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Fundamental solution in elasto-thermodiffusive (ETNP) semiconductor materials

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In this paper, the fundamental solution of system of differential equations in the theory of elasto-thermodiffusive(ETNP) semiconductor materials in case of steady oscillations in terms of elementary functions is constructed. Some basic properties of the fundamental solution are also established.

Key words: fundamental solution, elasto-thermodiffusion, steady oscillations.

1. Introduction

In the classical theory of thermoelasticity, Fourier's heat conduction theory assumes that the thermal disturbances propagate at infinite speed which is unrealistic from the physical point of view. Two different generalizations of the classical theory of thermoelasticity have been developed which predict only finite velocity of propagation for heat and displacement fields. The first one is given by Lord and Shulman [1] which incorporates a flux rate term into the Fourier law of heat conduction and formulates a generalized theory admitting finite speed for thermal signals. The second is given by Green and Lindsay [2] which develops a temperature-rate-dependent thermoelasticity by including temperature rate among the constitutive variables, which does not violate the classical Fourier law of heat conduction. Lord and Shulman [1] theory of generalized thermoelasticity have been further extended to homogeneous anisotropic heat conducting materials recommended by Dhaliwal and Sherief [3]. All these theories predict a finite speed of heat propagation. Chandrasekhariah [4] refers to this wave-like thermal disturbance as second sound. A survey article of various representative theories in the range of generalized thermoelasticity have been brought out by HETNARSKI and IGNACZAK [5].

The interaction of elastic, thermal and diffusion of charge carrier's fields in semiconductors has been investigated after formulating the problem mathematically by MARUSZEWSKI [6]–[10] and MANY *et al.* [11]. The theory developed in these researches is phenomenological by its nature and its application led to phonons of mixed nature, which cannot be considered of a pure transversal, longitudinal or interface character. This provides a description of optical phonons in semiconductors of different kinds, achieving a relatively good coincidence with both experimental data and calculation based on microscopic (atomistic) approaches. In case of semiconductors, the presence of coupled oscillations together with uncoupled one of pure mechanical nature was also found. It is also mentioned that in the uncoupled cases, the velocities for thermal fields are observed at very low temperatures; then, for diffusion, the velocities are observed in semiconductors at room temperatures dealing with charge carriers. In order to explore the simultaneous interactions of elastic, thermal and diffusion of charge carrier's fields, MARUSZEWSKI [10] studied the propagation of thermodiffusive surface waves in semiconductor materials based on the phenomenological model developed by him that includes relaxation times of heat and charge carriers in addition to life times of the carriers. He also presented numerical solutions of his model under these specific situations. But his investigations were limited to some special and particular situations and remained departed from the general solution of the said model thereby ignoring the presence of some of the interacting fields included in the basic governing equations at a time. SHARMA and THAKUR [12] simplified the MARUSZEWSKI [10] model of governing equations by introducing non-dimensional quantities and studied the propagation of plane harmonic elasto-thermodiffusive (ETNP) waves in semiconductor materials.

To investigate the boundary value problems of the theory of elasticity and thermoelasticity by potential method, it is necessary to construct a fundamental solution of systems of partial differential equations and to establish their basic properties respectively. HETNARSKI [13, 14] was the first to study the fundamental solutions in the classical theory of coupled thermoelasticity. IESAN [15] presented the fundamental solution in the theory of thermoelasticity without energy dissipation. The fundamental solutions in the microcontinuum fields theories have been constructed by SVANADZE [16]–[19]. The information related to fundamental solutions of differential equations is contained in the books of HÖRMANDER [20, 21].

In this article, the fundamental solutions of system of equations in case of steady oscillations in the theory of elasto-thermodiffusive (ETNP) semiconductor material are constructed by means of elementary functions and basic properties are established.

2. Basic equations

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3, $|\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, $\boldsymbol{D}_{\boldsymbol{x}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ and let t denote the time variable. Following MARUSZEWSKI [10], the basic equations for homogeneous isotropic thermoelastic semiconductor material in the absence of body forces and heat sources are:

$$\mu\Delta\bar{\boldsymbol{u}} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\bar{\boldsymbol{u}} - \lambda^n\,\mathrm{grad}\,\bar{N} - \lambda^p\,\mathrm{grad}\,\bar{P} - \lambda^T\,\mathrm{grad}\,\bar{T} = \rho\ddot{\bar{\boldsymbol{u}}}, \tag{2.1}$$

$$\begin{aligned} &K\Delta\bar{T} + m^{nq}\Delta\bar{N} + m^{pq}\Delta\bar{P} \\ &- \left(1 + t^Q\frac{\partial}{\partial t}\right)(\rho C_e\dot{\bar{T}} + \rho T_0\alpha^n\dot{\bar{N}} + \rho T_0\alpha^p\dot{\bar{P}} + T_0\lambda^T\,\mathrm{div}\,\dot{\bar{\boldsymbol{u}}}) - \rho(a_1^n\dot{\bar{N}} + a_1^p\dot{\bar{P}}) \\ &= \left(a_1^n\left(\frac{\rho}{t_n^+}\right)\bar{N} + a_1^p\left(\frac{\rho}{t_p^+}\right)\bar{P}\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\rho D^n\Delta\bar{N} + m^{qn}\Delta\bar{T} - \rho\left[1 - a_2^nT_0\alpha^n + t^n\frac{\partial}{\partial t}\right]\dot{\bar{N}} \\ &- a_2^n(\rho C_e\dot{\bar{T}} + \rho T_0\alpha^p\dot{\bar{P}} + T_0\lambda^T\,\mathrm{div}\,\dot{\bar{\boldsymbol{u}}}) \\ &= -\left(1 + t^n\frac{\partial}{\partial t}\right)\left(\frac{\rho}{t_n^+}\right)\bar{N}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\rho D^p\Delta\bar{P} + m^{qp}\Delta\bar{T} - \rho\left[1 - a_2^pT_0\alpha^p + t^p\frac{\partial}{\partial t}\right]\dot{\bar{P}} \\ &- a_2^p(\rho C_e\dot{\bar{T}} + \rho T_0\alpha^n\dot{\bar{N}} + T_0\lambda^T\,\mathrm{div}\,\dot{\bar{\boldsymbol{u}}}) = -\left(1 + t^p\frac{\partial}{\partial t}\right)\left(\frac{\rho}{t_p^+}\right)\bar{P}, \end{aligned} \tag{2.4}$$

where

$$a_1^n = \frac{a^{Qn}}{a^Q}, \quad a_1^p = \frac{a^{Qp}}{a^Q}, \quad a_2^n = \frac{a^{Qn}}{a^n}, \quad a_2^p = \frac{a^{Qp}}{a^p},$$
$$\bar{P} = p - p_0, \quad \bar{N} = n - n_0, \quad T = T_1 - T_0, \quad \lambda^T = (3\lambda + 2\mu)\alpha_T.$$

Here $\bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ is the displacement vector, ρ, C_e are, respectively, the density of the semiconductor and specific heat, λ, μ are Lame's constants, T_0 is the reference temperature assumed to be such that $\left|\bar{T}/T_0\right| \ll 1$, $\bar{T}$ is the temperature change, K is the thermal conductivity, α_T is the coefficient of linear thermal expansion of the material, λ^n, λ^p are the elastodiffusive constants of electrons and holes, α^n, α^p are thermodiffusive constants of holes and electrons, a^{Qn}, a^Q, a^n, a^p are the flux-like constants, D^n, D^p are the diffusion coefficients of electrons and holes, $m^{nq}, m^{qn}, m^{pq}, m^{qp}$ are the Peltier–Seeback–Dufour–Soret-like constants, t^Q, t^n, t^p are the relaxation times of heat, electron and hole fields, t_n^+, t_p^+ denote the life times of carriers fields, n, p are the non-equilibrium and

n_0, p_0 equilibrium values of electrons and holes concentrations, respectively, Δ is the Laplacian operator. The comma notation is used for spatial derivatives and a superposed dot represents differentiation with respect to time. In addition, the field variables are also assumed here to satisfy all restrictions as described by Maruszewski [10].

We define the dimensionless quantities:

$$
\begin{aligned}
&\boldsymbol{x}' = \frac{w^*\boldsymbol{x}}{c_1}, \quad \bar{\boldsymbol{u}}' = \frac{\rho w^* c_1 \bar{\boldsymbol{u}}}{\lambda^T T_0}, \quad \bar{T}' = \frac{\bar{T}}{T_0}, \quad \bar{N}' = \frac{\bar{N}}{n_0}, \quad \bar{P}' = \frac{\bar{P}}{p_0},\\
&t^{Q'} = w^* t^Q, \quad t^{n'} = w^* t^n, \quad t_n^{+'} = w^* t_n^+, \quad t' = w^* t, \quad t^{p'} = w^* t^p,\\
&t_p^{+'} = w^* t_p^+, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad w^* = \frac{\rho C_e c_1^2}{K}, \quad \delta^2 = \frac{\mu}{\lambda + 2\mu},\\
&\epsilon_T = \frac{\lambda^{T^2} T_0}{\rho C_e(\lambda + 2\mu)}, \quad \chi = \frac{K}{\rho C_e}, \quad \bar{\lambda}_n = \frac{\lambda^n n_0}{\lambda^T T_0}, \quad \bar{\lambda}_p = \frac{\lambda^p p_0}{\lambda^T T_0},\\
&\varepsilon^{qn} = \frac{m^{qn} T_0}{\rho D^n n_0}, \quad \varepsilon^{qp} = \frac{m^{qp} T_0}{\rho D^p p_0}, \quad \epsilon_n = \frac{a_2^n K T_0}{\rho n_0 D^n}, \quad \epsilon_p = \frac{a_2^p C_e T_0 c_1^2}{\omega^* p_0 D^p},\\
&\varepsilon^{nq} = \frac{m^{nq} n_0}{K T_0}, \quad \varepsilon^{pq} = \frac{m^{pq} n_0}{K T_0}, \quad a_0^n = \frac{a_1^n n_0}{C_e T_0}, \quad a_0^p = \frac{a_1^p p_0}{C_e T_0},\\
&\alpha_0^n = \frac{\alpha^n n_0}{C_e}, \quad \alpha_0^p = \frac{\alpha^p p_0}{C_e},
\end{aligned}
\tag{2.5}
$$

where ϵ_T is the thermoelastic coupling parameter and χ is the thermal diffusivity.

Upon introducing the quantities (2.5) into the basic equations (2.1)–(2.4), after suppressing the primes, we obtain

$$
\delta^2 \Delta \bar{\boldsymbol{u}} + (1 - \delta^2) \text{ grad div } \bar{\boldsymbol{u}} - \bar{\lambda}_n \text{ grad } \bar{N} - \bar{\lambda}_p \text{ grad } \bar{P} - \text{grad } \bar{T} = \ddot{\bar{\boldsymbol{u}}},
\tag{2.6}
$$

$$
\begin{aligned}
&- \epsilon_T(\text{ div } \dot{\bar{\boldsymbol{u}}} + t^Q \text{ div } \ddot{\bar{\boldsymbol{u}}}) + \varepsilon^{nq} \Delta \bar{N} - \left((a_0^n + \alpha_0^n)\dot{\bar{N}} + t^Q \alpha_0^n \ddot{\bar{N}} + \frac{a_0^n}{t_n^+}\bar{N} \right)\\
&\qquad + \varepsilon^{pq} \Delta \bar{P} - ((a_0^p + \alpha_0^p)\dot{\bar{P}} + t^Q \alpha_0^p \ddot{\bar{P}} + \frac{a_0^p}{t_p^+}\bar{P}) + \Delta \bar{T} - (\dot{\bar{T}} + t^Q \ddot{\bar{T}}) = 0,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&- \epsilon_T \epsilon_n \text{ div } \dot{\bar{\boldsymbol{u}}} + \Delta \bar{N} + \frac{\chi}{D^n}\left[\frac{1}{t_n^+}\bar{N} - \left(1 - \frac{\epsilon_n \alpha_0^n D^n}{\chi} - \frac{t^n}{t_n^+} \right)\dot{\bar{N}} - t^n \ddot{\bar{N}} \right]\\
&\qquad - \epsilon_n \alpha_0^p \dot{\bar{P}} - \epsilon_n \dot{\bar{T}} + \varepsilon^{qn} \Delta \bar{T} = 0,
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&- \epsilon_T \epsilon_p \text{ div } \dot{\bar{\boldsymbol{u}}} - \epsilon_p \alpha_0^n \dot{\bar{N}} + \Delta \bar{P} + \frac{\chi}{D^p}\left[\frac{1}{t_p^+}\bar{P} - \left(1 - \frac{\epsilon_p \alpha_0^p D^p}{\chi} - \frac{t^p}{t_p^+} \right)\dot{\bar{P}} - t^p \ddot{\bar{P}} \right]\\
&\qquad - \epsilon_p \dot{\bar{T}} + \varepsilon^{qp} \Delta \bar{T} = 0.
\end{aligned}
\tag{2.9}
$$

3. Steady oscillations

Now, we consider the case of steady oscillations. We assume the functions of the displacement vector, temperature change and electrons and holes concentration change as

$$(\bar{\boldsymbol{u}}(\boldsymbol{x},t),\bar{N}(\boldsymbol{x},t),\bar{P}(\boldsymbol{x},t),\bar{T}(\boldsymbol{x},t))=\mathrm{Re}[(\boldsymbol{u},N,P,T)e^{-\iota\omega t}]. \tag{3.1}$$

Using equation (3.1) in equations (2.6)–(2.9), we obtain the system of equations of steady oscillations as

$$\delta^2\Delta\boldsymbol{u}+(1-\delta^2)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}-\bar{\lambda}_n\,\mathrm{grad}\,N-\bar{\lambda}_p\,\mathrm{grad}\,P-\mathrm{grad}\,T+\omega^2\boldsymbol{u}=\boldsymbol{0}, \tag{3.2}$$

$$\omega^2\epsilon_T\tau^Q\ \mathrm{div}\ \boldsymbol{u}+[\varepsilon^{nq}\Delta+\omega^2\tau_n']N+[\varepsilon^{pq}\Delta+\omega^2\tau_p']P+[\Delta+\omega^2\tau^Q]T=0, \tag{3.3}$$

$$\iota\omega\epsilon_n\epsilon_T\Delta\ \mathrm{div}\ \boldsymbol{u}+[\omega^2\tau_n^*+\Delta]N+\iota\omega\epsilon_n\alpha_0^pP+[\varepsilon^{qn}\Delta+\iota\omega\epsilon_n]T=0, \tag{3.4}$$

$$\iota\omega\epsilon_p\epsilon_T\Delta\ \mathrm{div}\ \boldsymbol{u}+\iota\omega\epsilon_p\alpha_0^nN+[\omega^2\tau_p^*+\Delta]P+[\varepsilon^{qp}\Delta+\iota\omega\epsilon_p]T=0, \tag{3.5}$$

where

$$\tau_n'=t^Q\alpha_0^n+\iota\omega^{-1}(\alpha_0^n+a_0^n)-a_0^n\omega^{-2}/t_n^+,\qquad \tau^Q=t^Q+\iota\omega^{-1},$$
$$\tau_n^*=\frac{\chi}{D^n}\left[t^n+\iota\omega^{-1}\left(1-\frac{\epsilon_n\alpha_0^nD^n}{\chi}-\frac{t^n}{t_n^+}\right)+\frac{1}{\omega^2t_n^+}\right],$$
$$\tau_p'=t^Q\alpha_0^p+\iota\omega^{-1}(\alpha_0^p+a_0^p)-a_0^p\omega^{-2}/t_p^+,$$
$$\tau_p^*=\frac{\chi}{D^p}\left[t^p+\iota\omega^{-1}\left(1-\frac{\epsilon_p\alpha_0^pD^p}{\chi}-\frac{t^p}{t_p^+}\right)+\frac{1}{\omega^2t_p^+}\right].$$

We introduce the matrix differential operator

$$\boldsymbol{F}(\boldsymbol{D_x})=\|F_{mn}(\boldsymbol{D_x})\|_{6\times6},$$

where

$$F_{mn}(\boldsymbol{D_x})=[\delta^2\Delta+\omega^2]\delta_{mn}+(1-\delta^2)\frac{\partial^2}{\partial x_m\partial x_n},$$
$$F_{m4}(\boldsymbol{D_x})=-\bar{\lambda}_n\frac{\partial}{\partial x_m},\qquad F_{m5}(\boldsymbol{D_x})=-\bar{\lambda}_p\frac{\partial}{\partial x_m},$$
$$F_{m6}(\boldsymbol{D_x})=-\frac{\partial}{\partial x_m},\qquad F_{4n}(\boldsymbol{D_x})=\omega^2\epsilon_T\tau^Q\frac{\partial}{\partial x_n},$$
$$F_{44}(\boldsymbol{D_x})=\varepsilon^{nq}\Delta+\omega^2\tau_n',\qquad F_{45}(\boldsymbol{D_x})=\varepsilon^{pq}\Delta+\omega^2\tau_p',$$
$$F_{46}(\boldsymbol{D_x})=\Delta+\omega^2\tau^Q,\qquad F_{5n}(\boldsymbol{D_x})=\iota\omega\epsilon_n\epsilon_T\frac{\partial}{\partial x_n},$$
$$F_{54}(\boldsymbol{D_x})=\omega^2\tau_n^*+\Delta,\qquad F_{55}(\boldsymbol{D_x})=\iota\omega\epsilon_n\alpha_0^p,$$
$$F_{56}(\boldsymbol{D_x})=\varepsilon^{qn}\Delta+\iota\omega\epsilon_n,\qquad F_{6n}(\boldsymbol{D_x})=\iota\omega\epsilon_p\epsilon_T\frac{\partial}{\partial x_n},$$
$$F_{64}(\boldsymbol{D_x})=\iota\omega\epsilon_p\alpha_0^n,\qquad F_{65}(\boldsymbol{D_x})=\omega^2\tau_p^*+\Delta,$$
$$F_{66}(\boldsymbol{D_x})=\varepsilon^{qp}\Delta+\iota\omega\epsilon_p,\qquad m,n=1,2,3.$$

The system of equations (3.2)–(3.5) can be written as

$$\boldsymbol{F}(\boldsymbol{D_x})\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{0},$$

where $\boldsymbol{U} = (\boldsymbol{u}, N, P, T)$ is a six-component vector function for E^3.

We assume that

$$1 - \varepsilon^{nq}\varepsilon^{qn} - \varepsilon^{pq}\varepsilon^{qp} \neq 0, \qquad \delta \neq 0. \tag{3.6}$$

If the condition (3.6) is satisfied, then $\boldsymbol{F}$ is an elliptic differential operator [20].

Definition. The fundamental solution of the system of equations (3.2)–(3.5) (the fundamental matrix of operator $\boldsymbol{F}$) is the matrix $\boldsymbol{G}(\boldsymbol{x})=\|G_{mn}(\boldsymbol{x})\|_{6\times6}$ satisfying condition [20]

$$\boldsymbol{F}(\boldsymbol{D_x})\boldsymbol{G}(\boldsymbol{x}) = \delta(\boldsymbol{x})\boldsymbol{I}(\boldsymbol{x}), \tag{3.7}$$

where $\delta(\boldsymbol{x})$ is the Dirac delta, $\boldsymbol{I} = \|\delta_{mn}\|_{6\times6}$ is the unit matrix and $\boldsymbol{x} \,\epsilon\, \mathrm{E}^3$.

Now, we construct $\boldsymbol{G}(\boldsymbol{x})$ in terms of elementary functions.

4. Fundamental solution of a system of equations of steady oscillations

Following Maruszewski [10], we consider the system of equations

$$\begin{aligned}\delta^2\Delta\boldsymbol{u} + (1-\delta^2)\ \mathrm{grad\ div}\ \boldsymbol{u} + \omega^2\epsilon_T\tau^Q\ \mathrm{grad}\, N \\ + \iota\omega\epsilon_n\epsilon_T\ \mathrm{grad}\ P + \iota\omega\epsilon_p\epsilon_T\ \mathrm{grad}\ T + \omega^2\boldsymbol{u} = \boldsymbol{H},\end{aligned} \tag{4.1}$$

$$-\bar{\lambda}_n\ \mathrm{div}\ \boldsymbol{u} + [\varepsilon^{nq}\Delta + \omega^2\tau_n']N + [\omega^2\tau_n^* + \Delta]P + \iota\omega\epsilon_p\alpha_0^n T = L, \tag{4.2}$$

$$-\bar{\lambda}_p\ \mathrm{div}\ \boldsymbol{u} + [\varepsilon^{pq}\Delta + \omega^2\tau_p']N + \iota\omega\epsilon_n\alpha_0^p P + [\omega^2\tau_p^* + \Delta]T = M, \tag{4.3}$$

$$-\mathrm{div}\ \boldsymbol{u} + [\Delta + \omega^2\tau^Q]N + [\varepsilon^{qn}\Delta + \iota\omega\epsilon_n]P + [\varepsilon^{qp}\Delta + \iota\omega\epsilon_p]T = D, \tag{4.4}$$

where $\boldsymbol{H}$ is three-component vector function on E^3; L, M and D are scalar functions on E^3.

The system of equations (4.1)–(4.4) may be written in the form

$$\boldsymbol{F}^{\mathrm{tr}}(\boldsymbol{D_x})\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{Q}(\boldsymbol{x}), \tag{4.5}$$

where $\boldsymbol{F}^{\mathrm{tr}}$ is the transpose of matrix $\boldsymbol{F}$, $\boldsymbol{Q} = (\boldsymbol{H}, L, M, D)$ and $\boldsymbol{x} \in \mathrm{E}^3$.

Applying the operator div to equation (4.1), we obtain

$$(\Delta + \omega^2)\ \mathrm{div}\ \boldsymbol{u} + \omega^2\epsilon_T\tau^Q\Delta N + \iota\omega\epsilon_n\epsilon_T\Delta P + \iota\omega\epsilon_p\epsilon_T\Delta T = \mathrm{div}\,\boldsymbol{H}. \tag{4.6}$$

The equations (4.2)–(4.4) and (4.6) may be written in the form

$$\boldsymbol{N}(\Delta)\boldsymbol{S} = \bar{\boldsymbol{Q}}, \tag{4.7}$$

where $\boldsymbol{S} = (\operatorname{div}\boldsymbol{u}, N, P, T)$, $\bar{\boldsymbol{Q}} = (d_1, d_2, d_3, d_4) = (\operatorname{div}\boldsymbol{H}, L, M, D)$ and

$$\boldsymbol{N}(\Delta) = \|N_{mn}(\Delta)\|_{4\times 4} = \left\| \begin{matrix} \Delta+\omega^2 & \omega^2\epsilon_T\tau^Q\Delta & \iota\omega\epsilon_n\epsilon_T\Delta & \iota\omega\epsilon_p\epsilon_T\Delta \\ -\bar{\lambda}_n & \varepsilon^{nq}\Delta+\omega^2\tau'_n & \omega^2\tau^*_n+\Delta & \iota\omega\epsilon_p\alpha_0^n \\ -\bar{\lambda}_p & \varepsilon^{pq}\Delta+\omega^2\tau'_p & \iota\omega\epsilon_n\alpha_0^p & \omega^2\tau^*_p+\Delta \\ -1 & \Delta+\omega^2\tau^Q & \varepsilon^{qn}\Delta+\iota\omega\epsilon_n & \varepsilon^{qp}\Delta+\iota\omega\epsilon_p \end{matrix} \right\|_{4\times 4}. \tag{4.8}$$

Equations (4.2)–(4.4) and (4.6) may be also written as

$$\mathit{\Gamma}(\Delta)\boldsymbol{S} = \boldsymbol{\Psi}, \tag{4.9}$$

where

$$\begin{gathered} \boldsymbol{\Psi} = (\mathit{\Psi}_1, \mathit{\Psi}_2, \mathit{\Psi}_3, \mathit{\Psi}_4), \mathit{\Psi}_n = \frac{1}{M_1}\sum_{m=1}^{4} N^*_{mn} d_m, \\ \mathit{\Gamma}(\Delta) = \frac{1}{M_1}\det\boldsymbol{N}(\Delta), \qquad n = 1,2,3,4 \end{gathered} \tag{4.10}$$

and N^*_{mn} is the cofactor of the elements N_{mn} of the matrix $\boldsymbol{N}$.

From Eqs. (4.7) and (4.9), we see that

$$\mathit{\Gamma}(\Delta) = \prod_{m=1}^{4}(\Delta + \lambda_m^2),$$

where

$$\begin{aligned} \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2 &= -\frac{M_2\omega^2}{M_1}, \\ \lambda_1^2(\lambda_2^2+\lambda_3^2+\lambda_4^2)+\lambda_2^2(\lambda_3^2+\lambda_4^2)+\lambda_3^2\lambda_4^2 &= \frac{M_3\omega^4}{M_1}, \\ \lambda_1^2(\lambda_2^2\lambda_3^2+\lambda_2^2\lambda_4^2+\lambda_3^2\lambda_4^2)+\lambda_2^2\lambda_3^2\lambda_4^2 &= -\frac{M_4\omega^6}{M_1}, \\ \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 &= \frac{M_5\omega^8}{M_1}. \end{aligned}$$

Here,

$$\begin{aligned} M_1 &= 1-\varepsilon^{nq}\varepsilon^{qn}-\varepsilon^{pq}\varepsilon^{qp}, \\ M_2 &= f_1+(1+\epsilon_T)f_2+f_3+f_4, \\ M_3 &= f_5+f_6+f_7+f_8+f_9+f_{10}+f_{11}, \\ M_4 &= f_{12}+f_{13}+f_{14}+f_{15}+f_{16}, \\ M_5 &= f_{17}+f_{18}+f_{19}, \\ f_1 &= 1-\varepsilon^{nq}\varepsilon^{qn}-\varepsilon^{pq}\varepsilon^{qp}, \\ f_2 &= \tau^Q-\iota\omega^{-1}(\epsilon_p\varepsilon^{pq}+\epsilon_n\varepsilon^{nq}), \end{aligned}$$

$$
\begin{aligned}
f_3 &= (1-\varepsilon^{qn}\varepsilon^{nq})(\tau_p^* + \iota\omega^{-1}\epsilon_p\epsilon_T\bar{\lambda}_p) - \varepsilon^{qp}(\tau_p' + \bar{\lambda}_p\tau^Q\epsilon_T \\
&\quad - \iota\omega^{-1}\epsilon_n\varepsilon^{nq}(\alpha_0^p + \bar{\lambda}_p\epsilon_T)),\\
f_5 &= \varepsilon^{nq}[-\tau_p^*\varepsilon^{qn} - \iota\omega^{-1}\epsilon_n - \omega^{-2}\epsilon_n(1+\epsilon_T)(\alpha_0^p\epsilon_p + \iota\omega\tau_p^*)],\\
f_7 &= \varepsilon^{qp}[-\tau_p'(1+\tau_n^*) + \iota\omega^{-1}\epsilon_n\epsilon_T(\tau_n'\bar{\lambda}_p - \tau_p'\bar{\lambda}_n)\\
&\quad + \iota\omega^{-1}\epsilon_n\alpha_0^p(\tau_n' + \varepsilon^{nq} + \epsilon_T\tau^Q\bar{\lambda}_n) - \bar{\lambda}_p\epsilon_T\tau^Q\tau_n^*],\\
f_9 &= \tau_n^* + (1+\epsilon_T)\tau^Q\tau_n^* - \iota\omega^{-1}\epsilon_n\tau_n'(1+\epsilon_T) + \omega^{-2}\epsilon_n\epsilon_T\bar{\lambda}_n(\alpha_0^p\epsilon_p + \iota\omega\tau_p^*),\\
f_{11} &= \tau^Q + \omega^{-2}[\tau_n^*\tau_p^* + \epsilon_n\epsilon_p\alpha_0^p\alpha_0^n],\\
f_{12} &= -\varepsilon^{nq}\omega^{-2}\epsilon_n(\epsilon_p\alpha_0^p + \iota\omega\tau_p^*) + \varepsilon^{qn}(\iota\omega^{-1}\tau_p'\epsilon_p\alpha_0^n - \tau_p^*\tau_n'),\\
f_{14} &= -\iota\omega\epsilon_n\tau_n' + \tau^Q\tau_n^* - (1+\epsilon_T)\omega^{-2}\tau_n'\epsilon_n(\epsilon_p\alpha_0^p + \iota\omega\tau_p^*),\\
f_{16} &= [1+\tau^Q(1+\epsilon_T)](\tau_n^*\tau_p^* + \omega^{-2}\epsilon_n\epsilon_p\alpha_0^n\alpha_0^p),\\
f_{17} &= -\omega^{-2}\epsilon_n\tau_n'(\epsilon_p\alpha_0^p + \iota\omega\tau_p^*),\\
f_{19} &= \tau^Q[\omega^{-2}\epsilon_n\epsilon_p\alpha_0^n\alpha_0^p + \tau_n^*\tau_p^*],
\end{aligned}
$$

and f_4, f_6, f_8, f_{10}, f_{13}, f_{15}, f_{18} can be obtained from f_3, f_5, f_7, f_9, f_{12}, f_{14}, f_{17}, respectively.

Applying the operator $\Gamma(\Delta)$ to the Eq. (4.1), we get

$$\Gamma(\Delta)(\Delta + \lambda_5^2)\boldsymbol{u} = \boldsymbol{\Psi}', \tag{4.11}$$

where $\lambda_5^2 = \omega^2/\delta^2$ and

$$\boldsymbol{\Psi}' = \frac{1}{\delta^2}\{\Gamma(\Delta)\boldsymbol{H} - \operatorname{grad}[(1-\delta^2)\Psi_1 + \omega^2\epsilon_T\tau^Q\Psi_2 + \iota\omega\epsilon_n\epsilon_T\Psi_3 + \iota\omega\epsilon_p\epsilon_T\Psi_4]\}. \tag{4.12}$$

From Eqs. (4.9) and (4.11), we obtain

$$\boldsymbol{\Theta}(\Delta)\boldsymbol{U}(\boldsymbol{x}) = \hat{\boldsymbol{\Psi}}(\boldsymbol{x}), \tag{4.13}$$

where $\hat{\boldsymbol{\Psi}} = (\boldsymbol{\Psi}', \Psi_2, \Psi_3, \Psi_4)$ and

$$\boldsymbol{\Theta}(\Delta) = \|\Theta_{qn}(\Delta)\|_{6\times 6},$$

$$\Theta_{mm}(\Delta) = \Gamma(\Delta)(\Delta + \lambda_5^2), \qquad \Theta_{qn}(\Delta) = 0, \qquad \Theta_{44} = \Theta_{55} = \Theta_{66} = \Gamma(\Delta),$$

with $m = 1, 2, 3,\ q, n = 1, \ldots, 6,\ q \neq n$.

Equations (4.10) and (4.12) can be rewritten in the form

$$
\begin{aligned}
\boldsymbol{\Psi}' &= \left[\frac{1}{\delta^2}\Gamma(\Delta)\boldsymbol{J} + q_{11}(\Delta)\operatorname{grad}\operatorname{div}\right]\boldsymbol{H}\\
&\qquad + q_{21}(\Delta)\operatorname{grad} L + q_{31}(\Delta)\operatorname{grad} M + q_{41}(\Delta)\operatorname{grad} D,\\
\Psi_2 &= q_{12}(\Delta)\operatorname{div}\boldsymbol{H} + q_{22}(\Delta)L + q_{32}(\Delta)M + q_{42}(\Delta)D,\\
\Psi_3 &= q_{13}(\Delta)\operatorname{div}\boldsymbol{H} + q_{23}(\Delta)L + q_{33}(\Delta)M + q_{43}(\Delta)D,\\
\Psi_4 &= q_{14}(\Delta)\operatorname{div}\boldsymbol{H} + q_{24}(\Delta)L + q_{34}(\Delta)M + q_{44}(\Delta)D,
\end{aligned}
\tag{4.14}
$$

where $\boldsymbol{J} = \|\delta_{mn}\|_{3\times 3}$ is the unit matrix.

In Eq. (4.14), we have used the following notations:

$$q_{m1}(\Delta) = -\frac{1}{M_1\delta^2}[(1-\delta^2)N^*_{m1} + \omega^2\epsilon_T\tau^Q N^*_{m2} + \iota\omega\epsilon_n\epsilon_T N^*_{m3} + \iota\omega\epsilon_p\epsilon_T N^*_{m4}],$$
$$q_{m2}(\Delta) = \frac{N^*_{m2}}{M_1}, \qquad q_{m3}(\Delta) = \frac{N^*_{m3}}{M_1}, \qquad q_{m4}(\Delta) = \frac{N^*_{m4}}{M_1}, \qquad m = 1,2,3,4.$$

Now, from Eq. (4.14), we have

$$\hat{\boldsymbol{\Psi}}(\boldsymbol{x}) = \boldsymbol{R}^{\mathrm{tr}}(\boldsymbol{D_x})\boldsymbol{Q}(\boldsymbol{x}), \tag{4.15}$$

where

$$\begin{gathered}
\boldsymbol{R} = \|R_{vw}\|_{6\times6} = \left\| \begin{matrix} \boldsymbol{R}^{(1)} & \boldsymbol{R}^{(2)} \\ \boldsymbol{R}^{(3)} & R^{(4)} \end{matrix} \right\|_{6\times6}, \\
\boldsymbol{R}^{(r)} = \|R^{(r)}_{lm}\|_{3\times3}, \qquad R^{(4)} = \|R^{(4)}_{lm}\|_{3\times3}, \\
R^{(1)}_{lm}(\mathbf{D_x}) = \frac{1}{\delta^2}\Gamma(\Delta)\delta_{lm} + q_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_m}, \\
R^{(2)}_{lm}(\mathbf{D_x}) = q_{1,m+1}(\Delta)\frac{\partial}{\partial x_l}, \qquad R^{(3)}_{lm}(\mathbf{D_x}) = q_{m+1,1}(\Delta)\frac{\partial}{\partial x_l}, \\
R^{(4)}_{lm}(\mathbf{D_x}) = q_{l+1,m+1}(\Delta), \qquad r = 1,2,3.
\end{gathered} \tag{4.16}$$

From Eqs (4.5), (4.13) and (4.15), we obtain

$$\boldsymbol{\Theta U} = \boldsymbol{R}^{\mathrm{tr}}\boldsymbol{F}^{\mathrm{tr}}\boldsymbol{U}.$$

It implies that

$$\boldsymbol{R}^{\mathrm{tr}}\boldsymbol{F}^{\mathrm{tr}} = \boldsymbol{\Theta}, \qquad \boldsymbol{F}(\boldsymbol{D_x})\boldsymbol{R}(\boldsymbol{D_x}) = \boldsymbol{\Theta}(\Delta). \tag{4.17}$$

We assume that

$$\lambda_l^2 \neq \lambda_m^2 \neq 0, \qquad l,m = 1,\dots,5, \qquad l \neq m.$$

Let

$$\begin{gathered}
\boldsymbol{Y}(\mathbf{x}) = \|Y_{rs}(\mathbf{x})\|_{6\times6}, \qquad Y_{ll}(\mathbf{x}) = \sum_{m=1}^{5} r_{1m}\varsigma_m(\mathbf{x}), \\
Y_{44}(\mathbf{x}) = Y_{55}(\mathbf{x}) = Y_{66}(\mathbf{x}) = \sum_{m=1}^{4} r_{2m}\varsigma_m(\mathbf{x}), \\
Y_{vw}(\mathbf{x}) = 0, \qquad l = 1,2,3, \qquad v,w = 1,\dots,6, \qquad v \neq w,
\end{gathered}$$

where

$$\varsigma_m(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\exp(\iota\lambda_m|\mathbf{x}|),$$

$$r_{1m} = \prod_{l=1,\, l\neq m}^{5}(\lambda_l^2-\lambda_m^2)^{-1}, \qquad m=1,\ldots,5,$$

$$r_{2v} = \prod_{l=1,\, l\neq v}^{4}(\lambda_l^2-\lambda_v^2)^{-1}, \qquad v=1,\ldots,4.$$

We will prove the following lemma:

Lemma. *The matrix $\boldsymbol{Y}$ defined above is the fundamental matrix of operator $\boldsymbol{\Theta}(\Delta)$, that is,*

$$\boldsymbol{\Theta}(\Delta)\boldsymbol{Y}(\mathbf{x}) = \delta(\mathbf{x})\boldsymbol{I}(\mathbf{x}). \tag{4.18}$$

Proof. To prove the lemma, it is sufficient to prove that

$$\mathit{\Gamma}(\Delta)(\Delta+\lambda_5^2)Y_{11}(\mathbf{x}) = \delta(\mathbf{x}), \qquad \mathit{\Gamma}(\Delta)Y_{44}(\mathbf{x}) = \delta(\mathbf{x}). \tag{4.19}$$

Consider

$$r_{11}+r_{12}+r_{13}+r_{14}+r_{15} = \frac{z_1-z_2+z_3-z_4+z_5}{z_6},$$

where

$$\begin{aligned}
z_1 &= (\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2),\\
z_2 &= (\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_5^2)(\lambda_3^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2),\\
z_3 &= (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2),\\
z_4 &= (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2),\\
z_5 &= (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_3^2-\lambda_4^2),\\
z_6 &= (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)\\
&\quad\times(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2).
\end{aligned}$$

Upon simplifying the right-hand side of the above relation, we obtain

$$r_{11}+r_{12}+r_{13}+r_{14}+r_{15} = 0. \tag{4.20}$$

Similarly, we find that

$$r_{12}(\lambda_1^2-\lambda_2^2)+r_{13}(\lambda_1^2-\lambda_3^2)+r_{14}(\lambda_1^2-\lambda_4^2)+r_{15}(\lambda_1^2-\lambda_5^2) = 0, \tag{4.21}$$

$$r_{13}(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)+r_{14}(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_4^2)+r_{15}(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)=0, \tag{4.22}$$

$$r_{14}(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_4^2)(\lambda_3^2-\lambda_4^2)+r_{15}(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2)=0. \tag{4.23}$$

Also,

$$\begin{aligned} r_{15}(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2) \\ = \frac{(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2)}{(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2)} = 1, \end{aligned} \tag{4.24}$$

$$(\Delta+\lambda_l^2)\varsigma_m(\mathbf{x}) = \delta(\mathbf{x}) + (\lambda_l^2-\lambda_m^2)\varsigma_m(\mathbf{x}), \qquad l,m=1,\ldots,5. \tag{4.25}$$

Now, let us consider the following:

$$\begin{aligned} &\Gamma(\Delta)(\Delta+\lambda_5^2)Y_{11}(\mathbf{x}) \\ &= (\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=1}^{5} r_{1m}\varsigma_m(\mathbf{x}) \\ &= (\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=1}^{5} r_{1m}[\delta(\mathbf{x})+(\lambda_1^2-\lambda_m^2)\varsigma_m(\mathbf{x})] \\ &= (\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)[\delta(\mathbf{x})\sum_{m=1}^{5} r_{1m} + \sum_{m=2}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)\varsigma_m(\mathbf{x})] \end{aligned} \tag{4.26}$$

Using Eq. (4.20) in the above relation (4.26), we obtain

$$\begin{aligned} &\Gamma(\Delta)(\Delta+\lambda_5^2)Y_{11}(\mathbf{x}) \\ &\quad = (\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=2}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)\varsigma_m(\mathbf{x}) \\ &\quad = (\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=2}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)[\delta(\mathbf{x})+(\lambda_2^2-\lambda_m^2)\varsigma_m(\mathbf{x})] \\ &\quad = (\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=3}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)(\lambda_2^2-\lambda_m^2)\varsigma_m(\mathbf{x}) \\ &\quad = (\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=3}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)(\lambda_2^2-\lambda_m^2)[\delta(\mathbf{x})+(\lambda_3^2-\lambda_m^2)\varsigma_m(\mathbf{x})] \\ &\quad = (\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{m=4}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)(\lambda_2^2-\lambda_m^2)(\lambda_3^2-\lambda_m^2)\varsigma_m(\mathbf{x}) \\ &\quad = (\Delta+\lambda_5^2)\sum_{m=4}^{5} r_{1m}(\lambda_1^2-\lambda_m^2)(\lambda_2^2-\lambda_m^2)(\lambda_3^2-\lambda_m^2)[\delta(\mathbf{x})+(\lambda_4^2-\lambda_m^2)\varsigma_m(\mathbf{x})] \\ &\quad = (\Delta+\lambda_5^2)\varsigma_5(\mathbf{x}) = \delta(\mathbf{x}). \end{aligned}$$

Similarly to Eqs (4.20)–(4.24), we find that

$$r_{21} + r_{22} + r_{23} + r_{24} = 0, \tag{4.27}$$

$$r_{21}(\lambda_2^2 - \lambda_1^2) + r_{23}(\lambda_2^2 - \lambda_3^2) + r_{24}(\lambda_2^2 - \lambda_4^2) = 0, \tag{4.28}$$

$$r_{23}(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2) + r_{24}(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2) = 0, \tag{4.29}$$

$$r_{24}(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2) = 1. \tag{4.30}$$

Now, we consider the second part of Eq. (4.19), that is,

$$\begin{aligned}
\mathit{\Gamma}(\Delta)Y_{44}(\mathbf{x}) &= (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2) \sum_{m=1}^{4} r_{2m}\varsigma_m(\mathbf{x}) \\
&= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2) \sum_{m=1}^{4} r_{2m}[\delta(\mathbf{x}) + (\lambda_1^2 - \lambda_m^2)\varsigma_m(\mathbf{x})] \\
&= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2) \sum_{m=2}^{4} r_{2m}(\lambda_1^2 - \lambda_m^2)\varsigma_m(\mathbf{x}) \\
&= (\Delta + \lambda_3^2)(\Delta + \lambda_4^2) \sum_{m=2}^{4} r_{2m}(\lambda_1^2 - \lambda_m^2)[\delta(\mathbf{x}) + (\lambda_2^2 - \lambda_m^2)\varsigma_m(\mathbf{x})] \\
&= (\Delta + \lambda_3^2)(\Delta + \lambda_4^2) \sum_{m=3}^{5} r_{2m}(\lambda_1^2 - \lambda_m^2)(\lambda_2^2 - \lambda_m^2)\varsigma_m(\mathbf{x}) \\
&= (\Delta + \lambda_4^2) \sum_{m=3}^{4} r_{2m}(\lambda_1^2 - \lambda_m^2)(\lambda_2^2 - \lambda_m^2)[\delta(\mathbf{x}) + (\lambda_3^2 - \lambda_m^2)\varsigma_m(\mathbf{x})] \\
&= (\Delta + \lambda_4^2)\varsigma_4(\mathbf{x}) = \delta(\mathbf{x}).
\end{aligned}$$

We introduce the matrix

$$\boldsymbol{G}(\mathbf{x}) = \boldsymbol{R}(\mathbf{D_x})\boldsymbol{Y}(\mathbf{x}). \tag{4.31}$$

From Eqs. (4.17), (4.18) and (4.31), we obtain

$$\boldsymbol{F}(\mathbf{D_x})\boldsymbol{G}(\mathbf{x}) = \boldsymbol{F}(\mathbf{D_x})\boldsymbol{R}(\mathbf{D_x})\boldsymbol{Y}(\mathbf{x}) = \boldsymbol{\Theta}(\Delta)\boldsymbol{Y}(\mathbf{x}) = \delta(\mathbf{x})\boldsymbol{I}(\mathbf{x}).$$

Hence, $\boldsymbol{G}(\boldsymbol{x})$ is a solution to Eq. (3.7).

Therefore, we have proved the following theorem:

Theorem. *The matrix* $\boldsymbol{G}(\mathbf{x})$ *defined by the equation* (4.31) *is the fundamental solution of system of equations* (3.2)–(3.5).

5. Basic properties of the matrix $\boldsymbol{G}(\boldsymbol{x})$

Property 1. Each column of the matrix $\boldsymbol{G}(\boldsymbol{x})$ is the solution of the system of equations (3.2)-(3.5) at every point $\boldsymbol{x} \in \mathrm{E}^3$ except the origin.

Property 2. The matrix $\boldsymbol{G}(\boldsymbol{x})$ can be written in the form

$$\boldsymbol{G} = \|G_{rs}\|_{6\times 6} = \left\| \begin{matrix} \boldsymbol{G}^{(1)} & \boldsymbol{G}^{(2)} \\ \boldsymbol{G}^{(3)} & G^{(4)} \end{matrix} \right\|_{6\times 6},$$

$$\begin{aligned} \boldsymbol{G}^{(l)}(\mathbf{x}) &= \boldsymbol{R}^{(l)}(\mathbf{D}_{\mathbf{x}})Y_{11}(\mathbf{x}), \qquad l = 1,3, \\ \boldsymbol{G}^{(2)}(\mathbf{x}) &= \boldsymbol{R}^{(2)}(\mathbf{D}_{\mathbf{x}})Y_{44}(\mathbf{x}), \\ G^{(4)}(\mathbf{x}) &= R^{(4)}(\mathbf{D}_{\mathbf{x}})Y_{44}(\mathbf{x}). \end{aligned} \tag{5.1}$$

6. Conclusions

The fundamental solution of a system of equations in the theory of elasto-thermodiffusive(ETNP) semiconductor materials in case of steady oscillations in terms of elementary functions has been constructed. The fundamental solution $\boldsymbol{G}(\boldsymbol{x})$ of the system of equations (3.2)–(3.5) makes it possible to investigate three-dimensional boundary value problems of generalized theories of thermoelastic diffusion by potential method [22].

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