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Brief Note

A variational analysis of the equilibrium turbulent closure models of a dry granular densified motion with weak turbulent intensity

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A variational principle is proposed to derive the equilibrium expressions of the turbulent closure models of an isothermal dry granular dense matter in weak turbulent motions. It is demonstrated that the equilibrium equations and the natural boundary conditions coincide with those derived by the thermodynamic analysis (Fang and Wu, in Acta Geotechnica, in press). The current work serves as a supplementary variational verification of the turbulent closure models proposed in Fang and Wu (in Acta Geotechnica, in press).

Key words: dry granular dense matter, turbulent closure models.

1. Introduction

When dry granular matter is set in motion, the interactions among the solid grains result essentially from short-term instantaneous elastic and inelastic collisions, and long-term enduring frictional contact and sliding at different degrees [1–6]. These grain-grain interactions cause fluctuations of the macroscopic quantities such as pressure or velocity about their time-averaged (mean) values in the context of continuum approach. Such phenomena are characterized as the turbulent motions, and are distinct from those of conventional Newtonian fluids by two perspectives: (i) they emerge from two-fold grain-grain interactions, in contrast to those from incoming flow instability, instability in transition region, or flow geometry of conventional Newtonian fluids [7–10], and (ii) they emerge equally at slow speed, in contrast to those of Newtonian fluids which are strongly velocity-dependent, characterized by the critical Reynolds number.

To account for the influence of the turbulent fluctuations on the mean flow features and the energy cascade between the turbulent kinetic energy and turbulent dissipation, conventional Reynolds-averaging (filter) process is applied to decompose the field quantities into the mean and fluctuating parts, by which the mean balance equations of the primitive mean fields are obtained, associated with ergodic turbulent fluctuating quantities. These turbulent fluctuating quantities need be specified as functions of the primitive mean fields, known as the turbulent closure models, in order to arrive at a mathematically likely well-posed problem [7–10]. By different prescriptions of the turbulent fluctuating quantities, turbulent closure models of different orders can be obtained [9, 10].

Most up-to-date turbulent closure models are established for rapid flows, in which the turbulent fluctuations result essentially from the short-term instantaneous inelastic collisions [11–21]. Recently, it has been shown that the concept of granular coldness can be extended to account for the influence of the turbulent fluctuations generated by the long-term enduring frictional contact and sliding between the grains for dense flows with weak turbulent intensity [22, 23]. In these works, the thermodynamic analysis, based on the Müller–Liu entropy principle, has been carried out to derive the equilibrium turbulent closure models, which satisfy the turbulence realizability conditions [24–26]. For convenience, the results in [22, 23] are summarized in the following:

- for the turbulent Helmholtz free energy ψ^T, the turbulent thermodynamic pressure $\bar{p}$ and the turbulent configurational pressure $\bar{\beta}$:

$$\psi^T = \begin{cases} \hat{\psi}^T_I(\nu_0, \bar{\nu}, \nabla\bar{\nu}, \bar{\gamma}, \vartheta^M, \vartheta^T, \bar{\mathbf{Z}}), \\ \hat{\psi}^T_{II}(\nu_0, \bar{\nu}, \bar{\gamma}, \vartheta^M, \vartheta^T, \bar{\mathbf{Z}}), \end{cases} \qquad \bar{p} = \bar{\gamma}^2 \psi^T_{,\bar{\gamma}}, \qquad \bar{\beta} = \bar{\gamma}\bar{\nu}\psi^T_{,\bar{\nu}}, \tag{1.1}$$

  in which $\bar{\nu}$ is the mean volume fraction defined as the ratio of the mean total solid content volume divided by the volume of a representative volume element (RVE); ν_0 the value of $\bar{\nu}$ in the reference configuration; $\bar{\gamma}$ the mean true mass density of the solid grains[1]; ϑ^M the granular material coldness; ϑ^T the granular coldness[2]; and $\bar{\mathbf{Z}}$ the mean internal friction. The notation ∇ stands for the nabla operator, and the subscripts I and II denote that the indexed quantities are applied to compressible and incompressible solid grains, respectively; quantities without these subscripts are applied for both cases; this notation rule is used in the forthcoming expressions;

[1]For dry granular systems, ρ denotes the overall density, with γ the (true mass) density of the solid grains. Thus, $\rho = \gamma\nu$. Since in turbulent motions all the quantities experience fluctuations, their mean values, followed by a Reynolds-filter process, are defined and investigated [7–10].

[2]For simple materials, the coldness ϑ is defined as the reciprocal of the material empirical temperature θ [27, 28]. For dry granular matter in turbulent motions, the turbulent kinetic energy of the solid grains is expressed, following the kinetic theory of gas in identifying the gas temperature, by the granular temperature θ^T [14, 29, 30]. In the study, the overall material empirical temperature θ^M and granular temperature θ^T are expressed, respectively by the material and granular coldnesses ϑ^M and ϑ^T for simplicity [14, 22, 23]. However, the simple relations that $\vartheta^i = 1/\theta^i$, $i = M, T$, do not hold in general for dry granular systems.

- for the turbulent kinetic energy $\bar{\gamma}\bar{\nu}k$ and turbulent dissipation $\bar{\gamma}\bar{\nu}\varepsilon$:

$$
\begin{aligned}
&\bar{\gamma}\bar{\nu}k = \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{,\vartheta^T}, \qquad \varepsilon|_{\mathrm{E}} = 0,\\
&\bar{\gamma}\bar{\nu}(\vartheta^M - \vartheta^T)\varepsilon_{,\dot{\bar{\nu}}}|_{\mathrm{E}} + \vartheta^M(\bar{p} - \bar{\beta}) - \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\dot{\bar{\nu}}}|_{\mathrm{E}} = 0,
\end{aligned}
\tag{1.2}
$$

where $\bar{\mathbf{\Phi}}$ denotes the mean production associated with $\bar{\mathbf{Z}}$, $\dot{\xi}$ stands for the material derivative with respect to the mean flow velocity $\bar{\mathbf{v}}$ for any quantity ξ, and the subscript E denotes that the indexed quantity is evaluated at an *thermodynamic equilibrium* state;
- for the mean flux $\bar{\mathbf{h}}$ and mean production $\bar{f}$ associated with $\bar{\nu}$:

$$
\begin{cases}
-\alpha\bar{\mathbf{h}}_I = \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{I,\nabla\bar{\nu}},\\
\bar{\mathbf{h}}_{II} = \mathbf{0},
\end{cases}
\qquad
\begin{cases}
\bar{f}_I|_{\mathrm{E}} = 0,\\
\bar{f}_{II} = 0,
\end{cases}
\tag{1.3}
$$

where α is an isotropic scalar function with $\alpha = \hat{\alpha}(\bar{\gamma}, \bar{\nu}, \vartheta^M, \vartheta^T)$[3];
- for the flux $\mathbf{K}$ associated with $\bar{\gamma}\bar{\nu}k$, and the mean production $\bar{\mathbf{\Phi}}$:

$$
\mathbf{K}|_{\mathrm{E}} = \bar{\gamma}\bar{\nu}(\vartheta^M - \vartheta^T)\varepsilon_{,\nabla\vartheta^T}|_{\mathrm{E}} - \bar{\gamma}\bar{\nu}\psi^T_{,\bar{\mathbf{Z}}}\bar{\mathbf{\Phi}}_{,\nabla\vartheta^T}|_{\mathrm{E}}, \qquad \bar{\mathbf{\Phi}}|_{\mathrm{E}} = \mathbf{0};
\tag{1.4}
$$

- and for the mean Cauchy stress $\bar{\mathbf{t}}$ and Reynolds' stress $\mathbf{R}$:

$$
\begin{cases}
\vartheta^M \bar{\mathbf{t}}_I|_{\mathrm{E}} + \vartheta^T \mathbf{R}_I|_{\mathrm{E}}\\
\qquad = -\bar{\nu}\vartheta^M \bar{p}\mathbf{I} + \alpha\bar{\mathbf{h}}_I \otimes \nabla\bar{\nu} - \bar{\gamma}\bar{\nu}(\vartheta^M - \vartheta^T)\varepsilon_{,\bar{\mathbf{D}}}|_{\mathrm{E}} + \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{I,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\bar{\mathbf{D}}}|_{\mathrm{E}},\\
\vartheta^M \bar{\mathbf{t}}_{II}|_{\mathrm{E}} + \vartheta^T \mathbf{R}_{II}|_{\mathrm{E}}\\
\qquad = -\bar{\nu}\vartheta^M \bar{p}\mathbf{I} - \bar{\gamma}\bar{\nu}(\vartheta^M - \vartheta^T)\varepsilon_{,\bar{\mathbf{D}}} + \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{II,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\bar{\mathbf{D}}}|_{\mathrm{E}},
\end{cases}
\tag{1.5}
$$

where $\mathbf{I}$ is the second-rank identity tensor, the notation $\otimes$ stands for the dyadic product, and $\bar{\mathbf{D}}$ is the symmetric part of the mean velocity gradient.

Since in laminar motions the Helmholtz free energy ψ at thermodynamic equilibrium corresponds to the stored energy function ϕ in static equilibrium [32], it is possible to derive the equilibrium turbulent closure models by extending the functional of the stored energy function for turbulent motions, associated with a properly formulated variational analysis[4].

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[3]Vanishing $\bar{\mathbf{h}}$ and f for incompressible grains are obtained due to the consistency between the mass balance and Wilmański's model for the time evolution of the volume fraction, see [31].

[4]One can start with a proposed functional and search for some definite conditions for which the proposed functional may have extreme values. This is known as the *direct problem*, resulting in the equations of motion and the associated natural boundary conditions. By contrast, the reverse procedure leads to the *inverse problem*. However, due to the *variational crisis*, this approach may sometimes fail, and various methods have been proposed to overcome this difficulty [33–39].

In this paper, a variational principle is proposed for the stored energy function in turbulent motions to derive the mean balance equations in static equilibrium, the equilibrium turbulent closure models, and the associated equilibrium natural boundary conditions for isothermal flows with compressible and incompressible solid grains. With the above principle, the proposed variational analysis is similar to the direct problem. However, the existence and the functional dependency of the stored energy function are *a priori* assumed, and the principle of virtual power is extended for the variational analysis in static equilibrium, as will be shown later. In Section 2, the definition of the variation in the formulation will be given, which will be applied in Section 3 to derive the equilibrium turbulent closure models for compressible and incompressible solid grains. This paper will be summarized and commented in Section 4.

2. Variation definition

By extending the definition of the variation used in laminar motions [40], the definition of variation for turbulent motions is given in the following:

Definition. Associated with the material point $\mathbf{X}$ let there exist a spatial position $\mathbf{x}(\mathbf{X})$, a mean volume fraction $\bar{\nu}(\mathbf{X})$, a granular coldness $\vartheta^T(\mathbf{X})$ and a mean internal friction $\bar{\mathbf{Z}}(\mathbf{X})$. Define equilibrium states, $\mathbf{x}(\mathbf{X})$, $\bar{\nu}(\mathbf{X})$, $\vartheta^T(\mathbf{X})$ and $\bar{\mathbf{Z}}(\mathbf{X})$, and the comparison states, $\mathbf{x}(\mathbf{X},\lambda)$, $\bar{\nu}(\mathbf{X},\lambda)$, $\vartheta^T(\mathbf{X},\lambda)$ and $\bar{\mathbf{Z}}(\mathbf{X},\lambda)$, parameterized by the scalar λ. Denote by the symbol δ the variation performed on these state variables by holding the particle $\mathbf{X}$ fixed. The variations of the equilibrium spatial position $\delta\mathbf{x}$, of the equilibrium mean volume fraction $\delta\bar{\nu}$, of the equilibrium granular coldness $\delta\vartheta^T$ and of the equilibrium mean internal friction $\delta\bar{\mathbf{Z}}$ on $\mathbf{X}$ are defined as

$$
\begin{aligned}
&\delta\mathbf{x} \equiv \left.\frac{d\mathbf{x}}{d\lambda}\right|_{\lambda\,=\,0,\,\mathbf{X}}, \qquad && \delta\bar{\nu} \equiv \left.\frac{d\bar{\nu}}{d\lambda}\right|_{\lambda\,=\,0,\,\mathbf{X}}, \\
&\delta\vartheta^T \equiv \left.\frac{d\vartheta^T}{d\lambda}\right|_{\lambda\,=\,0,\,\mathbf{X}}, \qquad && \delta\bar{\mathbf{Z}} \equiv \left.\frac{d\bar{\mathbf{Z}}}{d\lambda}\right|_{\lambda\,=\,0,\,\mathbf{X}},
\end{aligned}
\tag{2.1}
$$

with $(2.1)_{2-4}$ recast in the forms

$$
\delta\alpha = \left.\frac{\partial\alpha}{\partial\lambda}\right|_{\lambda\,=\,0,\,\mathbf{x}} + \left.\frac{\partial\alpha}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\lambda}\right|_{\lambda=0}, \qquad \alpha \in \{\bar{\nu}, \vartheta^T, \bar{\mathbf{Z}}\},
\tag{2.2}
$$

when the functional dependencies of $\bar{\nu}$, ϑ^T and $\bar{\mathbf{Z}}$ are represented by $\bar{\nu}(\mathbf{x},t)$, $\vartheta^T(\mathbf{x},t)$ and $\bar{\mathbf{Z}}(\mathbf{x},t)$, rather than $\nu(\mathbf{X},t)$ $\vartheta^T(\mathbf{X},t)$ and $\bar{\mathbf{Z}}(\mathbf{X},t)$, respectively. A variation of α performed by holding the spatial position $\mathbf{x}$ of the particle fixed rather than the particle itself is denoted by $\Delta\alpha$ and defined as

$$\Delta\alpha = \frac{\partial\alpha}{\partial\lambda}\bigg|_{\lambda=0,\,\mathbf{x}}, \qquad \alpha \in \{\bar{\nu}, \vartheta^T, \bar{\mathbf{Z}}\}. \tag{2.3}$$

In view of the definitions (2.1)–(2.3), it follows that

$$\delta\alpha = \Delta\alpha + \alpha_{,i}\delta x_i, \quad \delta(\alpha_{,i}) = (\delta\alpha)_{,i} - (\alpha_{,j})(\delta x_j)_{,i}, \quad \alpha \in \{\bar{\nu}, \vartheta^T, \bar{\mathbf{Z}}\}, \tag{2.4}$$

hold. The total mass M of a granular body $\mathcal{B}$ and its variation are given by

$$\begin{aligned} M &\equiv \int_{\mathcal{B}} \bar{\gamma}\bar{\nu}dv, \\ 0 = \delta M &= \delta\int_{\mathcal{B}} \bar{\gamma}\bar{\nu}dv = \int_{\mathcal{B}} \delta(\bar{\gamma}\bar{\nu}dv) = \int_{\mathcal{B}} \{\delta(\bar{\gamma}\bar{\nu}) + \bar{\gamma}\bar{\nu}\nabla\cdot(\delta\mathbf{x})\}dv, \end{aligned} \tag{2.5}$$

due to the conservation of mass. Equation $(2.5)_2$ can be violated unless

$$0 = \delta(\bar{\gamma}\bar{\nu}) + \bar{\gamma}\bar{\nu}\nabla\cdot(\delta\mathbf{x}) \quad \longrightarrow \quad \delta\bar{\gamma} = -\frac{\bar{\gamma}}{\bar{\nu}}\delta\bar{\nu} - \bar{\gamma}(\delta x_i)_{,i}, \tag{2.6}$$

is satisfied. Equation $(2.6)_2$ applies essentially for compressible grains; for incompressible grains $\delta\bar{\gamma}$ vanishes, and the equation (2.6) reduces to

$$\delta\bar{\nu} = -\bar{\nu}(\delta x_i)_{,i}. \tag{2.7}$$

3. The variational analysis: the principle of virtual power

3.1. Case of compressible grains

Let ϕ^T be the stored energy function in turbulent motions whose functional dependency in *static equilibrium* is given by $\phi^T = \hat{\phi}^T(\bar{\gamma}, \bar{\nu}, \nabla\bar{\nu}, \vartheta^T, \bar{\mathbf{Z}}, \nabla\bar{\mathbf{Z}})$ for isothermal flows. We require that ϕ^T satisfies, for all variations of $\mathbf{x}$, $\bar{\nu}$ and $\bar{\mathbf{Z}}$, the following identity:

$$\delta\int_{\mathcal{B}} \bar{\gamma}\bar{\nu}\phi^T dv = \int_{\partial\mathcal{B}} \big\{\sigma_i\delta x_i + H_{pq}\delta\bar{Z}_{pq} + S\delta\vartheta^T + H\delta\bar{\nu})\big\}da + \int_{\mathcal{B}} \bar{\gamma}\bar{\nu}b_i\delta x_i dv, \tag{3.1}$$

where σ_i is the surface traction (surface force on the boundary), b_i the specific body force, $\partial\mathcal{B}$ the surface of $\mathcal{B}$; H the self-equilibrated stress system which can be shown to be a center of dilatation/compression [41]. The quantity S denotes the surface energy flux induced by the variations of the granular coldness ϑ^T, a measure of the turbulent kinetic energy; and H_{pq} is a surface strain induced by the mean internal friction $\bar{\mathbf{Z}}$, which is frequently employed in higher-order microcontinuum/micropolar theory and statistical mechanics [32, 42–44]. Equation (3.1) is similar to the classical principle of virtual power, except that additional (postulating) contributions of the powers of working done by the surface terms S, H and H_{pq} are taken into account. This is done so, because the variations of ϑ^T, $\bar{\nu}$ and $\bar{\mathbf{Z}}$ are assumed to be independent of the motion of the body. On the other hand, since ϑ^T, $\bar{\nu}$ and $\bar{\mathbf{Z}}$ are introduced as internal variables without external supplies, no additional powers of working are included in the volume integral on the right-hand side of (3.1).

In view of $\phi^T = \hat{\phi}^T(\bar{\gamma}, \bar{\nu}, \nabla\bar{\nu}, \vartheta^T, \bar{\mathbf{Z}}, \nabla\bar{\mathbf{Z}})$, the left-hand side of (3.1) reduces to

$$
\begin{aligned}
&\delta \int_{\mathcal{B}} \bar{\gamma}\bar{\nu}\phi^T dv = \int_{\mathcal{B}} \bar{\gamma}\bar{\nu}\delta\phi^T dv,\\
&\delta\phi^T = \frac{\partial\phi^T}{\partial\bar{\gamma}}\delta\bar{\gamma} + \frac{\partial\phi^T}{\partial\bar{\nu}}\delta\bar{\nu} + \frac{\partial\phi^T}{\partial(\bar{\nu}_{,i})}\delta(\bar{\nu}_{,i})\\
&\qquad + \frac{\partial\phi^T}{\partial\vartheta^T}\delta\vartheta^T + \frac{\partial\phi^T}{\partial\bar{Z}_{pq}}\delta\bar{Z}_{pq} + \frac{\partial\phi^T}{\partial\bar{Z}_{pq,\,i}}\delta(\bar{Z}_{pq,\,i}).
\end{aligned}
\tag{3.2}
$$

Substituting $(2.4)_2$ and $(2.6)_2$ into $(3.2)_2$ results in

$$
\begin{aligned}
\bar{\gamma}\bar{\nu}\delta\phi^T &= (-\bar{p} + \bar{\beta})\delta\bar{\nu} + \bar{\gamma}\bar{\nu}k\delta\vartheta^T + \bar{h}_i(\delta\bar{\nu})_{,i}\\
&\quad + \bar{A}_{pq}\delta\bar{Z}_{pq} + \bar{B}_{pqi}(\delta\bar{Z}_{pq})_{,\,i} + \bar{t}_{ij}(\delta x_j)_{,\,i},
\end{aligned}
\tag{3.3}
$$

with the abbreviations,

$$
\begin{aligned}
&\bar{p} = \bar{\gamma}^2\frac{\partial\phi^T}{\partial\bar{\gamma}}, \qquad \bar{\beta} = \bar{\gamma}\bar{\nu}\frac{\partial\phi^T}{\partial\bar{\nu}}, \qquad \bar{h}_i = \bar{\gamma}\bar{\nu}\frac{\partial\phi^T}{\partial(\bar{\nu}_{,i})},\\
&\bar{\gamma}\bar{\nu}k = \bar{\gamma}\bar{\nu}\frac{\partial\phi^T}{\partial\vartheta^T}, \qquad \bar{A}_{pq} = \bar{\gamma}\bar{\nu}\frac{\partial\phi^T}{\partial\bar{Z}_{pq}}, \qquad \bar{B}_{pqi} = \bar{\gamma}\bar{\nu}\frac{\partial\phi^T}{\partial(\bar{Z}_{pq,\,i})},\\
&\bar{t}_{ij} = -\bar{\nu}\bar{p}\delta_{ij} - \bar{h}_i(\bar{\nu}_{,j}) - \bar{B}_{pqi}(\bar{Z}_{pq,\,j}),
\end{aligned}
\tag{3.4}
$$

where δ_{ij} is the Kronecker delta. With (3.4), the equation $(3.2)_1$ is recast in the form

$$
\begin{aligned}
&\delta \int_{\mathcal{B}} \bar{\gamma}\bar{\nu}\phi^T dv\\
&= \int_{\mathcal{B}} \left\{(-\bar{p} + \bar{\beta} - \bar{h}_{i,\,i})\delta\bar{\nu} + \bar{\gamma}\bar{\nu}k\delta\vartheta^T + (\bar{A}_{pq} - \bar{B}_{pqi,\,i})\delta\bar{Z}_{pq} - \bar{t}_{ij,\,i}\delta x_j\right\}dv\\
&\quad + \int_{\partial\mathcal{B}} \left\{\bar{h}_i n_i\delta\bar{\nu} + \bar{B}_{pqi}n_i\delta\bar{Z}_{pq} + \bar{t}_{ij}n_i\delta x_j\right\}da,
\end{aligned}
\tag{3.5}
$$

where n_i is the unit outward normal of da on $\partial\mathcal{B}$. In deriving equation (3.5), Gauss' divergence theorem has been used. Incorporating (3.5) into Eq. (3.1) leads to

$$\begin{aligned}(3.6)\qquad &\int_{\mathcal{B}} \left\{(\bar{p}-\bar{\beta}+\bar{h}_{i,\,i})\delta\bar{\nu}-\bar{\gamma}\bar{\nu}k\delta\vartheta^T+(\bar{B}_{pqi,\,i}-\bar{A}_{pq})\delta\bar{Z}_{pq}+(\bar{t}_{ij,\,j}+\bar{\gamma}\bar{\nu}b_i)\delta x_i\right\}dv\\ &+\int_{\partial\mathcal{B}} \left\{(H-\bar{h}_i n_i)\delta\bar{\nu}+S\delta\vartheta^T+(H_{pq}-\bar{B}_{pqi}n_i)\delta\bar{Z}_{pq}+(\sigma_i-\bar{t}_{ij}n_j)\delta x_i\right\}da=0.\end{aligned}$$

Since $\mathcal{B}$ and $\partial\mathcal{B}$ of the material body are arbitrarily chosen, we can choose it to be infinitesimal with vanishing volume but finite surface, a "pillbox". For such a volume, Eq. (3.6) reduces to

$$(3.7)\qquad \int_{\partial\mathcal{B}}\left\{(H-\bar{h}_i n_i)\delta\bar{\nu}+S\delta\vartheta^T+(H_{pq}-\bar{B}_{pqi}n_i)\delta\bar{Z}_{pq}+(\sigma_i-\bar{t}_{ij}n_j)\delta x_i\right\}da=0.$$

Since da can be arbitrarily chosen and $\delta\bar{\nu}$, $\delta\vartheta^T$, $\delta\bar{Z}_{pq}$ and δx_i are independent of one another, Eq. (3.7) can be violated unless

$$(3.8)\qquad \sigma_i=\bar{t}_{ij}n_j,\qquad H=\bar{h}_i n_i,\qquad H_{pq}=\bar{B}_{pqi}n_i,\qquad S=0,\qquad \text{on } \partial\mathcal{B},$$

hold. Substituting (3.8) into Eq. (3.6) results in

$$(3.9)\qquad \int_{\mathcal{B}}\left\{(\bar{p}-\bar{\beta}+\bar{h}_{i,\,i})\delta\bar{\nu}-\bar{\gamma}\bar{\nu}k\delta\vartheta^T+(\bar{B}_{pqi,\,i}-\bar{A}_{pq})\delta\bar{Z}_{pq}+(\bar{t}_{ij,\,j}+\bar{\gamma}\bar{\nu}b_i)\delta x_i\right\}dv=0.$$

Again, since dv can be arbitrarily chosen and $\delta\bar{\nu}$, $\delta\vartheta^T$, $\delta\bar{Z}_{pq}$ and δx_i are independent of one another, Eq. (3.9) might be violated unless

$$(3.10)\qquad \bar{t}_{ij,\,j}+\bar{\gamma}\bar{\nu}b_i=0,\quad \bar{p}-\bar{\beta}+\bar{h}_{i,\,i}=0,\quad \bar{B}_{pqi,\,i}-\bar{A}_{pq}=0,\quad \bar{\gamma}\bar{\nu}k=0,\quad \text{in } \mathcal{B},$$

hold. Whereas $(3.10)_1$ is the conventional mean balance of linear momentum in static equilibrium associated with its natural boundary condition given in $(3.8)_1$, Eq. $(3.10)_2$ emerges as an equilibrium statement for the flux associated with the time evolution of the mean volume fraction with a similar "traction" boundary condition given in $(3.8)_2$. Equation $(3.10)_3$ indicates the restrictions of the functional of the stored energy that should be satisfied, associated with the natural boundary condition given in $(3.8)_3$. No further information can be gained unless a specific form of ϕ^T is prescribed. In view of $(3.4)_6$ and $(3.8)_3$, H_{pq} is similar to the concept of surface strain in higher-order micropolar theory and statistical mechanics, as mentioned previously. Although in the context of the proposed variational approach the definition of the turbulent kinetic energy can be identified (see $(3.4)_4$), a vanishing turbulent kinetic energy and its vanishing natural boundary condition are obtained, as indicated by $(3.8)_4$ and $(3.10)_4$, respectively. This is due to the static equilibrium, in which all fluctuating quantities cease, yielding a vanishing turbulent kinetic energy. The results coincide with those by the thermodynamic analysis given in (1.2)–(1.5).

Of particular interest is the derived mean Cauchy stress at static equilibrium. In view of $(3.4)_{6,7}$, the mean Cauchy stress can be recast in the form

$$\bar{t}_{ij} = -\bar{\nu}\bar{p}\delta_{ij} - \bar{h}_i(\bar{\nu}_{,j}) - \bar{\gamma}\bar{\nu}\frac{\partial\phi^T}{\partial\bar{Z}_{mn}}\underbrace{\left(\frac{\partial\bar{Z}_{mn}}{\partial\bar{Z}_{pq,\,i}}\bar{Z}_{pq,\,j}\right)}_{\aleph}, \tag{3.11}$$

in which $\aleph$ is a fourth-rank tensor. It describes the evolution of the internal friction inside a granular microcontinuum and needs be prescribed for further identifications of the mean Cauchy stress (its prescription is accomplished by use of the hypoplastic model in the thermodynamic analysis [22, 23]). However, in view of $\aleph$, Eq. (3.11) coincides formally with (1.5) derived by the thermodynamic analysis[5].

3.2. Case of incompressible grains

For incompressible grains, the stored energy ϕ^T depends no longer on the mean true mass density of the solid grains $\bar{\gamma}$, thus Eq. (3.3) reduces to

$$\begin{aligned} \bar{\gamma}\bar{\nu}\delta\phi^T &= \bar{\gamma}\bar{\nu}\left\{\frac{\partial\phi^T}{\partial\bar{\nu}}\delta\bar{\nu} + \frac{\partial\phi^T}{\partial(\bar{\nu}_{,i})}\delta(\bar{\nu}_{,i}) + \frac{\partial\phi^T}{\partial\vartheta^T}\delta\vartheta^T + \frac{\partial\phi^T}{\partial\bar{Z}_{pq}}\delta\bar{Z}_{pq} + \frac{\partial\phi^T}{\partial\bar{Z}_{pq,\,i}}\delta(\bar{Z}_{pq,\,i})\right\} \\ &= \bar{t}_{ij}(\delta x_j)_{,\,i} + \bar{\gamma}\bar{\nu}k\delta\vartheta^T + \bar{h}_i(\delta\bar{\nu})_{,\,i} + \bar{A}_{pq}\delta\bar{Z}_{pq} + \bar{B}_{pqi}(\delta\bar{Z}_{pq})_{,\,i} \end{aligned} \tag{3.12}$$

with the revised mean Cauchy stress $\bar{t}_{ij}$ given by

$$\bar{t}_{ij} = -\bar{\nu}\bar{\beta}\delta_{ij} - \bar{h}_i(\bar{\nu}_{,j}) - \bar{B}_{pqi}(\bar{Z}_{pqi,\,j}). \tag{3.13}$$

In deriving (3.12), Eqs. $(2.4)_2$, (2.7) and $(3.4)_{2-6}$ have been used. Substituting (3.12) into (3.1) together with the Gauss divergence theorem results in

$$\begin{aligned} &\int_{\mathcal{B}} \left\{\bar{h}_{i,\,i}\delta\bar{\nu} - \bar{\gamma}\bar{\nu}k\delta\vartheta^T + (\bar{B}_{pqi,\,i} - \bar{A}_{pq})\delta\bar{Z}_{pq} + (\bar{t}_{ij,\,j} + \bar{\gamma}\bar{\nu}b_i)\delta x_i\right\}dv \\ &+ \int_{\partial\mathcal{B}} \left\{(H - \bar{h}_i n_i)\delta\bar{\nu} + S\delta\vartheta^T + (H_{pq} - \bar{B}_{pqi}n_i)\delta\bar{Z}_{pq} + (\sigma_i - \bar{t}_{ij}n_j)\delta x_i\right\}da = 0. \end{aligned} \tag{3.14}$$

[5]In static equilibrium both $\mathbf{R}$ and $\bar{\gamma}\bar{\nu}\varepsilon$ vanish.

Applying the "pillbox" analysis to equation (3.14) leads to

$$\begin{aligned}&\sigma_i = \bar{t}_{ij} n_j, && H = \bar{h}_i n_i, && H_{pq} = \bar{B}_{pqi} n_i, && S = 0 && \text{on } \partial\mathcal{B},\\ &\bar{t}_{ij,\,j} + \bar{\gamma}\bar{\nu} b_i = 0, && \bar{h}_{i,\,i} = 0, && \bar{B}_{pqi,\,i} - \bar{A}_{pq} = 0, && \bar{\gamma}\bar{\nu} k = 0 && \text{in } \mathcal{B}.\end{aligned} \tag{3.15}$$

While the interpretations of equations $(3.15)_{1-4,5,7-8}$ are the same as those given previously, Eq. $(3.15)_6$ states that the mean flux $\bar{\mathbf{h}}$ should vanish for incompressible grains. This result coincides to that derived by the thermodynamic analysis given in (1.3). If we consider the circumstance of incompressible grains as a special case of compressible grains, it follows from $(3.10)_2$ that $\bar{p} = \bar{\beta}$ holds for incompressible grains. Such a fact is also demonstrated by the replacement of $\bar{p}$ by $\bar{\beta}$ in the expression (3.13) in comparison with $(3.4)_7$ or (3.11).

With the vanishing $\bar{\mathbf{h}}$, Eq. (3.13) reduces to

$$\bar{t}_{ij} = -\bar{\nu}\bar{\beta}\delta_{ij} - \bar{B}_{pqi}(\bar{Z}_{pqi,\,j}). \tag{3.16}$$

This expression coincides with the result derived by the thermodynamic analysis, and also asserts the conclusion derived in the previous work that the internal friction needs be taken into account to generate a non-spherical Cauchy stress when a kinematic equation is used for the time evolution of the volume fraction [31].

4. Concluding remarks

In the present study, a variational principle was proposed for isothermal dry granular dense flows with weak turbulent intensity. It was applied to derive the equilibrium equations associated with the natural boundary conditions, and the equilibrium turbulent closure models for both compressible and incompressible grains. It was demonstrated that the results coincide with those of the thermodynamic approach. However, the equilibrium closure models of the turbulent dissipation, the Reynolds stress, the flux associated with the turbulent kinetic energy, and the mean production associated with the mean volume fraction were not present. This was due to the postulated stored energy function in static equilibrium, in which all fluctuating quantities cease, resulting in vanishing turbulent kinetic energy.

Nevertheless, the conclusions that the derived results from the variational approach coincide with those from the thermodynamic approach should be taken with caution, and some provisions need to be formulated. The two approaches are in fact not equivalent: the variational approach only yields equilibrium properties and cannot be used for thermodynamic process, whereas the thermodynamic approach allows construction of a dynamic model with dissipative properties. This difference is visible in the differences of the state space. In the present study, the stored energy function ϕ^T is only dependent upon a subset of the state space used in [22, 23]. The case when the variables which vanish in thermodynamic equilibrium and the material temperature are omitted from the state space of ϕ^T, is the expression of the fact that the variational formulation only furnishes equilibrium properties.

Despite these imperfections the two-fold derivation is helpful, because identical equations for equilibrium processes deduced with two different methods assign greater credibility to them. In addition, the variational approach provides a better insight into the natural boundary conditions that are associated with the internal friction. However, the evaluations of the flux terms, e.g., $\bar{\mathbf{h}}$ and H_{pq} on the boundaries cause some difficulties, since this involves prescriptions of the values of $\bar{\nu}$ and the gradient of $\bar{\mathbf{Z}}$ on those locations, respectively. Thus, comparison between the variational and thermodynamic approaches only provides a reassurance that the formulations are mutually consistent in their basic postulates. In view of these, the present study serves as a variational verification of the turbulent closure models proposed in [22, 23].

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