

On spatial behavior of the solution of a non-standard problem in linear thermoviscoelasticity with voids

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IN THIS PAPER WE STUDY THE CONSTRAINED MOTION of a prismatic cylinder made of a thermoviscoelastic material with voids and subjected to final given data that are proportional, but not identical, to their initial values. We show how certain cross-sectional integrals of the solution spatially evolve with respect to axial variable. Some conditions are derived upon the proportionality coefficients in order to show that the integrals exhibit alternative behavior.

Key words: spatial behavior, non-standard problems, growth and decay estimates, porous thermoviscoelasticity.

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1. Introduction

A MATERIAL WITH SMALL, DISTRIBUTED VOIDS may be called porous material or material with voids. The theory of elastic materials with voids is a recent generalization of the classical theory of elasticity. The intended applications of this theory are to geological materials such as rocks, soils and wood, and to biological and manufactured porous materials. Besides the usual elastic effects, these materials have a microstructure with an important property: the mass in each point can be obtained as the product of two fields: the density field of the matrix material and the volume fraction field. This representation of the bulk density is very important because it introduces an additional degree of kinematic freedom. Such a representation was previously used by GOODMAN and COWIN [1] in order to describe the flowing granular materials.

IEȘAN [2] developed a linear theory of thermoviscoelastic porous materials, in which the time derivative of the strain tensor, the time derivative of the volume fraction field and the time derivative of the gradient of the volume fraction field are included in the set of independent constitutive variables. This theory represents an extension of the theory of elastic materials with voids (see, COWIN and NUNZIATO [3]) and the theory of thermoelastic materials with voids (see,

IEȘAN [4]), because it takes into consideration the memory effects. For a review of the literature on thermoviscoelastic materials with voids the reader is referred to [5–8].

On the other hand, it is well known that the backward in time problems are improperly posed problems, because they fail to have a global solution, or the solution does not depend continuously on the data, or the solution is not unique. There are several methods to “solve” this kind of ill-posed problems. Some of them involve changing the initial and/or boundary conditions, while others involve constraining solutions to exist in some constraint set. A number of non-standard problems have attracted the attention of many researchers in the last two decades (see, for example, AMES *et al.* [9], PAYNE *et al.* [10], CHIRIȚĂ [11], CHIRIȚĂ and CIARLETTA [12], QUINTANILLA and STRAUGHAN [13], AMES and PAYNE [14], AMES *et al.* [15]).

KNOPS and PAYNE [16] considered a non-standard problem associated with the linear elasticity for a prismatic cylinder and established some decay and growth exponential estimates with respect to the axial variable for some time integrals of the cross-sectional energy, provided the elasticity tensor is positive definite. Similar problems were studied by CHIRIȚĂ and CIARLETTA [17, 18] when studying the theory of thermoelastic materials, and by BULGARIU [19] within the context of the linear theory of elastic materials with voids.

In this paper, we consider a prismatic cylinder occupied by an anisotropic inhomogeneous compressible linear thermoviscoelastic material with voids, subjected to zero body force and zero lateral boundary conditions. The motion is induced by a time-dependent displacement, porosity and temperature variation specified pointwise over the base. Moreover, we consider that the motion is constrained such that the displacement, the volume fraction, the temperature variation and their derivatives with respect to time are proportional, but not identical with their initial values. The spatial behavior of the solution is studied by means of certain time-weighted integrals of the cross-sectional energy terms. We derive some conditions upon the proportionality coefficients in order to see how certain integrals of the cross-sectional energy evolve with respect to the axial variable.

The problem studied in this article finds application in geology and structural engineering. Following KNOPS and PAYNE [16], we give an example of a pile driven into a rigid foundation that prevents movement of the lateral boundary. The time-dependent displacement, porosity and temperature variation prescribed over the excited end, constrains the motion such that the displacement, the temperature variation, the volume fraction and their derivatives with respect to time at some given time are proportional, but not identical with, their initial values. It is convenient to predict the deformation at each cross-section of the pile in terms of the base displacement, porosity, and temperature variation.

2. Basic equations and formulation of the problem

Throughout this paper, we refer the motion of a continuum to a fixed system of Cartesian axes Ox_i , ($i = 1, 2, 3$). We shall employ the usual summation and differentiation conventions: Latin subscripts have the range 1, 2, 3, Greek subscripts have the range 1, 2, summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes partial differentiation with respect to time. Throughout this section we assume that a regular region B is filled by an anisotropic and inhomogeneous thermoviscoelastic material with voids. According to Ieşan [2], the governing equations for the linear theory of anisotropic and inhomogeneous thermoviscoelastic materials with voids are given by:

(i) the equations of motion

$$(2.1) \quad t_{ji,j} + \rho f_i = \rho \ddot{u}_i,$$

$$(2.2) \quad H_{j,j} + g + \rho \ell = \rho \kappa \ddot{\varphi},$$

for $B \times (0, \infty)$,

(ii) the equation of energy

$$(2.3) \quad \rho T_0 \dot{\eta} = Q_{j,j} + \rho S,$$

for $B \times (0, \infty)$,

(iii) the geometrical relations

$$(2.4) \quad e_{rs} = \frac{1}{2}(u_{r,s} + u_{s,r}),$$

in $\overline{B} \times [0, \infty)$, and

(iv) the constitutive equations

$$(2.5) \quad \begin{aligned} t_{ij} &= C_{ijrs} e_{rs} + B_{ij} \varphi + D_{ijk} \varphi_{,k} - \beta_{ij} \theta + S_{ij}^*, \\ H_i &= A_{ij} \varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi - a_i \theta + H_i^*, \\ g &= -B_{ij} e_{ij} - \xi \varphi - d_i \varphi_{,i} + m \theta + g^*, \\ \rho \eta &= \beta_{ij} e_{ij} + a \theta + m \varphi + a_i \varphi_{,i}, \\ Q_i &= k_{ij} \theta_{,j} + f_{irs} \dot{e}_{rs} + b_i \dot{\varphi} + a_{ij} \dot{\varphi}_{,j}, \end{aligned}$$

with S_{ij}^* , H_i^* and g^* given by

$$(2.6) \quad \begin{aligned} S_{ij}^* &= C_{ijrs}^* \dot{e}_{rs} + B_{ij}^* \dot{\varphi} + D_{ijk}^* \dot{\varphi}_{,k} + M_{ijk}^* \theta_{,k}, \\ H_i^* &= A_{ij}^* \dot{\varphi}_{,j} + G_{rsi}^* \dot{e}_{rs} + d_i^* \dot{\varphi} + P_{ij}^* \theta_{,j}, \\ g^* &= -F_{ij}^* \dot{e}_{ij} - \xi^* \dot{\varphi} - \gamma_k^* \dot{\varphi}_{,k} - R_j^* \theta_{,j}, \end{aligned}$$

for $\overline{B} \times [0, \infty)$.

Here, u_i are the components of the displacement vector, φ is the void volume fraction, θ is the variation of temperature from the uniform reference absolute temperature $T_0 > 0$, t_{ij} are the components of stress tensor, H_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated force, Q_i are the components of the heat flux vector, κ is the equilibrated inertia, η is the entropy density per unit mass, ρ is the density mass, f_i are the components of the body force vector, ℓ is the extrinsic equilibrated body force per unit mass, and S is the heat supply per unit mass.

The constitutive coefficients are prescribed functions depending on the spatial variable \mathbf{x} , with the following symmetries

$$(2.7) \quad C_{ijrs} = C_{jirs} = C_{rsij}, \quad B_{ij} = B_{ji}, \quad D_{ijk} = D_{jik}, \quad \beta_{ij} = \beta_{ji}, \quad A_{ij} = A_{ji}, \\ C_{ijrs}^* = C_{jirs}^* = C_{rsij}^*, \quad B_{ij}^* = B_{ji}^*, \quad D_{ijk}^* = D_{jik}^*, \quad A_{ij}^* = A_{ji}^*, \\ (2.8) \quad k_{ij} = k_{ji}, \quad M_{ijk}^* = M_{jik}^*, \quad G_{rsi}^* = G_{sri}^*, \quad F_{ij}^* = F_{ji}^*, \\ P_{ij}^* = P_{ji}^*, \quad f_{irs} = f_{isr}, \quad a_{ij} = a_{ji}.$$

Furthermore, in view of the second law of thermodynamics, the Clausius–Duhem inequality must be satisfied, which provides the positive semi-definiteness of the total dissipation energy Λ , that is

$$(2.9) \quad \Lambda \equiv C_{ijrs}^* \dot{e}_{ij} \dot{e}_{rs} + \xi^* \dot{\varphi}^2 + A_{ij}^* \dot{\varphi}_{,i} \dot{\varphi}_{,j} + \frac{1}{T_0} k_{ij} \theta_{,i} \theta_{,j} + (B_{ij}^* + F_{ij}^*) \dot{e}_{ij} \dot{\varphi} \\ + (D_{ijk}^* + G_{ijk}^*) \dot{e}_{ij} \dot{\varphi}_{,k} + \left(M_{ijk}^* + \frac{1}{T_0} f_{kij} \right) \dot{e}_{ij} \theta_{,k} + (d_i^* + \gamma_i^*) \dot{\varphi} \dot{\varphi}_{,i} \\ + \left(R_j^* + \frac{1}{T_0} b_j \right) \dot{\varphi} \theta_{,j} + \left(P_{ij}^* + \frac{1}{T_0} a_{ji} \right) \dot{\varphi}_{,i} \theta_{,j} \geq 0.$$

Moreover, we consider that the displacement, temperature variation, and volume fraction are such that a classical solution exists for $\overline{B} \times [0, \infty)$. This means that: (i) u_i and φ are of class $C^{2,2}$ for $B \times (0, \infty)$, (ii) θ is of class $C^{2,1}$ for $B \times (0, \infty)$, (iii) $u_i, \dot{u}_i, \ddot{u}_i, \varphi, \dot{\varphi}, \ddot{\varphi}, u_{i,j}, \dot{u}_{i,j}, \varphi_{,i}, \dot{\varphi}_{,i}, \theta, \theta_{,i}$, and $\dot{\theta}$ are continuous for $\overline{B} \times [0, \infty)$.

In what follows, we consider that the region $B \subset \mathbb{R}^3$ is a prismatic cylinder whose bounded uniform cross-section $D \subset \mathbb{R}^2$ has piecewise continuously differentiable boundary ∂D . We suppose that the region B is made of an anisotropic and inhomogeneous thermoviscoelastic material with voids. The origin of the Cartesian coordinate system is located in the cylinder's base and the positive x_3 -axis is directed along that of the cylinder. For further convenience we introduce the following notation:

$$B(z) = \{\mathbf{x} \in B : z \leq x_3\},$$

and, moreover, we will use $D(x_3)$ to indicate that the respective quantities are evaluated over the cross-section whose distance from the origin is x_3 . We denote by Π the lateral surface of the cylinder, that is $\Pi = \partial D \times [0, L]$, where L is the length of the cylinder (see Fig. 1).

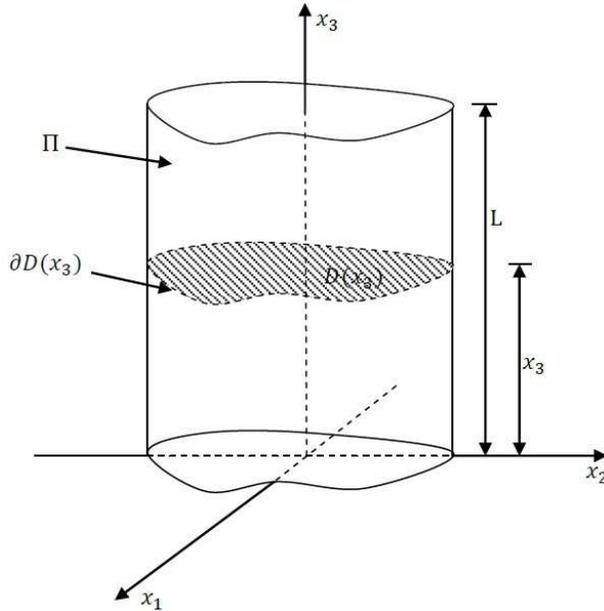


FIG. 1. The cylinder under study and its dimensions.

In what follows, we consider that the displacement, the temperature variation, and the volume fraction field on the base of the cylinder are pointwise prescribed, the supply terms are absent and zero lateral specific boundary conditions are considered. Therefore, we will consider the problem described by the following differential system:

$$(2.10) \quad t_{ji,j} = \rho \dot{u}_i,$$

$$(2.11) \quad H_{j,j} + g = \rho \kappa \ddot{\varphi},$$

$$(2.12) \quad \rho T_0 \dot{\eta} = Q_{j,j},$$

for $B \times [0, T]$, subject to the lateral boundary conditions

$$(2.13) \quad \begin{aligned} \dot{u}_i [C_{ijrs} u_{r,s} + B_{ij} \varphi + D_{ijk} \varphi_{,k} - \beta_{ij} \theta + S_{ij}^*] n_j &= 0, \\ \dot{\varphi} [A_{ji} \varphi_{,i} + D_{rsj} u_{r,s} + d_j \varphi - a_j \theta + H_j^*] n_j &= 0, \\ \theta [k_{ji} \theta_{,i} + f_{jrs} \dot{u}_{r,s} + b_j \dot{\varphi} + a_{ij} \dot{\varphi}_{,i}] n_j &= 0, \end{aligned}$$

for $(\mathbf{x}, t) \in \Pi \times [0, T]$, the base boundary conditions

$$(2.14) \quad u_i(\mathbf{x}, t) = f_i(x_1, x_2, t), \quad \varphi(\mathbf{x}, t) = g(x_1, x_2, t), \quad \theta(\mathbf{x}, t) = h(x_1, x_2, t),$$

for $(\mathbf{x}, t) \in D(0) \times [0, T]$, and the final conditions at time T given by

$$(2.15) \quad \begin{aligned} u_i(\mathbf{x}, T) &= \lambda u_i(\mathbf{x}, 0), \quad \varphi(\mathbf{x}, T) = \lambda \varphi(\mathbf{x}, 0), \quad \theta(\mathbf{x}, T) = \mu \theta(\mathbf{x}, 0), \\ \dot{u}_i(\mathbf{x}, T) &= \alpha \dot{u}_i(\mathbf{x}, 0), \quad \dot{\varphi}(\mathbf{x}, T) = \beta \dot{\varphi}(\mathbf{x}, 0), \quad \text{for } x \in B. \end{aligned}$$

Here, n_i are the components of the unit outward normal on Π , $f_i(x_1, x_2, t)$, $g(x_1, x_2, t)$, and $h(x_1, x_2, t)$ are prescribed differentiable functions compatible with the initial/final data and the lateral boundary data. The parameters λ , μ , α , and β are prescribed and satisfy the conditions

$$(2.16) \quad |\lambda| > 1, \quad |\mu| > 1, \quad |\alpha| > 1, \quad |\beta| > 1.$$

We are interested in the study of the spatial behavior of the solution $\{\mathbf{u}, \varphi, \theta\}(\mathbf{x}, t)$ of the non-standard problem (\mathcal{P}) defined by the evolution equations (2.10)–(2.12), the geometrical relations (2.4), the constitutive equations (2.5), the lateral boundary conditions (2.13), the base boundary conditions (2.14), and the initial-final conditions (2.15).

The conditions (2.15) are also called non-standard conditions, because when $\lambda = 0$, $\mu = 0$, $\alpha = 0$ and $\beta = 0$ this leads to an improperly posed problem. In what follows, we are interested in determining a range of values for the λ , μ , α and β for which the problem is well-posed and to obtain spatial estimates which describe how the solution evolves with respect to the distance from the cylinder's base.

Note that the standard problem (the forward in time problem) associated with the linear theory of thermoviscoelastic materials with voids has been studied by IEŞAN in [2].

3. Constitutive hypotheses and auxiliary estimates

In what follows, we assume some constitutive hypotheses that are necessary for the study of the spatial behavior of the solution to the problem (\mathcal{P}) (defined in Section 2). Therefore, we suppose that ρ , κ , and the constitutive coefficients are continuous and bounded fields on \overline{B} and

$$(3.1) \quad \rho(\mathbf{x}) \geq \rho_0 > 0, \quad a(\mathbf{x}) \geq a_0 > 0, \quad \kappa(\mathbf{x}) \geq \kappa_0 > 0,$$

with ρ_0 , a_0 , and κ_0 positive constants.

Moreover, we suppose that the dissipation energy density Λ is a positive quadratic form in terms of \dot{e}_{ij} , $\dot{\varphi}$, $\dot{\varphi}_{,i}$, and $\theta_{,i}$, so there exist μ_m^* , μ_M^* , ν_m^* , ν_M^* , γ_m^* , γ_M^* , k_m , and k_M such that

$$(3.2) \quad \Lambda \leq \mu_M^* \dot{e}_{ij} \dot{e}_{ij} + \nu_M^* \dot{\varphi}^2 + \gamma_M^* \kappa_0 \dot{\varphi}_{,i} \dot{\varphi}_{,i} + \frac{1}{T_0} k_M \theta_{,i} \theta_{,i},$$

and

$$(3.3) \quad \Lambda \geq \mu_m^* \dot{e}_{ij} \dot{e}_{ij} + \nu_m^* \dot{\varphi}^2 + \gamma_m^* \kappa_0 \dot{\varphi}_{,i} \dot{\varphi}_{,i} + \frac{1}{T_0} k_m \theta_{,i} \theta_{,i}.$$

We also assume that the specific internal energy W defined by

$$(3.4) \quad W = \frac{1}{2} C_{ijrs} e_{ij} e_{rs} + \frac{1}{2} \xi \varphi^2 + \frac{1}{2} A_{ij} \varphi_{,i} \varphi_{,j} + B_{ij} e_{ij} \varphi + D_{ijk} e_{ij} \varphi_{,k} + d_i \varphi \varphi_{,i},$$

is a positive quadratic form in terms of the variables e_{ij} , φ , and $\varphi_{,i}$. Therefore, there exist positive constants μ_m and μ_M such that

$$(3.5) \quad \mu_m (e_{ij} e_{ij} + \varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}) \leq 2W \leq \mu_M (e_{ij} e_{ij} + \varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}).$$

In what follows, we will establish some estimates that we will use further to prove the spatial behavior of the solution to the non-standard problem (\mathcal{P}) .

Following [8], we introduce the linear space \mathcal{D}_4 as the set of all four-dimensional displacements fields \mathbf{U} defined by

$$(3.6) \quad \mathbf{U} := \{u_i, \sqrt{\kappa_0} \varphi\}.$$

For every $\mathbf{U} \in \mathcal{D}_4$, we consider the state of strain $\mathbf{E}(\mathbf{U})$ defined by

$$(3.7) \quad \mathbf{E}(\mathbf{U}) := \{e_{ij}(\mathbf{U}), \varphi, \sqrt{\kappa_0} \varphi_{,i}(\mathbf{U})\}.$$

We will denote by \mathcal{E} the vector space of all objects of the form (3.7). For every state $\mathbf{E} \in \mathcal{E}$, we introduce the field

$$(3.8) \quad \mathbf{S}(\mathbf{E}) = \left\{ S_{ij}(\mathbf{E}), G(\mathbf{E}), \frac{1}{\sqrt{\kappa_0}} h_i(\mathbf{E}) \right\},$$

where

$$(3.9) \quad S_{ij}(\mathbf{E}) = C_{ijrs} e_{rs} + B_{ij} \varphi + D_{ijk} \varphi_{,k},$$

$$(3.10) \quad G(\mathbf{E}) = -B_{ij} e_{ij} - \xi \varphi - d_i \varphi_{,i},$$

$$(3.11) \quad h_i(\mathbf{E}) = A_{ij} \varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi.$$

Throughout the following, we consider the bilinear form defined by

$$(3.12) \quad \mathcal{G}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) = \frac{1}{2} \left[C_{ijrs} e_{ij}^{(1)} e_{rs}^{(2)} + \xi \varphi^{(1)} \varphi^{(2)} + A_{ij} \varphi_{,i}^{(1)} \varphi_{,j}^{(2)} \right. \\ \left. + B_{ij} (e_{ij}^{(1)} \varphi^{(2)} + e_{ij}^{(2)} \varphi^{(1)}) + D_{ijk} (e_{ij}^{(1)} \varphi_{,k}^{(2)} + e_{ij}^{(2)} \varphi_{,k}^{(1)}) \right. \\ \left. + d_i (\varphi^{(1)} \varphi_{,i}^{(2)} + \varphi^{(2)} \varphi_{,i}^{(1)}) \right],$$

for every $\mathbf{E}^{(\alpha)}(\mathbf{U}) \in \mathcal{E}$, $\alpha = 1, 2$, where

$$\mathbf{E}^{(\alpha)}(\mathbf{U}) = \left\{ e_{ij}(\mathbf{U}^{(\alpha)}), \varphi(\mathbf{U}^{(\alpha)}), \sqrt{\kappa_0} \varphi_{,i}(\mathbf{U}^{(\alpha)}) \right\}.$$

We can remark that

$$(3.13) \quad \mathcal{G}(\mathbf{E}, \mathbf{E}) = \mathbf{W}(\mathbf{E}), \quad \forall \mathbf{E} \in \mathcal{E}.$$

Moreover, by using the Cauchy–Schwarz inequality, we can write

$$(3.14) \quad \mathcal{G}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) \leq [W(\mathbf{E}^{(1)})]^{1/2} [W(\mathbf{E}^{(2)})]^{1/2}, \quad \forall \mathbf{E}^{(1)}, \mathbf{E}^{(2)} \in \mathcal{E}.$$

On the other hand, according to Eqs. (3.9)–(3.11), we have

$$(3.15) \quad |\mathbf{S}(\mathbf{E})|^2 = S_{ij}(\mathbf{E}) S_{ij}(\mathbf{E}) + G^2(\mathbf{E}) + \frac{1}{\kappa_0} h_i(\mathbf{E}) h_i(\mathbf{E}) \\ = C_{ijrs} e_{rs} S_{ij} + B_{ij} \varphi S_{ij} + D_{ijk} \varphi_{,k} S_{ij} - B_{ij} e_{ij} G - \xi \varphi G \\ - d_i \varphi_{,i} G + \frac{1}{\kappa_0} A_{ij} \varphi_{,j} h_i + \frac{1}{\kappa_0} D_{rsi} e_{rs} h_i + \frac{1}{\kappa_0} d_i \varphi h_i.$$

If we introduce the following notation:

$$(3.16) \quad \Gamma(\mathbf{E}) = \left\{ S_{ij}(\mathbf{E}), -G(\mathbf{E}), \frac{1}{\sqrt{\kappa_0}} h_i(\mathbf{E}) \right\},$$

then, from (3.15) we get

$$(3.17) \quad |\mathbf{S}(\mathbf{E})|^2 = 2\mathcal{G}(\mathbf{E}, \Gamma(\mathbf{E})).$$

From the assumption that $W(\mathbf{E})$ is a positive definite quadratic form, we obtain

$$(3.18) \quad 2W(\Gamma(\mathbf{E})) \leq \mu_M \left(S_{ij} S_{ij} + G^2 + \frac{1}{\kappa_0} h_i h_i \right),$$

and consequently, by means of the Cauchy–Schwarz inequality and relation (3.18), we get

$$(3.19) \quad S_{ij}(\mathbf{E}) S_{ij}(\mathbf{E}) + \frac{1}{\kappa_0} h_i(\mathbf{E}) h_i(\mathbf{E}) \leq 2\mu_M W(\mathbf{E}), \quad \forall \mathbf{E} \in \mathcal{E}.$$

In view of the relations (2.5), (2.6), and (3.9)–(3.11), we can write

$$(3.20) \quad t_{ij}t_{ij} + \frac{1}{\kappa_0}H_iH_i \leq 2 \left[(S_{ij} - \beta_{ij}\theta)(S_{ij} - \beta_{ij}\theta) + \frac{1}{\kappa_0}(h_i - a_i\theta)(h_i - a_i\theta) + S_{ij}^*S_{ij}^* + \frac{1}{\kappa_0}H_i^*H_i^* \right].$$

We recall that for all second-order tensors M_{ij}, N_{ij} and every positive number ε , the following inequality holds

$$(3.21) \quad (M_{ij} + N_{ij})(M_{ij} + N_{ij}) \leq (1 + \varepsilon)M_{ij}M_{ij} + \left(1 + \frac{1}{\varepsilon}\right)N_{ij}N_{ij}.$$

Therefore, applying (3.21) in (3.20), we obtain

$$(3.22) \quad t_{ij}t_{ij} + \frac{1}{\kappa_0}H_iH_i \leq 2 \left[(1 + \varepsilon) \left(S_{ij}S_{ij} + \frac{1}{\kappa_0}h_ih_i \right) + \left(1 + \frac{1}{\varepsilon}\right)M^2\theta^2 + S_{ij}^*S_{ij}^* + \frac{1}{\kappa_0}H_i^*H_i^* \right],$$

where

$$(3.23) \quad M^2 = \max_{\mathbf{x} \in \bar{B}} \left(\beta_{ij}\beta_{ij} + \frac{1}{\kappa_0}a_ia_i \right).$$

Moreover, with the aid of the inequality established in (3.19), we get

$$(3.24) \quad t_{ij}t_{ij} + \frac{1}{\kappa_0}H_iH_i \leq 4\mu_M(1 + \varepsilon)W(\mathbf{E}) + 2 \left(1 + \frac{1}{\varepsilon}\right)M^2\theta^2 + 2 \left(S_{ij}^*S_{ij}^* + \frac{1}{\kappa_0}H_i^*H_i^* \right),$$

for all $\varepsilon > 0$.

We proceed now to estimate $S_{ij}^*S_{ij}^* + \frac{1}{\kappa_0}H_i^*H_i^*$. According to the constitutive equations (2.6), we can write

$$(3.25) \quad S_{ij}^*S_{ij}^* + \frac{1}{\kappa_0}H_i^*H_i^* = C_{ijrs}^*\dot{e}_{rs}S_{ij}^* + B_{ij}^*\dot{\varphi}S_{ij}^* + D_{ijk}^*\dot{\varphi}_{,k}S_{ij}^* + M_{ijk}^*\theta_{,k}S_{ij}^* + \frac{1}{\kappa_0}A_{ij}^*\dot{\varphi}_{,j}H_i^* + \frac{1}{\kappa_0}G_{rsi}^*\dot{e}_{rs}H_i^* + \frac{1}{\kappa_0}d_i^*\dot{\varphi}H_i^* + \frac{1}{\kappa_0}P_{ij}^*\theta_{,j}H_i^*.$$

In order to obtain an estimation for $S_{ij}^*S_{ij}^* + \frac{1}{\kappa_0}H_i^*H_i^*$, we must evaluate every term of (3.25). For instance, we have

$$\begin{aligned}
(3.26) \quad C_{ijrs}^* \dot{e}_{rs} S_{ij}^* + \frac{1}{\kappa_0} G_{rsi}^* \dot{e}_{rs} H_i^* & \\
& \leq (C_{mnpq}^* C_{mnpq}^*)^{1/2} (\dot{e}_{rs} S_{ij}^* \dot{e}_{rs} S_{ij}^*)^{1/2} \\
& \quad + \left(\frac{1}{\kappa_0} G_{mnp}^* G_{mnp}^* \right)^{1/2} \left(\dot{e}_{rs} \dot{e}_{rs} \frac{1}{\kappa_0} H_i^* H_i^* \right)^{1/2} \\
& \leq (C_{mnpq}^* C_{mnpq}^*)^{1/2} (\dot{e}_{rs} \dot{e}_{rs})^{1/2} (S_{ij}^* S_{ij}^*)^{1/2} \\
& \quad + \left(\frac{1}{\kappa_0} G_{mnp}^* G_{mnp}^* \right)^{1/2} (\dot{e}_{rs} \dot{e}_{rs})^{1/2} \left(\frac{1}{\kappa_0} H_i^* H_i^* \right)^{1/2},
\end{aligned}$$

and hence we obtain

$$C_{ijrs}^* \dot{e}_{rs} S_{ij}^* + \frac{1}{\kappa_0} G_{rsi}^* \dot{e}_{rs} H_i^* \leq \alpha_1 (\dot{e}_{rs} \dot{e}_{rs})^{1/2} \left(S_{ij}^* S_{ij}^* + \frac{1}{\kappa_0} H_i^* H_i^* \right)^{1/2},$$

with

$$(3.27) \quad \alpha_1 = \max_{\mathbf{x} \in \overline{B}} \left[(C_{mnpq}^* C_{mnpq}^*)^{1/2} + \left(\frac{1}{\kappa_0} G_{mnp}^* G_{mnp}^* \right)^{1/2} \right].$$

Applying the same procedure to the other terms of (3.27), we get

$$\begin{aligned}
(3.28) \quad S_{ij}^* S_{ij}^* + \frac{1}{\kappa_0} H_i^* H_i^* & \leq \left[\alpha_1 (\dot{e}_{rs} \dot{e}_{rs})^{1/2} + \alpha_2 |\dot{\varphi}| + \alpha_3 (\kappa_0 \dot{\varphi}_{,j} \dot{\varphi}_{,j})^{1/2} \right. \\
& \quad \left. + \alpha_4 \left(\frac{1}{T_0} \theta_{,j} \theta_{,j} \right)^{1/2} \right] \left(S_{ij}^* S_{ij}^* + \frac{1}{\kappa_0} H_i^* H_i^* \right)^{1/2},
\end{aligned}$$

with

$$\begin{aligned}
(3.29) \quad \alpha_2 & = \max_{\mathbf{x} \in \overline{B}} \left[(B_{mn}^* B_{mn}^*)^{1/2} + \left(\frac{1}{\kappa_0} d_m^* d_m^* \right)^{1/2} \right], \\
\alpha_3 & = \max_{\mathbf{x} \in \overline{B}} \left[\left(\frac{1}{\kappa_0} D_{mnp}^* D_{mnp}^* \right)^{1/2} + \left(\frac{1}{\kappa_0^2} A_{mn}^* A_{mn}^* \right)^{1/2} \right], \\
\alpha_4 & = \max_{\mathbf{x} \in \overline{B}} \left[(T_0 M_{mnp}^* M_{mnp}^*)^{1/2} + \left(\frac{T_0}{\kappa_0} P_{mn}^* P_{mn}^* \right)^{1/2} \right],
\end{aligned}$$

and α_1 given by (3.27). Thus, the inequality (3.28) can be written in the following form:

$$S_{ij}^* S_{ij}^* + \frac{1}{\kappa_0} H_i^* H_i^* \leq 4 \left[\alpha_1^2 \dot{e}_{ij} \dot{e}_{ij} + \alpha_2^2 \dot{\varphi}^2 + \alpha_3^2 \kappa_0 \dot{\varphi}_{,i} \dot{\varphi}_{,i} + \alpha_4^2 \frac{1}{T_0} \theta_{,i} \theta_{,i} \right].$$

Hence, we obtain the following estimation:

$$(3.30) \quad S_{ij}^* S_{ij}^* + \frac{1}{\kappa} H_i^* H_i^* \leq \aleph_M A_1,$$

with

$$(3.31) \quad A_1 = \frac{\mu_m^*}{2} \dot{e}_{ij} \dot{e}_{ij} + \frac{\nu_m^*}{2} \dot{\varphi}^2 + \frac{\gamma_m^*}{2} \kappa_0 \dot{\varphi}_{,i} \dot{\varphi}_{,i} + \frac{1}{2T_0} k_m \theta_{,i} \theta_{,i},$$

and

$$\aleph_M = \max_{\mathbf{x} \in \bar{B}} \left\{ \frac{8\alpha_1^2}{\mu_m^*}, \frac{8\alpha_2^2}{\nu_m^*}, \frac{8\alpha_3^2}{\gamma_m^*}, \frac{8\alpha_4^2}{k_m} \right\}.$$

Therefore, a consequence of the relations (3.30) and (3.23) is the following:

$$(3.32) \quad t_{ij} t_{ij} + \frac{1}{\kappa_0} H_i H_i \leq 4\mu_M (1 + \varepsilon) W(\mathbf{E}) + 2 \left(1 + \frac{1}{\varepsilon} \right) M^2 \theta^2 + 2\aleph_M A_1.$$

In order to estimate $Q_i Q_i$, we introduce the following notations:

$$(3.33) \quad \begin{aligned} q_i &= k_{ij} \theta_{,j}, \\ \tilde{q}_i &= f_{irs} \dot{e}_{rs} + b_i \dot{\varphi} + a_{ij} \dot{\varphi}_{,j}. \end{aligned}$$

Thus, we can write

$$(3.34) \quad q_i q_i \leq (T_0 k_{rs} k_{rs})^{1/2} \left(\frac{1}{T_0} \theta_{,j} \theta_{,j} \right)^{1/2} (q_i q_i)^{1/2},$$

so that we derive

$$(3.35) \quad q_i q_i \leq 2K_M \left(\frac{1}{2T_0} k_m \theta_{,i} \theta_{,i} \right),$$

where

$$K_M = \frac{k_{rs} k_{rs} T_0}{k_m}.$$

In a similar way, from (3.33) it follows that

$$(3.36) \quad \begin{aligned} \tilde{q}_i \tilde{q}_i &= (f_{irs} \dot{e}_{rs} + b_i \dot{\varphi} + a_{ij} \dot{\varphi}_{,j}) \tilde{q}_i \\ &\leq \left[(f_{mnp} f_{mnp})^{1/2} (\dot{e}_{rs} \dot{e}_{rs})^{1/2} + (b_m b_m)^{1/2} |\dot{\varphi}| \right. \\ &\quad \left. + \left(\frac{1}{\kappa_0} a_{mn} a_{mn} \right)^{1/2} (\kappa_0 \dot{\varphi}_{,j} \dot{\varphi}_{,j})^{1/2} \right] (\tilde{q}_i \tilde{q}_i)^{1/2}. \end{aligned}$$

If we denote by

$$(3.37) \quad \begin{aligned} \vartheta_1 &= \max_{\mathbf{x} \in \overline{B}} (f_{mnp} f_{mnp})^{1/2}, & \vartheta_2 &= \max_{\mathbf{x} \in \overline{B}} (b_m b_m)^{1/2}, \\ \vartheta_3 &= \max_{\mathbf{x} \in \overline{B}} \left(\frac{1}{\kappa_0} a_{mn} a_{mn} \right)^{1/2}, \end{aligned}$$

then, in view of the relation (3.36) we obtain

$$(3.38) \quad \begin{aligned} \tilde{q}_i \tilde{q}_i &\leq 3(\vartheta_1^2 \dot{e}_{ij} \dot{e}_{ij} + \vartheta_2^2 \dot{\varphi}^2 + \vartheta_3^2 \dot{\varphi}_{,i} \dot{\varphi}_{,i}) \\ &\leq \beth_M \left(\frac{\mu_m^*}{2} \dot{e}_{ij} \dot{e}_{ij} + \frac{\nu_m^*}{2} \dot{\varphi}^2 + \frac{\gamma_m^*}{2} \kappa_0 \dot{\varphi}_{,i} \dot{\varphi}_{,i} \right), \end{aligned}$$

with

$$\beth_M = \max_{\mathbf{x} \in \overline{B}} \left\{ \frac{6\vartheta_1^2}{\mu_m^*}, \frac{6\vartheta_2^2}{\nu_m^*}, \frac{6\vartheta_3^2}{\gamma_m^*} \right\}.$$

By using relations (3.21) and (3.33) we obtain

$$(3.39) \quad Q_i Q_i \leq (1 + \varepsilon') q_i q_i + \left(1 + \frac{1}{\varepsilon'} \right) \tilde{q}_i \tilde{q}_i,$$

for every $\varepsilon' > 0$.

Consequently, by means of (3.35), (3.38) and (3.39), we obtain the following estimation for $Q_i Q_i$:

$$(3.40) \quad Q_i Q_i \leq 2K_M (1 + \varepsilon') A_2 + \beth_M \left(1 + \frac{1}{\varepsilon'} \right) A_3,$$

where

$$(3.41) \quad A_2 = \frac{k_m}{2T_0} \theta_{,i} \theta_{,i},$$

$$(3.42) \quad A_3 = \frac{\mu_m^*}{2} \dot{e}_{ij} \dot{e}_{ij} + \frac{\nu_m^*}{2} \dot{\varphi}^2 + \frac{\gamma_m^*}{2} \kappa_0 \dot{\varphi}_{,i} \dot{\varphi}_{,i}.$$

4. Spatial behavior

In this section we will study the spatial behavior of the solution to the non-standard problem (\mathcal{P}) defined previously. In order to do that, we introduce the following function:

$$(4.1) \quad I(x_3) = \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left(t_{3i} \dot{u}_i + H_3 \dot{\varphi} + \frac{1}{T_0} Q_3 \theta \right) da d\tau, \quad x_3 \in [0, L],$$

with σ a positive parameter at our disposal whose values will be explicitly given later. In the above relation we have used the notation $D(x_3, \tau)$ to indicate that

relevant quantities are to be evaluated at time τ over the cross-section of the cylinder whose distance from the origin is x_3 .

By differentiation with respect to x_3 in (4.1), and by using the evolution equations (2.10)–(2.12), we obtain

$$\begin{aligned}
 (4.2) \quad \frac{dI}{dx_3}(x_3) &= - \int_0^T \int_{\partial D(x_3, \tau)} e^{-\sigma\tau} \left(t_{\alpha i} n_\alpha \dot{u}_i + H_\alpha n_\alpha \dot{\varphi} + \frac{1}{T_0} Q_\alpha n_\alpha \theta \right) ds d\tau \\
 &\quad + \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\partial}{\partial \tau} \left(\frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{1}{2} \rho \kappa \dot{\varphi}^2 \right) + t_{ij} \dot{e}_{ij} + H_i \dot{\varphi}_{,i} \right. \\
 &\quad \left. - g \dot{\varphi} + \rho \dot{\eta} \theta + \frac{1}{T_0} Q_i \theta_{,i} \right] da d\tau.
 \end{aligned}$$

In view of the lateral boundary conditions (2.13) and the constitutive equations (2.5), we obtain

$$\begin{aligned}
 (4.3) \quad \frac{dI}{dx_3}(x_3) &= \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{1}{2} \frac{\partial}{\partial \tau} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a \theta^2 + 2W) + \Lambda \right] da d\tau \\
 &= \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a \theta^2 + 2W) + \Lambda \right] da d\tau \\
 &\quad + \int_{D(x_3, T)} \frac{1}{2} e^{-\sigma T} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a \theta^2 + 2W) da \\
 &\quad - \int_{D(x_3, 0)} \frac{1}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a \theta^2 + 2W) da.
 \end{aligned}$$

Therefore, if we use the constraint relations (2.15), we obtain

$$\begin{aligned}
 (4.4) \quad \frac{dI}{dx_3}(x_3) &= \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a \theta^2 + 2W) + \Lambda \right] da d\tau \\
 &\quad + \frac{\alpha^2 e^{-\sigma T} - 1}{2} \int_{D(x_3, 0)} \rho \dot{u}_i \dot{u}_i da + \frac{\beta^2 e^{-\sigma T} - 1}{2} \int_{D(x_3, 0)} \rho \kappa \dot{\varphi}^2 da \\
 &\quad + \frac{\mu^2 e^{-\sigma T} - 1}{2} \int_{D(x_3, 0)} a \theta^2 da + \frac{\lambda^2 e^{-\sigma T} - 1}{2} \int_{D(x_3, 0)} 2W da.
 \end{aligned}$$

Let us assume that the conditions (2.16) hold true. Then, it is possible to choose the parameter σ so that we have

$$(4.5) \quad \lambda^2 e^{-\sigma T} - 1 > 0, \quad \mu^2 e^{-\sigma T} - 1 > 0, \quad \alpha^2 e^{-\sigma T} - 1 > 0, \quad \beta^2 e^{-\sigma T} - 1 > 0.$$

Therefore, we will assume that σ ranges in the set

$$(4.6) \quad 0 < \sigma < \frac{2}{T} \min_{\mathbf{x} \in B} \{\ln|\lambda|, \ln|\mu|, \ln|\alpha|, \ln|\beta|\}.$$

Further, we establish

$$(4.7) \quad 0 < \varkappa_\sigma = \frac{1}{2} \min_B \{\lambda^2 e^{-\sigma T} - 1, \mu^2 e^{-\sigma T} - 1, \alpha^2 e^{-\sigma T} - 1, \beta^2 e^{-\sigma T} - 1\},$$

and note that

$$(4.8) \quad \frac{dI}{dx_3}(x_3) \geq \varkappa_\sigma \int_{D(x_3,0)} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) da \\ + \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) + \Lambda \right] da d\tau.$$

It is easy to observe that in view of assumptions of the positive definiteness of the internal energy density W and the dissipation energy density Λ , we obtain that $I(x_3)$ is a non-decreasing function with respect to x_3 on $[0, L]$.

In what follows, we want to obtain an appropriate estimate for the function $I(x_3)$. Therefore, by using the Schwarz inequality in (4.1), we obtain

$$(4.9) \quad |I(x_3)| \leq \frac{1}{2} \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left[\frac{\varepsilon_1}{\rho_0} \left(t_{ij} t_{ij} + \frac{1}{\kappa_0} H_i H_i \right) + \frac{1}{\varepsilon_1} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2) \right. \\ \left. + \frac{\varepsilon_2}{a_0 T_0} Q_i Q_i + \frac{1}{\varepsilon_2 T_0} a\theta^2 \right] da d\tau,$$

for every $\varepsilon_1, \varepsilon_2$ positive numbers. According to the estimates (3.24), (3.30), and (3.39), we can write

$$(4.10) \quad |I(x_3)| \leq \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left\{ \frac{1}{\varepsilon_1 \sigma} \frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2) + \frac{2\mu_M \varepsilon_1 (1 + \varepsilon)}{\rho_0 \sigma} \sigma W \right. \\ \left. + \left[\frac{2\varepsilon_1 M^2}{\rho_0 a_0 \sigma} \left(1 + \frac{1}{\varepsilon} \right) + \frac{1}{\varepsilon_2 T_0 \sigma} \right] \frac{\sigma}{2} a\theta^2 + \frac{\varepsilon_2 K_M (1 + \varepsilon')}{a_0 T_0} \Lambda_2 \right. \\ \left. + \frac{\varepsilon_2 \mathfrak{M}}{2a_0 T_0} \left(1 + \frac{1}{\varepsilon'} \right) \Lambda_3 \right\} da d\tau + \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \frac{\varepsilon_1 \mathfrak{N}_M}{\rho_0} \Lambda_1 da d\tau.$$

Further, we equate the coefficients of the various energy terms of the last integral:

$$\begin{aligned} \frac{1}{\varepsilon_1 \sigma} &= \frac{2\mu_M \varepsilon_1 (1 + \varepsilon)}{\rho_0 \sigma} = \frac{2\varepsilon_1 M^2}{\rho_0 a_0 \sigma} \left(1 + \frac{1}{\varepsilon}\right) + \frac{1}{\varepsilon_2 T_0 \sigma} \\ &= \frac{\varepsilon_2 K_M (1 + \varepsilon')}{a_0 T_0} = \frac{\varepsilon_2 \beth_M}{2a_0 T_0} \left(1 + \frac{1}{\varepsilon'}\right). \end{aligned}$$

Therefore, we set

$$(4.11) \quad \varepsilon_1 = \frac{1}{c}, \quad \varepsilon_2 = \frac{2a_0 T_0 c}{\sigma(2K_M + \beth_M)}, \quad \varepsilon' = \frac{\beth_M}{2K_M},$$

where

$$(4.12) \quad c = \sqrt{\frac{2\mu_M(1 + \varepsilon)}{\rho_0}},$$

and

$$(4.13) \quad \varepsilon = \frac{1}{2} \left[-1 + \frac{M^2}{\mu_M a_0} + \frac{\rho_0 \sigma \beth_M}{4\mu_M a_0 T_0^2} + \frac{\rho_0 \sigma K_M}{2\mu_M a_0 T_0^2} + \sqrt{\left(1 - \frac{M^2}{\mu_M a_0} - \frac{\rho_0 \sigma \beth_M}{4\mu_M a_0 T_0^2} - \frac{\rho_0 \sigma K_M}{2\mu_M a_0 T_0^2}\right)^2 + \frac{4M^2}{\mu_M a_0}} \right].$$

With these choices, the relation (4.10) becomes

$$(4.14) \quad |I(x_3)| \leq \frac{\omega}{\sigma} \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2) + \sigma W + 2\Lambda_1 \right] da \, d\tau,$$

where

$$(4.15) \quad \vartheta = \max_{\mathbf{x} \in B} \left\{ c, \frac{\sigma \alpha_M \varepsilon_1}{\rho_0} \right\}.$$

Therefore, we obtain the following first-order differential inequality:

$$(4.16) \quad \frac{\sigma}{\omega} |I(x_3)| \leq \frac{dI}{dx_3}(x_3), \quad \forall x_3 \in [0, L].$$

In order to discuss the implications of (4.16), we first have to observe that the non-decreasing function $I(x_3)$ implies only two possibilities:

(i) $I(x_3) \leq 0$ for all $x \in [0, L]$

or

(ii) there exists $x_3^* \in [0, L]$ so that $I(x_3^*) > 0$.

Let us consider the case (i). If we assume that $I(x_3) \leq 0, \forall x_3 \in [0, L]$, then the differential inequality (4.16) implies that

$$(4.17) \quad \frac{dI}{dx_3}(x_3) + \frac{\sigma}{\omega}I(x_3) \geq 0, \quad \text{for all } x_3 \in [0, L].$$

By integrating the above relation, we obtain the following decay estimate of Saint-Venant type:

$$(4.18) \quad 0 \leq -I(x_3) \leq -I(0)e^{-\frac{\sigma}{\omega}x_3}, \quad \text{for all } x_3 \in [0, L].$$

If we consider the case of a semi-infinite cylinder (i.e., the case when $L \rightarrow \infty$), then the relation (4.18) proves that $I(x_3) \rightarrow 0$ as $x_3 \rightarrow \infty$. Moreover, the relation (4.8) gives us the following decay estimate:

$$(4.19) \quad E(x_3) \leq -I(0)e^{-\frac{\sigma}{\omega}x_3}, \quad \text{for all } x_3 \in [0, \infty),$$

with

$$(4.20) \quad E(x_3) = \varkappa_\sigma \int_{D(x_3,0)} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) dv \\ + \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) + \Lambda \right] dv d\tau.$$

We consider now the case (ii). If we suppose that there exists $x_3^* \in [0, L]$ so that $I(x_3) > 0$, then, by taking into account that $I(x_3)$ is a non-decreasing function with respect to x_3 , we obtain that

$$I(x_3) \geq I(x_3^*) > 0, \quad \text{for all } x_3 \in [x_3^*, L].$$

Therefore, in view of the relation (4.16) we obtain the following differential inequality:

$$(4.21) \quad \frac{dI}{dx_3}(x_3) - \frac{\sigma}{\omega}I(x_3) \leq 0, \quad \text{for all } x_3 \in [x_3^*, L],$$

which after integration furnishes the following growth estimate:

$$(4.22) \quad I(x_3) \geq I(x_3^*)e^{\frac{\sigma}{\omega}(x_3-x_3^*)}, \quad \text{for all } x_3 \in [x_3^*, L].$$

For a semi-infinite cylinder the above relation proves that $I(x_3)$ becomes unbounded for asymptotically large values of x_3 , and hence $E(x_3)$ becomes unbounded for $L \rightarrow \infty$. Therefore, we have obtained for the semi-infinite cylinder, an alternative of Phragmén–Lindelöf type.

5. Further comments

Our analysis in this work was developed under the assumptions that the constraint parameters $|\lambda| > 1$, $|\mu| > 1$, $|\alpha| > 1$ and $|\beta| > 1$. When we use other values for the constraint parameters, we can observe that it is not possible to develop an analysis similar with the one obtained in the previous section. For example, if we take the conditions $|\lambda| < 1$, $|\mu| < 1$, $|\alpha| < 1$ and $|\beta| < 1$, then we will obtain an ill-posed problem (see, QUINTANILLA and STRAUGHAN [13]).

In Section 4 we have considered the non-standard problem (\mathcal{P}) described by the evolution equations (2.10)-(2.12), the lateral boundary conditions (2.13), the base boundary conditions (2.14), and the final conditions (2.15), in which the displacement and the volume fraction have the same proportionality coefficient.

In what follows, we consider a similar non-standard problem (\mathcal{P}), but instead of the initial-final conditions (2.15), we take the following more general conditions:

$$(5.1) \quad \begin{aligned} u_r(\mathbf{x}, T) &= \lambda u_r(\mathbf{x}, 0), & \varphi(\mathbf{x}, T) &= \gamma \varphi(\mathbf{x}, 0), & \theta(\mathbf{x}, t) &= \mu \theta(\mathbf{x}, 0), \\ \dot{u}_r(\mathbf{x}, T) &= \alpha \dot{u}_r(\mathbf{x}, 0), & \dot{\varphi}(\mathbf{x}, T) &= \beta \dot{\varphi}(\mathbf{x}, 0), & & \text{for } x \in B. \end{aligned}$$

We want to see what conditions we would have to take for the proportionality coefficients λ , γ , μ , α , and β , so that our just presented study may follow the same path.

According to the assumption that the internal energy density W is a positive definite quadratic form, we obtain

$$(5.2) \quad \int_{D(x_3, T)} 2W \, da \geq \mu_m \int_{D(x_3, T)} (e_{ij}e_{ij} + \varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}) \, da.$$

If we use now the conditions (5.1), we get

$$(5.3) \quad \int_{D(x_3, T)} 2W \, da \geq \mu_m \lambda^2 \int_{D(x_3, 0)} e_{ij}e_{ij} \, da + \mu_m \gamma^2 \int_{D(x_3, 0)} (\varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}) \, da.$$

On the other hand, from (3.5) we can write

$$(5.4) \quad \int_{D(x_3, 0)} 2W \leq \mu_M \int_{D(x_3, 0)} (e_{ij}e_{ij} + \varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}) \, da.$$

If we combine the relations (5.3) and (5.4), we obtain

$$\begin{aligned}
 (5.5) \quad \frac{1}{2}e^{-\sigma T} \int_{D(x_3,T)} 2W \, da - \frac{1}{2} \int_{D(x_3,0)} 2W \, da \\
 \geq \frac{1}{2} \left(\frac{\mu_m \lambda^2 e^{-\sigma T}}{\mu_M} - 1 \right) \int_{D(x_3,0)} \mu_M e_{ij} e_{ij} \, da \\
 + \frac{1}{2} \left(\frac{\mu_m \gamma^2 e^{-\sigma T}}{\mu_M} - 1 \right) \int_{D(x_3,0)} \mu_M (\varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}) \, da.
 \end{aligned}$$

Therefore, in this case, instead of relation (4.4), we will have

$$\begin{aligned}
 (5.6) \quad \frac{dI}{dx_3}(x_3) &\geq \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) + \Lambda \right] da \, d\tau \\
 &+ \frac{1}{2} (\alpha^2 e^{-\sigma T} - 1) \int_{D(x_3,0)} \rho \dot{u}_i \dot{u}_i \, da + \frac{1}{2} (\beta^2 e^{-\sigma T} - 1) \int_{D(x_3,0)} \rho \kappa \dot{\varphi}^2 \, da \\
 &+ \frac{1}{2} (\mu^2 e^{-\sigma T} - 1) \int_{D(x_3,0)} a\theta^2 \, da + \frac{1}{2} \left(\frac{\mu_m \lambda^2 e^{-\sigma T}}{\mu_M} - 1 \right) \int_{D(x_3,0)} \mu_M e_{ij} e_{ij} \, da \\
 &+ \frac{1}{2} \left(\frac{\mu_m \gamma^2 e^{-\sigma T}}{\mu_M} - 1 \right) \int_{D(x_3,0)} \mu_M (\varphi^2 + \kappa_0 \varphi_{,i} \varphi_{,i}) \, da.
 \end{aligned}$$

We will choose the parameter σ so that we have

$$0 < \varkappa_\sigma = \frac{1}{2} \min \left\{ \alpha^2 e^{-\sigma T} - 1, \beta^2 e^{-\sigma T} - 1, \mu^2 e^{-\sigma T} - 1, \right. \\
 \left. \frac{\mu_m \lambda^2 e^{-\sigma T}}{\mu_M} - 1, \frac{\mu_m \gamma^2 e^{-\sigma T}}{\mu_M} - 1 \right\}.$$

Thus, we can assume that σ ranges in the set

$$(5.7) \quad 0 < \sigma < \frac{1}{T} \min \left\{ \ln \alpha^2, \ln \beta^2, \ln \mu^2, \ln \frac{\lambda^2 \mu_m}{\mu_M}, \ln \frac{\gamma^2 \mu_m}{\mu_M} \right\},$$

if the following conditions hold true:

$$(5.8) \quad |\alpha| > 1, \quad |\beta| > 1, \quad |\mu| > 1, \quad |\lambda| > \sqrt{\frac{\mu_M}{\mu_m}} > 1, \quad |\gamma| > \sqrt{\frac{\mu_M}{\mu_m}} > 1.$$

With these choices, we can write (5.6) in the following form:

$$(5.9) \quad \frac{dI}{dx_3}(x_3) \geq \varkappa_\sigma \int_{D(x_3,0)} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) da \\ + \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} (\rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\varphi}^2 + a\theta^2 + 2W) + \Lambda \right] da d\tau.$$

So, we can continue to determine spatial estimates of Saint–Venant type or an alternative of Phragmén–Lindelöf type, following the same procedure as in the previous section.

In conclusion, if instead of conditions (2.15) we have (5.1), then the results obtained in Section 4 hold true if the parameters λ , γ , μ , α and β meet the conditions (5.8).

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