

Elastic buckling of Cassini ovaloidal shells under external pressure – theoretical study

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THE PAPER IS DEVOTED TO SHELLS of revolution with positive and negative Gaussian curvature. The meridian of shells is a plane curve in the Cassini oval form. Geometrical properties of the middle surface of the shell of revolution based on this curve are presented. The membrane state of stress of a family of shells with constant capacity and mass under uniform pressure is described analytically and numerically with the use of the FEM (the ANSYS system). The critical pressure, buckling modes and equilibrium paths of analysed shells are calculated numerically. The advantages of a pressure vessel made in the form of Cassini ovaloidal shell, such as the lack of edge effect and a stable post-buckling behaviour, are pointed out. The results of the analytical and numerical investigations are compared and presented in tables and figures.

Key words: shell of revolution, barrelled shell, elastic buckling, equilibrium path.

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1. Introduction

SHELLS OF REVOLUTION ARE DESCRIBED IN DETAILS in many monographs, e.g., by TIMOSHENKO and WOINOWSKY-KRIEGER [1], GRIGOLUK and KABANOV [2], VENTSEL and KRAUTHAMMER [3], TOVSTIK and SMIRNOV [4]. Such shells are basic components of thin-walled structures like pressure vessels, tanks, space crafts or submersibles. This kind of structures is usually subjected to internal or external pressure which, in most cases, acts uniformly on the whole surface area. For such a load the most favourable shape is a shell with the positive Gaussian curvature. Besides spherical shells, which are an ideal solution in this case, barrelled shells are worthy of further consideration. They are cylindrical shells with meridional curvature, positive or negative. Barrelled shells combine the advantages of cylindrical and spherical shells. Details concerning geometry and stability of barrelled shells can be found in papers by JASION and MAGNUCKI [5] and MAGNUCKI and JASION [6]. Results of experiments on a number of barrelled shells combined in one pressure hull are presented by BŁACHUT and SMITH [7].

Analysis of a shell structure should cover three states: pre-buckling, buckling and post-buckling state. When it comes to closed shells subjected to the external or internal pressure, in the pre-buckling analysis the efforts should be focused on the stress distribution. The edge effect, i.e., the stress concentration due to discontinuities of the curvature, should be avoided. This is possible if the meridional curvature of a shell is continuous along the whole meridian of the shells. This condition is fulfilled, for example, in the case of an ellipsoidal shell which can be treated as a special case of a barrelled shell. Ellipsoidal shells are analysed, e.g., by MA *et.al* [8]. The problem of closing of cylindrical shells with an ellipsoidal shell is investigated by Magnucki *et.al* [9]. Also in the paper by BŁACHUT and MAGNUCKI [10] ellipsoidal shells are mentioned as a closure for cylindrical shells. Other problems in designing pressure vessels are discussed as well. JASION and MAGNUCKI [11] proposed a closed shell of revolution composed of a clothoidal and a spherical part. The meridian of this shell is distinguished by a continuous curvature which ensures a smooth stress distribution. Non-conventional tanks for storage of liquid materials are presented by ZINGONI [12].

In the buckling analysis the goal is usually to increase the value of the buckling load or to obtain the most favourable buckling shape. This can be achieved by bowing out the meridian – a higher meridional curvature corresponds usually to a higher buckling load. This was shown by BŁACHUT and WANG [13], among others.

The last stage of the analysis – the post-buckling state – seems to be the most important one in terms of safety. The reason for this is that, differently than for bars and plates, the post-buckling behaviour of shells is often unstable. In practise, it may mean a disaster if the load reaches the critical value. However, at this stage of the design process there are possibilities to influence the behaviour of a structure. One of the first works in which this issue is mentioned are the ones by REITINGER and RAMM [14], GODOY [15], BOCHENEK [16] and MRÓZ and PIEKARSKI [17]. In the above papers authors discuss optimisation procedures in which the stability criterion is included. Other works devoted to stabilisation of a post-buckling behaviour of shells are by BOCHENEK [18], KRÓL *et.al* [19] and KRUŻELECKI and TRYBUŁA [20] in which authors achieve the goal by means of pre-loading forces or special support conditions. Other possibility to obtain a stable post-buckling behaviour is, again, the shaping of the meridian of a shell. Such approach is presented, e.g., by JASION [21] and SINGH *et.al* [22].

Summing up the above considerations a perfect closed shell structure should have a smooth meridian with a continuous curvature, should be characterised by a high buckling load and should behave stably in the post-critical range. Satisfying all the above requirements is usually impossible, especially in actual structures. However, there are some theoretical solutions in which at least two requirements can be fulfilled. An example is a shell of revolution the meridian

of which is a plane curve in the Cassini oval form. A family of such shells, called Cassini ovaloidal shells, is analysed in this paper. The geometry of such structure is described and the stress distribution is analysed analytically and numerically. The behaviour of Cassini ovaloidal shell in the critical and post-critical state is analysed with the use of the finite element method (FEM). It is pointed out that for some geometrical parameters the post-critical behaviour of the shell can have a stable character.

2. Geometry of the middle surface of the Cassini ovaloidal shell

The Cassini oval as a plane curve in Cartesian coordinate is defined in the following form [23]:

$$(2.1) \quad y(x) = \left[\sqrt{4c^2x^2 + a^2} - (c^2 + x^2) \right]^{1/2},$$

where a and c are the parameters of the above function.

The meridian shape of the middle surface of the barrelled shell is shown in Fig. 1.

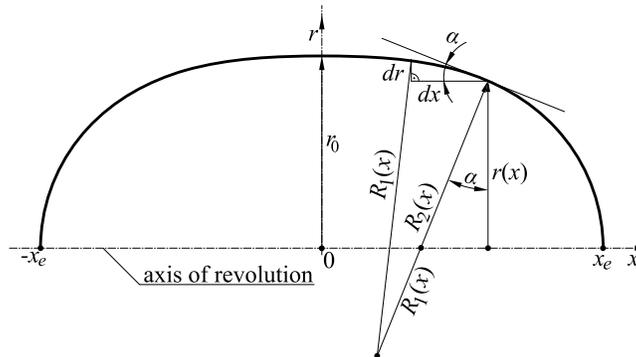


FIG. 1. The meridian of the shell – Cassini oval.

The radius of the parallel circle of the shell of revolution is as follows:

$$(2.2) \quad r(x) = y(x) = \left[\sqrt{4c^2x^2 + a^2} - (c^2 + x^2) \right]^{1/2}.$$

The value of the radius for $x = 0$ is equal to

$$(2.3) \quad r_0 = a\sqrt{1 - k_c^2}$$

and the x -coordinate for the end of the shell ($r = 0$) equals

$$(2.4) \quad x_e = a\sqrt{1 + k_c^2},$$

where $k_c = c/a$ is a dimensional parameter. The meridian shape of the shell depends on the value of this parameter. The following cases are possible:

- $0 < k_c < 1/\sqrt{2}$ – convex meridian,
- $k_c = 1/\sqrt{2}$ – plano-convex meridian,
- $1/\sqrt{2} < k_c < 1$ – concavo-convex meridian.

Knowing that (see Fig. 1)

$$\tan \alpha = -\frac{dr}{dx} \quad \text{and} \quad \cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}}$$

the middle surface of the shell of revolution can be characterised by principal radius of the meridian

$$(2.5) \quad R_1 = -\frac{\left[1 + \left(\frac{dr}{dx}\right)^2\right]^{3/2}}{\frac{d^2r}{dx^2}}$$

and principal radius of the parallel circle

$$(2.6) \quad R_2(x) = \frac{r(x)}{\cos \alpha} = \left[1 + \left(\frac{dr}{dx}\right)^2\right]^{1/2} r(x).$$

The principal radii (2.5) and (2.6) serve as a basis to determine the membrane state of stress.

3. Capacity and mass of the Cassini ovaloidal shell

The capacity of the shell (Fig. 1) is

$$(3.1) \quad V_s = 2\pi \int_0^{x_e} [r(x)]^2 dx$$

and after integration

$$(3.2) \quad V_s = \pi a^3 f_V,$$

where

$$(3.3) \quad f_V = \frac{1}{2k_c} \ln \left(1 + 2k_c^2 + 2k_c \sqrt{1 + k_c^2}\right) + \frac{1}{3} (1 - 2k_c^2) \sqrt{1 + k_c^2}.$$

The mass of the shell made of a material of the mass density ρ_s and of the thickness t_s (Fig. 1) is

$$(3.4) \quad m_s = A_s t_s \rho_s,$$

where lateral area

$$(3.5) \quad A_s = 4\pi \int_0^{x_e} \sqrt{1 + \left(\frac{dr}{dx}\right)^2} r(x) dx.$$

The integral (3.5) cannot be expressed by any analytical function, it may be determined only numerically.

For special case ($k_c = 0$) a spherical shell is obtained

$$\lim_{k_c \rightarrow 0} (V_s) = \frac{4}{3}\pi a^3 \quad \text{and} \quad \lim_{k_c \rightarrow 0} (A_s) = 4\pi a^2.$$

Taking into account the expressions (3.2) and (3.4) the following values are determined:

- parameter of the oval

$$(3.6) \quad a = \sqrt[3]{\frac{V_s}{\pi f_V}}$$

- thickness of the shell

$$(3.7) \quad t_s = \frac{m_s}{A_s \rho_s}.$$

The assumption of the values of the capacity V_s [m³] and the mass m_s [kg] of a shell enables calculating the parameter a and thickness t_s from (3.6) and (3.7).

The example calculations have been carried out for a Cassini ovaloidal steel shell of the capacity $V_s = 5$ m³ and the mass $m_s = 500$ kg. The following parameters have been assumed: the mass density of a steel $\rho_s = 7850$ kg/m³ and the dimensionless parameter $k_c = 0.60, 0.65, 1/\sqrt{2}, 0.75, 0.80, 0.85$. Results of calculations are shown in Table 1. The meridian shapes of the example steel Cassini ovaloidal shell are shown in Fig. 2.

Table 1. Values of basic quantities of the example steel Cassini ovaloidal shell.

k_c	0.60	0.65	$1/\sqrt{2}$	0.75	0.80	0.85
a [m]	1.1462	1.1668	1.1956	1.2220	1.2596	1.3071
c [m]	0.6877	0.7584	0.8454	0.9165	1.0077	1.1110
A_s [m ²]	14.498	14.646	14.875	15.103	15.448	15.910
t_s [mm]	4.393	4.349	4.282	4.217	4.123	4.003
r_0 [m]	0.9170	0.8867	0.8454	0.8083	0.7558	0.6885

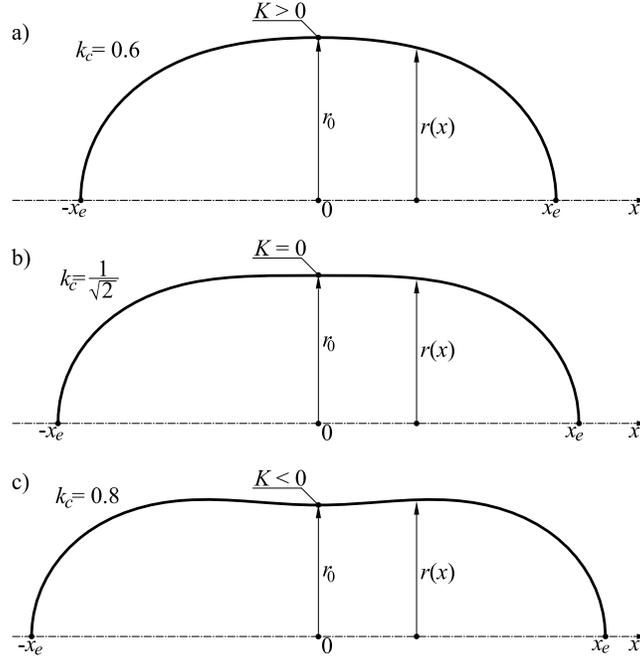


FIG. 2. Cassini ovals of different meridians: a) convex, b) plano-convex, c) concavo-convex (c). K – Gaussian curvature in the mid-length of the shell.

4. Analytical investigation of membrane state of stress – pre-buckling state

The membrane stress resultants for the Cassini ovaloidal shell loaded with uniform external pressure p_0 are defined as follows:

$$(4.1) \quad N_1(x) = \frac{1}{2}R_2(x)p_0, \quad N_2(x) = \left[1 - \frac{R_2(x)}{2R_1(x)}\right] R_2(x)p_0.$$

The normal stresses is

$$(4.2) \quad \sigma_1(x) = \tilde{\sigma}_1(x)p_0, \quad \sigma_2(x) = \tilde{\sigma}_2(x)p_0$$

and Huber-Mises-Hencky stress is

$$(4.3) \quad \sigma_{eq}(x) = \tilde{\sigma}_{eq}(x)p_0,$$

where dimensionless stresses are

$$\tilde{\sigma}_1(x) = \frac{1}{2} \frac{R_2(x)}{t_s}, \quad \tilde{\sigma}_2(x) = \left[1 - \frac{R_2(x)}{2R_1(x)}\right] \frac{R_2(x)}{t_s},$$

$$\tilde{\sigma}_{eq}(x) = \sqrt{\tilde{\sigma}_1^2 - \tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_2^2}.$$

The results of example calculations showing the distribution of the membrane stress resultants, meridional N_1 and circumferential N_2 , in selected shells are given in the next section in which the comparison of the results obtained analytically and with the use of FEM is presented.

5. Numerical calculation – FEM study

The model of a Cassini ovaloidal shell has been elaborated in the ANSYS code. The mid-surface of the shell has been modelled with the use of shell elements shell181 available in the system. The element is characterised by four nodes and six degrees of freedom in each node. The model has been subjected to uniform external pressure p_0 that is a typical load for submersibles, underground tanks or aboveground tanks during emptying process. Support conditions as well as the whole model are shown in Fig. 3.

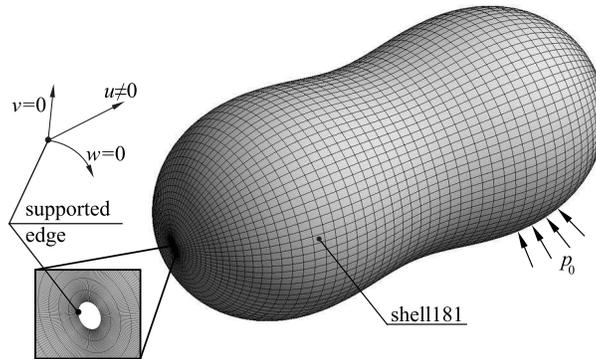


FIG. 3. FE model of the shell.

At the supported ends of the model circular holes have been cut the diameter of which equals 5 mm. Thanks to this a uniform quadrilateral mesh is possible to generate and a stress concentration at the supports can be avoided. Even though this solution causes small bending stresses near supported edges, they do not affect the results of the buckling and post-buckling analyses.

A family of shells of constant capacity $V_s = 5 \text{ m}^3$ and constant mass $m_s = 500 \text{ kg}$ has been analysed. The thickness of the shell t_s as well as the parameter a has been calculated from Eqs. (3.6) and (3.7). A linear-elastic material has been assumed with the following parameters: Young's modulus $E = 205000 \text{ MPa}$, Poisson's ratio $\nu = 0.3$. The mass density corresponds to steel – $\rho_s = 7850 \text{ kg/m}^3$.

5.1. The membrane state of stress

In the pre-buckling analysis the deformation and membrane stress resultants have been analysed. The dimensionless parameter k_c ranges from 0.55 to 0.95.

Detailed results are presented for $k_c = 0.65$ and 0.85 . The deformation, which is similar for all analysed shells, is shown in Fig. 4a. The highest deformation appears in the mid-length of the shell; when moving to the supporting edge the deformation decreases.

In Fig. 4b the distributions of dimensionless membrane stress resultants along the axis of revolution are presented. These are the values obtained based on the theoretical formulae (Eqs (4.1)) for pressure $p_0 = 1$ MPa. Thicknesses are given in Table 1. For convex shell both resultants reach the highest values in the mid-length of the shell and decrease with decreasing of x . Similar distribution has the circumferential resultant N_2 for the concave shell. However, the longitudinal resultant N_1 , according to Eqs. (4.1), follows the value of the cir-

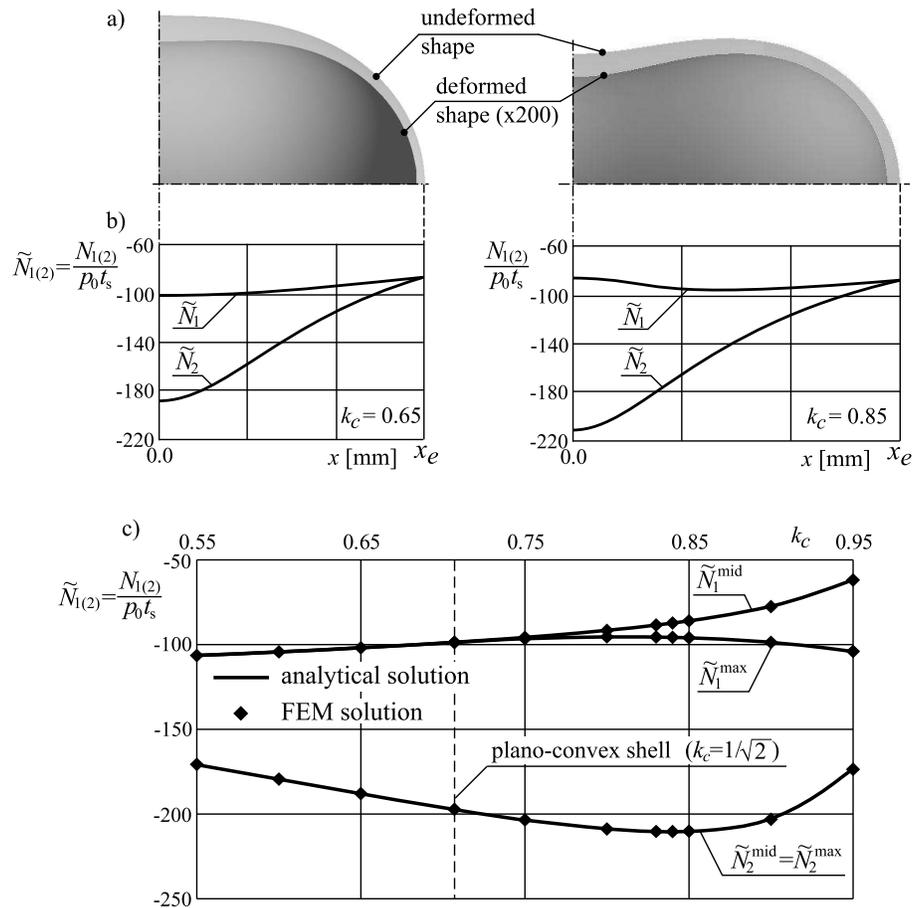


FIG. 4. a) Total deformation; b) membrane stress resultants for shell with $k_c = 0.85$ and $k_c = 0.65$; c) membrane stress resultants in the mid-length of shells and their maximum values.

cumferential radius and reaches the highest value for x corresponding to the biggest R_2 .

To compare the membrane stress resultants for different shells the plot shown in Fig. 4c has been prepared showing the influence of the parameter k_c on the dimensionless resultants. The values in the mid-length of shells are given ($\tilde{N}_{1(2)}^{mid}$) as well as the maximum values ($\tilde{N}_{1(2)}^{max}$). A good agreement is seen between the results obtained analytically (solid line) and with the use of FEM (diamonds). It can be seen that the maximum value of \tilde{N}_2^{max} corresponds to the shell described by the parameter $k_c = 0.84$. The minimum value of \tilde{N}_1^{max} corresponds to $k_c = 0.8$.

5.2. The critical state

The same family of shells for which the membrane stress resultants were determined in a previous subsection will be analysed here in terms of the buckling resistance. The results of the linear buckling analysis are shown in Fig. 5. On the plot the horizontal axis corresponds to the geometry of the shell and on the vertical axis the critical pressure p_{cr} is marked, normalised by the critical pressure p_{cr}^{p-c} for the shell with a plano-convex meridian. This way the value of

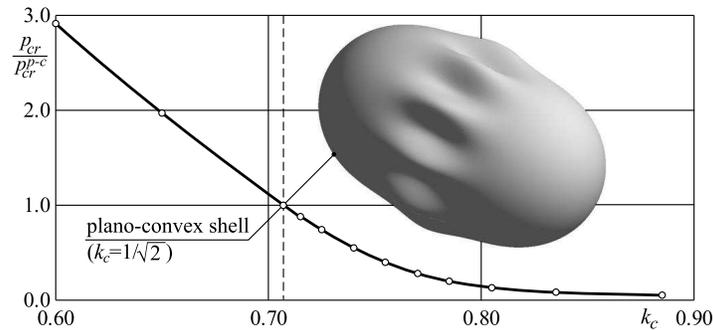


FIG. 5. Critical loads for Cassini ovaloidal shells of different geometries and an example of buckling shape (FEM results).

dimensionless critical pressure of the value equal to 1 corresponds to a plano-convex shell (marked with dashed line – $k_c = 1/\sqrt{2}$). The Gaussian curvature in the mid-length of this shell equals 0. When moving to the left from the dashed line ($k_c < 1/\sqrt{2}$) the positive Gaussian curvature of the meridian increases. At the same time the critical load increases monotonically. Moreover, the higher the buckling load the higher the number of circumferential waves. For shells with $k_c > 1/\sqrt{2}$ the Gaussian curvature in the mid-length is negative and the higher the curvature the smaller the buckling load. However, a detailed analysis of this part of the plot shows that the relation between the geometry, the number of

stable. Similar situation is observed for curve 'b' but in this case the slope of the path after the limit point is smaller. It can be said that the effect of stabilisation appeared. This effect is well visible in paths 'c' and 'd' which have a stable character. The next path 'e' is also stable but the effect of stabilisation is much smaller. The last path, 'f', is again unstable. The results suggest that nearby the minimum point of each curve shown in Fig. 6 the equilibrium path is stable.

To verify this conclusion the second analysis has been carried out for shells, geometry of which is marked with circles in Fig. 6. The results are shown in Fig. 7c. As can be seen all equilibrium paths have stable character – the load increases monotonically during the whole load history what is typical for bars and plates but not necessarily for shells. The effect of stabilization is more pronounced in the case of higher number of circumferential waves. For $n = 2$ the equilibrium path becomes nearly flat at the critical load. The phenomenon of stabilization of post-buckling behaviour of shells by introducing a negative Gaussian curvature has been also reported by Jasion [21] for single-layered shells and by Jasion [24] for sandwich cylindrical shells with variable thickness.

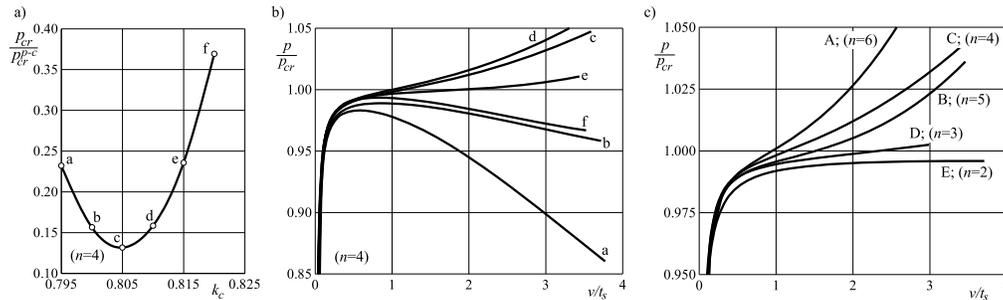


FIG. 7. Post-buckling behaviour of Cassini ovaloidal shells: a) results of buckling analysis for $n = 4$, b) equilibrium paths for shells marked in Fig. 7a, c) equilibrium paths for shells marked with circles in Fig. 6.

An important issue in the shell stability analysis is the sensitivity of a structure to the initial geometrical imperfections since even small disturbance of a perfect shape may influence the post-buckling behaviour significantly. For this reason the imperfection sensitivity analysis has been performed for two selected shells with very similar k_c parameters but for which the character of the post-buckling behaviour is different. The geometry of shells corresponds to points 'b' and 'c' in Fig. 7a.

The results of analyses are shown in Fig. 8. Six different imperfection amplitudes have been considered as a fraction of the shell thickness: $0.01t_s$, $0.02t_s$, $0.05t_s$, $0.1t_s$, $0.5t_s$, $1.0t_s$. The shape of imperfection corresponds to the first

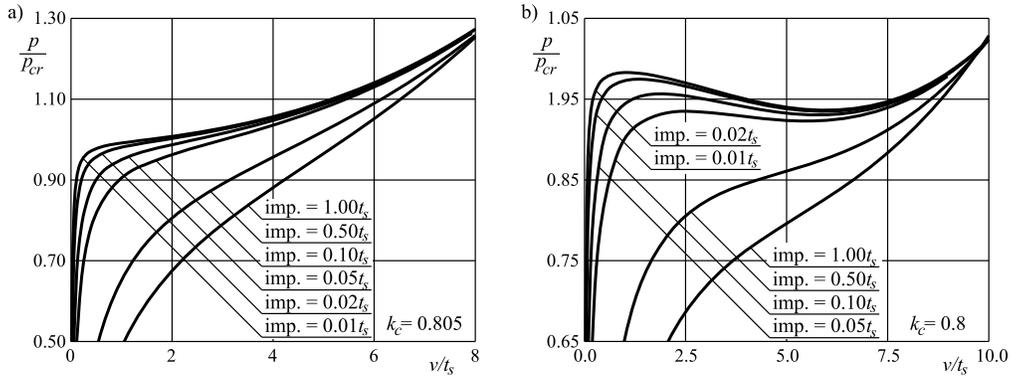


FIG. 8. Imperfection sensitivity of selected shells; a) $k_c = 0.805$, b) $k_c = 0.8$.

eigenmode. Both shells seem to be moderately sensitive to the initial geometrical imperfections. What is important is that the magnitude of imperfections does not influence the character of behaviour of shells in the post-buckling range.

6. Conclusions

In the present paper the results of the theoretical and numerical (FEM) study on the strength and stability of shells in the form of Cassini oval are presented. The study covers the pre-buckling, buckling and post-buckling state.

The pre-buckling state is described both analytically and numerically. The formulae for membrane stress resultants and membrane stresses are derived. A very good agreement is seen as to the value of membrane stress resultants obtained with both approaches. Since the middle surface of the shell is distinguished by the continuity of curvatures the stress distribution is smooth – no edge effect is present, differently than in classical cylindrical tanks closed with heads.

The critical state has been analysed with the use of FEM only. The buckling shape of analysed shells is typical for shells of revolution: one longitudinal half-wave and a number of circumferential waves. The waves are located in the mid-length of the shell where the circumferential resultant N_2 take the biggest value. The number of waves decreases with increasing of the parameter k_c . The buckling load depends strongly on this parameter. For $k_c < 1/\sqrt{2}$ (shells with the positive Gaussian curvature in the mid-length), the buckling load is higher for smaller values of k_c . For $k_c > 1/\sqrt{2}$ (shells with the negative K in the mid-length), the buckling load decreases with increasing of k_c .

A distinguishing feature of the Cassini ovaloidal shells is that they may behave in a stable way in the post-buckling range. Such behaviour was observed

in the post-critical analysis conducted with the use of FE method on shells with $K < 0$ – for some shells the obtained equilibrium paths had a stable character. When this post-critical behaviour is combined with a smooth stress distribution the Cassini ovaloidal shell seems to be a favourable shape for a shell structure subjected to external pressure.

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