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On the propagation of plane waves in piezoelectromagnetic monoclinic crystals

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In a piezoelectromagnetic crystalline medium belonging to the class 2 of the monoclinic crystallographic system we find some classes of piezoelectricity-induced electromagnetic waves. These are time harmonic plane waves propagating along the symmetry axis and depending only on the axial coordinate. There are two independent modes of propagation, one longitudinal and one transverse, with mechanical and electromagnetical couplings. The transverse mode admits as a particular case an electromagnetic wave with no associated elastic deformation.

Key words: piezoelectricity, piezoelectromagnetism, wave propagation, time harmonic plane waves, crystal class 2.

1. Introduction

As is well known, in the Voigt theory of linear piezoelectricity the electromagnetic equations are static and the electric and magnetic fields are not coupled. In this approximation, called quasi-static ([1]–[3]), no electromagnetic field evolves in time, and the wave behavior of electromagnetic fields cannot be described. In [4] we read: "This assumption implies that both the optical effect as well as the contribution from the rotational part of electric field are neglected. Although it is generally believed that the optical effect is minor, it is certainly of practical interest in accurately predict the piezoelectricity-induced electromagnetic radiation, which might be helpful in some engineering applications, such as optical detection, as well as nondestructive evaluation in general."

In fact, couplings between electromagnetic waves and acoustic waves are used in certain ultrasonics applications involving piezoelectric or ferroelectric crystals, for instance to predict the amount of electromagnetic energy radiated by vibrating piezoelectric bodies [5]–[9], in acoustic delay lines [10] and in wireless acoustic wave sensors [11]. Moreover, the conversion of electromagnetic energy into mechanical energy and vice versa can be realized by exploiting this coupling.

The interactions between the mechanical and electromagnetic fields in a piezoelectric material can be described by an extension of Voigt theory in which the simultaneous use of the equations of infinitesimal elastic waves and Maxwell's equations is allowed.

Such interactions are due to Voigt constitutive equations of linear piezoelectricity with full electromagnetic coupling, i.e., with a further linear coupling between the magnetic vector and the magnetic induction vector. The resulting field equations, which arise by substituting the constitutive equations into the balance equations, form the equations of a fully dynamic theory, called by some authors "piezoelectromagnetism" ([12]–[14]).

Electromagnetic wave propagation and acousto-optic interaction inside dielectrics has always attracted a great deal of interests (see, e.g., [5], [16]). In the last decades, the design of new anisotropic smart materials gave a further interest in modeling these wave phenomena (see, e.g., [3], [17], [18]). Papers [19]–[21] discuss the propagation of electromagnetic waves (Love waves, shear horizontal waves, etc.) in polarized ceramics using equations of linear piezoelectromagnetism. Iadonisi *et al.* in [9] study acoustic and electromagnetic modes in piezoelectric hexagonal ceramics; Weiss in [22] studies the generation of hypersound in piezoelectric quartz crystal by means of an incident plane electromagnetic wave.

The propagation of a monochromatic elastic and electromagnetic wave has been investigated by Nowacki in [23, pp. 153–157] in a crystal belonging to the tetragonal system of class $\overline{4}\,2m$ (amonium dihydrogen phosphate), and by [26] in Zinc Sulfide belonging to the cubic class 31.

Many authors (e.g., [15], [24]–[25]) have studied piezoelectric monoclinic crystals, because several monoclinic crystals show excellent piezoelectric properties and can be used for applications as electro-mechanic transducers. Nevertheless, it is difficult to find in the literature studies on optical effects and full electromagnetic coupling in such crystals. Therefore, the present paper develops, with regard to a monoclinic crystal, a work similar to the ones in the afore-mentioned papers [23, pp. 153–157], [26]. Furthermore, the study written here is parallel to [27], where time harmonic plane waves propagating along the symmetry axis of a thermo-piezoelectric crystal monoclinic of class 2 are studied within the quasi-static approximation; unlike [27] here we consider full coupling between electromagnetic fields and mechanical fields in the isothermal case.

In more detail, here we consider a piezoelectromagnetic crystalline medium belonging to the class 2 of the monoclinic crystallographic system. We study wave propagation of time harmonic plane waves propagating along the symmetry axis x_2, with the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$, electric vector $\mathbf{E} = (E_1, E_2, E_3)$, and magnetic vector $\mathbf{H} = (H_1, H_2, H_3)$ depending only on the axial coordinate x_2.

We find two independent modes of propagation, one longitudinal and one transverse, with mechanical and electromagnetical couplings. We show that:

(i) all such waves can propagate along the symmetry axis x_2 with $\mathbf{H}$ perpendicular to such axis;

(ii) there is an electro-mechanical longitudinal wave $B = (u_2, E_2)$ parallel to the symmetry axis, coupling displacement with electric vector;

(iii) there is a transverse mechanic-electromagnetic wave

$$C = [(u_1, u_3),\ (E_1, E_3),\ (H_1, H_3)]$$

coupling the parts of the displacement vector, electric vector, and magnetic vector that are perpendicular to the symmetry axis. This transverse mode admits as a particular case the existence of an electromagnetic wave with no associated elastic deformation: for $(u_1, u_3) = (0, 0)$, wave C degenerates into an electromagnetic wave $C_0 = [(E_1, E_3), (H_1, H_3)]$ which propagates without elastic deformation.

In the fully-dynamic theory of piezoelectromagnetism we have found some modes of propagation of electromagnetic waves, which of course do not exist within the quasi-static theory of piezoelectricity. They might be useful, for instance, to enlarge the class of waves used in the dynamic methods of determination of the elastic and piezoelectric coefficients of a monoclinic crystal [28] and in general in nondestructive evaluation.

2. On the linear thermo-piezoelectricity theory referred to a natural state

2.1. Balance laws

In the absence of external fields, the linear equations for a piezoelectric dielectric body with no magnetic effects, no electric currents and charges (field equations of linear piezoelectromagnetism) are the balance law of linear momentum (2.1) and Maxwell's equation (2.2)–(2.5) (see, e.g., [13], [14], [20]):

$$\sigma_{ji,j} = \rho_0 \ddot{u}_i, \tag{2.1}$$

$$D_{i,i} = 0, \tag{2.2}$$

$$\varepsilon_{ijk} H_{k,j} = \dot{D}_i, \tag{2.3}$$

$$\varepsilon_{ijk} E_{k,j} = -\dot{B}_i, \tag{2.4}$$

$$B_{i,i} = 0, \tag{2.5}$$

where σ_{ji} is the mechanical Cauchy stress tensor, u_i is the mechanical displacement vector, D_i is the electric displacement vector, E_i is the electric vector, H_i is the magnetic vector, B_i is the magnetic induction vector and ε_{ijk} is the Ricci tensor.

Note that (2.5) is a consequence of (2.4) and (2.2) is a consequence of (2.3). Hence the equations of piezoelectromagnetism are (2.1), (2.3), and (2.4).

Also note that in (2.1)–(2.5) we have assumed that there are no body forces, no free charges, no current density, and that the dielectric is nonmagnetic.

2.2. Linear constitutive equations in the natural state

The following constitutive equations are assumed:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \tag{2.6}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k, \tag{2.7}$$

$$B_i = \mu_0 H_i, \tag{2.8}$$

where

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{2.9}$$

is the linearized strain tensor, c_{ijkl} are the elastic stiffness coefficients at constant electric field, e_{kij} are the piezoelectric stress constants, ϵ_{ik} are the dielectric permittivities and μ_0 is the magnetic permeability of a vacuum; the symmetries

$$c_{ijkl} = c_{jikl} = c_{klij}, \qquad e_{kij} = e_{kji}$$

hold.

2.3. Differential equations

In the homogeneous and anisotropic case, substituting equations (2.6)–(2.8) into (2.1), (2.3), and (2.4) and making use of (2.9) we get the following system of differential equations

$$c_{ijkl}u_{k,lj} - e_{kij}E_{k,j} + f_i = \rho\ddot{u}_i, \tag{2.10}$$

$$-\varepsilon_{ijk}H_{k,j} + e_{ikl}\dot{\varepsilon}_{kl} + \epsilon_{ik}\dot{E}_k = 0, \tag{2.11}$$

$$\varepsilon_{ijk}E_{k,j} = -\mu_0\dot{H}_i. \tag{2.12}$$

Equations (2.10)–(2.12) form the set of differential equations of piezoelectromagnetism. The following unknown field quantities occur in them: the components of the displacement vector u_i, electric vector E_i and magnetic vector H_i.

2.4. Use of compressed notation and matrix arrays

Let us replace ij or kl by p or q, where i, j, k, l take the values $1, 2, 3$ and p, q take the values $1, 2, 3, 4, 5, 6$ according to the following prescriptions

ij or kl	11	22	33	23 or 32	31 or 13	12 or 21
p, q	1	2	3	4	5	6

By virtue of the above identifications the constitutive equations (2.6)–(2.7) become

$$T_p = c_{pq}S_q - e_{ip}E_i, \tag{2.13}$$

$$D_i = e_{ip}S_p + \epsilon_{ik}E_k, \tag{2.14}$$

where

$$c_{pq} = c_{ijkl}, \quad e_{iq} = e_{ikl}, \quad T_p = \sigma_{ij} = T_{ij}, \qquad \text{for } i = j,\ p = 1,2,3, \tag{2.15}$$

$$S_p = 2\varepsilon_{ij}, \quad \text{for } i \neq j,\ p = 4,5,6. \tag{2.16}$$

We can now write the elastic, piezoelectric and dielectric constants as matrices, since they are described by two indices.

3. Crystal belonging to class 2

3.1. Material constants

We consider a crystal belonging to class 2 with the symmetry axis parallel to the x_2-axis of the monoclinic crystallographic system. The arrays for such a material, for which the double symmetry axis parallel to the x_2-axis is characteristic, write as (see, e.g., [27, p. 101])

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & c_{15} & 0 \\ c_{12} & c_{11} & c_{13} & 0 & c_{25} & 0 \\ c_{13} & c_{13} & c_{33} & 0 & c_{35} & 0 \\ 0 & 0 & 0 & c_{44} & 0 & c_{46} \\ c_{15} & c_{25} & c_{35} & 0 & c_{55} & 0 \\ 0 & 0 & 0 & c_{46} & 0 & c_{66} \end{bmatrix}, \tag{3.1}$$

$$\mathbf{e} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & 0 & e_{16} \\ e_{21} & e_{22} & e_{23} & 0 & e_{25} & 0 \\ 0 & 0 & 0 & e_{34} & 0 & e_{36} \end{bmatrix}, \qquad \epsilon = \begin{bmatrix} \epsilon_{11} & 0 & \epsilon_{13} \\ 0 & \epsilon_{22} & 0 \\ \epsilon_{13} & 0 & \epsilon_{33} \end{bmatrix}. \tag{3.2}$$

The number of independent material constants appearing in the above matrices equals 23. Moreover, the constitutive relation (2.8) and equation (2.12) contain the magnetic permeability of free space μ_0. Hence we have together 24 independent material constants. The conditions

$$c_{22} + \frac{e_{22}}{\epsilon_{22}} > 0, \qquad \epsilon_{22} \neq 0,$$
$$c_{44}c_{66} > c_{46}^2, \qquad c_{66} + c_{44} > 0,$$
$$\epsilon_{11}\epsilon_{33} > \epsilon_{13}^2 \qquad \text{and} \qquad \epsilon_{13} \neq 0$$

are assumed in the sections 4.1.1, 4.1.2, 4.1.3 and 4.1.5, respectively.

3.2. Field equations

By substituting (2.13)–(3.2) into (2.10)–(2.12) we obtain the field equations

$$\begin{aligned}(3.3)\qquad & c_{11}u_{1,11}+c_{66}u_{1,22}+c_{55}u_{1,33}+c_{15}u_{3,11}+c_{46}u_{3,22}+c_{35}u_{3,33}+2c_{15}u_{1,31}\\ &+(c_{12}+c_{66})u_{2,21}+(c_{46}+c_{25})u_{2,32}+(c_{13}+c_{55})u_{3,31}=\rho\ddot{u}_1,\end{aligned}$$

$$\begin{aligned}(3.4)\qquad & c_{66}u_{2,11}+c_{22}u_{2,22}+c_{44}u_{2,33}+(c_{12}+c_{66})u_{1,12}+(c_{25}+c_{46})u_{1,32}\\ &+2c_{46}u_{2,31}+(c_{25}+c_{46})u_{3,21}+(c_{23}+c_{44})u_{3,32}\\ &-e_{16}E_{1,1}-e_{22}E_{2,2}-e_{34}E_{3,3}-(e_{36}+e_{14})E_{1,3}=\rho\ddot{u}_2,\end{aligned}$$

$$\begin{aligned}(3.5)\qquad & c_{15}u_{1,11}+c_{66}u_{1,22}+c_{35}u_{1,33}+c_{55}u_{3,11}+c_{44}u_{3,22}+c_{33}u_{3,33}\\ &+(c_{55}+c_{13})u_{1,31}+(c_{25}+c_{46})u_{2,21}+(c_{44}+c_{23})u_{2,32}+2c_{35}u_{3,31}\\ &-(e_{25}+e_{14})E_{2,1}-(e_{34}+e_{23})E_{2,3}=\rho\ddot{u}_3,\end{aligned}$$

$$(3.6)\qquad -H_{3,2}+H_{2,3}+e_{14}(\dot{u}_{2,3}+\dot{u}_{3,2})+e_{16}(\dot{u}_{1,2}+\dot{u}_{2,1})+\epsilon_{11}\dot{E}_1+\epsilon_{13}\dot{E}_3=0,$$

$$(3.7)\qquad -H_{1,3}+H_{3,1}+e_{21}\dot{u}_{1,1}+e_{22}\dot{u}_{2,2}+e_{23}\dot{u}_{3,3}+e_{25}(\dot{u}_{3,1}+\dot{u}_{1,3})+\epsilon_{22}\dot{E}_2=0,$$

$$(3.8)\qquad -H_{2,1}+H_{1,2}+e_{34}(\dot{u}_{2,3}+\dot{u}_{3,2})+e_{36}(\dot{u}_{1,2}+\dot{u}_{2,1})+\epsilon_{13}\dot{E}_1+\epsilon_{33}\dot{E}_3=0,$$

$$(3.9)\qquad E_{3,2}-E_{2,3}=-\mu_0\dot{H}_1,$$

$$(3.10)\qquad E_{1,3}-E_{3,1}=-\mu_0\dot{H}_2,$$

$$(3.11)\qquad E_{2,1}-E_{1,2}=-\mu_0\dot{H}_3.$$

4. Plane harmonic waves

4.1. Equations for waves depending only on symmetry axis coordinate x_2

We consider a plane wave moving in an infinite medium, of the type described in Section 3, that changes harmonically with time in the direction x_2 with a constant phase velocity c. The quantities that characterize the wave are

$$(4.1)\qquad u_i=u_i(x_2,t),\qquad E_i=E_i(x_2,t),\qquad H_i=H_i(x_2,t).$$

The field equations (3.3) to (3.11) for the plane waves (4.1) propagating in the medium described in Section 3 reduce to simpler equations, which we split in the following three systems of equations

$$(4.2)\qquad (A)\equiv\begin{cases}c_{46}u_{3,22}+c_{66}u_{1,22}=\rho\ddot{u}_1,\\ c_{44}u_{3,22}+c_{46}u_{1,22}=\rho\ddot{u}_3,\end{cases}$$

$$(4.3)\qquad (B)\equiv\begin{cases}c_{22}u_{2,22}-e_{22}E_{2,2}=\rho\ddot{u}_2,\\ e_{22}\dot{u}_{2,2}+\epsilon_{22}\dot{E}_2=0,\end{cases}$$

$$(C) \equiv \begin{cases} -H_{3,2} + e_{14}\dot{u}_{3,2} + e_{16}\dot{u}_{1,2} + \epsilon_{11}\dot{E}_1 + \epsilon_{13}\dot{E}_3 = 0, \\ H_{1,2} + e_{34}\dot{u}_{3,2} + e_{36}\dot{u}_{1,2} + \epsilon_{13}\dot{E}_1 + \epsilon_{33}\dot{E}_3 = 0, \\ E_{3,2} = -\mu_0\dot{H}_1, \\ 0 = -\mu_0\dot{H}_2, \\ E_{1,2} = \mu_0\dot{H}_3. \end{cases} \tag{4.4}$$

Note that $(4.4)_4$ is equivalent to $H_2 =$ constant, hence $H_2 = 0$ because we are concerned with waves.

PROPOSITION 1. *Each plane wave* (4.1)*, propagating along the symmetry axis x_2, has the magnetic vector* $\mathbf{H}$ *perpendicular to such axis.*

Now let us replace $H_2 = 0$ in system (4.4) and split it in the two subsystems

$$(C_1) \equiv \begin{cases} -H_{3,2} + e_{14}\dot{u}_{3,2} + e_{16}\dot{u}_{1,2} + \epsilon_{11}\dot{E}_1 + \epsilon_{13}\dot{E}_3 = 0, \\ H_{1,2} + e_{34}\dot{u}_{3,2} + e_{36}\dot{u}_{1,2} + \epsilon_{13}\dot{E}_1 + \epsilon_{33}\dot{E}_3 = 0, \end{cases} \tag{4.5}$$

$$(C_2) \equiv \begin{cases} E_{3,2} = -\mu_0\dot{H}_1, \\ E_{1,2} = \mu_0\dot{H}_3. \end{cases} \tag{4.6}$$

In order to solve the above field equations, we observe that

1. System (4.3) in the two variables u_2 and E_2 is independent from the others.
2. Equations (4.2), (4.4) are coupled through the variables u_1, u_3. System (4.2) in the two variables u_1, u_3 can be solved (just as in [27]), so one finds the expressions of u_1, u_3. These are then substituted in equations (4.5).
3. Lastly, E_1, E_3, H_1 and H_3 can be found by solving (4.5), (4.6).

For plane waves (4.1), we give the following definitions:

(1) *Wave A* is the mechanical wave (u_1, u_3) which is solution of (4.2).

(2) *Wave B* is the electro-mechanical wave (u_2, E_2) which is solution of (4.3).

(3) *Wave* $C = (u_1, u_3, E_1, E_3, H_1, H_3)$ is the electro-magneto-mechanical wave which is solution of (4.4), where $A = (u_1, u_3)$ is solution of (4.2).

(4) *Wave* C_0 is the pure electromagnetic wave (E_1, E_3, H_1, H_3) which is solution of (4.4), or of (4.5), (4.6), when $u_1 = 0 = u_3$.

Note that the waves B and C (or C_0) are not coupled.

4.1.1. Electro-mechanical wave B**.** Next we solve (4.3) in the variables u_2, E_2. From $(4.3)_2$, differentiating with respect to x_2 we find $E_{2,2} = -(e_{22}/\epsilon_{22})u_{2,22}$, and substituting the latter into $(4.3)_1$ we have the equation

$$Au_{2,22} - \rho\ddot{u}_2 = 0, \tag{4.7}$$

where

$$A := c_{22} + \frac{e_{22}}{\epsilon_{22}}. \tag{4.8}$$

Now substituting

$$u_2(x_2,t) = U_2^0 \exp[i(Kx_2 - \omega t)] \tag{4.9}$$

in (4.7) we obtain the characteristic equation

$$AK^2 - \rho\omega^2 = 0, \tag{4.10}$$

that is the algebraic equation for the wave number K; when $A > 0$ it has the roots

$$K_{1,2} = \pm\omega\sqrt{\frac{\rho}{A}}. \tag{4.11}$$

Hence, putting

$$C_i^{\pm} = \cos(K_i x \mp \omega t), \quad S_i^{\pm} = \sin(K_i x \mp \omega t) \qquad (x = x_2,\ i = 1,2), \tag{4.12}$$

the solutions of Eq. (4.7) are the functions

$$u_2 = U_+ C_1^+ + U_- C_1^- + V_+ C_2^+ + V_- C_2^-, \tag{4.13}$$

where U_+, U_-, V_+, V_- are real constants. Hence, by $(4.3)_2$ we find

$$E_2 = \frac{e_{22}}{\epsilon_{22}}[K_1(U_+ S_1^+ + U_- S_1^-) + K_2(V_+ S_2^+ + V_- S_2^-)]. \tag{4.14}$$

The phase velocities c_β are related with the wave numbers K_β by the equalities

$$c_\beta = \omega / K_\beta, \qquad \beta = 1,2. \tag{4.15}$$

Note that by (4.8) the existence of the roots (4.11) requires the conditions

$$c_{22} + \frac{e_{22}}{\epsilon_{22}} > 0, \qquad \epsilon_{22} \neq 0.$$

4.1.2. Mechanical wave A. Following [27] now we study the solutions of the system of equations

$$\begin{cases} c_{46}u_{3,22} + c_{66}u_{1,22} = \rho\ddot{u}_1, \\ c_{44}u_{3,22} + c_{46}u_{1,22} = \rho\ddot{u}_3, \end{cases} \tag{4.16}$$

which have the form (4.1). Substituting

$$(u_1,u_3)(x_2,t) = (U_1^0, U_3^0)\exp[i(Kx_2 - \omega t)] \tag{4.17}$$

in (4.16) we obtain the characteristic equation, the algebraic quartic equation for the wave number K

$$K^4(c_{44}c_{66} - c_{46}^2) - K^2\rho\omega^2(c_{66} + c_{44}) + \rho^2\omega^4 = 0 \tag{4.18}$$

(which coincides with [27, (3.9)]); putting

$$A = c_{44}c_{66} - c_{46}^2, \qquad B = -\rho\omega^2(c_{66} + c_{44}), \qquad C = \rho^2\omega^4, \tag{4.19}$$

we have

$$B^2 - 4AC > 0; \tag{4.20}$$

one can verify that when $c_{44}c_{66} > c_{46}^2$, $c_{66} + c_{44} > 0$ Eq. (4.18) has the four real roots

$$\begin{aligned} K_1 &= \sqrt{\frac{-B + \sqrt{B^2 - 4AC}}{2A}} = -K_3, \\ K_2 &= \sqrt{\frac{-B - \sqrt{B^2 - 4AC}}{2A}} = -K_4. \end{aligned} \tag{4.21}$$

Hence the solutions u_1, u_3 to Eqs. (4.16) have the form (cf. [27, (3.6) on p. 102])

$$\begin{cases} u_1 = A_+C_1^+ + A_-C_1^- + B_+C_2^+ + B_-C_2^-, \\ u_3 = A'_+C_1^+ + A'_-C_1^- + B'_+C_2^+ + B'_-C_2^-, \end{cases} \tag{4.22}$$

where the equalities (4.12) hold with the K_i given by (4.21) and A_+, A_-, B_+, B_-, A'_+, A'_-, B'_+, B'_- are real constants.

Of course, since the wave equations (4.16) are linear, the above 8-tuples $(A_\pm, \ldots, B'_\pm)$ form a linear space. In order to prove the existence of waves of type C (see Section 4.1.5 below) we show that there are independent solutions (4.22) of (4.16).

Remark 1. There are (at least) two linearly independent 8-tuples $(A_\pm, \ldots, B'_\pm)$ such that the equalities (4.22) give a solution to the system of equations (4.16).

To prove the remark we find the relations between the constants $A_\pm, \ldots, B'_\pm$ by substituting (4.22) into (4.16) (so completing the considerations in [27, on p. 102 below (3.7)]). First, we compute the derivatives of $C_i^\pm$ in (4.12) and u_1, u_3:

$$\begin{aligned} C_{i,x}^\pm &= -K_iS_i^\pm, \qquad & C_{i,xx}^\pm &= -K_i^2C_i^\pm, \\ C_{i,t}^\pm &= \omega S_i^\pm, \qquad & C_{i,tt}^\pm &= -\omega^2C_i^\pm, \end{aligned}$$

$$
\begin{aligned}
u_{1,xx} &= A_+(-K_1^2)C_1^+ + A_-(-K_1^2)C_1^- + B_+(-K_2^2)C_2^+ + B_-(-K_2^2)C_2^-,\\
u_{3,xx} &= A'_+(-K_1^2)C_1^+ + A'_-(-K_1^2)C_1^- + B'_+(-K_2^2)C_2^+ + B'_-(-K_2^2)C_2^-,\\
u_{1,tt} &= A_+(-\omega^2)C_1^+ + A_-(-\omega^2)C_1^- + B_+(-\omega^2)C_2^+ + B_-(-\omega^2)C_2^-,\\
u_{3,tt} &= A'_+(-\omega^2)C_1^+ + A'_-(-\omega^2)C_1^- + B'_+(-\omega^2)C_2^+ + B'_-(-\omega^2)C_2^-.
\end{aligned} \tag{4.23}
$$

Substituting the latter in (4.16) we obtain the two relations

$$
\begin{aligned}
&c_{46}[A'_+(-K_1^2)C_1^+ + A'_-(-K_1^2)C_1^- + B'_+(-K_2^2)C_2^+ + B'_-(-K_2^2)C_2^-]\\
&\quad + c_{66}[A_+(-K_1^2)C_1^+ + A_-(-K_1^2)C_1^- + B_+(-K_2^2)C_2^+ + B_-(-K_2^2)C_2^-]\\
&\quad = \rho[A_+(-\omega^2)C_1^+ + A_-(-\omega^2)C_1^- + B_+(-\omega^2)C_2^+ + B_-(-\omega^2)C_2^-],
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
&c_{44}[A'_+(-K_1^2)C_1^+ + A'_-(-K_1^2)C_1^- + B'_+(-K_2^2)C_2^+ + B'_-(-K_2^2)C_2^-]\\
&\quad + c_{46}[A_+(-K_1^2)C_1^+ + A_-(-K_1^2)C_1^- + B_+(-K_2^2)C_2^+ + B_-(-K_2^2)C_2^-]\\
&\quad = \rho[A'_+(-\omega^2)C_1^+ + A'_-(-\omega^2)C_1^- + B'_+(-\omega^2)C_2^+ + B'_-(-\omega^2)C_2^-].
\end{aligned} \tag{4.25}
$$

Now by equating to zero the coefficients of $C_i^{\pm}$ $(i = 1,\, 2)$ the latter gives the following eight relations for the constants $A_{\pm}, \dots, B'_{\pm}$:

$$
\begin{cases}
(c_{66}K_1^2 - \rho\omega^2)A_+ + c_{46}K_1^2 A'_+ = 0,\\
(c_{66}K_1^2 - \rho\omega^2)A_- + c_{46}K_1^2 A'_- = 0,\\
(c_{66}K_2^2 - \rho\omega^2)B_+ + c_{46}K_2^2 B'_+ = 0,\\
(c_{66}K_2^2 - \rho\omega^2)B_- + c_{46}K_2^2 B'_- = 0,\\
c_{46}K_1^2 A_+ + (c_{44}K_1^2 - \rho\omega^2)A'_+ = 0,\\
c_{46}K_1^2 A_- + (c_{44}K_1^2 - \rho\omega^2)A'_- = 0,\\
c_{46}K_2^2 B_+ + (c_{44}K_2^2 - \rho\omega^2)B'_+ = 0,\\
c_{46}K_2^2 B_- + (c_{44}K_2^2 - \rho\omega^2)B'_- = 0.
\end{cases} \tag{4.26}
$$

Putting

$$
L_i = c_{46}K_i^2, \qquad M_i = c_{66}K_i^2 - \rho\omega^2, \qquad N_i = c_{44}K_i^2 - \rho\omega^2 \qquad (i = 1, 2),
$$
$$
v = (A_+, A_-, A'_+, A'_-, B_+, B_-, B'_+, B'_-),
$$

$$
m := \begin{bmatrix}
M_1 & 0 & L_1 & 0 & 0 & 0 & 0 & 0\\
0 & M_1 & 0 & L_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & M_2 & 0 & L_2 & 0\\
0 & 0 & 0 & 0 & 0 & M_2 & 0 & L_2\\
L_1 & 0 & N_1 & 0 & 0 & 0 & 0 & 0\\
0 & L_1 & 0 & N_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & L_2 & 0 & N_2 & 0\\
0 & 0 & 0 & 0 & 0 & L_2 & 0 & N_2
\end{bmatrix}, \tag{4.27}
$$

the system of equations (4.26) rewrites as $m\,v^T = 0$, where v^T is the transpose of v. Now,

$$\operatorname{Det} m = (L_1^2 - M_1 N_1)^2 (L_2^2 - M_2 N_2)^2. \tag{4.28}$$

Hence, the equation $\operatorname{Det} m = 0$ (in the variables K_i) has the same roots of the characteristic equation (4.18). Consequently, if K_1, K_2 are roots of the characteristic equation (4.18), then there exist constants $A_\pm, \dots, B'_\pm$ such that the equalities (4.22) form a solution to Eqs. (4.16).

Note that

$$rank(m) = rank(m_1),$$

where

$$m_1 := \begin{bmatrix} m_{11} & (0) \\ (0) & m_{12} \end{bmatrix}, \qquad m_{1i} := \begin{bmatrix} M_i & 0 & L_i & 0 \\ 0 & M_i & 0 & L_i \\ L_i & 0 & N_i & 0 \\ 0 & L_i & 0 & N_i \end{bmatrix} \qquad (i = 1,\, 2) \tag{4.29}$$

and (0) denotes the 4×4 null matrix. Now, if K_i is a root of the characteristic equation (4.18), then $rank(m_{1i}) = 2$ and thus $rank(m) \le 6$.

4.1.3. Pure electromagnetic wave C_0. For $u_1 = 0 = u_3$, the subsystem (4.5) of (4.4) becomes

$$\begin{cases} -H_{3,2} + \epsilon_{11}\dot{E}_1 + \epsilon_{13}\dot{E}_3 = 0, \\ H_{1,2} + \epsilon_{13}\dot{E}_1 + \epsilon_{33}\dot{E}_3 = 0. \end{cases} \tag{4.30}$$

Taking the derivative of Eqs. (4.6) with respect to x_2 we find

$$\dot{H}_{1,2} = -\mu_0^{-1} E_{3,22}, \qquad \dot{H}_{3,2} = \mu_0^{-1} E_{1,22}. \tag{4.31}$$

Then take the derivative of Eqs. (4.30) with respect to t and substitute (4.31) into the latter equality; we have

$$\begin{cases} -\mu_0^{-1} E_{1,22} + \epsilon_{11}\ddot{E}_1 + \epsilon_{13}\ddot{E}_3 = 0, \\ -\mu_0^{-1} E_{3,22} + \epsilon_{13}\ddot{E}_1 + \epsilon_{33}\ddot{E}_3 = 0. \end{cases} \tag{4.32}$$

Assume the equalities

$$C^\pm = \cos(Kx \mp \omega t), \qquad S^\pm = \sin(Kx \mp \omega t) \qquad (x = x_2); \tag{4.33}$$

putting

$$E_1 = F_+ C^+, \qquad E_3 = F'_+ C^+, \tag{4.34}$$

we have

$$E_{1,xx} = -K^2 F_+ C^+, \tag{4.35}$$

$$E_{3,xx} = -K^2 F'_+ C^+, \tag{4.36}$$

$$E_{1,tt} = -\omega^2 F_+ C^+, \tag{4.37}$$

$$E_{3,tt} = -\omega^2 F'_+ C^+. \tag{4.38}$$

Substituting the latter in Eqs. (4.32) we have

$$\begin{cases} (\mu_0^{-1} K^2 - \epsilon_{11}\omega^2) F_+ - \epsilon_{13}\omega^2 F'_+ = 0, \\ -\epsilon_{13}\omega^2 F_+ + (-\epsilon_{33}\omega^2 + \mu_0^{-1} K^2) F'_+ = 0. \end{cases} \tag{4.39}$$

Equating to zero the determinant of the matrix of coefficients of (4.39) we obtain the characteristic equation

$$K^4 - K^2 \mu_0 (\epsilon_{11} + \epsilon_{33})\omega^2 + \mu_0^2 (\epsilon_{11}\epsilon_{33} - \epsilon_{13}^2)\omega^4 = 0; \tag{4.40}$$

one can verify that when $\epsilon_{11}\epsilon_{33} > \epsilon_{13}^2$ this equation has the four real roots

$$\begin{aligned} K_1 &= \sqrt{\frac{-B + \sqrt{B^2 - 4C}}{2}} = -K_3, \\ K_2 &= \sqrt{\frac{-B - \sqrt{B^2 - 4C}}{2}} = -K_4, \end{aligned} \tag{4.41}$$

where

$$B = -\mu_0(\epsilon_{11} + \epsilon_{33})\omega^2, \qquad C = \mu_0^2(\epsilon_{11}\epsilon_{33} - \epsilon_{13}^2)\omega^4, \tag{4.42}$$

and

$$B^2 - 4C = \mu_0^2 \omega^4 [(\epsilon_{11} - \epsilon_{33})^2 + 4\epsilon_{13}^2)] > 0.$$

Hence, the solutions to Eqs. (4.32) are the functions

$$\begin{cases} E_1 = F_+ C_1^+ + F_- C_1^- + G_+ C_2^+ + G_- C_2^-, \\ E_3 = F'_+ C_1^+ + F'_- C_1^- + G'_+ C_2^+ + G'_- C_2^-, \end{cases} \tag{4.43}$$

where the equalities (4.33) hold and F_+, F_-, G_+, G_-, F'_+, F'_-, G'_+, G'_- are real constants. The phase velocities c_β are related with the wave numbers K_β by the equalities

$$c_\beta = \omega / K_\beta, \qquad \beta = 1, 2. \tag{4.44}$$

Then H_1 and H_3 can be determined by Eqs. (4.6).

4.1.4. Electromagnetic wave C. Taking the derivative of Eqs. (4.6) with respect to x_2 we find

$$\dot{H}_{1,2} = -\mu_0^{-1} E_{3,22}, \qquad \dot{H}_{3,2} = \mu_0^{-1} E_{1,22}. \tag{4.45}$$

Then take the derivative of Eqs. (4.5) with respect to t and substitute (4.45) in the resulting equalities; we have

$$\begin{cases} -\mu_0^{-1} E_{1,22} + e_{14}\ddot{u}_{3,2} + e_{16}\ddot{u}_{1,2} + \epsilon_{11}\ddot{E}_1 + \epsilon_{13}\ddot{E}_3 = 0, \\ -\mu_0^{-1} E_{3,22} + e_{34}\ddot{u}_{3,2} + e_{36}\ddot{u}_{1,2} + \epsilon_{13}\ddot{E}_1 + \epsilon_{33}\ddot{E}_3 = 0. \end{cases} \tag{4.46}$$

Next we proceed in the following steps:

(A) We replace into equations (4.46) the mechanical wave (A), i.e., the found solutions for u_1 and u_3 to equations (4.16) (see Section 4.1.3). Then the resulting equations yield E_1 and E_3.

(C) Substituting the found E_1 and E_3 into (4.6) we find H_1, H_3 by solving the resulting system of equations.

4.1.5. Existence of waves of type C. Next we show that waves of type (C) exist by considering the particular case in which $E_3 = 0$, $H_1 = 0$. Any such a wave satisfies the system of equations

$$\begin{cases} c_{46}u_{3,22} + c_{66}u_{1,22} = \rho\ddot{u}_1, \\ c_{44}u_{3,22} + c_{46}u_{1,22} = \rho\ddot{u}_3, \\ -H_{3,2} + e_{14}\dot{u}_{3,2} + e_{16}\dot{u}_{1,2} + \epsilon_{11}\dot{E}_1 = 0, \\ e_{34}\dot{u}_{3,2} + e_{36}\dot{u}_{1,2} + \epsilon_{13}\dot{E}_1 = 0, \\ E_{1,2} = \mu_0\dot{H}_3. \end{cases} \tag{4.47}$$

By taking the derivatives, from $(4.47)_{3,5}$ we obtain

$$-\dot{H}_{3,2} + e_{14}\ddot{u}_{3,2} + e_{16}\ddot{u}_{1,2} + \epsilon_{11}\ddot{E}_1 = 0, \tag{4.48}$$

$$\dot{H}_{3,2} = \mu_0^{-1} E_{1,22}, \tag{4.49}$$

while from $(4.47)_4$ for $\epsilon_{13} \neq 0$ we obtain

$$E_1 = -\epsilon_{13}^{-1}(e_{34}u_{3,2} + e_{36}u_{1,2} + c), \tag{4.50}$$

where c is any differentiable function of x_2. By replacing (4.50) in (4.48), (4.49), since $\dot{c} = 0$, for $c_{22} = 0$ we obtain the equality

$$\mu_0^{-1}\epsilon_{13}^{-1}(e_{34}u_{3,222} + e_{36}u_{1,222}) + e_{14}\ddot{u}_{3,2} + e_{16}\ddot{u}_{1,2} - \epsilon_{11}\epsilon_{13}^{-1}(e_{34}\ddot{u}_{3,2} + e_{36}\ddot{u}_{1,2}) = 0. \tag{4.51}$$

Remind that there are ∞^n 8-tuples $(A_\pm, \ldots, B'_\pm)$, with $n \geq 2$, such that the equalities (4.12), (4.22), where the K_i are given by (4.21), give a solution to the system of equations $(4.47)_{1,2}$ (see Remark 1). This implies that there exists at least one such a 8-tuple that in addition satisfies (4.51).

Indeed, let $\mathcal{A}^{(i)} = (A_\pm^{(i)}, \ldots, B'^{(i)}_\pm)$, $i = 1, 2$, be two linearly independent 8-tuples that, when replaced in (4.22), generate two distinct A-wave solutions $u^{(1)} := (u_1^{(1)}, u_3^{(1)})$, $u^{(2)} := (u_1^{(2)}, u_3^{(2)})$ to $(4.47)_{1,2}$ (or (4.2)). Moreover, let $\mathcal{E}(u^{(i)})$ be the expression obtained by substituting the derivatives of such $u^{(i)}$ into the left-hand side of (4.51). Since for each real constant τ also the 8-tuple $\mathcal{A}^{(1)}+\tau\mathcal{A}^{(2)}$ by (4.22) generates a solution $w^{(\tau)} := u^{(1)}+\tau u^{(2)}$ to $(4.47)_{1,2}$, and since equation (4.51) is linear, then $\mathcal{E}(w^{(\tau)}) = \mathcal{E}(u^{(1)}) + \tau\mathcal{E}(u^{(2)})$. Now, if $\mathcal{E}(u^{(2)}) = 0$, then $u^{(2)}$ solves Eq. (4.51) too; if $\mathcal{E}(u^{(2)}) \neq 0$, for $\tau_0 := -\mathcal{E}(u^{(1)})/\mathcal{E}(u^{(2)})$ the 8-tuple $\mathcal{A}^{(1)} + \tau_0\mathcal{A}^{(2)}$ generates a solution $w^{(\tau_0)}$ to $(4.47)_{1,2}$ such that $\mathcal{E}(w^{(\tau_0)}) = 0$, i.e., solving Eq. (4.51) too.

5. Conclusions

In the thermo-piezoelectric medium considered in [27] the mechanical wave $A = (u_1, u_3)$ is not coupled to the temperature and electric fields. Instead in the piezoelectromagnetic elastic medium considered here the mechanical wave $A = (u_1, u_3)$ is coupled to the electrical and magnetic fields (E_1, E_3), (H_1, H_3). The existence of acousto-electromagnetic waves $C = (u_1, u_3, E_1, E_3, H_1, H_3)$ is then proved. When $u_1 = 0 = u_3$ (wave C_0) the general solution is shown. All the above waves constitute the transverse mode of propagation. The longitudinal mode of propagation is given by the mechano-electric wave $B = (u_2, E_2)$, which is not coupled with the previous one. Waves A, B and C_0 are not subject to dispersion and damping. Their phase velocities are calculated.

By discarding the quasi-static approximation, i.e., allowing the full electromagnetic coupling, this paper finds piezoelectricity-induced electromagnetic waves in a class of piezoelectric crystal, just as in [5], [23, pp. 153–157], [26], etc., for other classes of crystals.

The elastic and piezoelectric coefficients in a crystal can be measured by dynamical experiments, in which the longitudinal mode of propagation of small bars are measured, or square and rectangular plates of any orientation can be excited by an electric field normal to the plates in modes dependent on the contour [28]. In such experiments the quasi-static equations of piezoelectricity are used. Using the fully-dynamic equations of piezoelectromagnetism, rather than the quasi-static equations of piezoelectricity, gives further modes of propagation, connected with electromagnetic waves. These, in my opinion, may enlarge the possible experiments of dynamic methods of determination.

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