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Surface instability of a semi-infinite isotropic laminated plate under surface van der Waals forces

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By means of complex variable method, the present work demonstrates that the surface of a semi-infinite isotropic laminated plate that is being attracted to a rigid contactor through van der Waals forces is always unstable. Two distinct surface instability modes are identified, and their wavenumbers and wavelengths are presented in concise and simple expressions. Furthermore, the two wavenumbers and wavelengths are completely determined by three elastic parameters of the laminated plate, three parameters related to the interactions between the surface and the contactor, and three parameters related to surface energy.

Key words: surface instability, van der Waals force, surface energy, wavenumber, isotropic laminated plate.

1. Introduction

When a rigid contactor is in close proximity to a compliant solid, the van der Waals forces come into play. It has been demonstrated both experimentally and theoretically that the surface of an elastic film will become unstable when it is subject to van der Waals forces [1–11]. In particular, the wavelength of the surface instability is nearly independent of the nature and magnitude of the external force (or the interaction) but proportional to the film thickness [4, 5]. Apparently, the results for thin films are not directly applicable to a semi-infinite elastic body because an infinite wavelength will be predicted if the film thickness approaches infinity.

By using the complex variable technique, Ru [12] analyzed the surface instability of a semi-infinite elastic body under plane strain condition in which the thickness in the x_3-direction approaches infinity. By using a similar method, WANG *et al.* [13] investigated the surface instability of a semi-infinite harmonic solid under finite plane strain deformation. By using the Stroh method, WANG [14] analyzed the surface instability of a semi-infinite anisotropic elastic body under two-dimensional deformation (or the generalized plane strain condition). These studies have demonstrated that the surface of the semi-infinite elastic body attracted by van der Waals forces is always unstable, and a unique surface instability mode exists. FRIED and TODRES [15] investigated the combined effects of surface prestress, curvature dependence of the surface free-energy density and interactions between the surface and rigid contactor on the wrinkling instability of an incompressible half-space. Their results showed that the combined effects will lead to an increased number of linearly stable wrinkled configurations.

The present work aims to analyze the surface instability of a semi-infinite isotropic laminated thin plate under stretching and bending deformations due to van der Waals attraction. The surface instability studied in this work is quite different from other known wrinkling patterns in thin elastic sheets due to small compressive stress [16] or significant stretching [17–19].

2. Basic formulation

Consider an undeformed plate of uniform thickness h, to which a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$ is attached and of which the main plane is located at $x_3 = 0$. The plate is composed of an isotropic, linearly elastic material that can be inhomogeneous and laminated in the thickness direction. In this work, Greek subscripts take the values 1, 2. Summation over repeated subscripts is understood. The coordinate system is chosen in such a way that the two in-plane displacements u_α and the out-of-plane deflection w on the main plane are decoupled in the equilibrium equations [20]. We denote by h_0 the distance between the main plane and the lower surface of the plate [20] and introduce the integral operator

$$Q(\cdots) = \int\limits_{-h_0}^{h-h_0} (\cdots)\, dx_3.$$

Consequently, the membrane stress resultants and bending moments defined by $N_{\alpha\beta} = Q\sigma_{\alpha\beta}$, $M_{\alpha\beta} = Qx_3\sigma_{\alpha\beta}$ with $\sigma_{\alpha\beta}$ being the in-plane stress components, transverse shearing forces $R_\beta = M_{\alpha\beta,\alpha}$, in-plane displacements, deflection and slopes $\vartheta_\alpha = -w_{,\alpha}$ on the main plane of the plate, and the four stress functions φ_α and η_α can be concisely expressed in terms of four complex potentials $\phi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ [20–22]:

$$
\begin{aligned}
N_{11}+N_{22} &= 4\mathrm{Re}\left\{\phi'(z)+B\Phi'(z)\right\},\\
N_{22}-N_{11}+2\mathrm{i}N_{12} &= 2\left[\bar{z}\phi''(z)+\psi'(z)+B\bar{z}\Phi''(z)+B\Psi'(z)\right],\\
M_{11}+M_{22} &= 4D(1+\nu^D)Re\left\{\Phi'(z)\right\}\\
&\quad+\frac{B(\kappa^A-1)}{\mu}Re\left\{\phi'(z)\right\},\\
M_{22}-M_{11}+2\mathrm{i}M_{12} &= -2D(1-\nu^D)\left[\bar{z}\Phi''(z)+\Psi'(z)\right]\\
&\quad-\frac{B}{\mu}\left[\bar{z}\phi''(z)+\psi'(z)\right],\\
R_1-\mathrm{i}R_2 &= 4D\Phi''(z)+\frac{B(\kappa^A+1)}{2\mu}\phi''(z),
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
2\mu(u_1+\mathrm{i}u_2) &= \kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)},\\
\vartheta_1+\mathrm{i}\vartheta_2 &= \Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)},\\
w &= -\mathrm{Re}\left\{\bar{z}\Phi(z)+\chi(z)\right\},\\
\varphi_1+\mathrm{i}\varphi_2 &= \mathrm{i}\left[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}\right]+\mathrm{i}B\left[\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}\right],\\
\eta_1+\mathrm{i}\eta_2 &= \mathrm{i}D(1-\nu^D)\left[\kappa^D\Phi(z)-z\overline{\Phi'(z)}-\overline{\Psi(z)}\right]\\
&\quad+\mathrm{i}\frac{B}{2\mu}\left[\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}\right],
\end{aligned}
\tag{2.2}
$$

where $\Psi(z)=\chi'(z)$, and

$$
\begin{gathered}
\mu=\frac{1}{2}(A_{11}-A_{12}),\quad B=B_{12},\quad D=D_{11},\quad \nu^A=\frac{A_{12}}{A_{11}},\quad \nu^D=\frac{D_{12}}{D_{11}},\\
\kappa^A=\frac{3A_{11}-A_{12}}{A_{11}+A_{12}}=\frac{3-\nu^A}{1+\nu^A},\quad \kappa^D=\frac{3D_{11}+D_{12}}{D_{11}-D_{12}}=\frac{3+\nu^D}{1-\nu^D}.
\end{gathered}
\tag{2.3}
$$

Detailed definitions of the five elastic constants $A_{11},\ A_{12},\ B_{12},\ D_{11}$ and D_{12} can be found in BEOM and EARMME [20]. Moreover, the membrane stress resultants, bending moments, transverse shearing forces, and modified Kirchhoff transverse shearing forces $V_1=R_1+M_{12,2}$ and $V_2=R_2+M_{21,1}$, that exclusively apply to free edges, can be expressed in terms of the four stress functions φ_α and η_α [21]:

$$
\begin{aligned}
N_{\alpha\beta} &= -\epsilon_{\beta\omega}\varphi_{\alpha,\omega},\\
M_{\alpha\beta} &= -\epsilon_{\beta\omega}\eta_{\alpha,\omega}-\frac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega},\\
R_\alpha &= -\frac{1}{2}\in_{\alpha\beta}\eta_{\omega,\omega\beta},\qquad V_\alpha=-\epsilon_{\alpha\omega}\eta_{\omega,\omega\omega},
\end{aligned}
\tag{2.4}
$$

where $\epsilon_{\alpha\beta}$ are the components of the two-dimensional permutation tensor.

In addition, the explicit expressions of the 4×4 real matrices $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ and the 4×4 impedance matrix $\mathbf{M}$ introduced in [21] for an isotropic laminated plate have been obtained in [23]. $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ can be considered as the counterparts of Barnett–Lothe tensors in the Stroh sextic formalism for generalized plane strain elasticity [24]. In particular, $\mathbf{H}$ and $\mathbf{L}$ are positive definite real symmetric matrices and $\mathbf{M}$ is a positive definite Hermitian matrix [25].

3. Surface instability

Now we consider a semi-infinite isotropic laminated plate ($x_2 > 0$ and $-h_0 < x_3 < h - h_0$) attracted to a rigid contactor through van der Waals forces, as illustrated in Fig. 1.

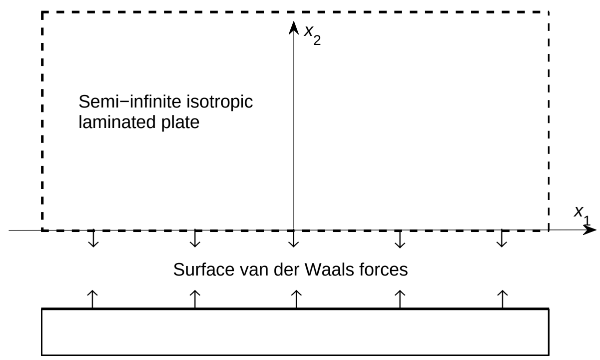

FIG. 1. A semi-infinite isotropic laminated plate ($x_2 > 0$ and $-h_0 < x_3 < h - h_0$) interacting with a rigid contactor through van der Waals-like forces.

The original surface conditions for the perturbed semi-infinite elastic body are given by [12]

$$
\begin{aligned}
&\sigma_{22} = -\beta\tilde{u}_2 - \gamma\tilde{u}_{2,11} = -\beta(u_2 + x_3\vartheta_2) - \gamma(u_{2,11} + x_3\vartheta_{2,11}),\\
&\sigma_{12} = 0, \quad x_2 = 0^+, \quad -h_0 < x_3 < h - h_0,
\end{aligned} \tag{3.1}
$$

where σ_{22} and σ_{12} are the perturbed surface normal and shear stresses, $\tilde{u}_2$ is the perturbed surface normal displacement, β (> 0) is the interaction coefficient [12], γ (> 0) is the surface energy of the semi-infinite plate [12]. Here, it is assumed that β and γ can be inhomogeneous in the plate thickness direction (i.e., β and γ are functions of x_3) to reflect the realistic scenario that the van der Waals interaction energy and the surface energy are material dependent.

Through integrating the stresses in Eq. (3.1), the surface conditions for the perturbed semi-infinite laminated plate take the following form:

$$
\begin{aligned}
N_{22} &= -\beta_{11}u_2 - \beta_{12}\vartheta_2 - \gamma_{11}u_{2,11} - \gamma_{12}\vartheta_{2,11},\\
M_{22} &= -\beta_{12}u_2 - \beta_{22}\vartheta_2 - \gamma_{12}u_{2,11} - \gamma_{22}\vartheta_{2,11},\\
N_{12} &= V_2 = 0, \qquad x_2 = 0^+,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
&\beta_{11} = Q\beta > 0, \qquad \beta_{12} = Qx_3\beta, \qquad \beta_{22} = Qx_3^2\beta > 0,\\
&\gamma_{11} = Q\gamma > 0, \qquad \gamma_{12} = Qx_3\gamma, \qquad \gamma_{22} = Qx_3^2\gamma > 0.
\end{aligned}
\tag{3.3}
$$

The Schwarz integral inequality gives rise to $\beta_{11}\beta_{22} > \beta_{12}^2$ and $\gamma_{11}\gamma_{22} > \gamma_{12}^2$.

It is stressed that the surface conditions in Eq. (3.2) are perturbed ones. It is enough to assume that there is a homogeneous deformation with flat surface due to remote tension and bending $N_{22} = N_{22}^{\infty}$ and $M_{22} = M_{22}^{\infty}$, and then examine if there exists a perturbed solution which can satisfy Eq. (3.2). The solution to the homogeneous deformation is simply given below:

$$
\phi(z) = \chi_1 z, \qquad \psi(z) = \chi_2 z, \qquad \Phi(z) = \eta_1 z, \qquad \Psi(z) = \eta_2 z,
$$

where the four real coefficients χ_1, χ_2, η_1 and η_2 are

$$
\begin{aligned}
\chi_1 &= \frac{\mu D(1+\nu^D)N_{22}^{\infty} - B\mu M_{22}^{\infty}}{4\mu D(1+\nu^D) - B^2(\kappa^A - 1)}, \qquad
\eta_1 = \frac{4\mu M_{22}^{\infty} - B(\kappa^A - 1)N_{22}^{\infty}}{16\mu D(1+\nu^D) - 4B^2(\kappa^A - 1)},\\
\chi_2 &= \frac{\mu D(1-\nu^D)N_{22}^{\infty} + B\mu M_{22}^{\infty}}{2\mu D(1-\nu^D) - B^2}, \qquad
\eta_2 = -\frac{2\mu M_{22}^{\infty} + BN_{22}^{\infty}}{4\mu D(1-\nu^D) - 2B^2}.
\end{aligned}
$$

In view of Eq. (2.4), the condition of $N_{12} = V_2 = 0$ on the surface $x_2 = 0$ is equivalent to $\varphi_1 = \eta_1 = 0$ on $x_2 = 0$. By using Eq. (2.2), this condition can be expressed in terms of the four complex potentials $\phi(z)$, $\Phi(z)$, $\Theta(z) = z\phi'(z) + \psi(z)$ and $\Omega(z) = z\Phi'(z) + \Psi(z)$ as

$$
\begin{aligned}
&\phi^+(z) - \Theta^+(z) - \bar{\phi}^-(z) + \bar{\Theta}^-(z) + B\left[\Phi^+(z) - \Omega^+(z) - \bar{\Phi}^-(z) + \bar{\Omega}^-(z)\right] = 0,\\
&2\mu D(1-\nu^D)\left[\kappa^D\Phi^+(z) + \Omega^+(z) - \kappa^D\bar{\Phi}^-(z) - \bar{\Omega}^-(z)\right]\\
&\qquad\qquad\qquad + B\left[\kappa^A\phi^+(z) + \Theta^+(z) - \kappa^A\bar{\phi}^-(z) - \bar{\Theta}^-(z)\right] = 0,\\
&\mathrm{Im}\{z\} = 0.
\end{aligned}
\tag{3.4}
$$

It can be conveniently derived from the above expressions that $\Theta(z)$ and $\Omega(z)$ can be given in terms of $\phi(z)$ and $\Phi(z)$ as

$$
\begin{aligned}
\Theta(z) &= \frac{\left[2\mu D(1-\nu^D) + B^2\kappa^A\right]\phi(z) + 8B\mu D\Phi(z)}{2\mu D(1-\nu^D) - B^2},\\
\Omega(z) &= \frac{-B(\kappa^A + 1)\phi(z) - \left[2\mu D(3+\nu^D) + B^2\right]\Phi(z)}{2\mu D(1-\nu^D) - B^2}.
\end{aligned}
\tag{3.5}
$$

Consequently, φ_2, η_2, u_1, u_2, ϑ_1 and ϑ_2 on the surface $x_2 = 0$ can be expressed in terms of $\phi(z)$ and $\Phi(z)$ as

$$
\begin{aligned}
\varphi_2 &= \phi(z) + B\Phi(z) + \bar{\phi}(z) + B\bar{\Phi}(z),\\
\eta_2 &= \frac{B\kappa^A}{2\mu}\phi(z) + D(3+\nu^D)\Phi(z) + \frac{B\kappa^A}{2\mu}\bar{\phi}(z) + D(3+\nu^D)\bar{\Phi}(z),\\
u_1 &= \frac{\left[\mu D(1-\nu^D)(\kappa^A-1) - B^2\kappa^A\right]\phi(z) - 4B\mu D\Phi(z)}{4\mu^2\tilde{D}(1-\tilde{\nu}^D)}\\
&\quad + \frac{\left[\mu D(1-\nu^D)(\kappa^A-1) - B^2\kappa^A\right]\bar{\phi}(z) - 4B\mu D\bar{\Phi}(z)}{4\mu^2\tilde{D}(1-\tilde{\nu}^D)},\\
u_2 &= -\,\mathrm{i}\frac{D(1-\nu^D)(\kappa^A+1)\phi(z) + 4BD\Phi(z)}{4\mu\tilde{D}(1-\tilde{\nu}^D)}\\
&\quad + \mathrm{i}\frac{D(1-\nu^D)(\kappa^A+1)\bar{\phi}(z) + 4BD\bar{\Phi}(z)}{4\mu\tilde{D}(1-\tilde{\nu}^D)},\\
\vartheta_1 &= -\,\frac{B(\kappa^A+1)\phi(z) + 2\left[2\mu D(1+\nu^D) + B^2\right]\Phi(z)}{4\mu\tilde{D}(1-\tilde{\nu}^D)}\\
&\quad - \frac{B(\kappa^A+1)\bar{\phi}(z) + 2\left[2\mu D(1+\nu^D) + B^2\right]\bar{\Phi}(z)}{4\mu\tilde{D}(1-\tilde{\nu}^D)},\\
\vartheta_2 &= -\,\mathrm{i}\frac{B(\kappa^A+1)\phi(z) + 8\mu D\Phi(z)}{4\mu\tilde{D}(1-\tilde{\nu}^D)} + \mathrm{i}\frac{B(\kappa^A+1)\bar{\phi}(z) + 8\mu D\bar{\Phi}(z)}{4\mu\tilde{D}(1-\tilde{\nu}^D)},\\
&\operatorname{Im}\{z\} = 0,
\end{aligned}
\tag{3.6}
$$

where $\tilde{\nu}^D$ and $\tilde{D}$ are defined by [20]

$$
\tilde{\nu}^D = \frac{\tilde{D}_{12}}{\tilde{D}_{11}}, \quad \tilde{D} = \tilde{D}_{11} = D_{11} - \frac{A_{11}B_{12}^2}{A_{11}^2 - A_{12}^2}, \quad \tilde{D}_{12} = D_{12} + \frac{A_{12}B_{12}^2}{A_{11}^2 - A_{12}^2}. \tag{3.7}
$$

Thus, the first two conditions in Eq. (3.2) can be expressed in terms of $\phi(z)$ and $\Phi(z)$ as

$$
\begin{aligned}
&2\mathrm{i}\mu\tilde{D}(1-\tilde{\nu}^D)\begin{bmatrix} 2 & 2B \\ B\kappa^A & 2\mu D(3+\nu^D)\end{bmatrix}\begin{bmatrix}\phi'(z)\\ \Phi'(z)\end{bmatrix}^+\\
&+ \begin{bmatrix} (\kappa^A+1)\left[\beta_{11}D(1-\nu^D) + \beta_{12}B\right] & 4D(\beta_{11}B + 2\beta_{12}\mu) \\ \mu(\kappa^A+1)\left[\beta_{12}D(1-\nu^D) + \beta_{22}B\right] & 4\mu D(\beta_{12}B + 2\beta_{22}\mu)\end{bmatrix}\begin{bmatrix}\phi(z)\\ \Phi(z)\end{bmatrix}^+\\
&+ \begin{bmatrix} (\kappa^A+1)\left[\gamma_{11}D(1-\nu^D) + \gamma_{12}B\right] & 4D(\gamma_{11}B + 2\gamma_{12}\mu) \\ \mu(\kappa^A+1)\left[\gamma_{12}D(1-\nu^D) + \gamma_{22}B\right] & 4\mu D(\gamma_{12}B + 2\gamma_{22}\mu)\end{bmatrix}\begin{bmatrix}\phi''(z)\\ \Phi''(z)\end{bmatrix}^+
\end{aligned}
\tag{3.8}
$$

$$= -2\mathrm{i}\mu\tilde{D}(1-\tilde{\nu}^D)\begin{bmatrix} 2 & 2B \\ B\kappa^A & 2\mu D(3+\nu^D)\end{bmatrix}\begin{bmatrix}\bar{\phi}'(z) \\ \bar{\Phi}'(z)\end{bmatrix}^-$$

$$+\begin{bmatrix} (\kappa^A+1)\left[\beta_{11}D(1-\nu^D)+\beta_{12}B\right] & 4D(\beta_{11}B+2\beta_{12}\mu) \\ \mu(\kappa^A+1)\left[\beta_{12}D(1-\nu^D)+\beta_{22}B\right] & 4\mu D(\beta_{12}B+2\beta_{22}\mu)\end{bmatrix}\begin{bmatrix}\bar{\phi}(z) \\ \bar{\Phi}(z)\end{bmatrix}^-$$

$$+\begin{bmatrix} (\kappa^A+1)\left[\gamma_{11}D(1-\nu^D)+\gamma_{12}B\right] & 4D(\gamma_{11}B+2\gamma_{12}\mu) \\ \mu(\kappa^A+1)\left[\gamma_{12}D(1-\nu^D)+\gamma_{22}B\right] & 4\mu D(\gamma_{12}B+2\gamma_{22}\mu)\end{bmatrix}\begin{bmatrix}\bar{\phi}''(z) \\ \bar{\Phi}''(z)\end{bmatrix}^-,$$

$$\mathrm{Im}\,\{z\}=0.$$

The left-hand side of Eq. (3.8) is analytic in the upper half-plane including the point at infinity, whilst its right-hand side is analytic in the lower half-plane including the point at infinity. By using Liouville's theorem, we arrive at the following set of coupled second-order differential equations:

$$(3.9)\qquad 2\mathrm{i}\mu\tilde{D}(1-\tilde{\nu}^D)\begin{bmatrix} 2 & 2B \\ B\kappa^A & 2\mu D(3+\nu^D)\end{bmatrix}\begin{bmatrix}\phi'(z) \\ \Phi'(z)\end{bmatrix}$$

$$+\begin{bmatrix} (\kappa^A+1)\left[\beta_{11}D(1-\nu^D)+\beta_{12}B\right] & 4D(\beta_{11}B+2\beta_{12}\mu) \\ \mu(\kappa^A+1)\left[\beta_{12}D(1-\nu^D)+\beta_{22}B\right] & 4\mu D(\beta_{12}B+2\beta_{22}\mu)\end{bmatrix}\begin{bmatrix}\phi(z) \\ \Phi(z)\end{bmatrix}$$

$$+\begin{bmatrix} (\kappa^A+1)\left[\gamma_{11}D(1-\nu^D)+\gamma_{12}B\right] & 4D(\gamma_{11}B+2\gamma_{12}\mu) \\ \mu(\kappa^A+1)\left[\gamma_{12}D(1-\nu^D)+\gamma_{22}B\right] & 4\mu D(\gamma_{12}B+2\gamma_{22}\mu)\end{bmatrix}\begin{bmatrix}\phi''(z) \\ \Phi''(z)\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}.$$

To solve the above set of equations, the unknown $\phi(z)$ and $\Phi(z)$ are assumed to take the following forms:

$$(3.10)\qquad \begin{aligned}\phi(z) &= \delta_1\exp(\mathrm{i}\lambda z),\\ \Phi(z) &= \delta_2\exp(\mathrm{i}\lambda z),\end{aligned}$$

where λ is a wavenumber. The real part of λ should be positive in order to ensure that $\phi(z)$ and $\Phi(z)$ are bounded as $x_2\to+\infty$.

Substitution of Eq. (3.10) into Eq. (3.9) yields the following eigenvalue problem:

$$(3.11)\qquad -2\lambda\mu\tilde{D}(1-\tilde{\nu}^D)\begin{bmatrix} 2 & 2B \\ B\kappa^A & 2\mu D(3+\nu^D)\end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix}$$

$$+\begin{bmatrix} (\kappa^A+1)\left[\beta_{11}D(1-\nu^D)+\beta_{12}B\right] & 4D(\beta_{11}B+2\beta_{12}\mu) \\ \mu(\kappa^A+1)\left[\beta_{12}D(1-\nu^D)+\beta_{22}B\right] & 4\mu D(\beta_{12}B+2\beta_{22}\mu)\end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix}$$

$$-\lambda^2\begin{bmatrix} (\kappa^A+1)\left[\gamma_{11}D(1-\nu^D)+\gamma_{12}B\right] & 4D(\gamma_{11}B+2\gamma_{12}\mu) \\ \mu(\kappa^A+1)\left[\gamma_{12}D(1-\nu^D)+\gamma_{22}B\right] & 4\mu D(\gamma_{12}B+2\gamma_{22}\mu)\end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix},$$

which leads to a quartic equation in λ given by

$$(\gamma_{11}\gamma_{22}-\gamma_{12}^2)\lambda^4+(\gamma_{11}L_{33}+\gamma_{22}L_{11}-2\gamma_{12}L_{13})\lambda^3 \\ +\left[L_{11}L_{33}-L_{13}^2-(\beta_{22}\gamma_{11}+\beta_{11}\gamma_{22}-2\beta_{12}\gamma_{12})\right]\lambda^2 \\ -(\beta_{11}L_{33}+\beta_{22}L_{11}-2\beta_{12}L_{13})\lambda+\beta_{11}\beta_{22}-\beta_{12}^2=0, \tag{3.12}$$

where L_{11}, L_{33} and L_{13} are defined as

$$\begin{aligned} L_{11}&=\mu(1+\nu^A)-\frac{B^2}{2D},\\ L_{33}&=\frac{D(1-\nu^D)(3+\nu^D)}{2}-\frac{B^2(3-\nu^A)}{4\mu},\\ L_{13}&=-\frac{B(\nu^D+\nu^A)}{2}. \end{aligned} \tag{3.13}$$

It is noted that L_{11}, L_{33} and L_{13} are elements of the following 4×4 positive definite real symmetric matrix $\mathbf{L}$ [23]

$$\mathbf{L}=\begin{bmatrix} L_{11} & 0 & L_{13} & 0\\ 0 & L_{11} & 0 & L_{13}\\ L_{13} & 0 & L_{33} & 0\\ 0 & L_{13} & 0 & L_{33} \end{bmatrix}. \tag{3.14}$$

The wavenumber λ can also be determined by solving the following eigenvalue problem:

$$(\mathbf{P}-\lambda\mathbf{M}-\lambda^2\mathbf{Q})\mathbf{v}=\mathbf{0}, \tag{3.15}$$

where $\mathbf{v}$ is the eigenvector associated with the eigenvalue λ, and

$$\mathbf{M}=\begin{bmatrix} \dfrac{4\mu}{3-\nu^A} & -\dfrac{2\mathrm{i}\mu(1-\nu^A)}{3-\nu^A} & 0 & \mathrm{i}B\\ \dfrac{2\mathrm{i}\mu(1-\nu^A)}{3-\nu^A} & \dfrac{4\mu}{3-\nu^A} & -\mathrm{i}B & 0\\ 0 & \mathrm{i}B & 2D & \mathrm{i}D(1+\nu^D)\\ -\mathrm{i}B & 0 & -\mathrm{i}D(1+\nu^D) & 2D \end{bmatrix}, \tag{3.16}$$

$$\mathbf{P}=\begin{bmatrix} 0&0&0&0\\ 0&\beta_{11}&0&\beta_{12}\\ 0&0&0&0\\ 0&\beta_{12}&0&\beta_{22} \end{bmatrix}, \qquad \mathbf{Q}=\begin{bmatrix} 0&0&0&0\\ 0&\gamma_{11}&0&\gamma_{12}\\ 0&0&0&0\\ 0&\gamma_{12}&0&\gamma_{22} \end{bmatrix}. \tag{3.17}$$

It is deduced from Eq. (3.15) that for a nontrivial solution of $\mathbf{v}$,

$$\left|\mathbf{P}-\lambda\mathbf{M}-\lambda^2\mathbf{Q}\right|=0, \tag{3.18}$$

which is a sextic equation in λ. There are six roots of λ in Eq. (3.18): two are zero, and the rest four nonzero eigenvalues are determined by Eq. (3.12).

The 4×4 Hermitian matrix $\mathbf{M}$ (or the so called edge-impedance matrix) is positive definite [25] and the 4×4 real symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$ are both positive semi-definite in view of the fact that $\beta_{11}>0$, $\beta_{22}>0$, $\beta_{11}\beta_{22}>\beta_{12}^2$ and $\gamma_{11}>0$, $\gamma_{22}>0$, $\gamma_{11}\gamma_{22}>\gamma_{12}^2$. If $\mathbf{v}$ is an eigenvector associated with a nonzero eigenvalue λ of Eq. (3.15), we will have $\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{P}\mathbf{v}>0$, $\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{Q}\mathbf{v}>0$ and $\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{M}\mathbf{v}>0$ (if $\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{P}\mathbf{v}=0$, we then have $\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{Q}\mathbf{v}=0$, as a result $\lambda=0$, which violates the assumption that λ is nonzero). Pre-multiplying Eq. (3.15) by $\bar{\mathbf{v}}^{\mathrm{T}}$, we obtain

$$\lambda^2\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{Q}\mathbf{v}+\lambda\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{M}\mathbf{v}-\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{P}\mathbf{v}=0, \tag{3.19}$$

from which we arrive at

$$\lambda=\frac{-\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{M}\mathbf{v}\pm\sqrt{(\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{M}\mathbf{v})^2+4(\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{P}\mathbf{v})(\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{Q}\mathbf{v})}}{2\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{Q}\mathbf{v}}. \tag{3.20}$$

The above expression together with Eq. (3.12) clearly indicates that the nonzero λ is always real, and the four real-valued nonzero λ contain both positive and negative numbers. It is further observed from Eq. (3.12) that two eigenvalues are positive, and the remaining two are negative because the product of the four eigenvalues $(\beta_{11}\beta_{22}-\beta_{12}^2)/(\gamma_{11}\gamma_{22}-\gamma_{12}^2)$ is positive. This fact implies that there always exist two distinct surface instability modes whenever $\beta>0$ and $\gamma>0$. In addition, it is observed from Eq. (3.12) that the two wavenumbers λ_1, λ_2, $(\lambda_1>\lambda_2>0)$ or the two wavelengths $2\pi/\lambda_1$, $2\pi/\lambda_2$ are completely determined by nine parameters: $(\beta_{11},\beta_{22},\beta_{12})$, $(\gamma_{11},\gamma_{22},\gamma_{12})$ and (L_{11},L_{33},L_{13}).

Meanwhile, it is obtained from Eq. (3.11) that δ_1 and δ_2 should satisfy the following restriction:

$$\frac{\delta_1}{\delta_2}=\frac{4\lambda B\mu\tilde{D}(1-\tilde{\nu}^D)-4D(\beta_{11}B+2\beta_{12}\mu)+4\lambda^2D(\gamma_{11}B+2\gamma_{12}\mu)}{(\kappa^A+1)\left[\beta_{11}D(1-\nu^D)+\beta_{12}B\right]-4\lambda\mu\tilde{D}(1-\tilde{\nu}^D)-\lambda^2(\kappa^A+1)\left[\gamma_{11}D(1-\nu^D)+\gamma_{12}B\right]}, \tag{3.21}$$

with $(\lambda=\lambda_1,\lambda_2)$.

The stretching and bending deformations of the laminated semi-infinite plate cannot be uniquely determined because δ_1 or δ_2 can still be arbitrary even though the ratio δ_1/δ_2 is uniquely determined from Eq. (3.21) for a given wavenumber. The decay rate of surface perturbation in the x_2-direction is determined by

$$\min\{\lambda_1,\lambda_2\}=\lambda_2. \tag{3.22}$$

In the following, we present five special cases to demonstrate the obtained solution.

Case I. When the plate is relatively stiff such that $\lambda \mathbf{Q}\mathbf{M}^{-1} \to 0$, Eq. (3.15) reduces to

$$(\mathbf{P} - \lambda \mathbf{M})\mathbf{v} = \mathbf{0}, \tag{3.23}$$

which is independent of the surface energy. In this case, Eq. (3.12) becomes

$$(L_{11}L_{33} - L_{13}^2)\lambda^2 - (\beta_{11}L_{33} + \beta_{22}L_{11} - 2\beta_{12}L_{13})\lambda + \beta_{11}\beta_{22} - \beta_{12}^2 = 0. \tag{3.24}$$

The two wavenumbers λ_1 and λ_2 can then be determined from Eq. (3.24) as

$$\begin{aligned} &\lambda_1 = \\ &\frac{\beta_{11}L_{33}+\beta_{22}L_{11}-2\beta_{12}L_{13}+\sqrt{(\beta_{11}L_{33}-\beta_{22}L_{11})^2+4(\beta_{11}L_{13}-\beta_{12}L_{11})(\beta_{22}L_{13}-\beta_{12}L_{33})}}{2(L_{11}L_{33}-L_{13}^2)}, \\ &\lambda_2 = \\ &\frac{\beta_{11}L_{33}+\beta_{22}L_{11}-2\beta_{12}L_{13}-\sqrt{(\beta_{11}L_{33}-\beta_{22}L_{11})^2+4(\beta_{11}L_{13}-\beta_{12}L_{11})(\beta_{22}L_{13}-\beta_{12}L_{33})}}{2(L_{11}L_{33}-L_{13}^2)}. \end{aligned} \tag{3.25}$$

If $\mathbf{v}$ is an eigenvector associated with a nonzero eigenvalue λ of Eq. (3.23), then

$$\lambda = \frac{\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{P}\mathbf{v}}{\bar{\mathbf{v}}^{\mathrm{T}}\mathbf{M}\mathbf{v}} > 0.$$

Consequently the two nonzero wavenumbers given by Eq. (3.24) are indeed positive. Considering the fact that

$$\begin{aligned} &(\beta_{11}L_{33} - \beta_{22}L_{11})^2 + 4(\beta_{11}L_{13} - \beta_{12}L_{11})(\beta_{22}L_{13} - \beta_{12}L_{33}) \\ &\quad = (\beta_{11}L_{33} + \beta_{22}L_{11} - 2\beta_{12}L_{13})^2 - 4(\beta_{11}\beta_{22} - \beta_{12}^2)(L_{11}L_{33} - L_{13}^2), \end{aligned} \tag{3.26}$$

the following inequalities can then be established from Eqs. (3.25) and (3.26):

$$\beta_{11}L_{33} + \beta_{22}L_{11} - 2\beta_{12}L_{13} \geq 2\sqrt{(\beta_{11}\beta_{22} - \beta_{12}^2)(L_{11}L_{33} - L_{13}^2)} > 0. \tag{3.27}$$

In addition, the two wavenumbers given by Eq. (3.25) are completely determined by six parameters: $(\beta_{11}, \beta_{22}, \beta_{12})$ and (L_{11}, L_{33}, L_{13}). The wavelengths of the two instability modes are given by

$$\begin{aligned} &\frac{2\pi}{\lambda_1} = \\ &\frac{\pi\left[\beta_{11}L_{33}+\beta_{22}L_{11}-2\beta_{12}L_{13}-\sqrt{(\beta_{11}L_{33}-\beta_{22}L_{11})^2+4(\beta_{11}L_{13}-\beta_{12}L_{11})(\beta_{22}L_{13}-\beta_{12}L_{33})}\right]}{\beta_{11}\beta_{22}-\beta_{12}^2}, \\ &\frac{2\pi}{\lambda_2} = \\ &\frac{\pi\left[\beta_{11}L_{33}+\beta_{22}L_{11}-2\beta_{12}L_{13}+\sqrt{(\beta_{11}L_{33}-\beta_{22}L_{11})^2+4(\beta_{11}L_{13}-\beta_{12}L_{11})(\beta_{22}L_{13}-\beta_{12}L_{33})}\right]}{\beta_{11}\beta_{22}-\beta_{12}^2}. \end{aligned} \tag{3.28}$$

When $B = 0$ for a homogeneous plate, $\beta_{12} = L_{13} = 0$. In this case, it is deduced from Eq. (3.25) that

$$\lambda_1 = \frac{\beta_{11}}{L_{11}} = \frac{\beta_{11}(\kappa^A + 1)}{4\mu}, \qquad \lambda_2 = \frac{\beta_{22}}{L_{33}} = \frac{2\beta_{22}}{D(1-\nu^D)(3+\nu^D)}. \tag{3.29}$$

In Eq. (3.29), λ_1 is just the one derived by Ru [12] when the surface energy γ is ignored. Equation (3.29) implies that there exist two distinct instability modes even when the thin plate is homogeneous in the thickness direction. The ratio of the two wavenumbers in Eq. (3.29) is

$$\frac{\lambda_1}{\lambda_2} = \frac{3+\nu}{1+\nu}$$

where ν is the Poisson's ratio of the homogeneous plate. For example, if ν=1/3, this ratio gives $\lambda_1/\lambda_2 = 2.5$.

Case II. If the plate is extremely compliant such that $\lambda\mathbf{Q}\mathbf{M}^{-1} \to \infty$, Eq. (3.15) reduces to

$$(\mathbf{P} - \lambda^2\mathbf{Q})\mathbf{v} = \mathbf{0}, \tag{3.30}$$

which is independent of the elastic properties of the plate. In this case, Eq. (3.12) becomes

$$(\gamma_{11}\gamma_{22} - \gamma_{12}^2)\lambda^4 - (\beta_{22}\gamma_{11} + \beta_{11}\gamma_{22} - 2\beta_{12}\gamma_{12})\lambda^2 + \beta_{11}\beta_{22} - \beta_{12}^2 = 0. \tag{3.31}$$

The two wavenumbers λ_1 and λ_2 are then determined from Eq. (3.31) as

$$\begin{aligned} &\lambda_1^2 = \\ &\frac{\beta_{11}\gamma_{22}+\beta_{22}\gamma_{11}-2\beta_{12}\gamma_{12}+\sqrt{(\beta_{11}\gamma_{22}-\beta_{22}\gamma_{11})^2+4(\beta_{11}\gamma_{12}-\beta_{12}\gamma_{11})(\beta_{22}\gamma_{12}-\beta_{12}\gamma_{22})}}{2(\gamma_{11}\gamma_{22}-\gamma_{12}^2)} > 0, \\ &\lambda_2^2 = \\ &\frac{\beta_{11}\gamma_{22}+\beta_{22}\gamma_{11}-2\beta_{12}\gamma_{12}-\sqrt{(\beta_{11}\gamma_{22}-\beta_{22}\gamma_{11})^2+4(\beta_{11}\gamma_{12}-\beta_{12}\gamma_{11})(\beta_{22}\gamma_{12}-\beta_{12}\gamma_{22})}}{2(\gamma_{11}\gamma_{22}-\gamma_{12}^2)} > 0. \end{aligned} \tag{3.32}$$

The positive values on the right-hand side of Eq. (3.32) are due to the fact that both

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}$$

are positive definite. The two wavenumbers given by Eq. (3.32) are completely determined by six parameters: $(\beta_{11}, \beta_{22}, \beta_{12})$ and $(\gamma_{11}, \gamma_{22}, \gamma_{12})$. In addition, the following inequality can be easily established from Eq. (3.32):

$$\beta_{22}\gamma_{11} + \beta_{11}\gamma_{22} - 2\beta_{12}\gamma_{12} \geq 2\sqrt{(\beta_{11}\beta_{22} - \beta_{12}^2)(\gamma_{11}\gamma_{22} - \gamma_{12}^2)} > 0. \tag{3.33}$$

Case III. If $\gamma_{\omega\rho}$ and $\beta_{\omega\rho}$ satisfy the following restriction:

$$\frac{\gamma_{11}}{\beta_{11}} = \frac{\gamma_{12}}{\beta_{12}} = \frac{\gamma_{22}}{\beta_{22}} = k > 0, \tag{3.34}$$

the two wavenumbers (denoted as $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$) can be simply obtained from λ_1 and λ_2 given by Eq. (3.25) in the absence of the surface energy as follows:

$$\begin{aligned} \tilde{\lambda}_1 &= \frac{2\lambda_1}{1+\sqrt{1+4k\lambda_1^2}} < \lambda_1, \\ \tilde{\lambda}_2 &= \frac{2\lambda_2}{1+\sqrt{1+4k\lambda_2^2}} < \lambda_2, \end{aligned} \tag{3.35}$$

which indicates that the surface energy will lower the values of the wavenumbers.

Case IV. If the plate is homogeneous in the thickness direction, we have

$$\beta_{12} = \gamma_{12} = L_{13} = 0, \qquad L_{11} = \mu(1+\nu^A), \qquad L_{33} = \frac{D(1-\nu^D)(3+\nu^D)}{2}.$$

In this case, Eq. (3.12) becomes

$$\begin{aligned} \gamma_{11}\gamma_{22}\lambda^4 + (\gamma_{11}L_{33} + \gamma_{22}L_{11})\lambda^3 + (L_{11}L_{33} - \beta_{22}\gamma_{11} - \beta_{11}\gamma_{22})\lambda^2 \\ - (\beta_{11}L_{33} + \beta_{22}L_{11})\lambda + \beta_{11}\beta_{22} = 0, \end{aligned} \tag{3.36}$$

or equivalently

$$(\gamma_{11}\lambda^2 + L_{11}\lambda - \beta_{11})(\gamma_{22}\lambda^2 + L_{33}\lambda - \beta_{22}) = 0. \tag{3.37}$$

The two wavenumbers can be determined as

$$\begin{aligned} \lambda_1 &= \frac{-L_{11} + \sqrt{L_{11}^2 + 4\beta_{11}\gamma_{11}}}{2\gamma_{11}}, \\ \lambda_2 &= \frac{-L_{33} + \sqrt{L_{33}^2 + 4\beta_{22}\gamma_{22}}}{2\gamma_{22}}. \end{aligned} \tag{3.38}$$

It is easily checked that λ_1 in Eq. (3.38) is just the one derived by RU [12]. In the presence of surface energy, there are two instability modes for a homogeneous plate given by Eq. (3.38): one is the in-plane mode observed in [12], the other one is the out-of-plane mode.

Case V. In the final example, we assume that the plate is made of two homogeneous layers of equal thickness. In addition, the Young's modulus of the top layer is just double that of the bottom layer, and the two layers have a constant Poisson's ratio $\nu = 0.25$. Both β and γ are constant in the thickness direction. In this example, it is calculated that $h_0 = 7h/12$, and that

$$
\begin{aligned}
&\gamma_{11}\gamma_{22} - \gamma_{12}^2 = \frac{\gamma^2 h^4}{12},\\
&\gamma_{11}L_{33} + \gamma_{22}L_{11} - 2\gamma_{12}L_{13} = \frac{13\times 71}{16\times 64}h^4\gamma C_{11},\\
&L_{11}L_{33} - L_{13}^2 - (\beta_{22}\gamma_{11} + \beta_{11}\gamma_{22} - 2\beta_{12}\gamma_{12}) = \frac{45\times 429}{64\times 512}h^4 C_{11}^2 - \frac{h^4\beta\gamma}{6},\\
&-(\beta_{11}L_{33} + \beta_{22}L_{11} - 2\beta_{12}L_{13}) = -\frac{13\times 71}{16\times 64}h^4\beta C_{11},\\
&\beta_{11}\beta_{22} - \beta_{12}^2 = \frac{\beta^2 h^4}{12},
\end{aligned}
\tag{3.39}
$$

where $C_{11} = E/(1-\nu^2)$ with E being the Young's modulus of the bottom layer.

The two wavenumbers can then be determined from Eq. (3.12). The dependence of the two wavelengths on both β and γ is shown in Fig. 2 for two values of $\beta = 10^{11}$, 2×10^{11} J/m^4 with $E = 0.5$ Mpa. First, it is clearly demonstrated in Fig. 2 that our theoretical prediction of the existence of two instability modes is numerically verified in this example. In addition, the two wavelengths of surface wrinkling are very sensitive to the interaction coefficient β but not to the surface energy γ. An increase in the interaction coefficient β will lower the values of the two wavelengths of surface wrinkling. This trend is in agreement with that observed by RU [12].

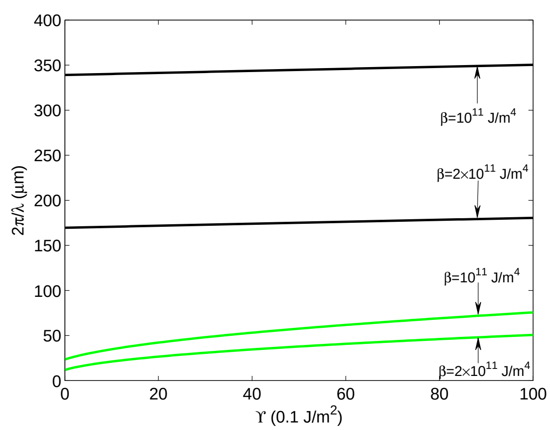

FIG. 2. The dependence of the two wavelengths of surface wrinkling on the interaction coefficient β and the surface energy γ for two values of $\beta = 10^{11}$, 2×10^{11} J/m^4 with $E = 0.5$ Mpa.

4. Further discussions

In Section 3, the results are obtained in the Cartesian coordinate system that is chosen such that $x_3 = 0$ is on the main plane. Next, the surface instability problem will be discussed by choosing a new coordinate system $\{\hat{x}_i\}$ $(i = 1, 2, 3)$ in which $\hat{x}_3 = 0$ is at an arbitrary distance of h_1 above the lower surface of the plate and $\hat{x}_\alpha = x_\alpha$.

In the new coordinate system, the surface conditions for the perturbed semi-infinite laminated plate take the following form:

$$
\begin{aligned}
\hat{N}_{22} &= -\hat{\beta}_{11}\hat{u}_2 - \hat{\beta}_{12}\hat{\vartheta}_2 - \hat{\gamma}_{11}\hat{u}_{2,11} - \hat{\gamma}_{12}\hat{\vartheta}_{2,11},\\
\hat{M}_{22} &= -\hat{\beta}_{12}\hat{u}_2 - \hat{\beta}_{22}\hat{\vartheta}_2 - \hat{\gamma}_{12}\hat{u}_{2,11} - \hat{\gamma}_{22}\hat{\vartheta}_{2,11},\\
\hat{N}_{12} &= \hat{V}_2 = 0, \qquad x_2 = 0^+,
\end{aligned}
\tag{4.1}
$$

where the symbol ˆ indicates the quantities in the new coordinate system, and

$$
\begin{aligned}
&\hat{\beta}_{11} = \hat{Q}\beta > 0, \qquad \hat{\beta}_{12} = \hat{Q}\hat{x}_3\beta, \qquad \hat{\beta}_{22} = \hat{Q}\hat{x}_3^2\beta > 0,\\
&\hat{\gamma}_{11} = \hat{Q}\gamma > 0, \qquad \hat{\gamma}_{12} = \hat{Q}\hat{x}_3\gamma, \qquad \hat{\gamma}_{22} = \hat{Q}\hat{x}_3^2\gamma > 0,
\end{aligned}
\tag{4.2}
$$

with $\hat{Q}(\cdots) = \int_{-h_1}^{h-h_1}(\cdots)\,d\hat{x}_3$.

It can be conveniently proved that

$$
\begin{aligned}
&\hat{\beta}_{11} = \beta_{11}, \qquad \hat{\beta}_{12} = \beta_{12} + \hat{h}\beta_{11}, \qquad \hat{\beta}_{22} = \beta_{22} + 2\hat{h}\beta_{12} + \hat{h}^2\beta_{11},\\
&\hat{\gamma}_{11} = \gamma_{11}, \qquad \hat{\gamma}_{12} = \gamma_{12} + \hat{h}\gamma_{11}, \qquad \hat{\gamma}_{22} = \gamma_{22} + 2\hat{h}\gamma_{12} + \hat{h}^2\gamma_{11},
\end{aligned}
\tag{4.3}
$$

where $\hat{h} = h_1 - h_0$.

In addition, $\hat{u}_\alpha$ and $\hat{\vartheta}_\alpha$ are related to u_α and ϑ_α, $\hat{\varphi}_\alpha$ and $\hat{\eta}_\alpha$ are related to φ_α and η_α through the following relationships:

$$
\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{\vartheta}_1\\ \hat{\vartheta}_2\end{bmatrix} = \begin{bmatrix}\mathbf{I} & -\hat{h}\mathbf{I}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ \vartheta_1\\ \vartheta_2\end{bmatrix}, \qquad \begin{bmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\\ \hat{\eta}_1\\ \hat{\eta}_2\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \hat{h}\mathbf{I} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \eta_1\\ \eta_2\end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.
\tag{4.4}
$$

Thus, the impedance matrix $\hat{\mathbf{M}} = \hat{\mathbf{H}}^{-1} + \mathrm{i}\hat{\mathbf{H}}^{-1}\hat{\mathbf{S}}$ and its inverse $\hat{\mathbf{M}}^{-1} = \hat{\mathbf{L}}^{-1} - \mathrm{i}\hat{\mathbf{S}}\hat{\mathbf{L}}^{-1}$ in the new coordinate system can be obtained from $\mathbf{M} = \mathbf{H}^{-1} + \mathrm{i}\mathbf{H}^{-1}\mathbf{S}$ and its inverse $\mathbf{M}^{-1} = \mathbf{L}^{-1} - \mathrm{i}\mathbf{S}\mathbf{L}^{-1}$ in the original coordinate system as follows:

$$
\hat{\mathbf{M}} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \hat{h}\mathbf{I} & \mathbf{I}\end{bmatrix}\mathbf{M}\begin{bmatrix}\mathbf{I} & \hat{h}\mathbf{I}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \qquad \hat{\mathbf{M}}^{-1} = \begin{bmatrix}\mathbf{I} & -\hat{h}\mathbf{I}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{M}^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\hat{h}\mathbf{I} & \mathbf{I}\end{bmatrix}.
\tag{4.5}
$$

Therefore, the three real matrices $\hat{\mathbf{H}}$, $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ in the new coordinate system $\{\hat{x}_i\}$ can be obtained from $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ in the original coordinate system $\{x_i\}$ as

$$\hat{\mathbf{H}} = \begin{bmatrix} \mathbf{I} & -\hat{h}\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\hat{h}\mathbf{I} & \mathbf{I} \end{bmatrix}, \qquad \hat{\mathbf{L}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{h}\mathbf{I} & \mathbf{I} \end{bmatrix} \mathbf{L} \begin{bmatrix} \mathbf{I} & \hat{h}\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$
$$\hat{\mathbf{S}} = \begin{bmatrix} \mathbf{I} & -\hat{h}\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{S} \begin{bmatrix} \mathbf{I} & \hat{h}\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{4.6}$$

As a result, $\hat{\mathbf{H}}$, $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ can be derived as

$$\hat{\mathbf{H}} = \begin{bmatrix} \hat{H}_{11} & 0 & \hat{H}_{13} & 0 \\ 0 & \hat{H}_{11} & 0 & \hat{H}_{13} \\ \hat{H}_{13} & 0 & \hat{H}_{33} & 0 \\ 0 & \hat{H}_{13} & 0 & \hat{H}_{33} \end{bmatrix}, \qquad \hat{\mathbf{L}} = \begin{bmatrix} \hat{L}_{11} & 0 & \hat{L}_{13} & 0 \\ 0 & \hat{L}_{11} & 0 & \hat{L}_{13} \\ \hat{L}_{13} & 0 & \hat{L}_{33} & 0 \\ 0 & \hat{L}_{13} & 0 & \hat{L}_{33} \end{bmatrix},$$
$$\hat{\mathbf{S}} = \begin{bmatrix} 0 & -\hat{S}_{21} & 0 & \hat{S}_{14} \\ \hat{S}_{21} & 0 & -\hat{S}_{14} & 0 \\ 0 & \hat{S}_{32} & 0 & \hat{S}_{34} \\ -\hat{S}_{32} & 0 & -\hat{S}_{34} & 0 \end{bmatrix}, \tag{4.7}$$

where

$$\begin{aligned} &\hat{H}_{11} = H_{11} + \hat{h}^2 H_{33}, \quad \hat{H}_{13} = -\hat{h} H_{33}, \quad \hat{H}_{33} = H_{33}, \\ &\hat{L}_{11} = L_{11}, \quad \hat{L}_{13} = L_{13} + \hat{h} L_{11}, \quad \hat{L}_{33} = L_{33} + 2\hat{h} L_{13} + \hat{h}^2 L_{11}, \\ &\hat{S}_{21} = S_{21} + \hat{h} S_{32}, \quad \hat{S}_{14} = S_{14} - \hat{h}(S_{21} + S_{34}) - \hat{h}^2 S_{32}, \\ &\hat{S}_{32} = S_{32}, \quad \hat{S}_{34} = S_{34} + \hat{h} S_{32}. \end{aligned} \tag{4.8}$$

In Eq. (4.8), L_{11}, L_{33} and L_{13} have been defined in Eq. (3.13), and

$$\begin{aligned} &H_{11} = \frac{3 - \nu^A}{4\mu}, \quad H_{33} = \frac{1}{2D}, \\ &S_{21} = \frac{1 - \nu^A}{2}, \quad S_{14} = \frac{B(3 - \nu^A)}{4\mu}, \quad S_{32} = \frac{B}{2D}, \quad S_{34} = \frac{1 + \nu^D}{2}. \end{aligned} \tag{4.9}$$

It is observed from Eq. (4.8) that H_{33}, L_{11} and S_{32} are invariants. Moreover, it is also found that the following quantities are invariants:

$$\begin{aligned} &\hat{\beta}_{11}\hat{H}_{11} + \hat{\beta}_{22}\hat{H}_{33} + 2\hat{\beta}_{12}\hat{H}_{13} = \beta_{11}H_{11} + \beta_{22}H_{33}, \\ &\hat{\beta}_{11}\hat{L}_{33} + \hat{\beta}_{22}\hat{L}_{11} - 2\hat{\beta}_{12}\hat{L}_{13} = \beta_{11}L_{33} + \beta_{22}L_{11} - 2\beta_{12}L_{13}, \\ &\hat{\beta}_{11}\hat{S}_{14} + \hat{\beta}_{12}(\hat{S}_{21} + \hat{S}_{34}) - \hat{\beta}_{22}\hat{S}_{32} = \beta_{11}S_{14} + \beta_{12}(S_{21} + S_{34}) - \beta_{22}S_{32}, \\ &\hat{H}_{11}\hat{L}_{11} + \hat{H}_{33}\hat{L}_{33} + 2\hat{H}_{13}\hat{L}_{13} = H_{11}L_{11} + H_{33}L_{33}, \\ &\hat{H}_{11}\hat{S}_{32} - \hat{H}_{33}\hat{S}_{14} + \hat{H}_{13}(\hat{S}_{21} + \hat{S}_{34}) = H_{11}S_{32} - H_{33}S_{14}, \\ &\hat{L}_{33}\hat{S}_{32} - \hat{L}_{11}\hat{S}_{14} + \hat{L}_{13}(\hat{S}_{21} + \hat{S}_{34}) = L_{33}S_{32} - L_{11}S_{14} + L_{13}(S_{21} + S_{34}), \\ &\hat{L}_{11}\hat{L}_{33} - \hat{L}_{13}^2 = L_{11}L_{33} - L_{13}^2, \quad \hat{H}_{11}\hat{H}_{33} - \hat{H}_{13}^2 = H_{11}H_{33}, \\ &\hat{S}_{21}\hat{S}_{34} + \hat{S}_{32}\hat{S}_{14} = S_{21}S_{34} + S_{32}S_{14}, \quad \hat{\beta}_{11}\hat{\beta}_{22} - \hat{\beta}_{12}^2 = \beta_{11}\beta_{22} - \beta_{12}^2, \end{aligned} \tag{4.10}$$

$$\hat\beta_{11}\hat\gamma_{22}+\hat\beta_{22}\hat\gamma_{11}-2\hat\beta_{12}\hat\gamma_{12}=\beta_{11}\gamma_{22}+\beta_{22}\gamma_{11}-2\beta_{12}\gamma_{12}. \tag{4.11}$$

More invariants can be obtained if β is replaced by γ in Eq. (4.10). Consequently Eq. (3.12) can also be expressed in terms of $\hat\beta_{11}$, $\hat\beta_{22}$, $\hat\beta_{12}$, $\hat\gamma_{11}$, $\hat\gamma_{22}$, $\hat\gamma_{12}$ and $\hat L_{11}$, $\hat L_{33}$, $\hat L_{13}$ in the new coordinate system as

$$\begin{aligned}(\hat\gamma_{11}\hat\gamma_{22}-\hat\gamma_{12}^2)\lambda^4+(\hat\gamma_{11}\hat L_{33}+\hat\gamma_{22}\hat L_{11}-2\hat\gamma_{12}\hat L_{13})\lambda^3 \\ +\left[\hat L_{11}\hat L_{33}-\hat L_{13}^2-(\hat\beta_{22}\hat\gamma_{11}+\hat\beta_{11}\hat\gamma_{22}-2\hat\beta_{12}\hat\gamma_{12})\right]\lambda^2 \\ -(\hat\beta_{11}\hat L_{33}+\hat\beta_{22}\hat L_{11}-2\hat\beta_{12}\hat L_{13})\lambda+\hat\beta_{11}\hat\beta_{22}-\hat\beta_{12}^2=0.\end{aligned} \tag{4.12}$$

In the following we confine our attention to Case I discussed in Section 3. The two wavenumbers given by Eq. (3.25) can now be expressed in terms of $\hat\beta_{11}$, $\hat\beta_{22}$, $\hat\beta_{12}$ and $\hat L_{11}$, $\hat L_{33}$, $\hat L_{13}$ in the new coordinate system as

$$\begin{aligned}\lambda_1&=\frac{\hat\beta_{11}\hat L_{33}+\hat\beta_{22}\hat L_{11}-2\hat\beta_{12}\hat L_{13}+\sqrt{(\hat\beta_{11}\hat L_{33}-\hat\beta_{22}\hat L_{11})^2+4(\hat\beta_{11}\hat L_{13}-\hat\beta_{12}\hat L_{11})(\hat\beta_{22}\hat L_{13}-\hat\beta_{12}\hat L_{33})}}{2(\hat L_{11}\hat L_{33}-\hat L_{13}^2)},\\ \lambda_2&=\frac{\hat\beta_{11}\hat L_{33}+\hat\beta_{22}\hat L_{11}-2\hat\beta_{12}\hat L_{13}-\sqrt{(\hat\beta_{11}\hat L_{33}-\hat\beta_{22}\hat L_{11})^2+4(\hat\beta_{11}\hat L_{13}-\hat\beta_{12}\hat L_{11})(\hat\beta_{22}\hat L_{13}-\hat\beta_{12}\hat L_{33})}}{2(\hat L_{11}\hat L_{33}-\hat L_{13}^2)}.\end{aligned} \tag{4.13}$$

If the coordinate system is chosen such that $\hat L_{13}=0$, h_1 can then be determined as

$$\hat h=h_1-h_0=-\frac{L_{13}}{L_{11}}=\frac{BD(\nu^D+\nu^A)}{2\mu D(1+\nu^A)-B^2}. \tag{4.14}$$

In this special coordinate system, the two wavenumbers can be more concisely given by

$$\begin{aligned}\lambda_1&=\frac{\hat\beta_{11}\hat L_{33}+\hat\beta_{22}\hat L_{11}+\sqrt{(\hat\beta_{11}\hat L_{33}-\hat\beta_{22}\hat L_{11})^2+4\hat\beta_{12}^2\hat L_{11}\hat L_{33}}}{2\hat L_{11}\hat L_{33}},\\ \lambda_2&=\frac{\hat\beta_{11}\hat L_{33}+\hat\beta_{22}\hat L_{11}-\sqrt{(\hat\beta_{11}\hat L_{33}-\hat\beta_{22}\hat L_{11})^2+4\hat\beta_{12}^2\hat L_{11}\hat L_{33}}}{2\hat L_{11}\hat L_{33}}.\end{aligned} \tag{4.15}$$

Furthermore, the following inequalities are established from the above expression:

$$\lambda_1\geq\max\left\{\frac{\hat\beta_{11}}{\hat L_{11}},\ \frac{\hat\beta_{22}}{\hat L_{33}}\right\}>\min\left\{\frac{\hat\beta_{11}}{\hat L_{11}},\ \frac{\hat\beta_{22}}{\hat L_{33}}\right\}\geq\lambda_2. \tag{4.16}$$

The two equalities in Eq. (4.16) are valid only when $\hat\beta_{12}=0$ in this special coordinate system, or more specifically when

$$\frac{\beta_{12}}{\beta_{11}}=\frac{L_{13}}{L_{11}}. \tag{4.17}$$

5. Conclusions

This work considers the surface instability of a semi-infinite isotropic laminated plate under surface van der Waals forces. The analytical results demonstrate that the surface of the semi-infinite isotropic laminated plate is always unstable whenever the van der Waals interaction coefficient $\beta > 0$. Furthermore, two distinct surface instability modes characterized by two positive wavenumbers λ_1 and λ_2 (or two wavelengths $2\pi/\lambda_1$ and $2\pi/\lambda_2$) are identified.

In general, the two wavenumbers are completely determined by $(\beta_{11}, \beta_{22}, \beta_{12})$, $(\gamma_{11}, \gamma_{22}, \gamma_{12})$ and (L_{11}, L_{33}, L_{13}). Interestingly, the two wavenumbers are completely determined by $(\beta_{11}, \beta_{22}, \beta_{12})$ and (L_{11}, L_{33}, L_{13}) when the plate is relatively stiff (in this case, the contribution from surface energy is ignored); and they are completely determined by $(\beta_{11}, \beta_{22}, \beta_{12})$ and $(\gamma_{11}, \gamma_{22}, \gamma_{12})$ if the plate is extremely compliant (in this case, the contribution from the elastic properties of the plate is ignored). The observation of two instability modes is quite different from the uniqueness of the surface instability mode observed for a semi-infinite elastic body under plane strain or generalized plane strain conditions [12–14]. It is pointed out that the observations of two surface instability modes by Ru [6] and Yoon *et al.* [8] are conditional: either when the thickness ratio exceeds a critical value for two mutually attracting films [6] or when the top layer is more compliant and much thinner than the bottom layer for a bilayer film interacting with another rigid contactor [8].

It is expected that these theoretical results can find application in the study of elastic behaviors of membranes [26].

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