

Crack nucleation in circular disk under mixed boundary conditions

V. M. MIRSALIMOV, N. M. KALANTARLY

*Institute of Mathematics and Mechanics
National Academy of Sciences of Azerbaijan
AZ-1141, B. Vahabzade 9
Baku, Azerbaijan
e-mails: mir-vagif@mail.ru, nailyak1975@gmail.com*

A MODEL OF CRACK NUCLEATION in a circular disk, based on consideration of cracking process zone is suggested. It is assumed that the cracking process zone is a finite-length layer containing a material with partially disturbed bonds between separate structural elements. Existence of bonds between the pre-fracture zone faces (the area of weakened interparticle bonds of the material) is simulated by application of cohesive forces caused by the existence of bonds to pre-fracture area surfaces. Analysis of limit equilibrium of the pre-fracture zone in a circular disk with mixed conditions on the boundary are fulfilled on the basis of ultimate stretching of material's bonds and includes: 1) setting up the dependence of cohesive forces on opening of pre-fracture area faces, 2) estimation of stress state near the pre-fracture zone with regard to external loads and cohesive forces, 3) determination of dependence of critical external loads on geometrical parameters of the disk, under which the crack appears.

Key words: circular disk, pre-fracture zones with interfacial bonds, cohesive forces, cracking, mixed boundary conditions.

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1. Introduction

CIRCULAR DISKS ARE WIDELY USED in the latest implements, in steam and gas turbines, in compressors, in chemical industry machines, etc. The disks are subjected to loading and this causes their stretching. Study of crack nucleation problem in circular disks is of great importance for engineering practice. Development of design models of cracking investigation in circular disks is an urgent problem of materials mechanics. Material constitutive equations and computational tools which have been recently developed to simulate ductile rupture are reviewed in [1]. In [2] it was demonstrated that an initial stage of fatigue damage can be described within the framework of model for fatigue-crack growth which is based on generalization of energy approach to fracture mechanics with inclusion of certain characteristics of the material's microdamage in computational scheme, and distribution functions of time to formation of a first macroscopic crack in a body of given dimensions, obtained from the known distribution of

nominal stresses and initial microdefects. Several issues in damage micromechanics for microcracked brittle or quasi-brittle solids are addressed in [3]. In particular, the methods for characterizing evolutionary damage of microcracks are discussed and concept of orientation domain of microcrack growth is introduced. An evolutionary fiber cracking process, governed by the internal stresses and fracture strength of fibers, is considered in [4], and Weibull's probabilistic distribution is used to describe the varying probability of fiber cracking. In [5, 6] crack nucleation model with cohesive forces in the most loaded, but still integrity zones of deformable solid is proposed. This initial fracture model was used in the calculation of various constructions [7–12], with various force and thermal-force effects. In [12] the model of crack nucleation in coating on an elastic foundation is proposed. In [13], the computational model describing the cracking in the brake drum during the car's braking process is developed and the effect of small deviations from the linear form of the zone of weakened interparticle material bonds on the cracks nucleation in the drum is investigated. Papers [14–21] are devoted to the study of fracture in composite materials and adhesive joints. For engineering practice the research of crack formation is important. At present, the research on the crack nucleation in a circular disk is practically absent. The development of computational models of the cracking process in the disks is an important problem in mechanics of materials. The development of investigation methods for cracking will be effective in improving the disks serviceability as well as for evidence-based selection of disks parameters at the design stage.

2. Formulation of the problem

Let the normal displacement $v_r(t)$ and tangential component of surface forces $N_\theta(t)$ be given on the boundary of a circular disk. In the course of loading, in the circular disk there will appear pre-fracture zones that will be modeled as the areas of weakened interparticle bonds of the material. Interaction of the faces of these areas is modelled by introducing the bonds possessing the given deformation diagram between the pre-fracture zone faces. Physical nature of such bonds and the sizes of pre-fracture areas where the interaction of the faces of weakened interparticle bonds zones is completed depend on the form of the material.

The embryonic crack is modelled by pre-fracture zones with interfacial bonds that are considered to be an area of the weakened interparticle bonds of the material. As the indicated zones (interlayers of overstrained material) are small compared with the remaining part of the disk, we can “mentally” remove them and replace with sections whose surfaces interact with each other by some rule corresponding to the action of the removed material. In the considered case, the

crack initiation is the process of passage of the pre-fracture zone to the area of the broken bonds between the surfaces of the material. At that point, the size of the pre-fracture zone is unknown beforehand and to be defined.

Investigations [22–24] of the onset of areas with disturbed structure of the material show that at the initial stage the pre-fracture zone is a narrow-stretched layer, and then by increasing the load, suddenly there appears the second system of zones containing a material with partially disturbed bonds. For mathematical description of crack nucleation in the disk, in the considered case we arrive at the mixed static problem of the plane theory of elasticity for a disk when the material has pre-fracture zones. The pre-fracture zone is oriented in the direction of the maximal tensile stresses arising in the disk. We define the disk in reference to the polar coordinates $r\theta$ with origin at the center of the circle L with radius R (Fig. 1).

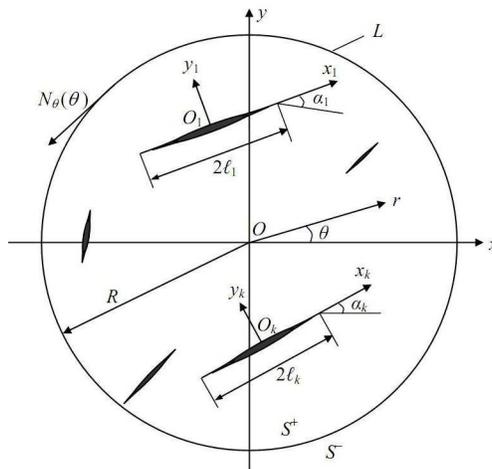


FIG. 1. Calculation scheme of fracture mechanics problem for a disk.

We will assume that the disk has N rectilinear pre-fracture zones of length $2\ell_k$ ($k = 1, 2, \dots, N$). At the centers of the pre-fracture zones is located the origin of the local systems of coordinates $x_k O_k y_k$, whose axes x_k coincide with the lines of pre-fracture zones and form the angles α_k with the axis x (Fig. 1). The pre-fracture zone faces interact so that this interaction (interfacial bonds) restrains the crack nucleation. Under the action of external loads on the disk, in the bonds connecting the pre-fracture zone faces there will arise normal $q_{y_k}(x_k)$ and tangential $q_{x_k y_k}(x_k)$ stresses ($k = 1, 2, \dots, N$). Consequently, the normal and tangential stresses numerically equal to $q_{y_k}(x_k)$ and $q_{x_k y_k}(x_k)$, respectively will be applied to the pre-fracture zone faces. The quantities of these stresses are not known in advance.

In the problem under consideration, the boundary conditions have the form:

$$(2.1) \quad \tau_{r\theta} = N_\theta(\theta), \quad v_r = f_r(\theta), \quad \text{on } L,$$

$$(2.2) \quad \sigma_{y_k} = q_{y_k}(x_k), \quad \tau_{x_k y_k} = q_{x_k y_k}(x_k), \quad \text{on } L_k \ (k = 1, 2, \dots, N),$$

where L is the circular bound of the disk and L_k are the faces of the k -th pre-fracture zone.

The equations connecting the opening of the pre-fracture zone faces and forces in the bonds may be represented in the form [6]:

$$(2.3) \quad (v_k^+(x_k, 0) - v_k^-(x_k, 0)) - i(u_k^+(x_k, 0) - u_k^-(x_k, 0)) \\ = C(x_k, \sigma_k)[q_{y_k}(x_k) - iq_{x_k y_k}(x_k)] \quad (k = 1, 2, \dots, N),$$

where the functions $C(x_k, \sigma_k)$ are the effective compliances of corresponding bonds that depend on the tension in the bonds, $\sigma_k = \sqrt{q_{y_k}^2 + q_{x_k y_k}^2}$ is the modulus of the vector of bond tractions, $(v_k^+ - v_k^-)$ are normal and $(u_k^+ - u_k^-)$ are tangential components of the opening of pre-fracture zone faces.

3. The case of a single pre-fracture zone

Denote the considered domain enclosed between the circumference L of radius R and rectilinear pre-fracture zone $L_1 = [a, b]$ ($\alpha_1 = 0$) along the abscissa axis by S^+ , and the domain supplemented to the complete complex plane by S^- (Fig. 2).

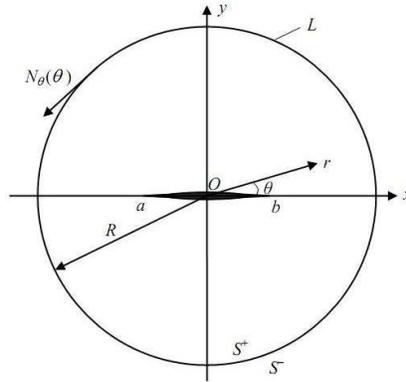


FIG. 2. Calculation scheme of fracture mechanics problem for the case of a single pre-fracture zone.

The stress-strain state of the disk under the plane problem conditions is described by two analytic functions $\Phi(z)$ and $\Omega(z)$ of a complex variable [25]:

$$\begin{aligned}
(3.1) \quad & \sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z), \\
& \sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \\
& 2\mu \frac{\partial}{\partial x}(u + iv) = \kappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}, \\
& \Omega(z) = \overline{\Phi}(z) + z\overline{\Phi'}(z) + \overline{\Psi}(z),
\end{aligned}$$

where $\kappa = (3 - \nu)/(1 + \nu)$ stands for plane stress state, ν is the Poisson ratio of the disk's material and μ is the material's shear modulus.

Under accepted assumptions, the problem is reduced to the definition of two analytical, in the domain S^+ , functions of complex variables $\Phi(z)$ and $\Psi(z)$, satisfying the following boundary conditions on the contour L :

$$(3.2) \quad \operatorname{Re} \left\{ \kappa\Phi(t) - \overline{\Phi(t)} + \frac{R^2}{t^2} [t\overline{\Phi'(t)} + \overline{\Psi(t)}] \right\} = 2\mu f_r'(t),$$

$$(3.3) \quad \operatorname{Im} \left\{ \Phi(t) + \overline{\Phi(t)} - \frac{t^2}{R^2} [t\overline{\Phi'(t)} + \overline{\Psi(t)}] \right\} = -N_\theta(t).$$

Here, t is affix of the points of the contour L .

The loading conditions on the pre-fracture zone faces L_1^+ and L_1^- will be

$$\begin{aligned}
(3.4) \quad & \Phi^+(x) + \overline{\Phi}^-(x) + x\overline{\Phi}^-(x) + \overline{\Psi}^-(x) = q_y^+(x) - iq_{xy}^+(x) \quad \text{on } L_1^+, \\
& \Phi^-(x) + \overline{\Phi}^+(x) + x\overline{\Phi}^+(x) + \overline{\Psi}^+(x) = q_y^-(x) - iq_{xy}^-(x) \quad \text{on } L_1^-.
\end{aligned}$$

Applying relations (3.2) and (3.3) to conjugated values, after some transformations on the contour L we get the boundary conditions in the form:

$$\begin{aligned}
(3.5) \quad & (\kappa - 1)[\Phi(t) + \overline{\Phi(t)}] + \frac{R^2}{t^2} \left[t\overline{\Phi'(t)} + \overline{\Psi(t)} \right] \\
& \quad + \frac{t^2}{R^2} \left[\frac{R^2}{t} \Phi'(t) + \Psi(t) \right] = 4\mu f_r'(t) \quad \text{on } L,
\end{aligned}$$

$$(3.6) \quad -\frac{R^2}{t^2} [t\overline{\Phi'(t)} + \overline{\Psi(t)}] + \frac{t^2}{R^2} \left[\frac{R^2}{t} \Phi'(t) + \Psi(t) \right] = 2iN_\theta(t) \quad \text{on } L.$$

Based on (3.5) and (3.6) we have on the circumference L :

$$(3.7) \quad (\kappa - 1)[\Phi(t) + \overline{\Phi(t)}] + \frac{2t^2}{R^2} \left[\frac{R^2}{t} \Phi'(t) + \Psi(t) \right] = 2[2\mu f_r'(t) + iN_\theta(t)].$$

We substitute, into the equality (3.7), the functions $f_r(t)$ and $N_\theta(t)$ in the form of Fourier series:

$$(3.8) \quad f_r(t) = \sum_{v=-\infty}^{\infty} V_v \left(\frac{t}{R} \right)^v, \quad iN_\theta(t) = \sum_{v=-\infty}^{\infty} T_v \left(\frac{t}{R} \right)^v,$$

where V_v, T_v ($v = 0, \pm 1, \pm 2, \dots$), generally speaking, are the known complex coefficients and are defined by the formulas:

$$V_k = \frac{1}{2\pi} \int_0^{2\pi} f_r(\theta) e^{ik\theta} d\theta, \quad T_k = \frac{1}{2\pi} \int_0^{2\pi} iN_\theta(\theta) e^{ik\theta} d\theta, \quad k = 0, \pm 1, \pm 2, \dots$$

As a result we get

$$(3.9) \quad (\kappa - 1)[\Phi(t) + \overline{\Phi(t)}] + 2\frac{t^2}{R^2} \left[\frac{R^2}{t} \Phi'(t) + \Psi(t) \right] \\ = 2 \left\{ \sum_{v=0}^{\infty} \left[T_v + \frac{2\mu(v+1)}{R} V_{v+1} \right] \left(\frac{t}{R} \right)^v + \sum_{v=1}^{\infty} \left[T_{-v} - \frac{2\mu(v-1)}{R} V_{-v+1} \right] \left(\frac{R}{t} \right)^v \right\}.$$

Next, we introduce on L new unknown auxiliary function $\omega(t) \in H$ (the Holder condition) in the form:

$$(3.10) \quad 2\omega(t) = (\kappa - 1)[\Phi(t) + \overline{\Phi(t)}] - \frac{2t^2}{R^2} \left[\frac{R^2}{t} \Phi'(t) + \Psi(t) \right].$$

Putting together relations (3.9) and (3.10), we have:

$$(3.11) \quad \Phi(t) = \frac{\omega(t)}{\kappa - 1} + \frac{1}{\kappa - 1} \sum_{v=0}^{\infty} \left[\frac{2\mu(v+1)}{R} V_{v+1} + T_v \right] \left(\frac{t}{R} \right)^v \\ + \frac{1}{\kappa - 1} \sum_{v=1}^{\infty} \left[T_{-v} - \frac{2\mu(v-1)}{R} V_{-v+1} \right] \left(\frac{R}{t} \right)^v \quad \text{on } L.$$

Substituting (3.11) in (3.10), we get

$$(3.12) \quad \Psi(t) = Q(t) + R_1(t) + R_2(t) \quad \text{on } L,$$

$$(3.13) \quad Q(t) = -\frac{R^2}{2t^2} [\omega(t) + \overline{\omega(t)}] - \frac{R^2}{(\kappa - 1)t} \omega'(t),$$

$$(3.14) \quad R_1(t) = \sum_{v=0}^{\infty} \left[\frac{1}{2} \left(1 - \frac{v-2}{\kappa-1} \right) T_{v+2} - \frac{1}{2} \bar{T}_{-v-2} + \frac{\mu(v+1)}{R} V_{-v-1} \right. \\ \left. + \frac{\mu(v+3)}{R} \left(1 - \frac{v-2}{\kappa-1} \right) V_{v+3} \right] \left(\frac{t}{R} \right)^v, \\ R_2(t) = -\sum_{v=2}^{\infty} \left[\frac{\mu(v-1)}{R} \bar{V}_{v-1} + \frac{1}{2} \bar{T}_{v-2} \right] \left(\frac{R}{t} \right)^v + \sum_{v=3}^{\infty} \left[\frac{1}{2} \left(1 + \frac{v-2}{\kappa-1} \right) T_{-v+2} \right. \\ \left. - \frac{\mu(v-3)}{R} \left(1 + \frac{v-2}{\kappa-1} \right) V_{-v+3} \right] \left(\frac{R}{t} \right)^v + \left[\frac{\mu}{R} V_1 + \frac{1}{2} T_0 \right] \frac{R^2}{t^2} \\ + \left[\frac{1}{2} \left(1 - \frac{1}{\kappa-1} \right) T_1 - \frac{1}{2} \bar{T}_{-1} - \frac{\mu}{(\kappa-1)R} V_1 + \frac{2\mu}{R} V_2 \right] \frac{R}{t}.$$

On the basis of the theorem on analytic continuation and properties of the Cauchy type integral, from relations (3.11) and (3.12) we have:

$$(3.15) \quad \Phi_*(z) = \begin{cases} \Phi(z) - \frac{1}{\kappa-1} \frac{1}{2\pi i} \int_L \frac{\omega(t)}{t-z} dt \\ - \frac{1}{\kappa-1} \sum_{v=0}^{\infty} \left[\frac{2\mu(v+1)}{R} V_{v+1} + T_v \right] \left(\frac{z}{R} \right)^v, & z \in S^+, \\ - \frac{1}{\kappa-1} \frac{1}{2\pi i} \int_L \frac{\omega(t)}{t-z} dt \\ + \frac{1}{\kappa-1} \sum_{v=1}^{\infty} \left[T_{-v} - \frac{2\mu(v-1)}{R} V_{-v+1} \right] \left(\frac{R}{z} \right)^v, & z \in S^-, \end{cases}$$

$$(3.16) \quad \Psi_*(z) = \begin{cases} \Psi(z) - \frac{1}{2\pi i} \int_L \frac{Q(t)}{t-z} dt - R_2(z), & z \in S^+, \\ - \frac{1}{2\pi i} \int_L \frac{Q(t)}{t-z} dt + R_2(z), & z \in S^-. \end{cases}$$

In relations (3.15) and (3.16) the functions $\Phi_*(z)$ and $\Psi_*(z)$ are analytic in the complete complex plane cut along the section $L_1 = [a, b]$ (the pre-fracture zone) and vanish at infinity, i.e., $\Phi_*(\infty) = 0$, $\Psi_*(\infty) = 0$.

We will look for the unknown function $\omega(t) \in H$ on L in the form of Fourier series

$$(3.17) \quad \omega(t) = \alpha_0^* + \sum_{v=1}^{\infty} \left[\alpha_v^* \left(\frac{t}{R} \right)^v + \alpha_{-v}^* \left(\frac{R}{t} \right)^v \right],$$

where α_v^* ($v = 0, \pm 1, \pm 2, \dots$) are the unknown complex coefficients.

Substituting relation (3.17) in the first formulas of (3.15) and (3.16), and using the Cauchy integral theorem, we get general formulas for the desired functions:

$$(3.18) \quad \begin{aligned} \Phi(z) &= \Phi_*(z) + \sum_{v=0}^{\infty} B_v \left(\frac{z}{R} \right)^v & z \in S^+, \\ \Psi(z) &= \Psi_*(z) + \sum_{v=0}^{\infty} D_v \left(\frac{z}{R} \right)^v & z \in S^+, \end{aligned}$$

$$(3.19) \quad B_v = \frac{1}{\kappa - 1} \left[\alpha_v^* + T_v + \frac{2\mu(v+1)}{R} V_{v+1} \right],$$

$$D_v = - \left(\frac{1}{2} + \frac{v+2}{\kappa - 1} \right) \alpha_{v+2}^* - \frac{1}{2} \bar{\alpha}_{-v-2} + \frac{1}{2} \left(1 - \frac{v+2}{\kappa - 1} \right) T_{v+2}$$

$$- \frac{1}{2} \bar{T}_{-v-2} + \frac{\mu(v+3)}{R} \left(1 - \frac{v+2}{\kappa - 1} \right) V_{v+3} + \frac{\mu(v+1)}{R} \bar{V}_{-v-1}.$$

For determining the function $\Phi_*(z)$, and consequently the function $\Psi_*(z)$ by the known method [25], we arrive at the linear conjunction problem:

$$(3.20) \quad [\Phi_*(t) - \Omega_*(t)]^+ - [\Phi_*(t) - \Omega_*(t)]^- = 0,$$

$$[\Phi_*(t) + \Omega_*(t)]^+ + [\Phi_*(t) + \Omega_*(t)]^- = f(t).$$

Here,

$$\Omega_*(z) = \bar{\Phi}_*(z) + z\bar{\Phi}'_*(z) + \bar{\Psi}_*(z),$$

$$f(t) = 2(q_y - iq_{xy}) + \sum_{k=0}^{\infty} (\beta_k + p_k) \left(\frac{t}{R} \right)^k,$$

$$(3.21) \quad \beta_k = -2 \left[\frac{1}{\kappa - 1} \alpha_k^* + \frac{k+1}{\kappa - 1} \bar{\alpha}_k^* - \frac{1}{2} \alpha_{-k-2}^* - \left(\frac{1}{2} + \frac{k+2}{\kappa - 1} \right) \bar{\alpha}_{k+2}^* \right],$$

$$p_k = -2 \left[\frac{1}{\kappa - 1} T_k + \frac{k+1}{\kappa - 1} \bar{T}_k - \frac{1}{2} T_{-k-2} + \left(\frac{1}{2} - \frac{k+2}{\kappa - 1} \right) \bar{T}_{k+2} \right.$$

$$+ \frac{2\mu(k+1)}{(\kappa - 1)R} V_{k+1} + \frac{2\mu(k+1)^2}{(\kappa - 1)R} \bar{V}_{k+1}$$

$$\left. + \frac{\mu(k+1)}{R} V_{-k-1} + \frac{\mu(k+3)}{R} \left(1 + \frac{k+2}{\kappa - 1} \right) \bar{V}_{k+3} \right].$$

Since the stresses in the disk are restricted, we should look for the solution of boundary value problem (3.20) in the class of everywhere bounded functions. The sought for solution to boundary value problem (3.20) is written in the form

$$(3.22) \quad \Phi_*(z) = \Omega_*(z) = \frac{\sqrt{(z-a)(z-b)}}{2\pi i} \int_a^b \frac{f(t)dt}{\sqrt{(t-a)(t-b)(t-z)}}.$$

Moreover, all the following solvability conditions of the boundary value problem should be fulfilled:

$$(3.23) \quad \int_a^b \frac{f(t)dt}{\sqrt{(t-a)(b-t)}} = 0, \quad \int_a^b \frac{tf(t)dt}{\sqrt{(t-a)(b-t)}} = 0.$$

These relations serve for determination of the unknown parameters a and b of the pre-fracture zone.

The obtained relations contain unknown stresses in the pre-fracture zone. Now, let's move to formulate an integral equation for determining the unknown forces $q_y - iq_{xy}$. The condition that determines the unknown stresses in the bonds between the pre-fracture zone faces is the additional relation (2.3) for $k = 1$, $\alpha_1 = 0$, $x_1 = x$. In the considered problem it is more convenient to write this additional condition for the derivative of opening of displacement of pre-fracture zone faces. Taking into account formula (3.1) and the boundary values of the functions $\Phi_*(z)$ and $\Omega_*(z)$, on the segment $y = 0$, $a \leq x \leq b$ we get the following equality:

$$(3.24) \quad \Phi^+(x) - \Phi^-(x) = \frac{2\mu}{1 + \kappa} \left[\frac{\partial}{\partial x}(u^+ - u^-) + i \frac{\partial}{\partial x}(v^+ - v^-) \right].$$

Using the Sokhotski–Plemelj formulas [25] and taking into account formula (3.22), we find

$$(3.25) \quad \Phi^+(x) - \Phi^-(x) = -\frac{i\sqrt{(x-a)(b-x)}}{\pi} \left[\int_a^b \frac{f(t)dt}{\sqrt{(t-a)(b-t)(t-x)}} \right].$$

We substitute the obtained expression (3.25) into the left side of (3.24), and by taking into account relation (2.3) $k = 1$, $\alpha_1 = 0$, $x_1 = x$, after some transformations we get the system of nonlinear integro-differential equations with respect to the unknown functions q_y and q_{xy} :

$$(3.26) \quad -\frac{1}{\pi} \sqrt{(x-a)(b-x)} \left[\int_a^b \frac{q_y(t)dt}{\sqrt{(t-a)(b-t)(t-x)}} + \int_a^b \frac{f_y(t)dt}{\sqrt{(t-a)(b-t)(t-x)}} \right] = \frac{2\mu}{1 + \kappa} \frac{d}{dx} (C(x, \sigma)q_y(x)),$$

$$(3.27) \quad -\frac{1}{\pi} \sqrt{(x-a)(b-x)} \left[\int_a^b \frac{q_{xy}(t)dt}{\sqrt{(t-a)(b-t)(t-x)}} + \int_a^b \frac{f_{xy}(t)dt}{\sqrt{(t-a)(b-t)(t-x)}} \right] = \frac{2\mu}{1 + \kappa} \frac{d}{dx} (C(x, \sigma)q_{xy}(x)).$$

Here

$$f_y(t) = \text{Re } f_1(t), \quad f_{xy}(t) = \text{Im } f_1(t),$$

$$f_1(t) = -\sum_{k=0}^{\infty} [B_k + (k + 1)\bar{B}_k + \bar{D}_k] \left(\frac{t}{R} \right)^k.$$

Each of the equations (3.26) or (3.27) is a nonlinear integro-differential equation with the Cauchy kernel and may be solved only numerically. In order to solve them one can use the collocation scheme with approximation of unknown functions [22, 26, 27].

Using formulas (3.22) and (3.23), and equations (3.26) and (3.27), the obtained relations (3.18) and (3.19) allow to get the terminal solution of the problem if the coefficients α_k^* ($k = 0, \pm 1, \pm 2, \dots$) will be determined.

In order to compose an infinite system of linear algebraic equations with respect to the unknowns α_k^* , we substitute the relations (3.18) and (3.19) into condition (3.10) with regard to (3.22) and the expansions

$$\sqrt{(t-a)(t-b)} = t \sum_{r=0}^{\infty} M_r \left(\frac{R}{t}\right)^r, \quad \frac{1}{\sqrt{(t-a)(t-b)}} = \sum_{r=0}^{\infty} M_r^* \left(\frac{R}{t}\right)^{r+1}.$$

After several transformations, condition (3.10) is reduced to the form

$$(3.28) \quad \sum_{m=0}^{\infty} A_m \left(\frac{t}{R}\right)^m + \sum_{m=0}^{\infty} A_m^* \left(\frac{R}{t}\right)^m = \sum_{m=0}^{\infty} C_m \left(\frac{t}{R}\right)^m + \sum_{m=0}^{\infty} C_m^* \left(\frac{R}{t}\right)^m.$$

Because of awkwardness of expressions for A_m , A_m^* , C_m , C_m^* ($m = 0, 1, 2, \dots$) we do not cite them.

Making comparison in the both sides of the obtained relation (3.28) the coefficients with identical powers t/R and R/t , we get the following infinite systems of linear algebraic equations:

$$(3.29) \quad A_0 + A_0^* = C_0 + C_0^* \quad (m = 0), \quad A_m = C_m \quad A_m^* = C_m^* \quad (m = 1, 2, \dots).$$

Now let us conduct algebraization of integro-differential equations (3.26) and (3.27) with additional conditions (3.23). At first, in equations (3.29) and (3.27) and in additional conditions (3.23) all integration segments are reduced to one interval $[-1, 1]$.

First we make a change of variables

$$(3.30) \quad t = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\tau, \quad x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\eta.$$

At such a change of variables, the left side of integro-differential equation (3.26) permits for the following form:

$$-\frac{1}{\pi} \sqrt{1-\eta^2} \left[\int_{-1}^1 \frac{q_y(\tau) d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} + \int_{-1}^1 \frac{f_y(\tau) d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} \right].$$

Respectively, for the left side of equation (3.27) we get

$$-\frac{1}{\pi} \sqrt{1-\eta^2} \left[\int_{-1}^1 \frac{q_{xy}(\tau) d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} + \int_{-1}^1 \frac{f_{xy}(\tau) d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} \right].$$

By changing the derivative contained on the right side of equation (3.26) for arbitrary internal node by finite-difference approximation we obtain:

$$\frac{d}{dx} [C(x, \sigma) q_y(x)]_i = \frac{C(x_{i+1}, \sigma(x_{i+1})) q_y(x_{i+1}) - C(x_{i-1}, \sigma(x_{i-1})) q_y(x_{i-1})}{2\Delta x},$$

where $\Delta x = (b-a)/M$.

We do the same with the right-hand side of equation (3.27). We take into account boundary conditions for $\eta = \pm 1$, $q_y(a) = q_y(b) = 0$, $q_{xy}(a) = q_{xy}(b) = 0$ (this corresponds to the conditions $v^+(a, 0) - v^-(a, 0) = 0$, $v^+(b, 0) - v^-(b, 0) = 0$, $u^+(a, 0) - u^-(a, 0) = 0$, $u^+(b, 0) - u^-(b, 0) = 0$).

Using the quadrature formula

$$\frac{1}{2\pi} \int_{-1}^1 \frac{g(\tau) d\tau}{\sqrt{1-\tau^2}(\tau-\eta)} = \frac{1}{M \sin \theta} \sum_{k=1}^M g_k \sum_{m=0}^{M-1} \cos \theta_k \sin m\theta,$$

$$\tau = \cos \theta, \quad \eta_m = \cos \theta_m, \quad \theta_m = \frac{2m-1}{2M} \pi \quad (m = 1, 2, \dots, M),$$

all the integrals in (3.26) and (3.27) are changed by finite sums, and the derivatives on the right sides of equations (3.26) and (3.27) are changed by finite difference approximations. The reduced formulas enable to change each integro-differential equation by a system of algebraic equations with respect to approximate values of the sought for function, respectively at nodal points. As a result we get

$$(3.31) \quad -\frac{2}{M} \left[\sum_{v=1}^M q_{y,v} \sum_{k=0}^{M-1} \cos k\theta_k \sin k\theta_m + \sum_{v=1}^M f_{y,v} \sum_{k=0}^{M-1} \cos k\theta_k \sin k\theta_m \right]$$

$$= [C(x_{m+1}, \sigma(x_{m+1})) q_{y,m+1} - C(x_{m-1}, \sigma(x_{m-1})) q_{y,m-1}] \frac{(1+\kappa)M}{4\mu(b-a)},$$

$$(3.32) \quad -\frac{2}{M} \left[\sum_{v=1}^M q_{xy,v} \sum_{k=0}^{M-1} \cos k\theta_k \sin k\theta_m + \sum_{v=1}^M f_{xy,v} \sum_{k=0}^{M-1} \cos k\theta_k \sin k\theta_m \right]$$

$$= \frac{(1+\kappa)M}{4\mu(b-a)} [C(x_{m+1}, \sigma(x_{m+1})) q_{xy,m+1} - C(x_{m-1}, \sigma(x_{m-1})) q_{xy,m-1}],$$

where $m = 1, 2, \dots, M$.

If we take into account the equality

$$2 \sum_{k=0}^{M-1} \cos k\theta_v \sin k\theta_m = \cot \frac{\theta_m \mp \theta_v}{2},$$

the systems will take the following forms:

$$(3.33) \quad \sum_{v=1}^M A_{mv}(q_{y,v} + f_{y,v}) = \frac{(1 + \kappa)M}{4\mu(b-a)} [C(x_{m+1}, \sigma)q_{y,m+1} - C(x_{m-1}, \sigma)q_{y,m-1}],$$

$$(3.34) \quad \sum_{v=1}^M A_{mv}(q_{xy,v} + f_{xy,v}) = \frac{(1 + \kappa)M}{4\mu(b-a)} [C(x_{m+1}, \sigma)q_{xy,m+1} - C(x_{m-1}, \sigma)q_{xy,m-1}],$$

where $m = 1, 2, \dots, M$, $q_{y,v} = q_y(\tau_v)$, $q_{xy,v} = q_{xy}(\tau_v)$, $f_{y,v} = f_y(\tau_v)$, $f_{xy,v} = f_{xy}(\tau_v)$, $x_{m+1} = \frac{a+b}{2} + \frac{b-a}{2}\eta_{m+1}$, $A_{mv} = -\frac{1}{M} \cot \frac{\theta_m \mp \theta_v}{2}$. The upper sign is taken when the number $|m - v|$ is odd, the lower sign when it is even.

Now let's move to algebraization of the solvability conditions of boundary value problem (3.23). Separating the real and imaginary parts in them and using the change of variables and Gauss' quadrature formula, we get the solvability conditions of the problem in the following form:

$$(3.35) \quad \sum_{v=1}^M f_y^*(\cos \theta_v) = 0, \quad \sum_{v=1}^M \tau_v f_y^*(\tau_v) = 0,$$

$$(3.36) \quad \sum_{v=1}^M f_{xy}^*(\cos \theta_v) = 0, \quad \sum_{v=1}^M \tau_v f_{xy}^*(\tau_v) = 0,$$

where $f_y^* = q_y + f_y$, $f_{xy}^* = q_{xy} + f_{xy}$.

As a result of algebraization, instead of each integro-differential equation with corresponding additional conditions, we get $M + 2$ algebraic equations for determining stresses at nodal points and the pre-fracture zone sizes. Even in the special case of linear elastic bonds, the obtained system of equations becomes nonlinear because of unknown size of the pre-fracture zone. In this connection, for solving the obtained systems, in the case of linear bonds the successive approximations method was used.

In the case of nonlinear law of deformation of bonds, the iteration algorithm similar to elastic solutions method [28] was also used to determine the forces in

the pre-fracture zone. It is assumed that the law of deformation of interparticle bonds (cohesive forces) is linear for $V = |(u^+ - u^-) - i(v^+ - v^-)| \leq V_*$. The first step of iterative calculation process consists of solving the combined system for linear interparticle elastic bonds. The subsequent iterations are performed only in the case when $V(x) > V_*$ holds on the parts of the pre-fracture zone, where V_* is the value of opening of pre-fracture zone faces, at which a transition from linear to non-linear bonds deformation law takes place. For such iterations the system of resolving equations is solved at each approximation for quasielastic bonds with effective compliance variable along the pre-fracture zone and depending on the quantity of force vector modulus in bonds obtained in previous calculation step. The calculation of effective compliance is conducted as in definition of the secant modulus in the method of variable parameters of elasticity [29]. It is accepted that the successive approximations process ends once the forces along the pre-fracture zone obtained in two successive iterations differ a little from each other.

The nonlinear part of the curve of bond deformation is represented in the form of bilinear dependence whose upward segment corresponds to elastic deformation of bonds ($0 < V(x_k) < V_*$) with maximum tension of bonds. For $V(x_k) > V_*$ the deformation law was described by nonlinear dependence defined by two points (V_*, σ_*) and (δ_c, σ_c) , moreover for $\sigma_c \geq \sigma_*$ we have an increasing linear dependence (linear hardening corresponding to elastic-plastic deformation of bonds), where σ_* are maximum elastic stresses in bonds, δ_c is the characteristics of the disk's material resistance to cracking and σ_c is tension of bonds, corresponding to limited opening of the pre-fracture zone faces.

For determining the ultimate equilibrium of the pre-fracture zone it is necessary to introduce an additional critical condition. In place of such condition we accept the condition of limit opening of the pre-fracture zone faces. It is assumed that the breaking of bonds on the pre-fracture zone faces ($x = x_0$) will happen subject to the condition

$$(3.37) \quad V(x_0) = \sqrt{[u^+(x_0, 0) - u^-(x_0, 0)]^2 + [v^+(x_0, 0) - v^-(x_0, 0)]^2} = \delta_c.$$

The joint solution of the obtained equations and condition (3.37) allows to determine the critical value of external loads, forces in bonds and the size of the pre-fracture zone for the limit equilibrium state under the given characteristics of bonds.

Distribution of normal stresses q_y/N_0 in the pre-fracture zone is depicted in Fig. 3. Here N_0 is the typical value of given external tangential load. In the computations we used the dimensionless coordinates $x = \frac{1}{2}(a + b) + \frac{1}{2}x'(b - a)$.

The compliances of bonds at normal and tangential directions were accepted to be equal and constant along the pre-fracture zone. The law of variation of

tangential forces along the pre-fracture zone is similar to the variation of normal forces with a difference that the absolute values of tangential forces are substantially lower and the maximum values of tangential forces are attained for small sizes of the pre-fracture zone.

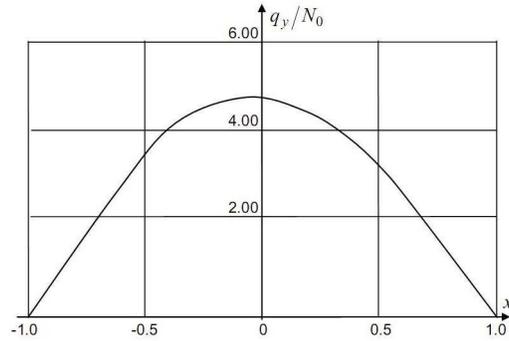


FIG. 3. Distribution of normal stresses q_y/N_0 in the pre-fracture zone.

The graph of dependence of relative length of the pre-fracture zone $(b-a)/R$ on the dimensionless load N_0/σ_* is given in Fig. 4.

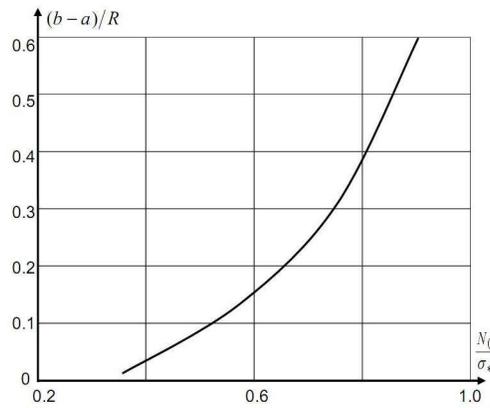


FIG. 4. The graph of dependence of relative length of the pre-fracture zone $(b-a)/R$ on the dimensionless load N_0/σ_* .

4. The case of an arbitrary number of prefracture zones

Now, let's assume that in the disk's operation process there exist N rectilinear pre-fracture zones of length $2\ell_k$ ($k = 1, 2, \dots, N$) (Fig. 1). It is assumed that the pre-fracture zones are oriented in the direction of action of maximal tensile stresses arising in the disk. The pre-fracture zone sizes are not known beforehand and should be defined in the course of solution of the boundary value problem.

The solution of the problem for this case is similar to the solution in the case with a single pre-fracture zone with a difference that for finding the functions $\Phi_*(z)$ and $\Psi_*(z)$ the method with the explicit form of Kolosov–Muskhelishvili potentials corresponding to unknown displacements along the pre-fracture zone is used. The problem is reduced to definition of two analytic functions of complex variables $\Phi(z)$ and $\Psi(z)$ and satisfying boundary conditions (3.2), (3.3) and

$$(4.1) \quad \Phi(x_k) + \overline{\Phi(x_k)} + x_k \overline{\Phi'(x_k)} + \overline{\Psi(x_k)} = F_k, \quad (k = 1, 2, \dots, N),$$

where $F_k = q_{y_k}(x_k) - iq_{x_k y_k}(x_k)$.

We look for the functions $\Phi_*(z)$ and $\Psi_*(z)$ in the form [30, 31]

$$(4.2) \quad \begin{aligned} \Phi_*(z) &= \frac{1}{2\pi} \sum_{k=1}^N \int_{-\ell_k}^{\ell_k} \frac{g_k(t)}{t - z_k} dt, \\ \Psi_*(z) &= \frac{1}{2\pi} \sum_{k=1}^N e^{-2i\alpha_k} \int_{-\ell_k}^{\ell_k} \left[\frac{\overline{g_k(t)}}{t - z_k} - \frac{\overline{T_k} e^{i\alpha_k}}{(t - z_k)^2} g_k(t) \right] dt, \\ T_k &= t e^{i\alpha_k} + z_k^0, \quad z_k^0 = x_k^0 + iy_k^0, \quad z_k = e^{-i\alpha_k} (z - z_k^0). \end{aligned}$$

Here $g_k(x_k)$ are the desired functions characterizing the opening of displacements of pre-fracture zone faces

$$(4.3) \quad g_k(x_k) = \frac{2\mu}{i(1 + \kappa)} \frac{\partial}{\partial x} [u_k^+(x_k, 0) - u^-(x_k, 0) + i(v_k^+(x_k, 0) - v_k^-(x_k, 0))],$$

($k = 1, 2, \dots, N$).

For determining the unknown functions $g_k(x_k)$ we use the boundary conditions (4.1) on the segments $y_k = 0, -\ell_k \leq x_k \leq \ell_k$ ($k = 1, 2, \dots, N$).

Satisfying the boundary conditions (4.1) on the pre-fracture zone faces by the functions (3.18) and (4.2), we get a system of N singular equations with respect to the unknown functions $g_k(x_k)$ ($k = 1, 2, \dots, N$):

$$(4.4) \quad \sum_{k=1}^N \int_{-\ell_k}^{\ell_k} [K_{nk}(t, x) g_k(t) + L_{nk}(t, x) \overline{g_k(t)}] dt = \pi(F_n(x) + F_n^*(x)),$$

$|x| \leq \ell_n, \quad n = 1, 2, \dots, N.$

Here,

$$\begin{aligned} K_{nk}(t, x) &= \frac{e^{i\alpha_k}}{2} \left[\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\overline{T_k} - \overline{X_n}} \right], \\ L_{nk}(t, x) &= \frac{e^{-i\alpha_k}}{2} \left[\frac{1}{\overline{T_k} - \overline{X_n}} - \frac{T_k - X_n}{(\overline{T_k} - \overline{X_n})^2} e^{-2i\alpha_n} \right], \end{aligned}$$

$$F_n = q_{y_n}(x) - iq_{x_n y_n}(x), \quad F^*(x) = -[\Phi_0(x) + \overline{\Phi_0(x)} + x\overline{\Phi'_0(x)} + \overline{\Psi_0(x)}],$$

$$\Phi_0(z) = \sum_{v=0}^{\infty} B_v \left(\frac{z}{R}\right)^v, \quad \Psi_0(z) = \sum_{v=0}^{\infty} D_v \left(\frac{z}{R}\right)^v.$$

To the system of singular integral equations (4.4) we should add additional equalities expressing the condition of uniqueness of displacements in tracing the pre-fracture zone contours

$$(4.5) \quad \int_{-\ell_k}^{\ell_k} g_k(t) dt = 0, \quad k = 1, 2, \dots, N.$$

Under additional conditions (4.5), by means of procedure for converting (see [22], appendix) a system to an algebraic system, the system of singular integral equations (4.4) is reduced to the system of $N \times M$ algebraic equations for determining the $N \times M$ unknown values $g_k(t_m)$ ($k = 1, 2, \dots, N$, $m = 1, 2, \dots, M$):

$$(4.6) \quad \frac{1}{M} \sum_{m=1}^M \sum_{k=1}^N \ell_k [g_k(t_m) K_{nk}(\ell_k t_m, \ell_n x_r) + \overline{g_k(t_m)} L_{nk}(\ell_k t_m, \ell_n x_r)]$$

$$= F_n(x_r) + F^*(x_r), \quad n = 1, 2, \dots, N, \quad r = 1, 2, \dots, M - 1,$$

$$\sum_{m=1}^M g_n(t_m) = 0,$$

where $t_m = \cos \frac{2m-1}{2M} \pi$, $m = 1, 2, \dots, M$, $x_r = \cos \frac{\pi r}{M}$, $r = 1, 2, \dots, M - 1$.

If in algebraic equations (4.6) we pass to complexly- conjugated values, we get one more $N \times M$ algebraic equations. The right sides of algebraic systems (4.6) contain the unknown values of normal q_{y_n} and tangential $q_{x_n y_n}$ forces at the nodal points of appropriate pre-fracture zones.

Using the obtained solution, we find

$$(4.7) \quad g_k(x_k) = \frac{2\mu}{i(1 + \kappa)} \frac{d}{dx_k} [C(x_k, \sigma_k)(q_{y_k}(x_k) - iq_{x_k y_k}(x_k))],$$

$$k = 1, 2, \dots, N.$$

These complex equations help to find the forces q_{y_k} and $q_{x_k y_k}$ ($k = 1, 2, \dots, N$) in the bonds between the appropriate pre-fracture zones. For obtaining missing algebraic equations to determine approximate values of the forces $q_{y_k}(t_m)$ and $q_{x_k y_k}(t_m)$ ($m = 1, 2, \dots, M$) at the nodal points it is required to fulfill conditions (4.7) at the nodal points located in pre-fracture zones. Therefore, we use the finite difference method. As a result we get a complex algebraic system of $N \times M$ equations to determine the values $q_{y_k}(t_m)$, $q_{x_k y_k}(t_m)$ ($k = 1, 2, \dots, N$;

$m = 1, 2, \dots, M$) at the nodal points of pre-fracture zones. Since the stresses in the disk are restricted, we look for the solution of singular integral equations in the class of everywhere bounded functions. Such a solution exists subject to solvability conditions of integral equations. Therefore the obtained algebraic systems are not still closed. For accurateness of the obtained algebraic equations we omit $2 \times N$ equations expressing the solvability conditions of integral equations (the condition of finiteness of stresses in the vicinity of pre-fracture zone vertices $x_k = \pm \ell_k$ ($k = 1, 2, \dots, N$)). These conditions are in the form:

$$(4.8) \quad \begin{aligned} \sum_{m=1}^M (-1)^{M+m} g_k(t_m) \cot \frac{2m-1}{4M} \pi &= 0, \\ \sum_{m=1}^M (-1)^m g_k(t_m) \tan \frac{2m-1}{4M} \pi &= 0, \quad (k = 1, 2, \dots, N). \end{aligned}$$

The obtained relations (3.18), (3.19), (4.2), (4.6), (4.7), (4.8) allow to get the final solution of the problem if the coefficients α_k^* ($k = 0, \pm 1, \dots$) are determined. To determine the system of infinite system of linear algebraic equations with respect to the unknowns α_k^* , we behave similarly as in the case of a single pre-fracture zone. As a result we get infinite linear algebraic systems of type (3.29).

Under the given characteristics of bonds, the obtained system of equations with respect to $g_k(t_m)$, $q_{y_k}(t_m)$, $q_{x_k y_k}(t_m)$ ($k = 1, 2, \dots, N$; $m = 1, 2, \dots, M$), α_v^* ($v = 0, \pm 1, \dots$) enables to determine the stress strain state of the disk based on the availability of arbitrary number of pre-fracture zones in the disk's material.

The combined resolving system of equations became nonlinear even for linear-elastic bonds because of unknown quantities ℓ_k ($k = 1, 2, \dots, N$). For solving it we use the successive approximations method. We solve the combined system at some definite values of ℓ_k^* ($k = 1, 2, \dots, N$) with respect to the unknowns α_v^* , $g_k(t_m)$, q_{y_k} , $q_{x_k y_k}$. The values of ℓ_k^* and the found quantities are substituted into (4.8), i.e., into the omitted equations of the combined system. The taken values of the parameters of ℓ_k^* and their respective values α_v^* , $g_k(t_m)$, q_{y_k} , $q_{x_k y_k}$ will not, generally speaking, satisfy equations (4.8). Therefore, choosing the values of the parameters ℓ_k^* , we will repeat the calculations until the equations (4.8) of the combined system will be satisfied within a given accuracy. At each approximation the combined system of equations was solved by the Gauss method choosing the principal element for different values of M . In the case of nonlinear law of deformation of bonds for determining the forces in pre-fracture zones the iteration method similar to the method of elastic solutions [28] was used. It is assumed that the law of deformation of interparticle bonds in pre-fracture zones is linear for $V_k = |(u_k^+ - u_k^-) - i(v_k^+ - v_k^-)| \leq V_*$. The first iteration step consists of solving the system of resolving equations for interparticle linear-elastic bonds.

The next iterations are fulfilled only in the case when $V_k > V_*$ holds on the parts of the pre-fracture zone. For such iterations the system of resolving equations is solved at each approximation for quasibrittle bonds (cohesive forces) with effective compliance variable along the pre-fracture zone and dependent of the size of the force vector modulus in bonds, obtained in the previous calculation step. The effective compliance calculation is carried out similarly to finding of the secant modulus in variable elasticity parameters [29]. It is assumed that the successive approximations process comes to an end as the forces in pre-fracture zones, obtained at two successive steps, differ a little from each other.

The nonlinear part of the bonds deformation curve was introduced in the form of bilinear dependence whose ascending segment corresponded to elastic deformation of bonds ($0 < V(x_k) < V_*$) with maximal tension of bonds. For $V(x_k) > V_*$ the deformation law was described by nonlinear dependence determined by two points (V_*, σ_*) and (δ_c, σ_c) ; moreover, for $\sigma_c \geq \sigma_*$ we have the increasing linear dependence (linear hardening corresponding to elastic-plastic deformation of bonds).

The obtained solution of the problem permits to predict the occurrence of cracks in a circular disk. For that the problem statement should be complemented with a crack nucleation condition (a criterion, breaking of interparticle bonds of the material). In place of such condition we accept the criterion of critical opening of the pre-fracture zone faces

$$V_k(x_k^*) = |(v_k^+ - v_k^-) - i(u_k^+ - u_k^-)| = \delta_c, \quad k = 1, 2, \dots, N,$$

where x_k^* are the coordinates of the pre-fracture zone point where the breaking of interparticle bonds of the material occurs.

Using the obtained solution, we can write the limit condition of crack nucleation in the form:

$$(4.9) \quad C(x_k^*, \sigma_k(x_k^*))\sigma_k(x_k^*) = \delta_c, \quad k = 1, 2, \dots, N.$$

The results of calculations are depicted in Figs. 4, 5, 6 and 7.

The dependences of the length of the pre-fracture zone ℓ_k/R ($k = 1, 2, 3$) on dimensionless value N_0/σ_* for different orientation angles ($\alpha_1 = 15^\circ$, $\alpha_2 = 30^\circ$, $\alpha_3 = 45^\circ$) are represented in Fig. 5. The distribution graphs of normal q_{y_k}/N_0 and tangential $q_{x_k y_k}/N_0$ forces in pre-fracture zones are given in Figs. 6, 7 and 8. Here, the curves indicated by 1 correspond to linear law of deformation of bonds, the curves described by 2 to bilinear law of deformation of bonds. The dimensionless coordinates $x_k' = x_k/\ell_k$ were used in calculations. While increasing the pre-fracture zone size, the level of stresses q_{y_k} and $q_{x_k y_k}$ in the bonds decreases. The location of the pre-fracture zone plays an important role on essential influence on stresses q_{y_k} and $q_{x_k y_k}$.

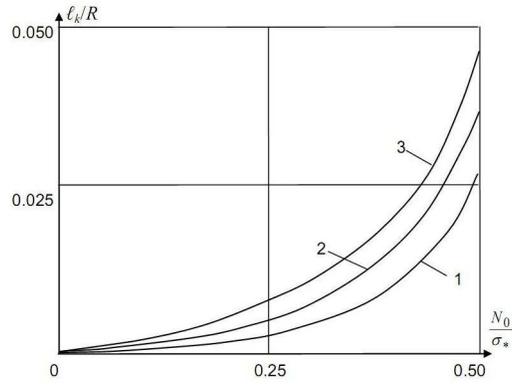


FIG. 5. The dependence of the length of the pre-fracture zone ℓ_k/R ($k = 1, 2, 3$) on dimensionless value N_0/σ_* for different orientation angles ($\alpha_1 = 15^\circ$, $\alpha_2 = 30^\circ$, $\alpha_3 = 45^\circ$).

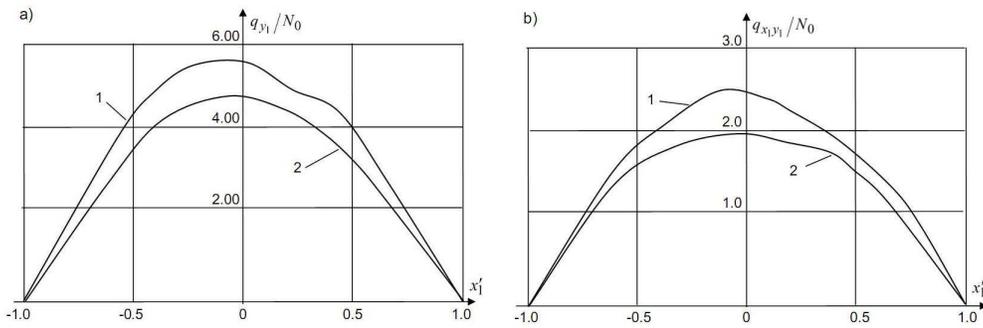


FIG. 6. The graph of distribution of normal q_{y1}/N_0 and tangential q_{x1y1}/N_0 forces in pre-fracture zones.

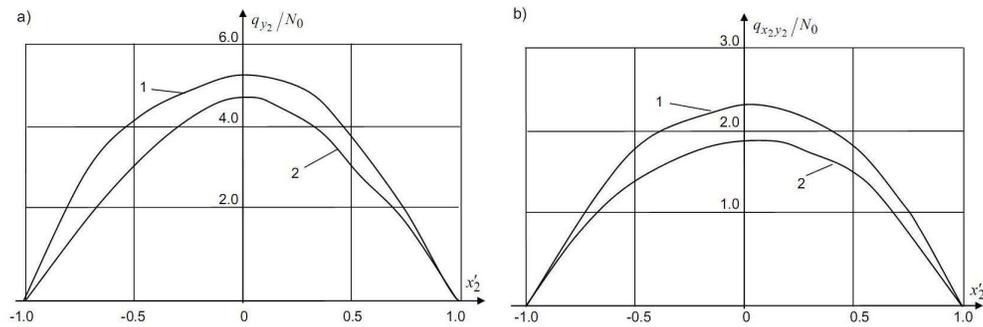


FIG. 7. The graph of distribution of normal q_{y2}/N_0 and tangential q_{x2y2}/N_0 forces in pre-fracture zones.

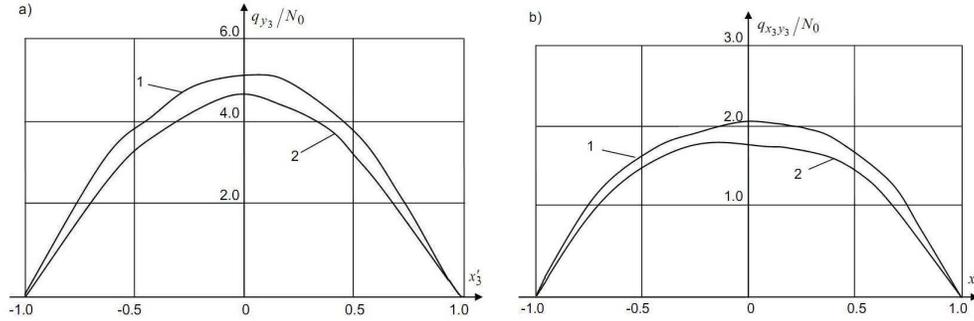


FIG. 8. The graph of distribution of normal q_{y_3}/N_0 and tangential $q_{x_3 y_3}/N_0$ forces in pre-fracture zones.

When the pre-fracture zones are located close to each other, the calculations show both an increase of the pre-fracture zones sizes and stresses in the bonds and also decrease of forces in the bonds and the pre-fracture zone sizes. Difference of the forms of mutual influence of damages (of the zones of material's weakened interparticle bonds) is explained by difference in their location. Thus, the joint solution of the obtained algebraic systems and limit condition (4.9) enables (at the given characteristics of the material) to determine the critical value of the external load and the pre-fracture zone sizes for the limit equilibrium state at which the crack occurs.

At some loading stage the simultaneous existence of pre-fracture zones and the generated cracks in the disk is possible. The solution method in this case combines the simultaneous consideration of damages and end zone cracks with interfacial bonds.

5. Conclusions

The use of operation of circular disks in practice shows that at the design stage it is necessary to take into account the cases when there may arise cracks in the disk. In this connection it is necessary to realize the limit analysis of disks in order to establish the ultimate loads at which the cracking in the disk occurs. The size of the minimal zone of weakened interparticle bonds of the material at which the cracking takes place, should be considered as the design characteristics of the disk's material. On the basis of the suggested design model, taking into account the existence of damages (pre-fracture zones), a method for calculation of the disks parameters at which the cracking appears, is developed.

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Received May 25, 2014; revised version October 13, 2014.
