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Natural vibrations of thick circular plate based on the modified Mindlin theory

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Outline of the modified Mindlin theory is presented in which the Mindlin mathematical model with three differential equations of motion for total deflection and rotations is decomposed into a single equation for pure bending vibrations with transverse shear and rotary inertia effects and two differential equations for in-plane shear vibrations. The governing equations are transformed from orthogonal to polar coordinate system for the purpose of circular plate vibration analysis. The fourth order differential equation of flexural vibrations is split further into two second order equations of Bessel type. Also, the in-plane shear differential equations are transformed to Bessel equation by introducing displacement potential functions. The exact values of natural frequencies are listed and compared with FEM results.

Key words: thick circular plate, modified Mindlin theory, flexural vibration, in-plane shear vibration, analytical solutions, FEM.

1. Introduction

Thick circular plates are extensively used as structural elements in many engineering structures encountered in mechanical, civil and marine fields. Vibration analysis is based on the well-known Mindlin theory of rectangular plates, which incorporates the effects of transverse shear deformation and rotary inertia [1, 2]. This first-order shear deformable plate theory improves the accuracy of thin plate theory, especially for moderately thick plates [3]. In addition, some higher-order shear deformable plate theories have been developed [4–6]. A comprehensive review of the relevant developments is presented in [7].

In order to increase accuracy of thick plate vibration analysis, three-dimensional theory has been developed and applied for circular plates [8–10]. A short review of this research can be found in introduction of [11], where 3D plate theory is applied for vibration analysis of circular plates by using the Chebyshev–Ritz method.

Special problem of Mindlin circular plate is axisymmetric vibration, which can be solved in different ways. For instance, application of the differential quadrature method is used in [12]. Circular plates with non-uniform thickness have wide practical use due to saving of material and weight reduction. Many papers have been published regarding non-linear thickness variation and step-wise thickness variation [13–17].

The Mindlin plate theory operates with three independent variables, i.e., total deflection and angles of cross-section rotations [1, 2]. The finite elements developed based on this general theory manifest shear locking phenomenon. Overcoming of that problem requires additional effort. Since there is no unique solution, different procedures have been worked out [18].

Recently, the Mindlin theory was modified in such a way that total deflection and rotations are split into pure bending deflection and shear deflection, and bending rotations and in-plane shear angles respectively [19, 20]. Since shear deflection and bending rotations depend on bending deflection, the Mindlin mathematical model with three DOFs is actually decomposed into a single DOF bending and double DOF shear model. Shear locking-free finite elements can be formulated based on the modified Mindlin theory as shown in [21].

In this paper, differential equations of flexural and in-plane shear vibrations of circular plate, by taking advantage of the modified Mindlin theory, are derived. Application of the developed procedure is illustrated in case of simply supported, clamped and free circular plate.

The paper presents some contribution to the specific classical problem of thick circular plate vibrations. An overview of shear deformation plate and shell theories is given in [22]. Nowadays, investigation is focused on multilayered, anisotropic, composite plate and shell theories as static problem, [23]. Carrera's unified formulation and generalized unified formulation [24–26], as well as Todd Wiliams' global-local models [27, 28], as new techniques for derivation of sophisticated finite elements have been developed.

2. Outline of the modified Mindlin theory

A rectangular plate, with aspect ratio a/b and thickness h, is considered in Cartesian coordinate system with corresponding displacements, Fig. 1. Total deflection $w(x, y, t)$ consists of pure bending deflection $w_b(x, y, t)$ and shear deflection $w_s(x, y, t)$, while total cross-section rotation angles $\psi_x(x, y, t)$ and $\psi_y(x, y, t)$ are split into pure bending rotation $\phi_x(x, y, t)$ and $\phi_y(x, y, t)$, and in-plane shear

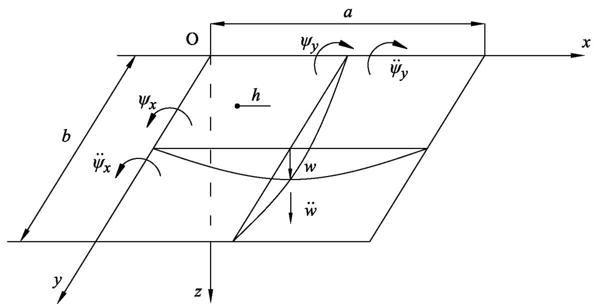

Fig. 1. Rectangular plate.

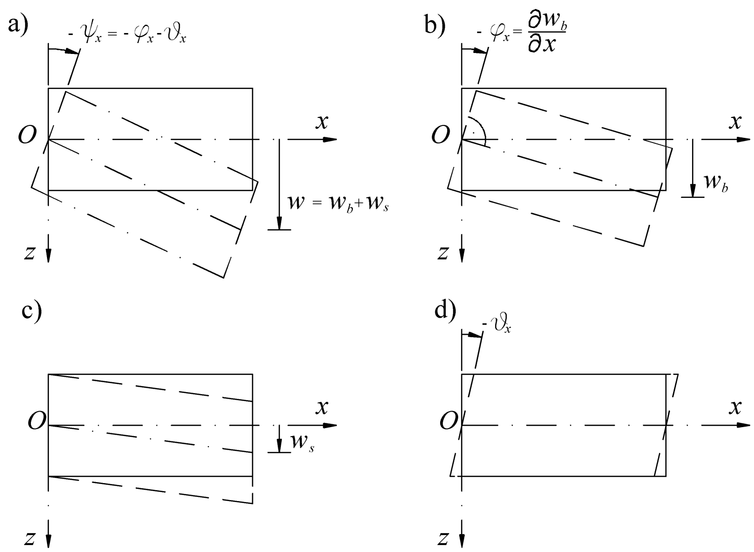

Fig. 2. Displacement components.

angles $\vartheta_x(x, y, t)$ and $\vartheta_y(x, y, t)$ respectively, Fig. 2, [21], i.e.,

$$w = w_b + w_s, \qquad \psi_x = \phi_x + \vartheta_x, \qquad \psi_y = \phi_y + \vartheta_y. \tag{2.1}$$

Shear deflection depends on bending deflection yielding total deflection as

$$w = w_b - \frac{D}{S}\Delta w_b + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}, \tag{2.2}$$

where

$$\Delta(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2}$$

is two-dimensional Laplace differential operator. Furthermore,

$$J = \frac{\rho h^3}{12}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \qquad S = kGh, \tag{2.3}$$

is mass moment of inertia, bending stiffness and shear stiffness, respectively, which depend on mass density ρ, Young's modulus E, shear modulus G, Poisson's coefficient ν, and shear coefficient k. Bending rotation angles are function of bending deflection

$$\phi_x = -\frac{\partial w_b}{\partial x}, \qquad \phi_y = -\frac{\partial w_b}{\partial y}. \tag{2.4}$$

Differential equation of flexural vibrations is related to the bending deflection as the only variable

$$\Delta\Delta w_b - \frac{J}{D}\left(1 + \frac{mD}{JS}\right)\frac{\partial^2}{\partial t^2}\Delta w_b + \frac{m}{D}\frac{\partial^2}{\partial t^2}\left(w_b + \frac{J}{mS}\frac{\partial^2 w_b}{\partial t^2}\right) = 0. \tag{2.5}$$

Sectional forces, i.e., bending moments, torsional moments, shear forces and effective shearing forces are determined as follows:

$$\begin{aligned}
M_x &= -D\left(\frac{\partial^2 w_b}{\partial x^2} + \nu\frac{\partial^2 w_b}{\partial y^2}\right),\\
M_y &= -D\left(\frac{\partial^2 w_b}{\partial y^2} + \nu\frac{\partial^2 w_b}{\partial x^2}\right),\\
M_{xy} &= M_{yx} = -(1-\nu)D\frac{\partial^2 w_b}{\partial x \partial y},\\
Q_x &= S\frac{\partial w_s}{\partial x} = -D\left(\frac{\partial^3 w_b}{\partial x^3} + \frac{\partial^3 w_b}{\partial x \partial y^2}\right) + J\frac{\partial^3 w_b}{\partial x \partial t^2},\\
Q_y &= S\frac{\partial w_s}{\partial y} = -D\left(\frac{\partial^3 w_b}{\partial y^3} + \frac{\partial^3 w_b}{\partial x^2 \partial y}\right) + J\frac{\partial^3 w_b}{\partial y \partial t^2},\\
\bar{Q}_x &= Q_x + \frac{\partial M_{xy}}{\partial y} = -D\left[\frac{\partial^3 w_b}{\partial x^3} + (2-\nu)\frac{\partial^3 w_b}{\partial x \partial y^2}\right] + J\frac{\partial^3 w_b}{\partial x \partial t^2},\\
\bar{Q}_y &= Q_y + \frac{\partial M_{yx}}{\partial x} = -D\left[\frac{\partial^3 w_b}{\partial y^3} + (2-\nu)\frac{\partial^3 w_b}{\partial x^2 \partial y}\right] + J\frac{\partial^3 w_b}{\partial y \partial t^2}.
\end{aligned} \tag{2.6}$$

For determination of the in-plane shear angles two differential equations are extracted from the Mindlin theory [21],

$$\begin{aligned}
D\left[\frac{\partial^2 \vartheta_x}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 \vartheta_x}{\partial y^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 \vartheta_x}{\partial x \partial y}\right] - S\vartheta_x - J\frac{\partial^2 \vartheta_x}{\partial t^2} = 0,\\
D\left[\frac{\partial^2 \vartheta_y}{\partial y^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 \vartheta_y}{\partial x^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 \vartheta_x}{\partial x \partial y}\right] - S\vartheta_y - J\frac{\partial^2 \vartheta_y}{\partial t^2} = 0.
\end{aligned} \tag{2.7}$$

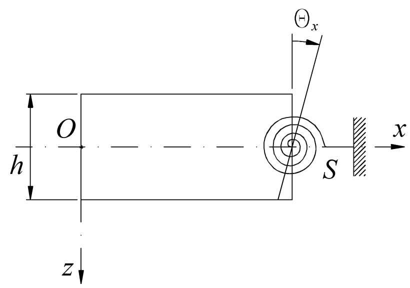

FIG. 3. Angular elastic support.

Terms in the square brackets and last terms are elastic and inertia forces, respectively, while the middle terms are reactions of the angular elastic support, Fig. 3.

The in-plane shear moments are

$$\begin{aligned} M_{sx} &= D\left(\frac{\partial\vartheta_x}{\partial x} + \nu\frac{\partial\vartheta_y}{\partial y}\right), \\ M_{sy} &= D\left(\frac{\partial\vartheta_y}{\partial y} + \nu\frac{\partial\vartheta_x}{\partial x}\right), \\ M_{sxy} = M_{syx} &= \frac{1}{2}(1-\nu)D\left(\frac{\partial\vartheta_x}{\partial y} + \frac{\partial\vartheta_y}{\partial x}\right). \end{aligned} \tag{2.8}$$

Actually, the Mindlin system of three differential equations with three general variables w, ψ_x and ψ_y is transformed into two independent systems, i.e., one single equation with pure bending deflection w_b, and another one with in-plane shear angles ϑ_x and ϑ_y.

If a plate is not elastically supported, i.e., $S = 0$, Eqs. (2.7) became analogical to those of in-plane stretching (membrane), [29], i.e.,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 v}{\partial x \partial y} - (1-\nu^2)\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} &= 0, \\ \frac{\partial^2 v}{\partial y^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 u}{\partial x \partial y} - (1-\nu^2)\frac{\rho}{E}\frac{\partial^2 v}{\partial t^2} &= 0. \end{aligned} \tag{2.9}$$

This fact enables to analyze the in-plane shear vibrations indirectly by in-plane stretching.

3. Flexural vibrations of circular plate

Natural vibrations are harmonic function, i.e., $w(x, y, t) = W(x, y)\sin\omega t$, where $W(x, y)$ is vibration amplitude (mode) and ω is natural frequency.

Harmonic function is expressed in polar coordinate system as $w(r,\phi,t) = W(r,\phi)\sin\omega t$ and differential equation (2.5) for natural vibrations of circular plate takes the form

$$\Delta_r\Delta_r W_b + \omega^2\frac{J}{D}\left(1+\frac{mD}{JS}\right)\Delta_r W_b + \omega^2\frac{m}{D}\left(\frac{\omega^2 J}{S}-1\right)W_b = 0, \tag{3.1}$$

where [30]

$$\Delta_r(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r}\frac{\partial(\cdot)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial\phi^2}.$$

Eq. (3.1) can be presented in a condensed form

$$(\Delta_r + \lambda^2)(\Delta_r - \chi^2)W_b = 0, \tag{3.2}$$

where, based on the identity of (3.1) and (3.2), one finds

$$\lambda^2(\omega), \chi^2(\omega) = \frac{1}{2}\left[\sqrt{\omega^4\left(\frac{J}{D}\right)^2\left(1+\frac{mD}{JS}\right)^2 + 4\omega^2\frac{m}{D}\left(1-\omega^2\frac{J}{S}\right)} \pm \omega^2\frac{J}{D}\left(1+\frac{mD}{JS}\right)\right]. \tag{3.3}$$

In that way the fourth order Eq. (2.9) is decomposed into two second order equations, i.e., $(\Delta_r + \lambda^2)W_b = 0$ and $(\Delta_r - \chi^2)W_b = 0$. Their solution can be assumed in the form $W_b(r,\phi) = W_b(r)\sin n\phi$ that leads to

$$(\Delta_r^n + \lambda^2)W_b = 0, \qquad (\Delta_r^n - \chi^2)W_b = 0, \tag{3.4}$$

where

$$\Delta_r^n(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r}\frac{\partial(\cdot)}{\partial r} - \frac{n^2}{r^2}.$$

By substituting $\lambda = \xi/r$ and $\chi = \eta/r$, Eqs. (3.4) are transformed into Bessel's differential equation and modified Bessel's differential equation respectively

$$\begin{aligned} &\frac{d^2 W_b}{d\xi^2} + \frac{1}{\xi}\frac{dW_b}{d\xi} + \left(1-\frac{n^2}{\xi^2}\right)W_b = 0, \\ &\frac{d^2 W_b}{d\eta^2} + \frac{1}{\eta}\frac{dW_b}{d\eta} - \left(1+\frac{n^2}{\eta^2}\right)W_b = 0, \end{aligned} \tag{3.5}$$

with special functions as their solution. The total solution for plate bending deflection reads

$$W_b = W_b^{(1,2)} + W_b^{(3,4)} = C_1 J_n(\xi) + C_2 Y_n(\xi) + C_3 I_n(\eta) + C_4 K_n(\eta), \tag{3.6}$$

where $J_n(\xi)$ is Bessel function of the first kind of order n, $Y_n(\xi)$ is Bessel function of the second kind of order n, $I_n(\eta)$ is modified Bessel function of the first kind of order n and $K_n(\eta)$ is modified Bessel function of the second kind of order n.

Having a solution of differential equation (3.4), the total deflection function, according to (2.2), reads (3.6)

$$
\begin{aligned}
(3.7)\qquad W &= \Big(1-\omega^2\frac{J}{S}\Big)W_b - \frac{D}{S}\Big(\frac{d^2W_b}{dr^2}+\frac{1}{r}\frac{dW_b}{dr}-\frac{n^2}{r^2}W_b\Big)\\
&= \Big(1-\omega^2\frac{J}{S}\Big)\,[C_1J_n(\xi)+C_2Y_n(\xi)+C_3I_n(\eta)+C_4K_n(\eta)]\\
&\quad - C_1\frac{D}{S}\lambda^2\Big[\frac{d^2J_n(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{dJ_n(\xi)}{d\xi}-\frac{n^2}{\xi^2}J_n(\xi)\Big]\\
&\quad - C_2\frac{D}{S}\lambda^2\Big[\frac{d^2Y_n(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{dY_n(\xi)}{d\xi}-\frac{n^2}{\xi^2}Y_n(\xi)\Big]\\
&\quad - C_3\frac{D}{S}\chi^2\Big[\frac{d^2I_n(\eta)}{d\eta^2}+\frac{1}{\eta}\frac{dI_n(\eta)}{d\eta}-\frac{n^2}{\eta^2}I_n(\eta)\Big]\\
&\quad - C_4\frac{D}{S}\chi^2\Big[\frac{d^2K_n(\eta)}{d\eta^2}+\frac{1}{\eta}\frac{dK_n(\eta)}{d\eta}-\frac{n^2}{\eta^2}K_n(\eta)\Big].
\end{aligned}
$$

Radial cross-section rotation angle is then given as

$$
\begin{aligned}
(3.8)\qquad \varphi_r &= -\frac{dW_b}{dr}\\
&= -\Big[C_1\lambda\frac{dJ_n(\xi)}{d\xi}+C_2\lambda\frac{dY_n(\xi)}{d\xi}+C_3\chi\frac{dI_n(\eta)}{d\eta}+C_4\chi\frac{dK_n(\eta)}{d\eta}\Big].
\end{aligned}
$$

Cross-sectional forces, necessary for satisfaction of boundary conditions, according to (2.6), take the following form:

$$
\begin{aligned}
(3.9)\qquad M_r &= -D\Big[\frac{d^2W_b}{dr^2}+\nu\Big(\frac{1}{r}\frac{dW_b}{dr}-\frac{n^2}{r^2}W_b\Big)\Big]\\
&= -C_1D\lambda^2\Big\{\frac{d^2J_n(\xi)}{d\xi^2}+\nu\Big(\frac{1}{\xi}\frac{dJ_n(\xi)}{d\xi}-\frac{n^2}{\xi^2}J_n(\xi)\Big)\Big\}\\
&\quad - C_2D\lambda^2\Big\{\frac{d^2Y_n(\xi)}{d\xi^2}+\nu\Big(\frac{1}{\xi}\frac{dY_n(\xi)}{d\xi}-\frac{n^2}{\xi^2}Y_n(\xi)\Big)\Big\}\\
&\quad - C_3D\chi^2\Big\{\frac{d^2I_n(\eta)}{d\eta^2}+\nu\Big(\frac{1}{\eta}\frac{dI_n(\eta)}{d\eta}-\frac{n^2}{\eta^2}I_n(\eta)\Big)\Big\}\\
&\quad - C_4D\chi^2\Big\{\frac{d^2K_n(\eta)}{d\eta^2}+\nu\Big(\frac{1}{\eta}\frac{dK_n(\eta)}{d\eta}-\frac{n^2}{\eta^2}K_n(\eta)\Big)\Big\},
\end{aligned}
$$

$$\begin{aligned}
\bar{Q}_r = \\
&-D\left[\frac{d^3W_b}{dr^3}+\frac{1}{r}\frac{d^2W_b}{dr^2}-\frac{(2-\nu)n^2+1}{r^2}\frac{dW_b}{dr}+\frac{(3-\nu)n^2}{r^3}W_b\right]-\omega^2 J\frac{dW_b}{dr}\\
&-C_1D\lambda^3\left[\frac{d^3J_n(\xi)}{d\xi^3}+\frac{1}{\xi}\frac{d^2J_n(\xi)}{d\xi^2}-\frac{(2-\nu)n^2+1}{\xi^2}\frac{dJ_n(\xi)}{d\xi}+\frac{(3-\nu)n^2}{\xi^3}J_n(\xi)\right]\\
&-C_2D\lambda^3\left[\frac{d^3Y_n(\xi)}{d\xi^3}+\frac{1}{\xi}\frac{d^2Y_n(\xi)}{d\xi^2}-\frac{(2-\nu)n^2+1}{\xi^2}\frac{dY_n(\xi)}{d\xi}+\frac{(3-\nu)n^2}{\xi^3}Y_n(\xi)\right]\\
&-C_3D\chi^3\left[\frac{d^3I_n(\eta)}{d\eta^3}+\frac{1}{\eta}\frac{d^2I_n(\eta)}{d\eta^2}-\frac{(2-\nu)n^2+1}{\eta^2}\frac{dI_n(\eta)}{d\eta}+\frac{(3-\nu)n^2}{\eta^3}I_n(\eta)\right]\\
&-C_4D\chi^3\left[\frac{d^3K_n(\eta)}{d\eta^3}+\frac{1}{\eta}\frac{d^2K_n(\eta)}{d\eta^2}-\frac{(2-\nu)n^2+1}{\eta^2}\frac{dK_n(\eta)}{d\eta}+\frac{(3-\nu)n^2}{\eta^3}K_n(\eta)\right]\\
&-\omega^2 J\left[C_1\lambda\frac{dJ_n(\xi)}{d\xi}+C_2\lambda\frac{dY_n(\xi)}{d\xi}+C_3\chi\frac{dI_n(\eta)}{d\eta}+C_4\chi\frac{dK_n(\eta)}{d\eta}\right].
\end{aligned} \tag{3.10}$$

4. In-plane shear vibrations of circular plate

Solution of differential equations (2.7) is rather a complex problem, especially in the polar coordinate system. Therefore, the Helmholtz decomposition method is applied [31–33], assuming the rotation angles in a form

$$\vartheta_x = \frac{\partial\psi}{\partial x}+\frac{\partial\gamma}{\partial y}, \qquad \vartheta_y = \frac{\partial\psi}{\partial y}-\frac{\partial\gamma}{\partial x}, \tag{4.1}$$

where $\psi(x,y,t)$ and $\phi(x,y,t)$ are displacement potential functions. Substituting (4.1) into (3.8) and by grouping the terms of the same potential, one arrives at

$$\begin{aligned}
&D\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)-S\frac{\partial\psi}{\partial x}-J\frac{\partial^3\psi}{\partial x\partial t^2}\\
&\qquad+GI\left(\frac{\partial^3\gamma}{\partial x^2\partial y}+\frac{\partial^3\gamma}{\partial y^3}\right)-S\frac{\partial\gamma}{\partial y}-J\frac{\partial^3\gamma}{\partial y\partial t^2}=0,\\
&D\left(\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right)-S\frac{\partial\psi}{\partial y}-J\frac{\partial^3\psi}{\partial y\partial t^2}\\
&\qquad-GI\left(\frac{\partial^3\gamma}{\partial x^3}+\frac{\partial^3\gamma}{\partial x\partial y^2}\right)+S\frac{\partial\gamma}{\partial x}+J\frac{\partial^3\gamma}{\partial x\partial t^2}=0,
\end{aligned} \tag{4.2}$$

where $I = h^3/12$ is moment of inertia of unit breadth cross-section. If Eqs. (4.2) are derivated once per x and y, respectively, and summed up, and another time per y and x, respectively, and subtracted, one arrives at

$$D\Delta\psi - S\psi - J\frac{\partial^2\psi}{\partial t^2} = 0,$$
$$GI\Delta\gamma - S\gamma - J\frac{\partial^2\gamma}{\partial t^2} = 0. \tag{4.3}$$

In that way coupled equations (2.7) for angles ϑ_x and ϑ_y are decomposed into two independent equations (4.3), with two potential functions ψ and γ. After solving Eq. (4.3) shear angles are determined by (4.1) and shear moments by Eqs. (2.8) which take the following form:

$$M_{sx} = D\Big[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\gamma}{\partial x\partial y} + \nu\Big(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\gamma}{\partial x\partial y}\Big)\Big],$$
$$M_{sy} = D\Big[\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\gamma}{\partial x\partial y} + \nu\Big(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\gamma}{\partial x\partial y}\Big)\Big], \tag{4.4}$$
$$M_{sxy} = \frac{1}{2}(1-\nu)D\Big[2\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\gamma}{\partial y^2} - \frac{\partial^2\gamma}{\partial x^2}\Big].$$

Equations (4.3), derived in orthogonal coordinate system, can be transformed into polar coordinate system for in-plane shear vibration analysis of a circular plate. By taking into account at the same time that natural vibrations are harmonic, as well as variation of displacements in circular direction, i.e.,

$$\psi(r,\phi,t) = \varPsi(r)\sin n\phi\cdot\sin\omega t,$$
$$\gamma(r,\phi,t) = \varGamma(r)\cos n\phi\cdot\sin\omega t, \tag{4.5}$$

differential equations (4.3) are transformed into Bessel's equations

$$\frac{d^2\varPsi}{d\xi^2} + \frac{1}{\xi}\frac{d\varPsi}{d\xi} + \Big(1-\frac{n^2}{\xi^2}\Big)\varPsi = 0,$$
$$\frac{d^2\varGamma}{d\eta^2} + \frac{1}{\eta}\frac{d\varGamma}{d\eta} + \Big(1-\frac{n^2}{\eta^2}\Big)\varGamma = 0, \tag{4.6}$$

where $\xi = \alpha r$, $\eta = \beta r$ and

$$\alpha = \sqrt{\frac{S}{D}\Big(\omega^2\frac{J}{S} - 1\Big)} = \frac{1}{h}\sqrt{6(1-\nu)k\Big[\frac{1}{72(1-\nu)k}\Big(\frac{h}{R}\Big)^4\varOmega^2 - 1\Big]},$$
$$\beta = \sqrt{\frac{S}{GI}\Big(\omega^2\frac{J}{S} - 1\Big)} = \frac{1}{h}\sqrt{12k\Big[\frac{1}{72(1-\nu)k}\Big(\frac{h}{R}\Big)^4\varOmega^2 - 1\Big]}. \tag{4.7}$$

Solutions of Bessel's equations (4.6) are expressed with Bessel functions

$$\begin{aligned} \Psi &= A_1 J_n(\xi) + A_2 Y_n(\xi), \\ \Gamma &= B_1 J_n(\eta) + B_2 Y_n(\eta). \end{aligned} \tag{4.8}$$

Displacement functions according to (4.1) read

$$\vartheta_r = \frac{\partial \psi}{\partial r} + \frac{\partial \gamma}{r\partial \phi}, \qquad \vartheta_\phi = \frac{\partial \psi}{r\partial \phi} - \frac{\partial \gamma}{\partial r}. \tag{4.9}$$

Their amplitudes are

$$\begin{aligned} \Theta_r &= \frac{d\Psi}{dr} - \frac{n}{r}\Gamma = A_1 \alpha \frac{dJ_n(\xi)}{d\xi} + A_2 \alpha \frac{dY_n(\xi)}{d\xi} - B_1 \frac{n}{r} J_n(\eta) - B_2 \frac{n}{r} Y_n(\eta), \\ \Theta_\phi &= \frac{n}{r}\Psi - \frac{d\Gamma}{dr} = A_1 \frac{n}{r} J_n(\xi) + A_2 \frac{n}{r} Y_n(\xi) - B_1 \beta \frac{dJ_n(\eta)}{d\eta} - B_2 \beta \frac{dY_n(\eta)}{d\eta}. \end{aligned} \tag{4.10}$$

Amplitude of shear moments (4.4) can be also expressed with potentials Ψ and Γ – Eqs. (4.8).

5. Numerical examples

5.1. Flexural vibrations

Natural flexural vibrations of a circular plate are considered for three cases of boundary conditions, i.e., simply supported ($W(R) = 0$, $M_r(R) = 0$), clamped ($W(R) = 0$, $\Phi_r(R) = 0$) and free ($M_r(R) = 0$, $\bar{Q}_r(R) = 0$). Displacement and forces terms with constants C_1 and C_3 are relevant, while $C_2 = C_4 = 0$ since Bessel functions $Y_n(0) = K_n(0) = \infty$ in the center of the plate. Hence, by using Eqs. (3.7)–(3.10) for boundary conditions, the eigenvalue problem in each of three cases takes the following form:

$$\begin{aligned} &[A(\Omega)]\{C\} \\ &\quad = \begin{bmatrix} a_{11}(\Omega, \lambda_R(\Omega), \xi_R(\Omega)) & a_{12}(\Omega, \chi_R(\Omega), \eta_R(\Omega)) \\ a_{21}(\Omega, \lambda_R(\Omega), \xi_R(\Omega)) & a_{22}(\Omega, \chi_R(\Omega), \eta_R(\Omega)) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \end{aligned} \tag{5.1}$$

where

$$\Omega = \omega R^2 \sqrt{\frac{m}{D}} \tag{5.2}$$

is a frequency parameter (nondimensional frequency). Parameters of Bessel functions Eq. (3.2), can be expressed with the frequency parameter Ω Eq. (5.2). By employing Eq. (1.3) one obtains

$$\lambda^2(\Omega), \chi^2(\Omega) = \frac{1}{2R^2}\left[\sqrt{\Omega^4 \frac{1}{144}\left(\frac{h}{R}\right)^2 \left[1 - \frac{2}{(1-\nu)k}\right]^2 + 4\Omega^2} \pm \Omega^2 \frac{1}{12}\left(\frac{h}{R}\right)^2 \left[1 + \frac{2}{(1-\nu)k}\right]\right]. \tag{5.3}$$

Values of Ω are determined from the condition $\mathrm{Det}[A(\Omega)] = 0$.

Table 1. Frequency parameter $\Omega = \omega R^2 \sqrt{\rho h/D}$, simply supported circular plate, $k = 5/6$.

h/R	i/n	0	1	2	3	4	5
0.001	1	4.935	13.898	25.613	39.957	56.841	76.202
		(4.931)*	(13.880)	(25.535)	(39.748)	(56.410)	(75.435)
	2	29.720	48.478	70.116	94.547	121.699	151.514
		(29.678)	(48.341)	(69.743)	(93.744)	(120.133)	(149.125)
	3	74.155	102.772	134.294	168.669	205.843	245.767
		(73.953)	(102.305)	(133.297)	(166.832)	(202.865)	(241.234)
	4	138.315	176.795	218.194	262.472	309.590	359.508
		(137.652)	(175.702)	(216.248)	(259.133)	(304.509)	(359.508)
	5	222.206	270.552	321.822	375.986	433.014	492.874
		(220.784)	(268.414)	(318.228)			
0.1	1	4.894	13.567	24.518	37.388	51.863	67.672
		(4.902)	(13.576)	(24.492)	(37.312)	(51.726)	(67.454)
	2	28.253	44.766	62.767	81.943	102.037	122.841
		(28.513)	(45.283)	(63.507)	(82.815)	(102.903)	(123.511)
	3	66.011	88.146	110.961	134.273	157.942	181.853
		(67.154)	(89.796)	(112.911)	(136.240)	(159.601)	(182.769)
	4	113.761	139.567	165.783	191.438	217.380	243.264
		(116.207)	(142.498)	(168.418)	(193.786)	(218.606)	
	5	167.902	195.996	223.856	251.479	278.865	306.015
		(171.397)	(199.419)	(226.355)			
0.2	1	4.778	12.723	22.023	32.173	42.841	53.814
		(4.817)	(12.849)	(22.245)	(32.509)	(43.278)	(54.293)
	2	25.031	37.671	50.444	63.219	75.921	88.514
		(25.703)	(38.782)	(51.831)	(64.678)	(77.240)	(89.471)
	3	52.649	67.189	81.363	95.216	108.781	122.090
		(54.384)	(69.188)	(83.296)	(96.772)	(109.671)	(122.017)
	4	83.050	98.275	112.998	127.310	141.277	154.953
		(85.248)	(100.259)	(114.386)	(127.722)	(140.373)	
	5	114.340	129.769	144.693	159.234	170.289	
		(115.949)	(130.531)	(144.192)			

$(\cdot)^*$ – FEM

Numerical calculation is performed for ratio of plate thickness and radius $h/R = 0.001$, 0.1 and 0.2, and the shear coefficient $k = 5/6$. The obtained values of frequency parameter Ω are listed in Tables 1–3 for simply supported, clamped and free plate, respectively. Indices i and n indicate radial and circular mode number. Generally, value of Ω is decreasing with increasing plate thickness due to higher stiffness. For simply supported and clamped plate Ω is monotonically

Table 2. Frequency parameter $\Omega = \omega R^2\sqrt{\rho h/D}$, clamped circular plate, $k = 5/6$.

h/R	i/n	0	1	2	3	4	5
0.001	1	10.216	21.260	34.877	51.029	69.665	90.737
		(10.201)*	(21.209)	(34.720)	(50.675)	(69.004)	(89.646)
	2	39.771	60.828	84.581	111.019	140.104	171.797
		(39.685)	(60.603)	(84.058)	(109.987)	(138.323)	(168.996)
	3	89.102	120.076	153.810	190.296	229.507	271.413
		(88.784)	(119.436)	(152.543)	(188.067)	(226.068)	(266.401)
	4	158.179	199.045	242.708	289.162	338.388	390.358
		(157.249)	(197.575)	(240.284)	(285.285)	(332.828)	
	5	246.994	297.742	351.311	407.696	466.881	528.846
		(245.097)	(295.081)	(347.065)			
0.1	1	9.944	20.185	32.212	45.764	60.594	76.792
		(9.956)	(20.258)	(32.389)	(46.088)	(61.098)	(77.179)
	2	36.511	53.903	72.291	91.496	111.361	131.757
		(36.810)	(54.489)	(73.170)	(92.611)	(112.595)	(132.934)
	3	75.780	98.044	120.653	143.536	166.630	189.883
		(76.898)	(99.609)	(122.484)	(145.385)	(168.228)	(190.843)
	4	123.581	149.019	174.383	199.678	224.902	250.052
		(125.735)	(151.515)	(176.819)	(201.536)	(225.707)	
	5	176.880	204.360	231.526	258.421	285.075	311.512
		(179.728)	(207.023)	(233.249)			
0.2	1	9.250	17.770	26.965	36.645	46.687	57.004
		(9.312)	(18.028)	(27.540)	(37.613)	(48.044)	(58.667)
	2	30.283	42.467	54.529	66.538	78.522	90.484
		(30.902)	(43.499)	(55.956)	(68.246)	(80.328)	(92.158)
	3	56.870	70.685	84.121	97.294	110.269	123.085
		(58.330)	(72.431)	(85.969)	(99.003)	(111.564)	(123.648)
	4	85.906	100.498	114.624	128.393	141.883	
		(87.793)	(102.286)	(116.037)	(125.486)	(141.533)	
	5	116.059	130.994	145.406	159.371	172.970	
		(117.525)	(131.789)	(145.226)			

$(\cdot)^*$ – FEM

increasing with (i, n), while for free plate the monotony is related only to i (Tables 1–3).

In order to evaluate the results obtained by the modified Mindlin theory, a comparison with those determined by direct application of the Mindlin theory [12] is performed. The first four natural frequency parameters of axisymmetric modes, and $h/R = 0.001$ and 0.25 for all boundary conditions are considered, Table 4. Almost the same results are obtained.

Table 3. Frequency parameter $\Omega = \omega R^2\sqrt{\rho h/D}$, free circular plate, $k = 5/6$.

h/R	i/n	0	1	2	3	4	5
0.001	1	9.003 (8.976)*	20.474 (20.366)	5.358 (5.349)	12.439 (12.393)	21.835 (21.713)	33.495 (33.246)
	2	38.443 (38.156)	59.811 (59.170)	35.260 (34.956)	53.007 (52.344)	73.542 (72.321)	96.754 (94.739)
	3	87.749 (86.653)	118.955 (117.049)	84.365 (83.108)	111.943 (109.731)	142.427 (138.864)	175.730 (170.358)
	4	156.814 (153.947)	197.865 (193.610)	153.302 (150.117)	190.685 (185.679)	231.021 (223.668)	274.239 (264.004)
	5	245.622 (239.877)	296.524	242.025 (235.720)	289.222	339.391	392.477
0.1	1	8.869 (8.909)	19.771 (19.880)	5.318 (5.295)	12.227 (12.127)	21.188 (20.960)	31.994 (31.586)
	2	36.059 (36.530)	54.360 (55.185)	33.277 (33.413)	48.770 (48.876)	65.839 (65.831)	84.168 (83.922)
	3	76.758 (78.349)	100.129 (102.275)	74.189 (75.265)	95.173 (96.3031)	117.030 (117.942)	139.547 (139.898)
	4	126.482 (129.451)	153.097 (156.449)	124.196 (126.546)	148.753 (150.863)	173.650 (175.074)	198.774 (198.946)
	5	181.862 (171.243)	210.460	179.852 (182.971)	206.663	233.467	260.223
0.2	1	8.508 (8.716)	18.087 (18.686)	5.203 (5.195)	11.671 (11.605)	19.635 (19.469)	28.719 (28.387)
	2	31.156 (32.676)	44.610 (46.881)	29.079 (30.052)	40.835 (42.061)	53.017 (54.293)	65.432 (66.493)
	3	59.794 (63.031)	74.625 (78.364)	58.197 (60.918)	71.786 (74.593)	85.316 (87.824)	98.765 (100.584)
	4	90.360 (94.603)	105.486 (109.792)	89.115 (92.918)	103.294 (106.751)	117.203 (119.940)	130.887 (132.509)
	5	121.047 (125.484)	135.763	120.050 (124.082)	134.078	147.597	160.642

$(\cdot)^*$ – FEM

Table 4. Comparison of frequency parameter $\Omega = \omega R^2\sqrt{\rho h/D}$, for circular plate, $k = \pi^2/12$, $n = 0$.

Boundary condition	Mode number i	$h/R = 0.001$ Ref. [8]	$h/R = 0.001$ PS	$h/R = 0.25$ Ref. [8]	$h/R = 0.25$ PS
SS	1	4.935	4.935	4.696	4.696
	2	29.720	29.720	23.254	23.254
	3	74.156	74.155	46.775	46.775
	4	138.318	138.314	71.603	71.603
C	1	10.216	10.216	8.807	8.807
	2	39.771	39.771	27.253	27.253
	3	89.104	89.102	49.420	49.420
	4	158.184	158.179	73.054	73.054
F	1	9.003	9.003	8.267	8.267
	2	38.443	38.443	28.605	28.605
	3	87.750	87.748	52.584	52.584
	4	156.818	156.813	76.936	76.936

PS – present solution

Vibrations of circular plate are also analyzed by the finite element method by employing NASTRAN [34]. The obtained results for simply supported, clamped

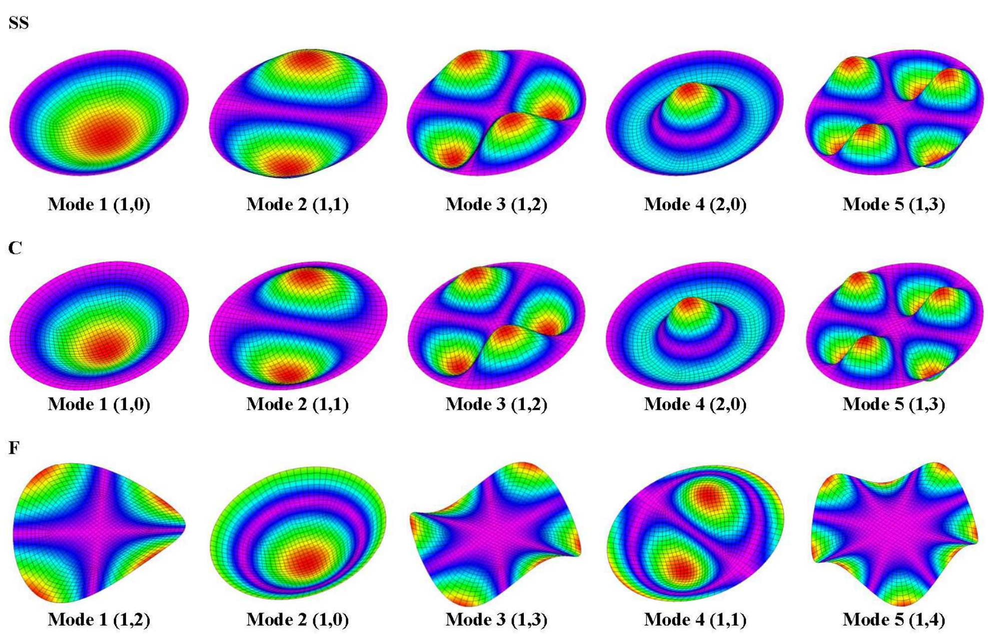

Fig. 4. The first five natural modes of simply supported (SS), clamped (C) and free (F) circular plate.

and free plate are listed in Tables 1–3 in brackets, respectively. Differences between analytical and numerical solutions are quite small and acceptable from engineering point of view. The first five natural modes of flexural vibrations for simply supported, clamped and free plate are shown in Fig. 4. It can be noted that for free plate mode (1, 0) has higher frequency than mode (1, 2), as well as mode (1, 1) than mode (1, 3).

5.2. In-plane shear vibrations

Let us consider in-plane shear vibrations of circular plate without central hole, fixed at the boundary. Boundary conditions $\Theta_r(R) = 0$ and $\Theta_\phi(R) = 0$ lead to the following system of algebraic equations at $r = R$:

$$\begin{bmatrix} \alpha \dfrac{dJ_n(\xi)}{d\xi} & -\dfrac{n}{r} J_n(\eta) \\ \dfrac{n}{r} J_n(\xi) & -\beta \dfrac{dJ_n(\eta)}{d\eta} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{5.4}$$

since $A_2 = B_2 = 0$ due to $Y_n(\xi = 0) = Y_n(\eta = 0) = \infty$. The frequency equation reads

$$\alpha\beta \frac{dJ_n(\xi)}{d\xi} \frac{dJ_n(\eta)}{d\eta} - \left(\frac{n}{r}\right)^2 J_n(\xi) J_n(\eta) = 0, \tag{5.5}$$

where α and β are specified by Eq. (4.7). Such form of the frequency equation is obtained in [33] for in-plane stretching vibrations of circular plate.

Problem of in-plane shear can be solved indirectly based on the analogy with in-plane stretching (membrane), Eqs. (2.7) and (2.9). In that case, by neglecting the stiffness of elastic support parameters, Eqs. (4.7) take the values

$$\alpha_m = \frac{1}{h\sqrt{12}} \left(\frac{h}{R}\right)^2 \Omega_m, \qquad \beta_m = \frac{1}{\sqrt{6(1-\nu)}} \frac{1}{h} \left(\frac{h}{R}\right)^2 \Omega_m. \tag{5.6}$$

A relation between natural frequencies of in-plane shear Ω and in-plane stretching Ω_m yields from (4.7)

$$\Omega = \sqrt{72(1-\nu)k\left(\frac{R}{h}\right)^4 + \Omega_m^2}. \tag{5.7}$$

Numerical calculation of fixed circular plate vibrations is performed for $h/R = 0.2$. The obtained results are shown in Table 5 for both in-plane stretching and in-plane shear. They are compared with those obtained by the finite element

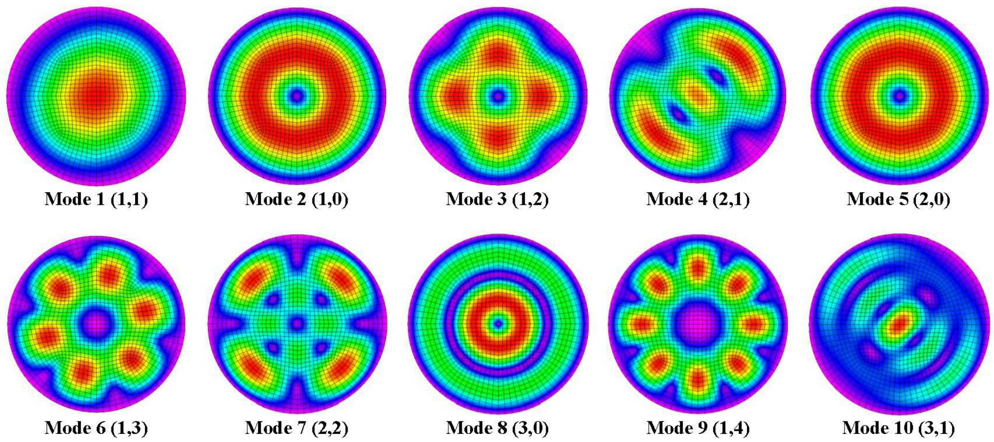

FIG. 5. The first 10 natural modes of the fixed circular membrane.

Table 5. Frequency parameter $\Omega = \omega R^2\sqrt{\rho h/D}$ for fixed circular plate, $h/R = 0.2$.

Mode	i, n	In-plane stretching		In-plane shear	
		PS	FEM	PS	FEM
1	1, 1	33.898	33.890	165.527	165.525
2	1, 0	39.264	39.204	166.708	166.694
3	1, 2	52.782	52.675	170.399	170.366
4	2, 1	55.053	54.878	171.116	171.060
5	2, 0	66.367	66.233	175.084	175.034
6	1, 3	68.642	68.345	175.959	175.843
7	2, 2	70.770	70.381	176.800	176.645
8	3, 0	71.888	71.513	177.251	177.099
9	1, 4	82.736	82.100	181.921	181.633
10	3, 1	86.740	86.107	183.776	183.478

method, NASTRAN [34]. Quite small differences can be noticed. The first 10 natural modes of in-plane stretching, valid for plate layers of in-plane shear, are shown in Fig. 5.

Problem of in-plane shear for free circular plate can be also solved analytically for relevant boundary conditions $M_{sr}(R) = 0$ and $M_{sr\phi}(R) = 0$ and Eqs. (4.4) expressed in polar coordinates. However, the expressions are rather complex, and therefore the problem is solved only numerically. The results of FEM analysis of in-plane stretching and their transform to in-plane shear by (5.7) are listed in Table 6. The corresponding natural modes are shown in Fig. 6.

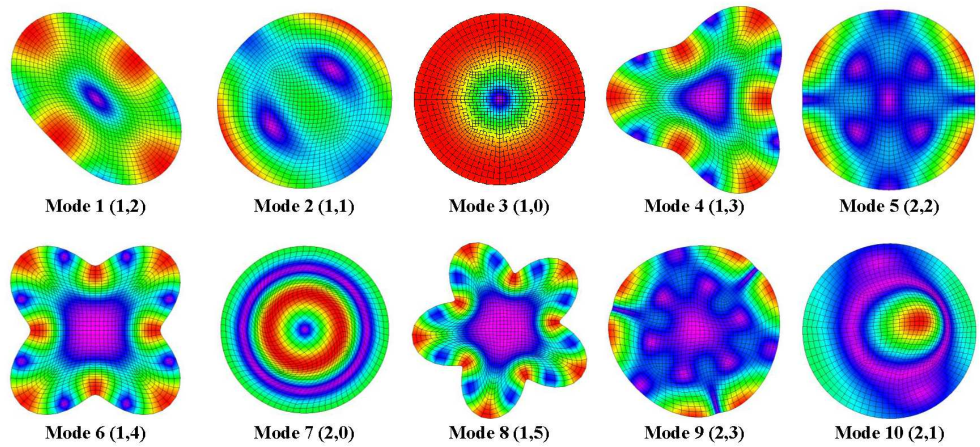

FIG. 6. The first 10 natural modes of the free circular membrane.

Table 6. Frequency parameter $\Omega = \omega R^2\sqrt{\rho h/D}$ for free circular plate, $h/R = 0.2$, FEM.

Mode	i, n	In-plane stretching	In-plane shear
1	$1, 2$	24.030	163.791
2	$1, 1$	27.975	164.416
3	$1, 0$	35.485	165.859
4	$1, 3$	36.831	166.152
5	$2, 2$	43.359	167.720
6	$1, 4$	47.829	168.931
7	$2, 0$	52.486	170.308
8	$1, 5$	58.072	172.111
9	$2, 3$	59.504	172.560
10	$2, 1$	61.041	173.136

5.3. Discussion

Comparison of the analytically determined natural flexural frequencies by the modified Mindlin theory and those obtained by the original Mindlin theory, Table 4, shows that both theories give the same results. This confirms that the former theory is a modification and not simplification of the latter one. Numerically determined natural frequencies manifest some discrepancies with respect to the analytical solutions. Discrepancies are larger in case of flexural vibrations than in case of in-plane shear vibrations. Maximum value of 5.4% is found for the free circular plate, Table 3, $h/R = 0.2$, (the third axially symmetric natural mode $(3, 0)$), and -0.8% for circular membrane (mode $(1, 4)$), Table 5. Reason is that thick finite element for flexural vibrations is more complex than one for in-plane vibrations. Accuracy of the former finite element is reduced due to shear-locking problem, which arises in transition from thick to thin plate since it is not possible to capture pure bending modes and zero shear strain constraints. There are few procedures for solving shear-locking problem, as for instance reduced integration for shear terms [35, 36], mixed formulation for hybrid finite element [37, 38], assumed natural strain [39] and discrete shear gap [40].

If the development of a thick plate finite element for flexural vibrations is based on the modified Mindlin theory, shear-locking problem is avoided. Such a four node rectangular finite element derived in [21] is used for vibration analysis of a simply supported square plate. Values of natural frequency parameter are listed in Table 7 and compared with those obtained by NASTRAN. The obtained results bound the analytically determined ones and accuracy of both FEM analyses is of the same order of magnitude.

Table 7. Frequency parameter $\Omega = \omega a^2\sqrt{\rho h/D}$ of simply supported plate, $k = 0.86667$, $h/a = 0.2$.

Mode (m, n)	(11)		(12)(21)		(22)		(13)(31)		(23)(32)	
Analytical, [21, 41]	17.506		38.385		55.586		65.719		79.476	
FEM-MMT*	17.795	1.65%	39.718	3.47%	57.512	3.46%	69.795	6.20%	83.561	5.14%
FEM-NASTRAN	16.604	−5.15%	37.414	−2.53%	51.803	−6.80%	63.932	−2.72%	72.406	−8.89%

*Modified Mindlin theory

6. Conclusion

The Mindlin thick plate theory with three differential equations of motion, related to total deflection and rotation angles, specified in Cartesian coordinate system, is used in literature as starting point for development of analytical and numerical methods for vibration analysis of rectangular and circular plates with different boundary conditions. Here, the modified Mindlin theory with decomposed flexural and in-plane shear vibrations is used for vibration analysis of circular plates, due to reason of simplicity. The fourth order differential equation of flexural vibrations is split into two second order differential equations of Bessel type. By introducing displacement potentials, in-plane shear vibration problem is also described by two Bessel differential equations. The exact vibration solutions of illustrated examples may be used as a benchmark to check the validity and accuracy of numerical methods for vibration analysis of isotropic plates as for instance it is done in this paper for FEM results. As a next step the theory can be extended to vibration problems of orthotropic and composite plates which are a subject of investigation nowadays.

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