperature in modern technology, the theoretical study and application of viscoelastic materials has become important for solid mechanics. The theory of thermo-viscoelasticity and the solutions of some BVPs of thermo-viscoelasticity are investigated by Ilioushin and Pobedria [18]. In the last decade, the works of Tanner [19] and Huilgol and Phan-Thien [20] have helped in finding solutions for the BVPs of linear viscoelastic materials including temperature variations in both quasi-static and dynamic problems.

Kovalenko and Karanaukhov [21] have presented a generalized linearized theory of thermo-viscoelasticity with the inclusion of the heat formation effect. Drozdov [22] has derived the constitutive relations for the non-isothermal viscoelastic behavior of polymers at finite strains. Kundu and Mukhopadhyay [23] have considered the distribution of field quantities in a viscoelastic medium with a spherical cavity in the context of generalized thermoelasticity with the relaxation effect. Baksi *et al.* [24] have derived the fundamental equations of an infinite rotating magneto-thermo-viscoelastic media due to heat sources with one relaxation parameter. Kanoria and Mallik [25] have studied the thermoviscoelastic interaction in an infinite Kelvin–Voigt-type viscoelastic medium subjected to periodically varying heat sources. Ezzat *et al.* [26] have applied the coupled fractional relaxation equations in the frame of thermo-viscoelasticity to the 1D problem with heat sources. Kar and kanoaria [27] have studied a problem dealing with thermo-visco-elastic interaction due to a step input of temperature on the stress free boundaries of a homogeneous viscoelastic isotropic spherical shell in the context of generalized theories of thermo-elasticity. Deswal and Kalkal [28] have presented a paper dealing with the problem of thermo-viscoelastic interactions in a homogeneous, isotropic 3D medium whose surface suffers a time-dependent thermal shock and based on a three-phase-lag model with two temperatures.

The contents and structure of the present paper are as follows. We initially present a conducting orthotropic body of variable thermal conductivity with a cylindrical cavity. We assume that the boundaries of the cylinder are subjected to a time-dependent thermal shock and its surface is traction free. We investigate the thermoelastic interactions in this body in the context of generalized thermoelasticity with DPL [29–32]. The present DPL model initially developed by Tzou [8, 9] is an extension of the well-known generalized thermoelasticity theory [2, 3]. Next, we graphically present some numerical results for the field quantities of the thermoviscoelastic body. Some special cases are considered and explained when the viscosity field and variability of thermal conductivity are neglected.

The main purpose of this study is to extend the linear theory of viscoelasticity to include the effects of a larger set of environmental factors on a larger class of materials.

2. Basic equations

Consider a generalized plane strain (where the non-zero strain components act in one plane only) thermoviscoelastic orthotropic body with a cylindrical cavity at uniform temperature T_0 whose surface is traction-free and subjected to a time-dependent thermal shock. We can employ the Kelvin–Voigt model of linear viscoelasticity to describe the viscoelastic nature of the material.

The theories of viscoelasticity, which include the Maxwell model, the Kelvin–Voigt model and the standard linear solid model, are used to predict a material's response under different loading conditions. One of the simplest mathematical models constructed to describe viscoelastic effects is the classical Kelvin–Voigt model [33]. The basic idea concerning this model is that stress is dependent on the deformation tensor and deformation-rate tensor. This model consists of a Newtonian damper and Hooke's elastic spring connected in parallel.

The cylindrical coordinates system (r, ξ, z) is used with z-axis lying along the axis of the cylinder. It is assumed that disturbances are small and are confined to the neighborhood of the interface $r = R$, and hence vanish as $r \to \infty$.

For an axially symmetric problem, the displacements are reduced to

$$u_r = u(r,t), \qquad u_\xi(r,t) = u_z(r,t) = 0, \tag{2.1}$$

and their radial ε_{rr} and hoop $\varepsilon_{\xi\xi}$ strains are given by

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{\xi\xi} = \frac{u}{r}. \tag{2.2}$$

The constitutive relations for a Kelvin–Voigt type solid are [33]

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\xi\xi} \\ \sigma_{zz} \end{Bmatrix} = \begin{bmatrix} \tau_m c_{11} & \tau_m c_{12} & -\beta_{11} \\ \tau_m c_{12} & \tau_m c_{22} & -\beta_{22} \\ \tau_m c_{13} & \tau_m c_{23} & -\beta_{33} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \theta \end{Bmatrix}, \tag{2.3}$$

where σ_{rr}, $\sigma_{\xi\xi}$ and σ_{zz} are the normal mechanical stresses, c_{ij} are the isothermal elastic constants, β_{ij} are the thermal elastic coupling components, $\tau_m = 1 + t_0 \frac{\partial}{\partial t}$ and t_0 is the mechanical relaxation time due to the viscosity. $\theta = T - T_0$ denotes the thermodynamical temperature, in which T is the temperature and T_0 is the reference temperature such that $|\theta/T_0| \ll 1$.

The dynamic equation of motion of the cylindrical cavity, in which the body forces are neglected, is expressed as

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\xi\xi}}{r}=\rho\frac{\partial^2 u}{\partial t^2}. \tag{2.4}$$

With the aid of Eq. (1.3), the above equation of motion becomes

$$\tau_m c_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-\tau_m c_{22}\frac{u}{r^2}=\rho\frac{\partial^2 u}{\partial t^2}+\beta_{11}\frac{\partial\theta}{\partial r}+(\beta_{11}-\beta_{22})\frac{\theta}{r}, \tag{2.5}$$

where ρ denotes the material density. The modified Fourier's law is given by

$$\left(1+\tau_q\frac{\partial}{\partial t}\right)\mathbf{q}=-K_r\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\nabla\theta, \tag{2.6}$$

where $\mathbf{q}$ is the heat flux vector, K_r is the thermal conductivity, and τ_θ and τ_q denote the finite times. The first delay time τ_θ is said to be the PL of the temperature gradient, while the second delay time τ_q denotes the PL of the heat flux. The aim of the delay time τ_q is to ensure that the heat conduction equation will predict finite speeds of heat propagation. Now, the energy conservation equation can be given by

$$-\nabla\cdot\mathbf{q}=\rho C_E\frac{\partial\theta}{\partial t}+T_0\frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r}+\beta_{22}\frac{u}{r}\right), \tag{2.7}$$

where C_E is the specific heat at constant strain. By eliminating $\mathbf{q}$ using Eqs. (2.6) and (2.7), the heat conduction equation with DPLs and without any heat sources will be

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)(K_r\theta_{,r})_{,r}=\left(1+\tau_q\frac{\partial}{\partial t}\right)\left[\rho C_E\frac{\partial\theta}{\partial t}+T_0\frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r}+\beta_{22}\frac{u}{r}\right)\right]. \tag{2.8}$$

The governing field equations in the context of linear generalized thermoelasticity with one relaxation time can be written using Eqs. (2.1)–(2.8) by setting mechanical PL parameters $\tau_\theta=0$ and $\tau_\theta=\tau_0$ (τ_0 is the thermal relaxation time). Upon taking the thermal PLs $\tau_\theta=\tau_q=0$, we obtain the governing field equations for a coupled theory of thermoelasticity. When setting thermal PLs as $\tau_\theta=\tau_q=0$, and the thermomechanical coupling parameters as $\beta_{11}=\beta_{11}=0$, the governing field equations for uncoupled thermoelasticity can also be obtained.

3. Variable thermal conductivity

The thermal properties of the thermosensitive body should vary with temperature and lead to a nonlinear heat conduction problem. This problem can be solved by simply assuming nonlinear properties of the material. This means that the thermal material coefficient K_r and the specific heat C_E are linearly dependent on the temperature [34], but the thermal diffusivity k ($k = K_r/\rho C_E$) is assumed constant. That is

$$K_r = K_r(\theta) = k_0 + k_1^*\theta, \tag{3.1}$$

where k_0 is the thermal conductivity at T_0 and k_1^* ($\equiv k_1/k_0$) is the slope of the thermal conductivity-temperature curve. Now, a new function ψ is considered to express heat conduction in the Kirchhoff transformation [34]

$$\psi = \frac{1}{k_0}\int_0^\theta K_r(\theta)d\theta. \tag{3.2}$$

The above equation, with the aid of Eq. (3.1), gives

$$\psi = \theta(1 + \tfrac{1}{2}k_1\theta). \tag{3.3}$$

From Eq. (3.3), it follows that

$$\nabla\psi = \frac{K_r(\theta)}{k_0}\nabla\theta, \qquad \frac{\partial\psi}{\partial t} = \frac{K_r(\theta)}{k_0}\frac{\partial\theta}{\partial t}. \tag{3.4}$$

The final form of the general heat equation with variable thermal conductivity, after substituting Eq. (3.4) into Eq. (2.8), is

$$\begin{aligned}\left(1 + \tau_\theta\frac{\partial}{\partial t}\right)&\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\ &= \left(1 + \tau_q\frac{\partial}{\partial t}\right)\left[\rho C_E\frac{\partial\psi}{\partial t} + \frac{T_0}{k_0}\frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r} + \beta_{22}\frac{u}{r}\right)\right].\end{aligned} \tag{3.5}$$

From Eqs. (3.3), the equation of motion will be as follows:

$$\begin{aligned}\tau_m c_{11}&\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) - \tau_m c_{22}\frac{u}{r^2}\\ &= \rho\frac{\partial^2 u}{\partial t^2} + \frac{\beta_{11}}{1 + 2k_1\theta}\frac{\partial\psi}{\partial r} + (\beta_{11} - \beta_{22})\left(\frac{-1 + \sqrt{1 + 2k_1\psi}}{k_1 r}\right).\end{aligned} \tag{3.6}$$

For linearity, where $\theta = T - T_0$ such that $|\theta/T_0| \ll 1$, the governing equations will be as follows:

$$\tau_m c_{11}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \tau_m c_{22}\frac{u}{r^2} = \rho\frac{\partial^2 u}{\partial t^2} + \beta_{11}\frac{\partial\psi}{\partial r} + (\beta_{11} - \beta_{22})\frac{\psi}{r}, \tag{3.7}$$

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\xi\xi}\\ \sigma_{zz}\end{Bmatrix} = \begin{bmatrix}\tau_m c_{11} & \tau_m c_{12} & -\beta_{11}\\ \tau_m c_{12} & \tau_m c_{22} & -\beta_{22}\\ \tau_m c_{13} & \tau_m c_{23} & -\beta_{33}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{u}{r}\\ \psi\end{Bmatrix}. \tag{3.8}$$

Let us consider the following non-dimensional variables

$$\begin{aligned}(3.9)\qquad &(r', u', R_i') = \frac{c_0}{k}(r, u, R_i), \qquad (t', t_0', \tau_q', \tau_\theta') = \frac{c_0^2}{k}(t, t_0, \tau_q, \tau_\theta),\\ &\psi' = \frac{\psi}{T_0}, \qquad \sigma_{ij}' = \frac{\sigma_{ij}}{c_{11}}, \qquad k_1' = T_0 k_1, \qquad c_0^2 = \frac{c_{11}}{\rho}.\end{aligned}$$

Using the above quantities in the governing equations given in Eqs. (3.5)–(3.8) and suppressing dashes, we obtain

$$(3.10)\qquad \tau_m \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \tau_m c_2 \frac{u}{r^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon_1 \frac{\partial \psi}{\partial r} + \varepsilon_3 \frac{\psi}{r},$$

$$(3.11)\qquad \left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\nabla^2\psi = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\left[\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial t}\left(\varepsilon_4 \frac{\partial u}{\partial r} + \varepsilon_5 \frac{u}{r}\right)\right],$$

$$(3.12)\qquad \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\xi\xi} \\ \sigma_{zz} \end{Bmatrix} = \begin{bmatrix} \tau_m & \tau_m c_1 & -\varepsilon_1 \\ \tau_m c_1 & \tau_m c_2 & -\varepsilon_2 \\ \tau_m c_3 & \tau_m c_4 & -\varepsilon_6 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \psi \end{Bmatrix},$$

where

$$\begin{aligned}(3.13)\qquad &c_1 = \frac{c_{12}}{c_{11}}, \qquad c_2 = \frac{c_{22}}{c_{11}}, \qquad c_3 = \frac{c_{13}}{c_{11}}, \quad c_4 = \frac{c_{23}}{c_{11}},\\ &\varepsilon_1 = \frac{\beta_{11} T_0}{c_{11}}, \quad \varepsilon_2 = \frac{\beta_{22} T_0}{c_{11}}, \quad \varepsilon_3 = \frac{(\beta_{11} - \beta_{22}) T_0}{c_{11}},\\ &\varepsilon_4 = \frac{\beta_{11}}{\rho C_E}, \qquad \varepsilon_5 = \frac{\beta_{22}}{\rho C_E}, \qquad \varepsilon_6 = \frac{\beta_{33} T_0}{c_{11}}.\end{aligned}$$

4. Conditions of the problem

The initial and regularity conditions should be considered to solve the present problem. They are given by

$$\begin{aligned}(4.1)\qquad &u(r,t)\Big|_{t=0} = \frac{\partial u(r,t)}{\partial t}\Big|_{t=0} = 0, \qquad \theta(r,t)\Big|_{t=0} = \frac{\partial \theta(r,t)}{\partial t}\Big|_{t=0} = 0,\\ &\psi(r,t)\Big|_{t=0} = \frac{\partial \psi(r,t)}{\partial t}\Big|_{t=0} = 0,\end{aligned}$$

$$(4.2)\qquad u(r,t) = \theta(r,t) = \psi(r,t) = 0 \quad \text{at } r \to \infty.$$

To solve Eqs. (3.10) and (3.11), we will consider that the medium described above is quiescent and the surface of the cylinder is subjected to a time-dependent thermal shock and it is traction free. Therefore, the corresponding boundary conditions will be

$$\theta(R,t) = \theta_0 H(t), \qquad t > 0, \tag{4.3}$$

where θ_0 is constant,

$$\bar{\sigma}_{rr}(R,t) = 0. \tag{4.4}$$

Using Eq. (3.3), we obtain

$$\psi(R,t) = \theta_0 H(t) + \frac{k_1}{2}[\theta_0 H(t)]^2. \tag{4.5}$$

5. Solution of the problem

The Laplace transform is applied to Eqs. (3.10)–(3.12), taking into account the initial conditions given in Eq. (4.1) and assuming that $\beta_{11} = \beta_{22}$ (i.e., $\varepsilon_4 = \varepsilon_5 = \varepsilon$) and $c_{11} = c_{22}$. This gives the following equations:

$$\frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r^2} - \frac{s^2}{1+t_0 s}\bar{u} = \frac{\varepsilon_1}{1+t_0 s}\frac{d\bar{\psi}}{dr}, \tag{5.1}$$

$$\nabla^2\bar{\psi} = \frac{s(1+\tau_q s)}{1+\tau_\theta s}\left[\bar{\psi} + \varepsilon\left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right)\right], \tag{5.2}$$

$$\begin{Bmatrix} \bar{\sigma}_{rr} \\ \bar{\sigma}_{\xi\xi} \\ \bar{\sigma}_{zz} \end{Bmatrix} = \begin{bmatrix} 1 & c_1 & -\varepsilon_1 \\ c_1 & 1 & -\varepsilon_1 \\ c_3 & c_4 & -\varepsilon_6 \end{bmatrix} \begin{Bmatrix} (1+t_0 s)\dfrac{d\bar{u}}{dr} \\ (1+t_0 s)\dfrac{\bar{u}}{r} \\ \bar{\psi} \end{Bmatrix}, \tag{5.3}$$

where $\bar{\zeta}$ is the Laplace transform of quantity ζ and s is the Laplace parameter. Equations (5.2) and (5.3) can be written as

$$\left(DD_1 - \frac{s^2}{1+t_0 s}\right)\bar{u} = \frac{\varepsilon_1}{1+t_0 s}D\bar{\psi}, \tag{5.4}$$

$$\varepsilon q D_1 \bar{u} = (D_1 D - q)\bar{\psi}, \tag{5.5}$$

where

$$D = \frac{d}{dr}, \qquad D_1 = \frac{d}{dr} + \frac{1}{r}, \qquad q = \frac{s(1+\tau_q s)}{1+\tau_\theta s}. \tag{5.6}$$

Now, the displacement u is given as a first derivative of a new thermoelastic potential function ϕ as

$$u = \frac{d\phi}{dr}, \tag{5.7}$$

then, the above relation is introduced into Eqs. (5.4) and (5.5); thus, we get

$$\left(D_1 D - \frac{s^2}{1+t_0 s}\right)\bar{\phi} = \frac{\varepsilon_1}{1+t_0 s}\bar{\psi}, \tag{5.8}$$

$$\varepsilon q D_1 D\bar{\phi} = (D_1 D - q)\bar{\psi}. \tag{5.9}$$

Eliminating $\bar{\psi}$ from Eqs. (5.8) and (5.9), we obtain

$$\left(\nabla^4 - \left[\frac{s^2}{1+t_0 s} + q\left(\frac{\varepsilon_1\varepsilon}{1+t_0 s}+1\right)\right]\nabla^2 + \frac{qs^2}{1+t_0 s}\right)\bar{\phi} = 0, \tag{5.10}$$

which can be rewritten as

$$(\nabla^2 - m_1^2)(\nabla^2 - m_2^2)\bar{\phi} = 0, \tag{5.11}$$

where m_1^2 and m_2^2 are the roots of the equation

$$m^4 - \left[\frac{s^2}{1+t_0 s} + q\left(\frac{\varepsilon_1\varepsilon}{1+t_0 s}+1\right)\right]m^2 + \frac{qs^2}{1+t_0 s} = 0. \tag{5.12}$$

The roots of the above characteristic equation are given by

$$m_1^2 = \tfrac{1}{2}[2A + \sqrt{A^2-4B}], \qquad m_2^2 = \tfrac{1}{2}[2A - \sqrt{A^2-4B}], \tag{5.13}$$

where

$$A = \frac{s^2 + q\varepsilon_1\varepsilon}{1+t_0 s} + q, \qquad B = \frac{qs^2}{1+t_0 s}. \tag{5.14}$$

Equation (5.11) leads to the modified Bessel equation for $\bar{\phi}$ of order zero

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - m_1^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - m_2^2\right)\bar{\phi} = 0. \tag{5.15}$$

The solutions of Eq. (5.15), under the regularity conditions that u, θ, $\psi \to 0$ as $r \to \infty$, can be written as

$$\bar{\phi} = \sum_{i=1}^{2} A_i K_0(m_i r), \tag{5.16}$$

where K_0 is the modified Bessel function of the first kind of order zero and A_i $(i = 1, 2)$ are two parameters depending on the parameter s of the Laplace transform. Using Eqs. (5.8) and (5.16) we obtain

$$\frac{\varepsilon_1}{1+t_0 s}\bar{\psi} = \sum_{i=1}^{2} (m_i^2 - s^2) A_i K_0(m_i r). \tag{5.17}$$

In addition, according to Eqs. (5.7) and (5.16) the radial displacement will be

$$\bar{u} = -\sum_{i=1}^{2} m_i A_i K_1(m_i r), \tag{5.18}$$

where K_1 is the modified Bessel function of the first kind of order one. The well-known relations of the Bessel function

$$xK_n'(x) = -xK_{n\pm1}(x) \pm nK_n(x), \tag{5.19}$$

are used to derive the stresses with the aid of the displacement $\bar{u}$ and the function $\bar{\psi}$. They are given by

$$\bar{\sigma}_{rr} = \sum_{i=1}^{2} \left(s^2 K_0(m_i r) + \frac{m_i(1-c_1)}{r} K_1(m_i r) \right) A_i, \tag{5.20}$$

$$\bar{\sigma}_{\xi\xi} = \sum_{i=1}^{2} \left([s^2 + m_i^2(c_1-1)]K_0(m_i r) + \frac{m_i(c_1-1)}{r} K_1(m_i r) \right) A_i, \tag{5.21}$$

$$\begin{aligned} \bar{\sigma}_{zz} = \sum_{i=1}^{2} \bigg[& \left(\frac{m_i^2 c_3}{2} - \frac{\varepsilon_6}{\varepsilon_1}(m_i^2 - s^2) \right) K_0(m_i r) \\ & - \frac{m_i c_4}{r} K_1(m_i r) + \frac{m_i^2 c_3}{2} K_2(m_i r) \bigg] A_i, \end{aligned} \tag{5.22}$$

where K_2 is the modified Bessel function of the first kind of order two. The boundary conditions, after using the Laplace transform and Eqs. (4.4) and (4.5), are

$$\psi(R, s) = \theta_0 \left(\frac{1}{s} + \frac{k_1}{2s} \right) = \bar{G}(s), \tag{5.23}$$

$$\bar{\sigma}_{rr}(R, s) = 0. \tag{5.24}$$

The substitution of Eqs. (5.17) and (5.20) into the above conditions gives two equations with the unknown parameters A_i as

$$\sum_{i=1}^{2} (m_i^2 - s^2) A_i K_0(m_i R) = \frac{\varepsilon_1}{1+t_0 s} \bar{G}(s), \tag{5.25}$$

$$\sum_{i=1}^{2} \left(s^2 K_0(m_i R) + \frac{m_i(1-c_1)}{R} K_1(m_i R) \right) A_i = 0. \tag{5.26}$$

Therefore, the solution of the problem will be completed in the Laplace transform domain. In addition, the temperature $\bar{\theta}$ can be obtained by solving Eq. (3.3) after applying the Laplace transform as

$$\bar{\theta}(r, s) = \frac{-1 + \sqrt{1 + 2k_1\bar{\psi}}}{k_1}. \tag{5.27}$$

6. Numerical results and discussion

Here, the distributions of the field quantities such as temperature, radial displacement, and stresses will be obtained inside the medium in their inverted forms. To invert the Laplace transform in Eqs. (5.18) and (5.20)–(5.22), a numerical inversion method based on a Fourier series expansion [8, 9] should be adopted. Any expression in Laplace domain can be inverted in this method to the time domain as

$$f(t) = \frac{e^{ct}}{t}\left(\frac{1}{2}\bar{f}(c) + \mathrm{Re}\sum_{n=1}^{N}(-1)^n \bar{f}(c + in\pi/t)\right). \tag{6.1}$$

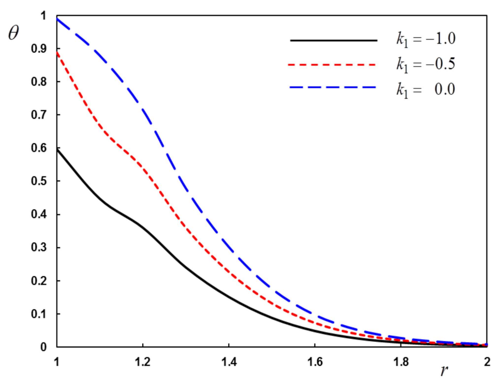
The value of c should satisfy the relation $ct \approx 4.7$ as mentioned in numerous numerical experiments [35]. So, we will use the same value of c for the purpose of numerical evaluation. Numerical evaluations are made by choosing an orthotropic material such as *cobalt*. The properties of such material are thus given in SI units [36] as

$$\begin{aligned}
&c_{11} = 3.071 \times 10^{11}\ \mathrm{N\cdot m^{-1}}, \quad c_{12} = 1.650 \times 10^{11}\ \mathrm{N\cdot m^{-1}},\\
&c_{22} = 3.071 \times 10^{11}\ \mathrm{N\cdot m^{-1}}, \quad T_0 = 298\ \mathrm{K}, \quad \rho = 8836\ \mathrm{kg\cdot m^{-3}},\\
&C_E = 427\ \mathrm{J\cdot kg^{-1}\cdot K^{-1}}, \qquad K_r = 69\ \mathrm{W\cdot m^{-1}\cdot K^{-1}\cdot s^{-1}},\\
&\beta_{11} = \beta_{22} = 7.04 \times 10^{6}\ \mathrm{N\cdot m^{-2}K^{-1}}, \quad \beta_{33} = 6.90 \times 10^{6}\ \mathrm{N\cdot m^{-2}\cdot K^{-1}}.
\end{aligned} \tag{6.2}$$

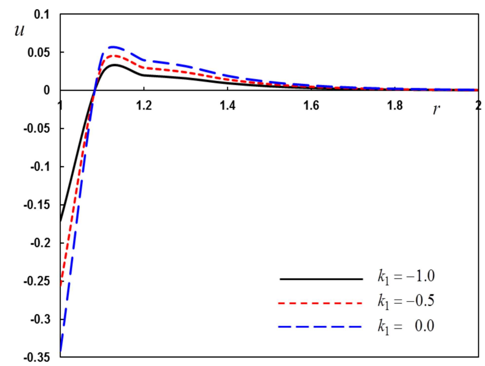
The numerically computed non-dimensional temperature θ, radial displacement u and distributions of thermal stresses σ_{rr}, $\sigma_{\xi\xi}$ and σ_{zz} have been presented graphically for thermoviscoelastic (TVE) and thermoelastic (TE) cylinders at different values of R, $(R \geq 1)$.

The results are graphically presented in Figs. 1–3 for three cases. The first case is devoted to discussing non-dimensional temperature, displacement and thermal stresses with the variable thermal conductivity parameter k_1 when τ_q and τ_θ remain constant ($\tau_q = 0.2$, $\tau_\theta = 0.1$). Three different values of k_1 for viscous solids are used ($t_0 = 0.1$). The values $k_1 = -1, -0.5$ are taken for variable thermal conductivity and $k_1 = 0$ for temperature-independent thermal conductivity. Variations of the spatial coordinate r can be observed in Fig. 1. It should be noted that the parameter k_1 has significant effects on all field quantities.The following important facts can also be observed:

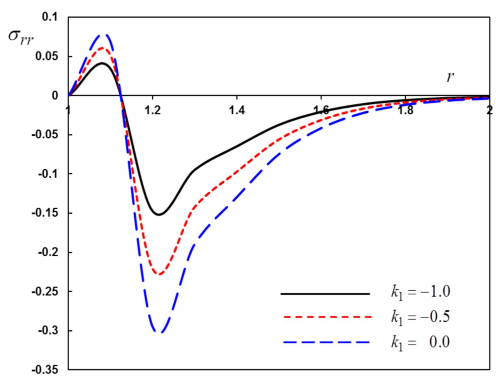
- Figure 1a shows that the temperature θ decreases along the radial direction. It also decreases as k_1 decreases.
- Figure 1b shows that variations of the radial displacement u start with negative values in all cases, and u increases continuously to attain its highest values at $r \approx 1.12$ and then it gradually diminishes to zero. The displace-



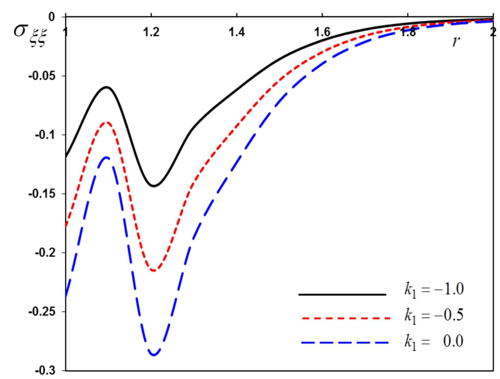
(a) θ versus r



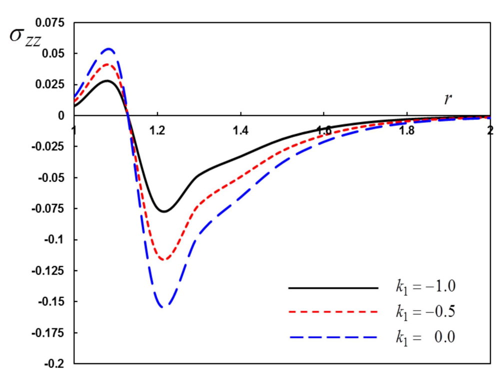
(b) u versus r



(c) σ_{rr} versus r



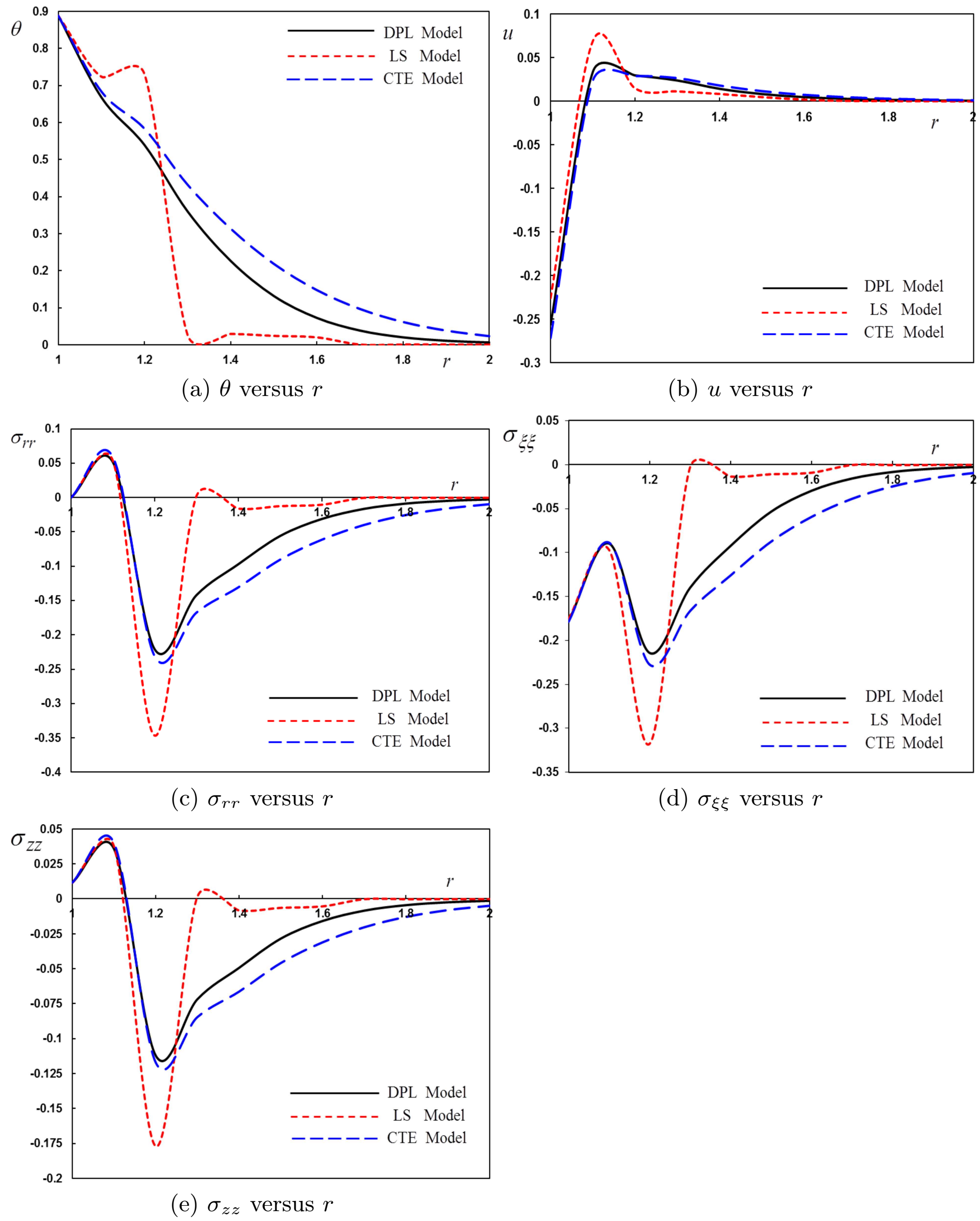
(d) $\sigma_{\xi\xi}$ versus r



(e) σ_{zz} versus k_1

Fig. 1. Effects of the thermal conductivity parameter k_1 on the field quantities along the radial direction of the cylindrical hole.

ment u vanishes twice, firstly at $r \approx 1.09$ and secondly at $r = 2$. As k_1 decreases the displacement u increases in the interval $1 \leq r \leq 1.09$ and decreases in the interval $1.09 < r \leq 2$.



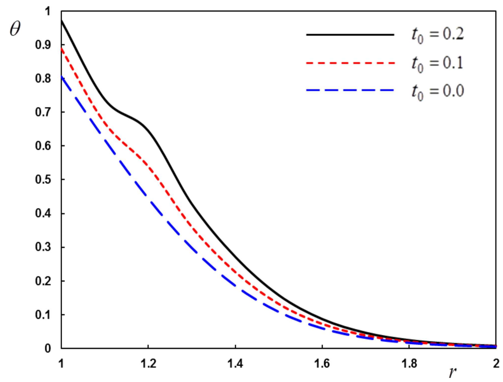
(a) θ versus r (b) u versus r

(c) σ_{rr} versus r (d) $\sigma_{\xi\xi}$ versus r

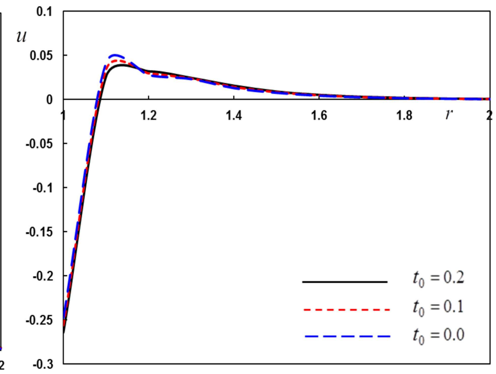
(e) σ_{zz} versus r

Fig. 2. Distributions of the field quantities along the radial direction of the cylindrical hole r for different theories of thermoelasticity.

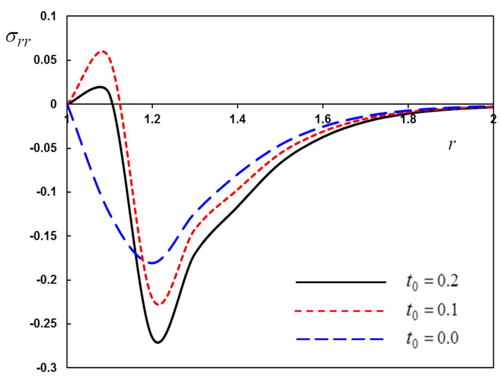
- Figure 1c shows that variations of the thermal stress σ_{rr} start with a zero value at $r = 1$ for all cases which agree with the boundary condition. The thermal stress σ_{rr} continuously increases to attain its highest values at



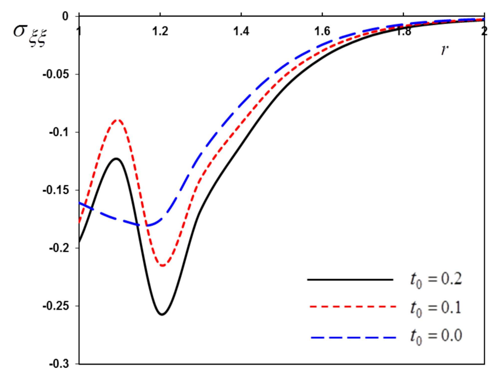
(a) θ versus r



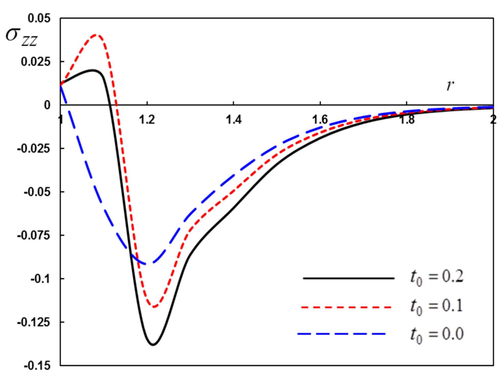
(b) u versus r



(c) σ_{rr} versus r



(d) $\sigma_{\xi\xi}$ versus r



(e) σ_{zz} versus r

FIG. 3. Effects of the viscosity parameter t_0 on the field quantities along the radial direction of the cylindrical hole.

$r \approx 1.07$ then it decreases to attain its lowest values at $r \approx 1.22$. It should be noted that the increase of k_1 increases the magnitude of the wave of thermal stress σ_{rr}.

- The thermal stress $\sigma_{\xi\xi}$ starts with negative values and continuously vibrates along the radial direction as shown in Fig. 1d. It can also be noticed that $\sigma_{\xi\xi}$ increases as the parameter k_1 decreases.
- The thermal stress σ_{zz} has similar behavior as that of σ_{rr} with different magnitudes as shown in Fig. 1e.
- The two stresses σ_{rr} and σ_{zz} are compressive at some part of the cylinder and tensile at another portion of it. However, the hoop stress $\sigma_{\xi\xi}$ is always compressive.
- The variable thermal conductivity parameter k_1 has a significant effect on all the fields which add importance to our consideration about thermal conductivity being variable.

The second case is devoted to the investigation of non-dimensional temperature, displacement and thermal stresses versus the PLs τ_q and τ_θ when the variable thermal conductivity parameter k_1 remains constant ($k_1 = -0.5$). The variations of temperature change θ, radial displacement u and thermal stresses σ_{rr}, $\sigma_{\xi\xi}$ and σ_{zz} are respectively plotted in Fig. 2 for different theories of thermoelasticity obtained as special cases of the present DPL model. We have the following theories: the coupled theory (CT) ($\tau_q = \tau_\theta = 0$), the Lord and Shulman (LS) theory ($\tau_\theta = 0$, $\tau_q = 0.2$) and the generalized theory of thermoelasticity proposed by Tzou (DPL) ($\tau_q = 0.2$, $\tau_\theta = 0.1$). From these figures it can be observed that:

- The fact that thermal waves in the coupled theory travel with an infinite speed of propagation as opposed to a finite speed in the generalized case is satisfied here. The coupled and generalized thermoelasticity theories give very close results near the surface of the cylinder where the boundary conditions dominate. However, the behaviors inside the cylinder are markedly different.
- All figures show that the variations of all field quantities in the context of the DPL and CT theories of thermoelasticity follow similar trends while the LS theory may be different.
- The difference between the three curves at any fixed point for the three theories is clearly visible in these figures.
- The fact that in generalized thermoelasticity theories (DPL and LS), the waves propagate with finite speeds is evident.
- With an increase in distance, the results are quite close to each other, which is in agreement with the generalized theories of thermoelasticity.
- The effects of the DPL parameters are very noticeable in the distributions of field quantities.

Finally, the third case is devoted to discussing how non-dimensional temperature, displacement and stresses vary with mechanical relaxation time due to the viscosity t_0 when $\tau_q = 0.2$, $\tau_\theta = 0.1$ and $k_1 = -0.5$. Comparisons of the dimensionless physical quantities are shown in Fig. 3 for two different cases:

(i) a thermoviscoelastic solid (TVE) when $t_0 = 0.2, 0.1$ and (ii) a thermoelastic solid (TE) when $t_0 = 0.0$. We can also observe the following important facts from Fig. 3:

- The influence of the viscosity parameter is very pronounced for temperature and thermal stresses.
- It can be seen in Fig. 3a that the viscosity parameter increases the magnitude of the temperature distribution. The temperature distribution in TE case has small behavior as compared to TVE case.
- In Fig. 3b we can see that, when the value of viscosity increases, the absolute values of the radial displacement u decrease and the peak occurs when $r = 1.18$.
- In Figs. 3c and 3e, when the value of the viscosity parameter increases, the absolute values of the stresses σ_{rr} and σ_{zz} increase along the radial direction.
- The difference in the values of $\sigma_{\xi\xi}$ at a particular point for three different values of viscosity parameter can easily be observed in Fig. 3d.
- All of the stress distributions in TE case have different behaviors than those of TVE case.

7. Conclusions

In this work, we constructed, based on the DPL model, the equations of generalized thermo-viscoelasticity for a homogeneous orthotropic infinite unbounded body containing a cylindrical cavity with variable thermal conductivity. The outer surface of the body was taken to be traction-free and subjected to a time-dependent thermal shock. The numerical solution of the problem has been presented with the aid of the Laplace transform technique. Results for all fields have been graphically presented. Comparisons between thermoelasticity theories have been made and the effects of different parameters were discussed. It can be observed that the viscous effect plays an important role and its effect is more pronounced in thermo-viscoelasticity. The variable thermal conductivity parameter has a significant effect on speed of the wave propagation in all fields. In the generalized thermoelasticity theory with phase-lags heat propagates in the medium as a wave with finite velocity instead of infinite velocity. The phase-lag of the heat flux and phase-lag of the temperature gradient have great influence on the field quantities. The results presented in this paper should prove useful to researchers in the development of the mechanics of solids, as well as to researchers in material science, designers of new materials, low temperature physicists and those working on the development of a theory of hyperbolic thermoelasticity. In addition, the results presented here may provide interesting information for experimental scientists and researchers working on this subject.

References

1. M.A. Biot, *Thermoelasticity and irreversible thermodynamics*, Journal of Applied Physics, **27**, 240–253, 1956.
2. H.W. Lord, Y. Shulman, *A generalized dynamical theory of thermoelasticity*, Journal of the Mechanics and Physics of Solids, **15**, 299–307, 1967.
3. A.E. Green, K.A. Lindsay, *Thermoelasticity*, Journal of Elasticity, **2**, 1–7, 1972.
4. D.S. Chandarasekharaia, *Hyperbolic thermoelasticity: a review of recent literature*, Applied Mechanics Reviews, **51**, 705–729, 1998.
5. M. Aouadi, *Generalized thermoelastic-piezoelectric problem by hybrid Laplace transform-finite element method*, International Journal for Computational Methods in Engineering Science and Mechanics, **8**, 137–147, 2007.
6. R.B. Hetnarski, M.R. Eslami, *Thermal Stresses–Advanced Theory and Applications*, Springer Science & Business Media, **158**, 2009.
7. A.M. Zenkour, *Three-dimensional thermal shock plate problem within the framework of different thermoelasticity theories*, Composite Structures, **132**, 1029–1042, 2015.
8. D.Y. Tzou, *A unified approach for heat conduction from macro- to micro-scales*, Journal of Heat Transfer, **117**, 8–16, 1995.
9. D.Y. Tzou, *Macro- to Microscale Heat Transfer: the Lagging Behavior, Series in Chemical and Mechanical Engineering*, Taylor & Francis, Washington, DC, 1997.
10. D.Y. Tzou, *Experimental support for the Lagging behavior in heat propagation*, Journal of Thermophysics Heat Transfer, **9**, 686–693, 1995.
11. S.K. Roychoudhuri, *One-dimensional thermoelastic waves in elastic half-space with dual-phase-lag effects*, Journal of Mechanics of Materials and Structures, **2**, 489–503, 2007.
12. S.K. Roychoudhuri, *On a thermoelastic three-phase-lag model*, Journal of Thermal Stresses, **30**, 231–238, 2007.
13. A.H. Akbarzadeh, D. Pasini, *Phase-lag heat conduction in multilayered cellular media with imperfect bonds*, International Journal of Heat and Mass Transfer, **75**, 656–667, 2014.
14. R. Lakes, *Viscoelastic Materials*, Cambridge University Press, UK, 2009.
15. R. Quintanilla, *Existence and exponential decay in the linear theory of viscoelastic mixtures*, European Journal of Mechanics A/Solids, **24**, 311–324, 2005.
16. D. Iesan, L. Nappa, *On the theory of viscoelastic mixtures and stability*, Mathematics and Mechanics of Solids, **13**, 55–80, 2008.
17. M.M. Svanadze, *Potential method in the linear theories of viscoelasticity and thermoviscoelasticity for Kelvin–Voigt materials*, Technische Mechanik, **32**, 554–563, 2012.
18. A.A. Ilioushin, B.E. Pobedria, *Fundamentals of the Mathematical Theory of Thermal Viscoelasticity*, Nauka, Moscow, 1970.
19. R.I. Tanner, *Engineering Rheology*, Oxford University Press, Oxford, 1988.
20. R. Huilgol, N. Phan-Thien, *Fluid Mechanics of Viscoelasticity*, Elsevier, Amsterdam, 1997.
21. A.D. Kovalenko, V.G. Karanaukhov, *A linearized theory of thermoviscoelasticity*, Polymer Mechanics, **8**, 194–199, 1972.
22. A.D. Drozdov, *A constitutive model in finite thermoviscoelasticity based on the concept of transient networks*, Acta Mechanica, **133**, 13–37, 1999.

23. M.R. Kundu, B. Mukhopadhyay, *A thermoviscoelastic problem of an infinite medium with a spherical cavity using generalized theory of thermoelasticity*, Mathematical and Computer Modelling, **41**, 25–32, 2005.

24. A. Baksi, B.K. Roy, R.K. Bera, *Eigenvalue approach to study the effect of rotation and relaxation time in generalized magneto-thermo-viscoelastic medium in one dimension*, Mathematical and Computer Modelling, **44**, 1069–1079, 2006.

25. M. Kanoria, S.H. Mallik, *Generalized thermoviscoelastic interaction due to periodically varying heat source with three-phase-lag effect*, European Journal of Mechanics A/Solids, **29**, 695–703, 2010.

26. M.A. Ezzat, A.S. El-Karamany, A.A. El-Bary, M.A. Fayik, *Fractional calculus in one-dimensional isotropic thermo-viscoelasticity*, Comptes Rendus Mecanique, **341**, 553–566, 2013.

27. A. Kar, M. Kanoaria, *Generalized thermo-visco-elastic problem of a spherical shell with three-phase-lag effect*, Applied Mathematical Modelling, **33**, 3287–3298, 2009.

28. S. Deswal, K.K. Kalkal, *Three-dimensional half-space problem within the framework of two-temperature thermo-viscoelasticity with three-phase-lag effects*, Applied Mathematical Modelling, In press, 2015.

29. A.E. Abouelregal, *Generalized thermoelasticity for an isotropic solid sphere indual-phase-lag of heat transfer with surface heat flux*, International Journal for Computational Methods in Engineering Science and Mechanics, **12**, 96–105, 2011.

30. A.M. Zenkour, D.S. Mashat, A.E. Abouelregal, *The effect of dual-phase-lag model on reflection of thermoelastic waves in a solid half space with variable material properties*, Acta Mechanica Solida Sinica, **26**, 659–670, 2013.

31. I.A. Abbas, A.M. Zenkour, *Dual-phase-lag model on thermoelastic interactions in a semi-infinite medium subjected to a ramp-type heating*, Journal of Computational and Theoretical Nanoscience, **11**, 642–645, 2014.

32. A.E. Abouelregal, A.M. Zenkour, *Effect of phase lags on thermoelastic functionally graded microbeams subjected to ramp-type heating*, IJST, Transactions of Mechanical Engineering, **38**, 321–335, 2014.

33. A.C. Eringen, *Mechanics of Continua*, Wiley, New York, 1967.

34. N. Noda, *Thermal stresses in materials with temperature-dependent properties, Thermal Stresses I*, R.B. Hetnarski (Editor), North-Holland, Amsterdam, 1986.

35. G. Honig, U. Hirdes, *A method for the numerical inversion of Laplace transform*, Journal of Computational and Applied Mathematics, **10**, 113–132, 1984.

36. J.C. Misra, N.C. Chattopadhyay, S.C. Samanta, *Study of the thermoelastic interactions in an elastic half space subjected to a ramp-type heating—a state-space approach*, International Journal of Engineering Sciences, **34**, 579–596, 1996.

Received February 18, 2015; revised version November 7, 2015.