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# Influence of mixed boundary conditions and heterogeneity on the vibration behavior of orthotropic truncated conical shells

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This paper presents the vibration behavior analysis of heterogeneous orthotropic conical shells with mixed boundary conditions. Basic equations of heterogeneous orthotropic truncated conical shells are derived using Donnell–Mushtari shell theory. Employing the separation of variables and Galerkin's method, the expressions for frequency of heterogeneous orthotropic conical shells with two mixed boundary conditions are obtained. The results are validated through numerical comparisons with available results in the literature. The influences of truncated shell characteristics, heterogeneity, material orthotropy and mixed boundary conditions on dimensionless frequency parameters are investigated.

**Key words:** vibration, frequency parameter, heterogeneity, orthotropic material, mixed boundary conditions, truncated conical shell.



## 1. Introduction

In accordance with the recent developments of science and modern technology, structural elements consisting of different heterogeneous materials are becoming more widely used in the aerospace industry and in other high-technology fields. This is connected with their high specific strength and specific stiffness, light weight, fatigue resistance and improving the behavior of the structure. Heterogeneous materials are generally isotropic, but they can be anisotropic as well. There are numerous studies on the behavior of structural elements consisting of heterogeneous or functionally graded (FG) isotropic materials. Reviews of the major developments in the research on FG isotropic materials, published since 1990, can be found in the works of Koizumi [1], Birman and Byrd [2] and Shen [3]. Because of the combined effects of anisotropy and heterogeneity, it is extremely difficult to obtain analytical solutions for the vibration problem of shells made of heterogeneous materials with anisotropic properties. One of the first works that proposed the theory of FG or heterogeneous orthotropic materials was published by Pan [4], which determined the shape of the heterogeneity of orthotropic materials as the exponential function and presented the exact solution for FG anisotropic plate. Since this field is relatively new, there have been few studies on the thermomechanical and vibrational behavior of FG or heterogeneous orthotropic shells. Chen, Bian and Ding [5] presented the solution to free vibration problems of simply supported, fluid-filled orthotropic FG cylindrical shells based on three-dimensional equations of elasticity. Batra and Jin [6] obtained natural frequencies of an FG graphite/epoxy rectangular plate using the first-order shear deformation. Pelletier and Vel [7] investigated an exact solution for the steady-state thermoelastic response of FG orthotropic cylindrical shells using Flügge and Donnell shell theories. Ootao and Tanigawa [8] presented solution for transient thermal stresses of an orthotropic FG rectangular plate based on the three-dimensional elasticity theory using Laplace and finite cosine transformation methods. Chalivendra [9] developed quasi-static mixed mode stress fields for a crack in orthotropic inhomogeneous medium using asymptotic analysis coupled with Westergaard stress function approach. Vel [10] obtained exact elasticity solution for the vibration of FG anisotropic cylindrical shells. Baron [11] analyzed propagation of elastic waves in the FG anisotropic hollow cylinder based on the Stroh's sextic formalism and an analytical solution, the matricant, explicitly expressed under the Peano series expansion form. Lal and Kumar [12] solved a transverse vibration problem of non-homogeneous rectangular orthotropic plates of bilinearly varying thickness using characteristic orthogonal polynomials. Najafov, Sofiyev and Kuruoglu [13] examined torsional vibration and stability of FG orthotropic cylindrical shells on elastic foundations. Sofiyev and Kuruoglu [14] solved vibration and buckling problems of simply supported FG orthotropic cylindrical shells based on the first-order shear deformation theory.

The orthotropic conical shells subjected to various loading and boundary conditions are widely used in various engineering applications such as hoppers, vessel heads, components of missiles and spacecrafts, and other areas, and their analysis has become an important research area in applied mechanics. These shells have a great potential to withstand the external pressure with. However, they can display complex dynamic behavior which is mainly due to orthotropy and sensitivity to heterogeneity. Although a large number of works have been published on the vibration behaviors of homogeneous orthotropic conical shells [15–28], only a small number of investigations are concerned with the analysis of heterogeneous or functionally graded isotropic and orthotropic shells with different boundary conditions [29–37].

From the review of the literature, the available solutions for the vibration problems of isotropic and orthotropic conical shells are relatively scarce, and most of the previous studies regarding the conical shells are confined to the simply supported and clamped boundary conditions. To the best knowledge of the authors, it is the first time that the vibration analysis of heterogeneous orthotropic truncated conical shells with the following mixed boundary conditions is conducted. At one end of heterogeneous orthotropic truncated conical shell is a sleeve that prevents its longitudinal displacement and rotation, and the other end is a free support. The main objective of this work is to develop analytical formulations and solutions for the vibration analysis of heterogeneous orthotropic conical shell with mixed boundary conditions using the Donnell–Mushtari shell theory [38, 39].

## 2. Formulation of the problem

Figure 1 shows a heterogeneous orthotropic truncated conical shell with a thickness $h$, half apex angle $\gamma$, length $L$, radii at the two ends $R_1$ and $R_2$, respectively, distance along the generator from the top to the small and large ends of the cone $S_1$ and $S_2$, respectively. Let the coordinate system $(S\theta z)$ be chosen such that the origin O is at the vertex of the whole cone, on the reference surface of the conical shell, and $S$ axis lies on the reference surface of the cone, $z$ axis is in a direction normal to the reference surface of the cone, and $\theta$ axis is in the normal direction to the $S$–$z$ plane. The displacement components of

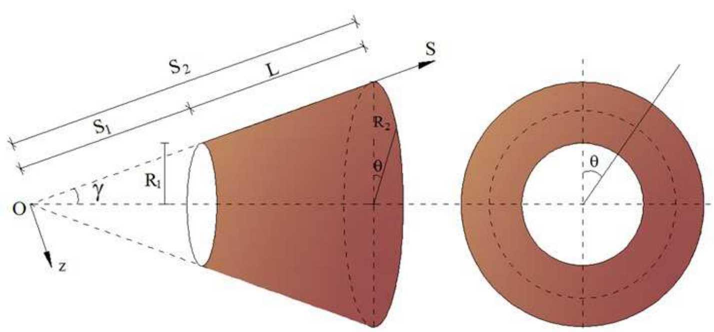


Fig. 1. The geometry of heterogeneous orthotropic truncated conical shell and notations.

the reference surface are $u$, $v$ and $w$ along the meridian, tangential and radial directions, respectively. The axes of orthotropy are parallel to the curvilinear coordinates $S$ and $\theta$.

We assume that the Young moduli, shear modulus and density of the orthotropic material are functions of the coordinate in the thickness direction. Hence, the material properties of heterogeneous orthotropic conical shell is expressed as a function of $Z$, the normalized coordinate in the thickness direction, and it is as follows:

$$\begin{aligned} &E_S(Z) = E_{0S}\varphi_1(Z), \qquad E_\theta(Z) = E_{0\theta}\varphi_1(Z), \qquad G(Z) = G_0\varphi_1(Z), \\ &\rho(Z) = \rho_0\varphi_2(Z), \qquad Z = z/h, \end{aligned} \tag{2.1}$$

where $E_{0S}$ and $E_{0\theta}$ are Young's moduli of the homogeneous orthotropic material along $S$ and $\theta$ directions, respectively, $G_0$ is shear modulus, $\rho_0$ is the density of the homogeneous orthotropic material. Furthermore, gradient functions $\varphi_1(Z)$ and $\varphi_2(Z)$ are the exponential function giving the variations of the Young moduli, shear modulus and density, respectively, and are expressed as follows [4, 8, 13]:

$$\varphi_1(Z) = e^{\eta_1(Z-0.5)}, \qquad \varphi_2(Z) = e^{\eta_2(Z-0.5)}, \tag{2.2}$$

in which $\eta_i$ $(i = 1, 2)$ are the exponential factors characterizing the degree of Young's moduli and shear modulus, and density of the orthotropic material, respectively, and satisfying $-1 \leq (\eta_1, \eta_2) \leq 1$. We remark that $\eta_1 = \eta_2 = 0$ corresponds to the homogeneous orthotropic material, $(\eta_1, \eta_2) < 0$ to the graded soft material and $(\eta_1, \eta_2) > 0$ to the graded stiff material. To make analysis tractable, we assume that Poisson's ratios ($\nu_{S\theta}$ and $\nu_{\theta S}$) in the heterogeneous orthotropic materials are taken as a constant. This is reasonable for most situations of heterogeneous orthotropic materials due to very slight variation of Poisson's ratios.

## 3. Fundamental relations and basic equations

For the derivation of the basic equations Donnell–Mushtari shell theory is used [40]–[42]. The stress-strain relations are given by considering the continuous change of orthotropic material properties of heterogonous truncated conical shell along the thickness direction [29]:

$$\begin{pmatrix} \sigma_S \\ \sigma_\theta \\ \sigma_{S\theta} \end{pmatrix} = \begin{bmatrix} Q_{11}(Z) & Q_{12}(Z) & 0 \\ Q_{21}(Z) & Q_{22}(Z) & 0 \\ 0 & 0 & Q_{66}(Z) \end{bmatrix} \begin{bmatrix} e_S - z\dfrac{\partial^2 w}{\partial S^2} \\ e_\theta - z\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial \varphi^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ e_{S\theta} - z\left(\dfrac{1}{S}\frac{\partial^2 w}{\partial S\partial \varphi} - \dfrac{1}{S^2}\dfrac{\partial w}{\partial \varphi}\right) \end{bmatrix}, \tag{3.1}$$

where $\sigma_S$, $\sigma_\theta$, $\sigma_{S\theta}$ are the stresses and $e_S$, $e_\theta$, $e_{S\theta}$ are the strains on the reference surface with respect to the axes and $\varphi = \theta \sin\gamma$. The stiffness coefficients are defined in terms of $Q_{ij}$, $(i,j = 1,2,6)$ as follows:

$$
\begin{aligned}
&Q_{11}(Z) = \frac{E_S(Z)}{1-\nu_{S\theta}\nu_{\theta S}}, \qquad Q_{22}(Z) == \frac{E_\theta(Z)}{1-\nu_{S\theta}\nu_{\theta S}},\\
&Q_{12}(Z) = \frac{\nu_{\theta S}E_S(Z)}{1-\nu_{S\theta}\nu_{\theta S}} = \frac{\nu_{S\theta}E_\theta(Z)}{1-\nu_{S\theta}\nu_{\theta \mathrm{S}}} = Q_{21}(Z), \qquad Q_{66}(Z) = 2G(Z).
\end{aligned}
\tag{3.2}
$$

The stress resultants are related to the stresses by the equations [34]:

$$
(N_S, N_\theta, N_{S\theta}) = \int\limits_{-h/2}^{h/2} (\sigma_S, \sigma_\theta, \sigma_{S\theta})\, dz,
\tag{3.3}
$$

$$
(M_S, M_\theta, M_{S\theta}) = \int\limits_{-h/2}^{h/2} (\sigma_S, \sigma_\theta, \sigma_{S\theta}) z\, dz.
\tag{3.4}
$$

The force resultants may be expressed with the Airy stress function $\Psi$ with the following partial derivatives [41]:

$$
(N_S,\ \mathrm{N}_\theta,\ \mathrm{N}_{\mathrm{S}\theta}) = \left(\frac{1}{S^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{1}{S}\frac{\partial\Psi}{\partial S}, \frac{\partial^2\Psi}{\partial S^2}, -\frac{1}{S}\frac{\partial^2\Psi}{\partial S\partial\varphi} + \frac{1}{S^2}\frac{\partial\Psi}{\partial\varphi}\right).
\tag{3.5}
$$

Inserting Eqs. (3.1) into (3.3), (3.4) and using the resulting relations together with the relations (3.5) in the basic equations for truncated conical shells [41], and then, for simplicity of mathematical operations introducing a new parameter $x = \ln(S/S_2)$, it is possible to obtain the motion and compatibility equations of heterogeneous orthotropic truncated conical shells, in the matrix form as follows:

$$
\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \Psi \\ w \end{bmatrix} = 0,
\tag{3.6}
$$

where $L_{ij}$ $(i,j = 1,2)$ are differential operators and the following definitions apply:

$$
\begin{aligned}
L_{11} = {} & \delta_1 e^{-4x}\frac{\partial^4}{\partial x^4} + \delta_2 e^{-4x}\frac{\partial^3}{\partial x^3} + \delta_3 e^{-4x}\frac{\partial^2}{\partial x^2} + \delta_4 e^{-4x}\frac{\partial}{\partial x}\\
& + S_2 e^{-3x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\cot\gamma + \delta_5 e^{-4x}\frac{\partial^4}{\partial\varphi^4} + \delta_6 e^{-4x}\frac{\partial^4}{\partial x^2\partial\varphi^2}\\
& + \delta_7 e^{-4x}\frac{\partial^3}{\partial x\partial\varphi^2} + \delta_8 e^{-4x}\frac{\partial^2}{\partial\varphi^2},
\end{aligned}
\tag{3.7}
$$

$$\begin{aligned}
L_{12} &= -\delta_9 e^{-4x}\frac{\partial^4}{\partial\varphi^4} - \delta_{10}e^{-4x}\frac{\partial^4}{\partial x^2\partial\varphi^2} + \delta_{11}e^{-4x}\frac{\partial^3}{\partial x\partial\varphi^2} \\
&\quad - \delta_{12}e^{-4x}\frac{\partial^2}{\partial\varphi^2} - \delta_{13}e^{-4x}\frac{\partial^4}{\partial x^4} + \delta_{14}e^{-4x}\frac{\partial^3}{\partial x^3} \\
&\quad + \delta_{15}e^{-4x}\frac{\partial^2}{\partial x^2} + \delta_{16}e^{-4x}\frac{\partial}{\partial x} - \rho_1 S_2^4\frac{\partial^2}{\partial t^2}, \\
L_{21} &= \Delta_1 e^{-4x}\frac{\partial^4}{\partial\varphi^4} + \Delta_2 e^{-4x}\frac{\partial^4}{\partial x^2\partial\varphi^2} - \Delta_3 e^{-4x}\frac{\partial^3}{\partial x\partial\varphi^2} \\
&\quad + \Delta_4 e^{-4x}\frac{\partial^2}{\partial\varphi^2} + \Delta_5 e^{-4x}\frac{\partial^4}{\partial x^4} + \Delta_6 e^{-4x}\frac{\partial^3\Psi}{\partial x^3} \\
&\quad + \Delta_7 e^{-4x}\frac{\partial^2\Psi}{\partial x^2} + \Delta_8 e^{-4x}\frac{\partial\Psi}{\partial x}, \\
L_{22} &= -\Delta_9 e^{-4x}\frac{\partial^4}{\partial\varphi^4} + \Delta_{10}e^{-4x}\frac{\partial^4}{\partial x^2\partial\varphi^2} + \Delta_{11}e^{-4x}\frac{\partial^3}{\partial x\partial\varphi^2} \\
&\quad + \Delta_{12}e^{-4x}\frac{\partial^2}{\partial\varphi^2} - \Delta_{13}e^{-4x}\frac{\partial^4}{\partial x^4} + \Delta_{14}e^{-4x}\frac{\partial^3}{\partial x^3} + \Delta_{15}e^{-4x}\frac{\partial^2}{\partial x^2} \\
&\quad + \Delta_{16}e^{-4x}\frac{\partial}{\partial x} + S_2 e^{-3x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\cot\gamma
\end{aligned}$$

in which $t$ is a time, and $\rho_1$, $\delta_j$ and $\Delta_j$ $(j = 1, 2, \ldots, 16)$ are given in Appendix A.

Equations (3.6) are the governing equations for the free vibration of heterogeneous orthotropic truncated circular conical shells.

## 4. The solution of basic equations

Let $w$ and $\Psi$ be defined by the following relations [42]:

$$w = e^{\lambda x} w_1(x,t)\cos(n_1\varphi), \tag{4.1}$$

$$\Psi = S_2 e^{(\lambda+1)x}\Psi_1(x,t)\cos(n_1\varphi), \tag{4.2}$$

where $n_1 = n/\sin\gamma$, $n$ is the circumferential wave number and $\lambda$ is parameter that will be determined from minimum conditions of frequencies.

Multiplying the first equation of the set (3.6) by $wS_2^2 e^{2x}\,d\varphi\,dx$ and second equation of the set by $\Psi S_2^2 e^{2x}\,d\varphi\,dx$ and applying the Galerkin method in the ranges $(-x_0 \le x \le 0)$ and $(0 \le \varphi \le 2\pi\sin\gamma)$, respectively, then integrating with respect to the coordinate $\varphi$, one finds the following characteristic equation:

$$
\begin{aligned}
(4.3)\qquad &\int\limits_{-x_0}^{0}\Big\{\delta_1 S_2 e^{(\lambda-2)x} w_1 \frac{\partial^4[e^{(\lambda+1)x}\Psi_1]}{\partial x^4} + \delta_2 S_2 e^{(\lambda-2)x} w_1 \frac{\partial^3[e^{(\lambda+1)x}\Psi_1]}{\partial x^3}\\
&+ \delta_3 S_2 e^{(\lambda-2)x} w_1 \frac{\partial^2[e^{(\lambda+1)x}\Psi_1]}{\partial x^2} + (\delta_4 - \delta_7 n_1^2) S_2 e^{(\lambda-2)x} w_1 \frac{\partial[e^{(\lambda+1)x}\Psi_1]}{\partial x}\\
&+ S_2^2 e^{(\lambda-1)x} w_1 \Big\{\frac{\partial^2[e^{(\lambda+1)x}\Psi_1]}{\partial x^2} - \frac{\partial[e^{(\lambda+1)x}\Psi_1]}{\partial x}\Big\}\cot\gamma\\
&+ (\delta_5 n_1^2 - \delta_8) n_1^2 S_2 e^{(2\lambda-1)x} w_1 \Psi_1 - \delta_6 n_1^2 S_2 e^{(\lambda-2)x} w_1 \frac{\partial^2[e^{(\lambda+1)x}\Psi_1]}{\partial x^2}\\
&+ (\delta_{12} - \delta_9 n_1^2) n_1^2 e^{2(\lambda-1)x} w_1^2 + n_1^2 e^{(\lambda-2)x} w_1 \Big[\delta_{10}\frac{\partial^2(e^{\lambda x}w_1)}{\partial x^2} - \delta_{11}\frac{\partial(e^{\lambda x}w_1)}{\partial x}\Big]\\
&- e^{(\lambda-2)x} w_1 \Big[\delta_{13}\frac{\partial^4(e^{\lambda x}w_1)}{\partial x^4} - \delta_{14}\frac{\partial^3(e^{\lambda x}w_1)}{\partial x^3} - \delta_{15}\frac{\partial^2(e^{\lambda x}w_1)}{\partial x^2} - \delta_{16}\frac{\partial(e^{\lambda x}w_1)}{\partial x}\Big]\\
&\qquad\qquad - \rho_1 S_2^4 e^{2(\lambda+1)x} w_1 \frac{\partial^2 w_1}{\partial t^2}\Big\} dx = 0,
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\qquad &\int\limits_{-x_0}^{0}\Big\{\varDelta_1 n_1^4 \mathrm{S}_2\, e^{2\lambda x}\, \Psi_1^2 - \varDelta_2 n_1^2 \mathrm{S}_2 e^{(\lambda-1)x}\Psi_1 \frac{\partial^2[e^{(\lambda+1)x}\Psi_1]}{\partial x^2}\\
&- \varDelta_3 n_1^2 \mathrm{S}_2 e^{(\lambda-1)x}\Psi_1 \frac{\partial[e^{(\lambda+1)x}\Psi_1]}{\partial x} - \varDelta_4 n_1^2 \mathrm{S}_2 e^{2\lambda x}\, \Psi_1^2\\
&+ \varDelta_5 \mathrm{S}_2 e^{(\lambda-1)x}\Psi_1 \frac{\partial^4[e^{(\lambda+1)x}\Psi_1]}{\partial x^4} + \varDelta_6 \mathrm{S}_2 e^{(\lambda-1)x}\Psi_1 \frac{\partial^3[e^{(\lambda+1)x}\Psi_1]}{\partial x^3}\\
&+ \varDelta_7 \mathrm{S}_2 e^{(\lambda-1)x}\Psi_1 \frac{\partial^2[e^{(\lambda+1)x}\Psi_1]}{\partial x^2} + \varDelta_8 \mathrm{S}_2 e^{(\lambda-1)x}\Psi_1 \frac{\partial[e^{(\lambda+1)x}\Psi_1]}{\partial x}\\
&- \varDelta_9 n_1^4 e^{(2\lambda-1)x}\Psi_1 w_1 - \varDelta_{10} n_1^2 e^{(\lambda-1)x}\Psi_1 \frac{\partial^4(e^{\lambda x}w_1)}{\partial x^2} - \varDelta_{11} n_1^2 e^{(\lambda-1)x}\Psi_1 \frac{\partial(e^{\lambda x}w_1)}{\partial x}\\
&- \varDelta_{12} n_1^2 e^{(2\lambda-1)x}\Psi_1 w_1 - \varDelta_{13} e^{(\lambda-1)x}\Psi_1 \frac{\partial^4(e^{\lambda x}w_1)}{\partial x^4} + \varDelta_{14} e^{(\lambda-1)x}\Psi_1 \frac{\partial^3(e^{\lambda x}w_1)}{\partial x^3}\\
&+ \varDelta_{15} e^{(\lambda-1)x}\Psi_1 \frac{\partial^2(e^{\lambda x}w_1)}{\partial x^2} + \varDelta_{16} e^{(\lambda-1)x}\Psi_1 \frac{\partial(e^{\lambda x}w_1)}{\partial x}\\
&\qquad\qquad + \mathrm{S}_2 e^{\lambda x}\Psi_1 \Big[\frac{\partial^2(e^{\lambda x}w_1)}{\partial x^2} - \frac{\partial(e^{\lambda x}w_1)}{\partial x}\Big]\cot\gamma\Big\} dx = 0.
\end{aligned}
$$

Now we describe the mixed boundary conditions for the problem. At one end of the truncated conical shell the following boundary condition is satisfied:

$$
(4.5)\qquad v = w = N_S = M_S = 0,
$$

which corresponds to the free support and the other end is satisfied by the following boundary condition:

$$u = \frac{\partial w}{\partial x} = Q_S = N_{S\theta} = 0. \tag{4.6}$$

In practice, this means that at this end there is a sleeve that prevents shell's rotation and longitudinal displacement [42].

**Case 1.** At the large and small ends of the truncated conical shell the boundary conditions (4.5) and (4.6) are satisfied respectively, so that the functions $w_1(x,t)$ and $\Psi_1(x,t)$, can be chosen as follows:

$$w_1 = C_1 \sin\left(\frac{m_1 x}{2}\right)\cos(\omega t), \qquad \Psi_1 = C_2 \sin\left(\frac{m_1 x}{2}\right)\cos(\omega t), \tag{4.7}$$

where $C_1$ and $C_2$ are unknown constants to be determined, $\omega$ (in radians per second) is the natural frequency and the following definitions apply:

$$m_1 = \frac{m\pi}{x_0}, \qquad x_0 = \ln\frac{S_1}{S_2}; \qquad m = 1, 3, 5, \dots, \tag{4.8}$$

in which $m$ is the longitudinal wave number.

Introducing Eqs. (4.7) into Eqs. (4.3) and (4.4), after integrating according to $x$ and eliminating $f_2(t)$ from the resulting equations, the following expression for the natural frequency of heterogeneous orthotropic truncated conical shells is obtained:

$$\omega = \sqrt{\frac{q_{13}q_{21} + (q_{11} + q_{12})q_{22}}{\rho_1 q_{21}}}, \tag{4.9}$$

where $q_{ij}$ $(i, j = 1, 2, 3)$ are parameters depending on heterogeneous orthotropic material properties, truncated conical shell characteristics and boundary conditions. The explicit form of $q_{ij}$ $(i, j = 1, 2, 3)$ is too long so it is omitted [35, 43].

Remembering that for a homogeneous orthotropic truncated conical shell, $\eta_1 = \eta_2 = 0$ and an analogue of expression (4.9) is considered as a special case.

For the non-dimensional frequency of heterogeneous orthotropic truncated conical shell, the following expression is used:

$$\omega_1 = \omega \frac{R_1^2}{h}\sqrt{\frac{(1 - \nu_{S\theta}\nu_{\theta S})\rho_0}{E_{0S}}}. \tag{4.10}$$

**Case 2.** At the small and large ends of the heterogeneous orthotropic truncated conical shell the boundary conditions (4.5) and (4.6) are satisfied respectively, so that the functions $w_1(x,t)$ and $\Psi_1(x,t)$, can be chosen as follows:

$$w_1 = C_1 \sin\left(\frac{m_1 x}{2}\right)\cos(\omega t), \qquad \Psi_1 = C_2 \sin\left(\frac{m_1 x}{2}\right)\cos(\omega t), \tag{4.11}$$

where the following definitions apply:

$$x = \ln \frac{S}{S_1}; \qquad m = 1, 3, 5, \ldots . \tag{4.12}$$

Introducing (4.11) into Eqs. (4.3) and (4.4), after integrating with respect to $x$ and eliminating $f_2(t)$ from the resulting equations, then performing some simplifications, the expressions for dimensional and non-dimensional natural frequencies of heterogeneous orthotropic truncated conical shells, which satisfied (4.5) and (4.6) at the small and large ends, respectively, are found. The resulting expressions will be similar to the expression (4.10). For this case, $S_1$ is written instead of $S_2$ and the boundaries of integrals varied from 0 to $x_0$ in Eqs. (4.3) and (4.4).

The dimensional and non-dimensional fundamental frequencies of heterogeneous orthotropic truncated conical shell may be obtained by means of minimizing the functions (4.9) and (4.10) with respect to $m$, $n$ and $\lambda$.

## 5. Numerical results and discussion

### 5.1. Comparison

In order to confirm the accuracy of the current study, the values of the natural frequency of isotropic truncated conical shells for different $L/R_1$ and $R_1/h$ ratios are compared with those of Agenosov and Sachenkov [42] and tabulated in Table 1. The other geometric parameters of the truncated conical shells are taken to be $h = 0.01$ m, $h = 0.01$, $\gamma = 45°$, $m = 1$ and $\lambda = 1.2$. The Young modulus, mass density and Poisson's ratio for the isotropic material are $E_0 = 1.93 \times 10^{11}$ Pa, $\nu_0 = 0.3$ and $\rho_0 = 8000$ kg/m$^3$. Here $E_0$, $\nu_0$ and $\rho_0$ are the Young modulus, Poisson's ratio and density of homogeneous isotropic material, respectively. These values were taken from the study of Agenosov and

**Table 1. Comparison of the natural frequency of isotropic truncated conical shells with mixed boundary conditions.**

| $\omega$ (rad/s), (n) | | | | | | |
|---|---|---|---|---|---|---|
| | Agenosov and Sachenkov [42] | | | Present study | | |
| | $L/R_1$ | | | $L/R_1$ | | |
| $R_1/h$ | 1 | 2 | 3 | 1 | 2 | 3 |
| 150 | 324.517(7) | 169.478(6) | 115.16(6) | 327.68(7) | 171.77(6) | 116.91(6) |
| 200 | 213.078(7) | 112.30(6) | 74.108(6) | 214.894(7) | 113.615(6) | 75.089(6) |
| 250 | 151.519(8) | 79.451(7) | 54.146(6) | 152.66(8) | 80.279(7) | 54.784(6) |
| 300 | 115.715(8) | 60.471(7) | 40.687(6) | 116.515(8) | 61.048(7) | 41.125(6) |

SACHENKOV [42]. The boundary conditions at the small and large ends are defined by Eqs. (4.5) and (4.6), respectively. It is evident that obtained a good agreement with the results from [42] was obtained.

### 5.2. Free vibration analysis of heterogeneous orthotropic conical shell with mixed boundary conditions

In this subsection, to study influences of heterogeneity and orthotropy on the dimensionless frequency parameters of truncated conical shells under mixed boundary conditions, different geometric relations $L/R_1 = 1, 2, 3$, $R_1/h = 50, 75, 100$; $\gamma = 15°, 30°, 45°$ and material gradient index $(\mu_1, \mu_2) = -1, 0, +1$ are selected. The numerical results are tabulated in Table 2 and illustrated in Figs. 2–4.

In order to discuss the influences of the heterogeneity, the stiffness ratio $(E_{01}/E_{02})$ and mixed boundary conditions for the minimum values of dimensionless frequency parameters, and the associated number of circumferential waves $(n)$ and parameter $(\lambda)$ for homogeneous and heterogeneous orthotropic truncated conical shells, the following material properties and shell characteristics are taken to be: $E_{0S} = 138 \times 10^{11}$ (Pa), $E_{0\theta} = E_{0S}/q$; $q = 10, 25, 40$; $G_0 = 0.5E_{0\theta}$; $\nu_{S\theta} = 0.25$; $\nu_{\theta S} = \nu_{S\theta}E_{0S}/E_{0\theta}$ and $R_1/h = 5$, $L/R_1 = 1$ (see Table 2). The minimum values of dimensionless frequency parameters of homogeneous and heterogeneous orthotropic truncated conical shells decrease as $E_{0S}$ is kept constant and the stiffness ratio $E_{0S}/E_{0\theta}$ increases, whereas the associated number of circumferential waves does not change in the mixed boundary conditions 1 and 2. In addition, the parameter $\lambda$ is zero in Case 1 and changes irregularly in Case 2. Comparing the heterogeneous orthotropic conical shells with the homogeneous orthotropic conical shells, the importance of the difference between the values of dimensionless frequency parameters can be seen. It should be mentioned that the effect of heterogeneity is not changed as $E_{0S}/E_{0\theta}$ is increased from 10 to 40 by step 15 for fixed $\eta_1$ and $\eta_2$ in Case 1, although sometimes this influence in Case 2 is higher than in Case 1, it changes with an increasing of $E_{0S}/E_{0\theta}$. One can see that influences of heterogeneity, stiffness ratio and mixed boundary conditions for the minimum values of dimensionless frequency parameters for truncated conical shells are apparent. It is noticed that the minimum values of dimensionless frequency parameter of heterogeneous orthotropic truncated conical shell are identical as $\eta_1 = \eta_2 = \pm 1$ for all $E_{0S}/E_{0\theta}$. The minimum values of the dimensionless frequency parameter of heterogeneous orthotropic conical shell are obtained at $\lambda = 0$ in Case 1, while at the various values of the parameter $\lambda$, in Case 2. The values of dimensionless frequency parameter of heterogeneous orthotropic conical shells are smaller than the other profiles as $\eta_1 = -1$ and $\eta_1 = 0$, whereas they are larger as $\eta_1 = 1$ and $\eta_2 = 0$ for the all $E_{0S}/E_{0\theta}$ ratio.

**Table 2. Influences of the heterogeneity, the stiffness ratio and the mixed boundary conditions on the minimum values of $\omega_1$.**

| $\omega_1$ (n, $\lambda$) | | | | | | |
|---|---|---|---|---|---|---|
| | Mixed boundary conditions: Case 1 | | | Mixed boundary conditions: Case 2 | | |
| $E_{01}/E_{02}$ | 10 | 25 | 40 | 10 | 25 | 40 |
| $\eta_1 = \eta_2 = 0$ | 3.414(5,0) | 2.278(5,0) | 1.835(5,0) | 0.523(1,3.1) | 0.262(5,3.2) | 0.202(5,3.1) |
| $\eta_1 = \eta_2 = \pm 1$ | 3.385(5,0) | 2.261(5,0) | 1.821(5,0) | 0.518(2,3.1) | 0.252(5,3.2) | 0.209(5,3.1) |
| $\mu_1 = +1; \mu_2 = 0$ | 2.691(5,0) | 1.797(5,0) | 1.448(5,0) | 0.412(2,3.1) | 0.200(5,3.2) | 0.166(5,3.1) |
| $\eta_1 = -1; \eta_2 = 0$ | 4.437(5,0) | 2.963(5,0) | 2.387(5,0) | 0.679(2,3.1) | 0.330(5,3.2) | 0.274(5,3.1) |

Figures 2 and 3 display the variation of minimum values of dimensionless frequency parameters for homogeneous ($\eta_1 = \eta_2 = 0$) and heterogeneous ($\eta_1 =$

a)

b)

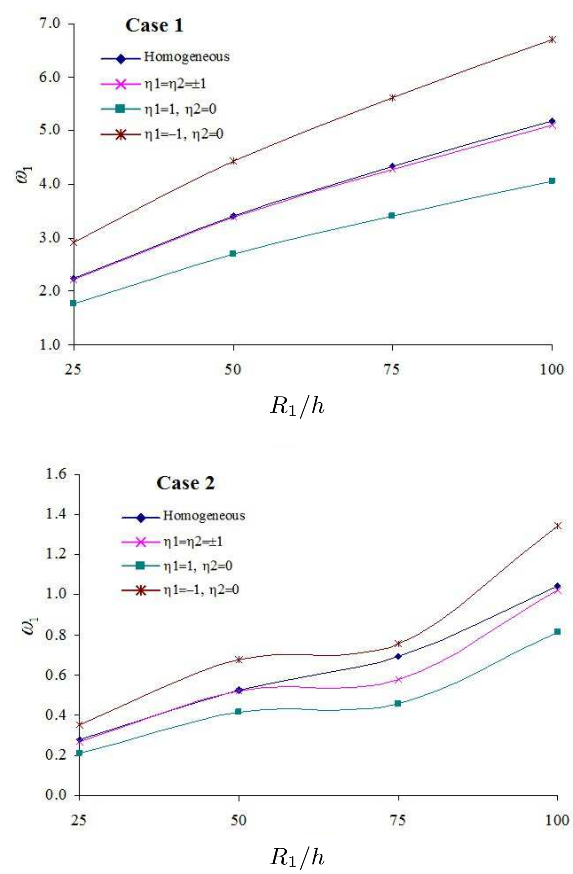


FIG. 2. Influences of the heterogeneity on the minimum values of $\omega_1$ for orthotropic conical shells depending on the ratio, $R_1/h$, in a) Case 1 and b) Case 2.

$\eta_2 = \pm 1$ or $\eta_1 = \pm 1$, $\eta_2 = 0$) orthotropic truncated conical shells depending on the $R_1/h$ and $L/R_1$ ratios, respectively, in two mixed boundary conditions: (a) Case 1 and (b) Case 2. The homogeneous orthotropic material properties are: $E_{0S} = 138 \times 10^{11}$ (Pa), $E_{0\theta} = E_{0S}/q$; $q = 10, 25, 40$; $G_0 = 0.5E_{0\theta}$; $\nu_{S\theta} = 0.25$; $\nu_{\theta S} = \nu_{S\theta} E_{0S}/E_{0\theta}$. The conical shell characteristics are taken to be $L/R_1 = 1$, $\gamma = 30°$, $R_1/h = 50, 75, 100$ in Fig. 2, $R_1/h = 50$, $L/R_1 = 2$ and $\gamma = 15°, 30°, 45°$ in Fig. 3. With a decrease in the ratio $R_1/h$, the minimum values of dimensionless frequency parameters increase, whereas with a decrease in the ratio $L/R_1$, these values decrease appreciably. The values of dimensionless frequency parameters for Case 2 are significantly lower than for Case 1 for all $R_1/h$ and $L/R_1$ ratios. The influence of heterogeneity on the dimensionless frequency parameter of orthotropic truncated conical shell is less influential as $\eta_1 = \eta_2 = \pm 1$, it is significant as $\eta_1 = +1$; $\eta_2 = 0$, and it is more pronounced as $\eta_1 = -1$; $\eta_2 = 0$ in Case 1 for all $R_1/h$ and $L/R_1$ ratios.

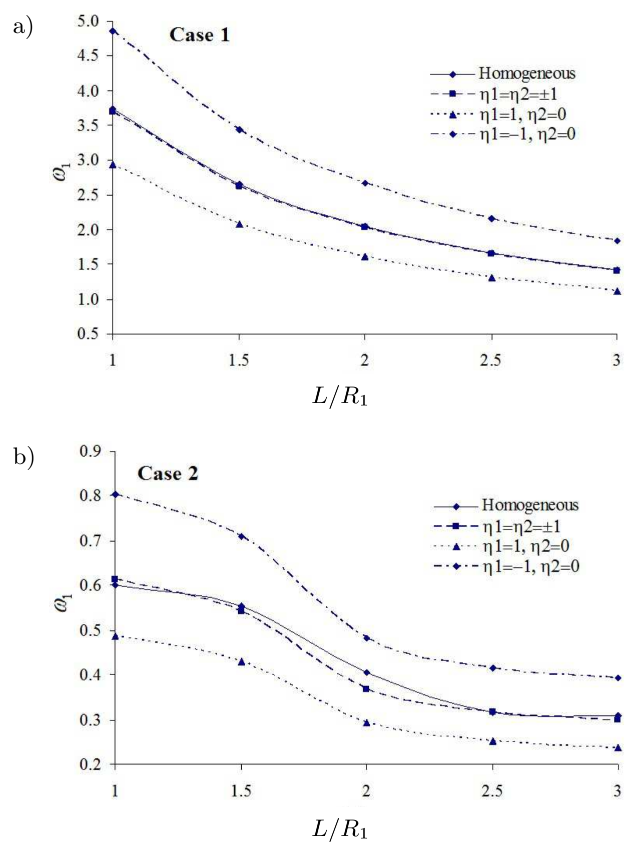


Fig. 3. Influences of the heterogeneity on the minimum values of $\omega_1$ for orthotropic conical shells depending on the ratio, $L/R_1$, in a) Case 1 and b) Case 2.

The curves pertaining to minimum values of dimensionless frequency parameters for homogeneous ($\eta_1 = \eta_2 = 0$) and heterogeneous ($\eta_1 = \eta_2 = \pm 1$ or $\eta_1 = \pm 1$, $\eta_2 = 0$) orthotropic truncated conical shells depending on the semi-vertex angle $\gamma$, in two mixed boundary conditions a) Case 1 and b) Case 2, are illustrated in Fig. 4. Homogeneous orthotropic material properties are identical to the previous calculation (see Figs. 2 and 3) and the truncated conical shell characteristics are taken to be $R_1/h = 50$, $L/R_1 = 2$ and $\gamma = 15°, 30°, 45°$. The minimum values of dimensionless frequency parameter of homogeneous and heterogeneous orthotropic truncated conical shells decrease, as the semi-vertex angle $\gamma$ increases for Cases 1 and 2. The influence of heterogeneity on the dimensionless frequency parameter of orthotropic truncated conical shell is more pronounced in the second mixed boundary condition than in the first, when the Young moduli and density vary together (i.e., $\eta_1 = \eta_2 = \pm 1$), whereas this influence is more obvious when density of the material is kept constant and Young's moduli and

a)

b)

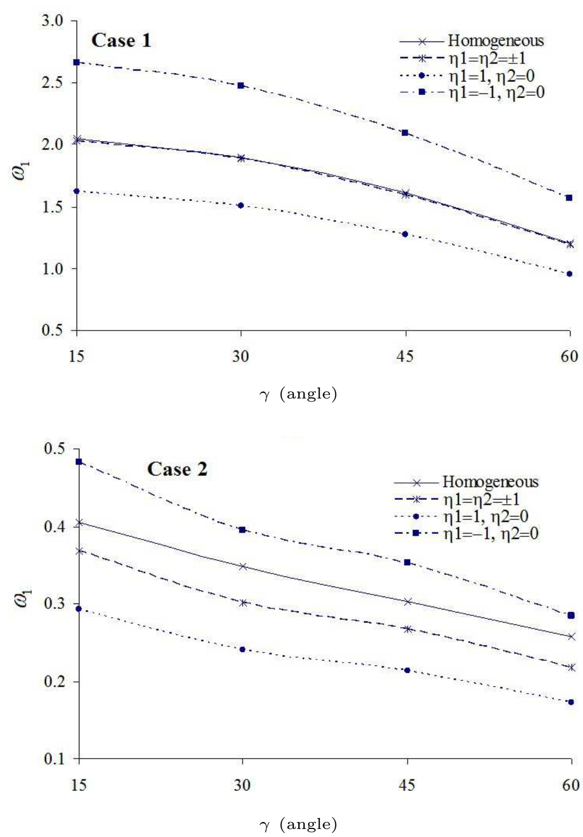


Fig. 4. Influences of the heterogeneity on the minimum values of $\omega_1$ for orthotropic conical shells depending on the semi-vertex angle $\gamma$ in a) Case 1 and b) Case 2.

shear modulus vary ($\eta_1 = \pm 1$; $\eta_2 = 0$), as the semi-vertex angle $\gamma$ increases, in Cases 1 and 2. The influence of heterogeneity on the dimensionless frequency parameter of orthotropic truncated conical shell is almost unchanged in Case 1, while this influence varies in Case 2, as the semi-vertex angle $\gamma$ increases.

## 6. Conclusions

An investigation has been carried out on the free vibration of heterogeneous orthotropic truncated conical shells with mixed boundary conditions. The basic equations of heterogeneous orthotropic truncated conical shells are derived using the Donnell–Mushtari shell theory. Using the separation of variables method and Galerkin's method, the expressions for natural frequency of heterogeneous orthotropic truncated conical shells with two mixed boundary conditions are successfully obtained in this study.

The numerical results support the following conclusions:

a) The minimum values of dimensionless frequency parameters of homogeneous and heterogeneous orthotropic truncated conical shell are obtained for $\lambda = 0$ for all the semi-vertex-angles, the length-to-radius and the radius-to-thickness ratios, and the stiffness ratio in Case 1, while these values are obtained for various $\lambda$ in Case 2.

b) The values of dimensionless frequency parameters of homogeneous and heterogeneous orthotropic truncated conical shell decrease, as the semi-vertex angle, the stiffness ratio and the length-to-radius ratio increase, whereas these values increase as the radius-to-thickness ratio increases.

c) The minimum values of dimensionless frequency parameters of heterogeneous orthotropic truncated conical shell are identical as $\eta_1 = \eta_2 = \pm 1$.

d) The minimum values of dimensionless frequency parameter of heterogeneous orthotropic conical shells are smaller than the other profiles as $\eta_1 < 0$ and $\eta_1 = 0$, whereas it is larger as $\eta_1 > 0$ and $\eta_2 = 0$.

e) The influence of heterogeneity on the dimensionless frequency parameter of orthotropic truncated conical shell is less influential as $\eta_1 = \eta_2 = \pm 1$, it is significant as $\eta_1 = +1$; $\eta_2 = 0$, and it is more effective as $\eta_1 = -1$; $\eta_2 = 0$ in Case 1.

f) The influence of heterogeneity is not changed, since $E_{0S}/E_{0\theta}$ increases for fixed $\eta_1$ and $\eta_2$ in Case 1, although sometimes this influence in Case 2 is higher than in Case 1, it changes with the increasing of stiffness ratio $E_{0S}/E_{0\theta}$.

g) The influence of heterogeneity does not affect the dimensionless frequency parameter of orthotropic truncated conical shell in Case 1, while this influence changes in Case 2.

## Appendix A

The coefficients $\delta_j$ and $\varDelta_j$ $(j = 1, \ldots, 16)$ of Eq. (3.7) are defined as follows:

$$\begin{aligned}
&\delta_1 = c_{12}; \quad \delta_2 = c_{11} - 4c_{12} - c_{22}; \quad \delta_3 = 5c_{12} + 3c_{22} - 3c_{11} - c_{21};\\
&\delta_4 = 2(c_{11} - c_{22} - c_{12} + c_{21}); \quad \delta_5 = c_{21}; \quad \delta_6 = c_{11} - 2c_{31} + c_{22};\\
&\delta_7 = 4c_{31} - 3c_{11} - c_{22}; \quad \delta_8 = 2(c_{11} - c_{31} + c_{21}); \quad \delta_9 = c_{24};\\
&\delta_{10} = c_{14} + c_{23} + 2c_{32}; \quad \delta_{11} = 3c_{14} + c_{23} + 4c_{32};\\
&\delta_{12} = 2(c_{14} + c_{32} + c_{24}); \quad \delta_{13} = c_{13}; \quad \delta_{14} = c_{23} - c_{14} + 4c_{13};\\
&\delta_{15} = c_{24} - 3c_{23} + 3c_{14} - 5c_{13}; \quad \delta_{16} = 2(c_{23} - c_{14} - c_{24} + c_{13});\\
&\varDelta_1 = b_{11}; \quad \varDelta_2 = 2b_{31} + b_{21} + b_{12}; \quad \varDelta_3 = 4b_{31} + 3b_{21} + b_{12};\\
&\varDelta_4 = 2(b_{31} + b_{21} + b_{11}); \quad \varDelta_5 = b_{22}; \quad \varDelta_6 = b_{21} * -4b_{22} - b_{12};\\
&\varDelta_7 = 5b_{22} + 3b_{12} - b_{11} - 3b_{21}; \quad \varDelta_8 = 2b_{21} - 2b_{22} - 2b_{12} + 2b_{11};\\
&\varDelta_9 = b_{14}; \quad \varDelta_{10} = 2b_{32} - b_{13} - b_{24}; \quad \varDelta_{11} = b_{13} + 3b_{24} - 4b_{32};\\
&\varDelta_{12} = 2b_{32} - 2b_{24} - 2b_{14}; \quad \varDelta_{13} = b_{23}; \quad \varDelta_{14} = b_{13} - b_{24} + 4b_{23};\\
&\varDelta_{15} = b_{14} - 3b_{13} + 3b_{24} - 5b_{23}; \quad \varDelta_{16} = 2b_{13} - 2b_{24} + 2b_{23} - 2b_{14},
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
&c_{11} = a_{11}^1 b_{11} + a_{12}^1 b_{21}, \quad c_{12} = a_{11}^1 b_{12} + a_{12}^1 b_{22},\\
&c_{13} = a_{11}^1 b_{13} + a_{12}^1 b_{23} + a_{11}^2, \quad c_{14} = a_{11}^1 b_{14} + a_{12}^1 b_{24} + a_{12}^2,\\
&c_{21} = a_{21}^1 b_{11} + a_{22}^1 b_{21}, \quad c_{22} = a_{21}^1 b_{12} + a_{22}^1 b_{22},\\
&c_{23} = a_{21}^1 b_{13} + a_{22}^1 b_{14} + a_{21}^2, \quad c_{24} = a_{21}^1 b_{14} + a_{22}^1 b_{13} + a_{22}^2,\\
&c_{31} = a_{66}^1 b_{31}, \qquad c_{32} = a_{66}^1 b_{32} + a_{66}^2,\\
&b_{11} = a_{22}^0/L_0, \quad b_{12} = -a_{12}^0/L_0, \quad b_{13} = (a_{12}^0 a_{21}^1 - a_{11}^1 a_{22}^0)/L_0,\\
&b_{14} = (a_{12}^0 a_{22}^1 - a_{12}^1 a_{22}^0)/L_0, \quad b_{21} = -a_{21}^0/L_0, b_{22} = a_{11}^0/L_0,\\
&b_{23} = (a_{21}^0 a_{11}^1 - a_{21}^1 a_{11}^0)/L_0, \quad b_{24} = (a_{21}^0 a_{12}^1 - a_{22}^1 a_{11}^0)/L_0,\\
&b_{31} = 1/a_{66}^0, \quad b_{32} = -a_{66}^1/a_{66}^0, \quad L_0 = a_{11}^0 a_{22}^0 - a_{12}^0 a_{21}^0,
\end{aligned} \tag{A.2}$$

in which

$$\begin{aligned}
a_{11}^k &= \frac{E_{0S}\mathrm{h}^{k+1}}{1 - \nu_{S\theta}\nu_{\theta S}} \int\limits_{-1/2}^{1/2} Z^k \varphi_1(Z)\, dZ,\\
a_{22}^k &= \frac{E_{0\theta}\mathrm{h}^{k+1}}{1 - \nu_{S\theta}\nu_{\theta S}} \int\limits_{-1/2}^{1/2} Z^k \varphi_1(Z)\, dZ, \quad k = 0, 1, 2,
\end{aligned} \tag{A.3}$$

$$a_{12}^k = \nu_{\theta S} a_{11}^k = \nu_{S\theta} a_{22}^k = a_{21}^k,$$

(A.3) [cont.]
$$a_{66}^k = 2G_0 h^{k+1} \int_{-1/2}^{1/2} Z^k \varphi_1(Z)\, dZ,$$

$$\rho_1 = \rho_0 \int_{h/2}^{-h/2} \varphi_2(Z)\, dZ.$$

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