Arch. Mech., **66**, 1, pp. 37–54, Warszawa 2014

Approximation of the Norton–Hoff plasticity model with isotropic hardening through Cosserat plasticity

P. M. BIES

Department of Mathematics and Information Sciences
Warsaw University of Technology
Koszykowa 75, 00-661 Warsaw, Poland
e-mail: biesp@mini.pw.edu.pl

In this paper the regularizing properties of Cosserat elasto-plastic models in a geometrically linear setting are investigated. For vanishing Cosserat effects it is shown that the Norton–Hoff model with isotropic hardening is approximated by the model with microrotations.

Key words: Cosserat model, Norton–Hoff model, isotropic hardening, elasto-plastic deformations.

1. Introduction

We investigate some regularizing properties of Cosserat elasto-plastic models. In general, this model was introduced by Cosserat brothers in [1]. Different cases of such a model were presented in the introduction of [2]. The infinitesimal elastic and elasto-plastic Cosserat models were introduced in the further part of the paper [2]. The purely elastic model can be obtained by dividing the macroscopic displacement gradient ∇u into infinitesimal microrotation and an infinitesimal non-symmetric micropolar stretch tensor $\bar{e} = \nabla u - A$. The theory of this model in the elastic case is then obtained from a variational principle.

The elasto-plastic case is obtained as an extension of the elastic model. This extension is itself non-dissipative. The basic idea of it is dividing the total micropolar stretch into elastic and plastic part and assuming that microrotational effects remain purely elastic. For more details see [2].

The Norton–Hoff model is an issue from theory of elasto-plastic deformations. Some special constitutive equation in this model is studied. Norton described this constitutive flow in [3]. A mathematical analysis of this model can be found in [4, 5].

The Norton–Hoff model with isotropic hardening is a Norton–Hoff model with one additional scalar function, which is called isotropic hardening. This model is well-posed. It can be shown by using an approximation procedure, which was proposed in [6]. Several models in the theory of inelastic deformations are listed in [7].

In this paper an elasto-plastic Cosserat model connected with the Norton–Hoff model with isotropic hardening is studied. The main goal of this article is to show that if the Cosserat effect vanishes, this model approximates Norton–Hoff model with isotropic hardening. The Prandtl-Reuss model and similar issues are investigated in [8].

The Cosserat elasto-plastic model is also studied in [9, 10]. The paper [11] is devoted to the study of dynamic Cosserat models. See also the article [12], where a poroplasticity model with Cosserat effects is investigated. The linear elastic Cosserat model is reconsidered in [13–17].

2. Formulation of the problem and the main result

We shall use the notation specified in Sec. 3 in this section. Let us denote by $\Omega \subset \mathbb{R}^3$ a bounded open set with smooth boundary $\partial\Omega$ and let $T > 0$. In order to describe a quasi-static deformation of an inelastic body with microrotations and with isotropic hardening we have to find the displacement vector $u^{\mu_c}\colon \Omega \times [0,\ T] \to \mathbb{R}^3$, the microrotation matrix $A^{\mu_c}\colon \Omega \times [0,\ T] \to \mathfrak{so}(3)$, the plastic deformation tensor $\varepsilon_p^{\mu_c}\colon \Omega \times [0,\ T] \to \mathrm{Sym}(3)$ and the isotropic hardening $y^{\mu_c}\colon \Omega \times [0,\ T] \to \mathbb{R}$ such that

$$\operatorname{div} \sigma^{\mu_c} = -f, \tag{2.1a}$$

$$\sigma^{\mu_c} = 2\mu(\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c}) + 2\mu_c(\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c}) + \lambda \operatorname{tr}[\varepsilon^{\mu_c}] \cdot \mathbb{I}, \tag{2.1b}$$

$$-l_c \Delta \operatorname{axl}(A^{\mu_c}) = \mu_c \operatorname{axl}(\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c}), \tag{2.1c}$$

$$\dot{\varepsilon}_p^{\mu_c} = F\left(T_E^{\mu_c}, -\frac{\gamma}{\alpha} y^{\mu_c}\right), \quad \dot{y}^{\mu_c} = g\left(T_E^{\mu_c}, -\frac{\gamma}{\alpha} y^{\mu_c}\right), \tag{2.1d}$$

$$T_E^{\mu_c} = 2\mu(\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c}), \tag{2.1e}$$

$$u^{\mu_c}|_{\partial\Omega} = u_d, \quad A^{\mu_c}|_{\partial\Omega} = A_d, \quad \varepsilon_p^{\mu_c}(0) = \varepsilon_p^0, \quad y^{\mu_c}(0) = y^0. \tag{2.1f}$$

Here, $\varepsilon^{\mu_c} = \operatorname{sym}(\nabla u^{\mu_c})$ denotes the infinitesimal elastic strain tensor. The numbers $\lambda,\ \mu$ are the positive Lame constants, μ_c is the Cosserat couple modulus and $l_c = \mu L_c^2 > 0$ is a material parameter, where L_c with the units of length defines an internal length scale. Constants γ and α are positive and $r > 1$. The functions $u_d,\ A_d$ are given Dirichlet boundary data, ε_p^0 and y^0 are given initial values and function f describes external body forces acting on the material. Functions F and g are given by $F(A, x) = (|\mathrm{dev}A| + \alpha x - k)_+^r \frac{\mathrm{dev}A}{|\mathrm{dev}A|}$ and $g(A, x) = \alpha(|\mathrm{dev}A| + \alpha x - k)_+^r$ for $(A,\ x) \in \mathrm{Sym}(3) \times \mathbb{R}$. It is easy to see that $(F,\ g)$ is a monotone field on $\mathrm{Sym}(3) \times \mathbb{R}$.

In this paper we want to investigate what will happen with the solution of (2.1), when $\mu_c \to 0^+$. It means that we will study the limit procedure of vanishing Cosserat effects. We predict then that $u^{\mu_c}, \varepsilon_p^{\mu_c}, y^{\mu_c}$ converge to the solution of

$$\begin{aligned} \operatorname{div} \sigma &= -f, \\ \sigma &= 2\mu(\varepsilon - \varepsilon_p) + \lambda \operatorname{tr}[\varepsilon] \cdot \mathbb{I}, \\ \dot{\varepsilon_p} &= (\operatorname{dev} T_E - \gamma y - k)_+^r \frac{\operatorname{dev} T_E}{|\operatorname{dev} T_E|}, \\ \dot{y} &= \alpha(\operatorname{dev} T_E - \gamma y - k)_+^r, \\ T_E &= 2\mu(\varepsilon - \varepsilon_p), \\ u|_{\partial\Omega} &= u_d, \quad \varepsilon_p(0) = \varepsilon_p^0, \quad y(0) = y^0, \end{aligned} \tag{2.2}$$

which is the Norton–Hoff model with isotropic hardening and A^{μ_c} converges to the solution of

$$\begin{aligned} -l_c \Delta \operatorname{axl}(A) &= 0, \\ A|_{\partial\Omega} &= A_d, \end{aligned} \tag{2.3}$$

which is the Laplace equation. The following theorem is a mathematical formulation of the last sentences. This theorem is the main result of this study.

THEOREM 2.1. *Let us assume that*

$$f \in W^{2,\infty}([0,T], L^2(\Omega, \mathbb{R}^3)), \quad u_d \in W^{3,\infty}([0,T], H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)),$$
$$A_d \in W^{3,\infty}([0,T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3))), \quad \varepsilon_p^0 \in L^2(\Omega, \mathrm{Sym}(3)), \quad y^0 \in L^2(\Omega),$$

and let u^{μ_c}, $\varepsilon_p^{\mu_c}$, y^{μ_c}, A^{μ_c} be the solutions of (2.1), $F(2\mu(\varepsilon(u^{\mu_c}(0)) - \varepsilon_p^0), -\frac{\gamma}{\alpha} y^0) \in L^2(\Omega, \mathrm{Sym}(3))$ *and* $g(2\mu(\varepsilon(u^{\mu_c}(0)) - \varepsilon_p^0), -\frac{\gamma}{\alpha} y) \in L^2(\Omega)$, *and that sequence* $(F(T_E^{\mu_c}(0), -\frac{\gamma}{\alpha} y^0), g(T_E^{\mu_c}(0), -\frac{\gamma}{\alpha} y^0))$ *is bounded in* $L^2(\Omega, \mathrm{Sym}(3)) \times L^2(\Omega)$, *then with* $\mu_c \to 0^+$

$$\begin{aligned} u^{\mu_c} &\overset{*}{\rightharpoonup} u && \text{in } W^{1,\infty}((0,T), H^1(\Omega, \mathbb{R}^3)), \\ A^{\mu_c} &\overset{*}{\rightharpoonup} A && \text{in } W^{1,\infty}((0,T), H^1(\Omega, \mathfrak{so}(3))), \\ \varepsilon_p^{\mu_c} &\overset{*}{\rightharpoonup} \varepsilon_p && \text{in } W^{1,\infty}((0,T), L^2(\Omega, \mathrm{Sym}(3))), \\ y^{\mu_c} &\overset{*}{\rightharpoonup} y && \text{in } W^{1,\infty}((0,T), L^2(\Omega)), \end{aligned}$$

where u, y, ε_p are solutions of (2.2), *and A is a solution of* (2.3).

It is easy to see that model (2.1) is coercive (similar issue is studied in [2]). We lose coerciveness, when Cosserat effects vanish. In [8] the authors investigate the limit procedure $\mu_c \to 0^+$ in conjunction with Cosserat model and Prandtl–Reuss model. They show that when μ_c vanishes, then $(u^{\mu_c}, \varepsilon_p^{\mu_c})$ converges in a measure sense to the solutions of the Prandtl–Reuss model. It can be shown that, if one does use a Cosserat model together with the Norton–Hoff model, then $(\varepsilon^{\mu_c}, \varepsilon_p^{\mu_c})$ does converge in the weak-* sense in $W^{1,\infty}((0,T), L^p(\Omega, \mathrm{Sym}(3)))$, when $\mu_c \to 0^+$, where $p \neq 2$. Our motivation to study Norton–Hoff with isotropic hardening model together with Cosserat model was to keep $(\varepsilon^{\mu_c}, \varepsilon_p^{\mu_c})$ in $W^{1,\infty}((0,T), L^2(\Omega, \mathrm{Sym}(3)))$ during the limit procedure.

3. Preliminaries and notations

In this section we shall recall some basic facts, which are used in this paper and make some remarks about the notation.

We denote by $\mathbb{R}^{3\times 3}$ the set of real 3×3 matrices. The sets $\mathrm{Sym}(3)$ and $\mathfrak{so}(3)$ are defined as follows:

$$\mathrm{Sym}(3) = \{A \in \mathbb{R}^{3\times 3} \colon A^T = A\} \qquad \text{and} \qquad \mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \colon A^T = -A\}.$$

For $A \in \mathbb{R}^{3\times 3}$ we define the symmetric part of a matrix as

$$\mathrm{sym}(A) = \frac{1}{2}(A + A^T),$$

and the skew-symmetric part of a matrix as

$$\mathrm{skew}(A) = \frac{1}{2}(A - A^T).$$

Now, it is easy to see that $A = \mathrm{sym}(A) + \mathrm{skew}(A)$, $\mathrm{sym}(A) \in \mathrm{Sym}(3)$ and $\mathrm{skew}(A) \in \mathfrak{so}(3)$. Let $B \in \mathfrak{so}(3)$, then there exist real numbers a, b, c such that

$$B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

We define $\mathrm{axl}(B) = (-c, b, -a)$. Let $A \in \mathrm{Sym}(3)$. We define the deviator of A as

$$\operatorname{dev} A = A - \frac{1}{3}\operatorname{tr}[A]\mathbb{I},$$

where $\operatorname{tr}[A]$ is the trace of A and it is defined by $\operatorname{tr}[A] = \sum_{i=1}^{3} A(i,i)$, $\mathbb{I}$ is the identity matrix. It is easy to see that $\operatorname{dev} A$ is a projection of A onto symmetric matrices with trace equal zero.

Now, let $\Omega \subset \mathbb{R}^3$ be an open set. Let us introduce the space $L^2_{\mathrm{div}}(\Omega)$:

$$L^2_{\mathrm{div}}(\Omega) = \{u \in L^2(\Omega;\mathbb{R}^3) \colon \operatorname{div} u \in L^2(\Omega)\},$$

where div means weak divergence. In this space the norm is defined as follows:

$$\|u\|_{L^2_{\mathrm{div}}(\Omega)} = \|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)}.$$

The subsequent fact holds:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with boundary of C^1-class. Then, there exists bounded linear operator $\gamma\colon L^2_{\mathrm{div}}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ such that*

(i) $$\|\gamma u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{L^2_{\mathrm{div}}(\Omega)} \qquad \text{for } u \in L^2_{\mathrm{div}(\Omega)},$$

(ii) $$\gamma u = u \cdot n|_{\partial\Omega} \qquad \text{for } u \in C(\bar{\Omega}),$$

where n denotes the exterior unit normal vector on $\partial\Omega$.

Moreover, if $w \in H^1(\Omega)$, such that $w|_{\partial\Omega} = \phi$ (in the sense of traces, see [18]*), then for $u \in L^2_{\mathrm{div}}(\Omega)$ the following equality is satisfied:*

$$\langle \gamma u, \phi\rangle = \int_\Omega u \cdot \nabla w\, dx + \int_\Omega \operatorname{div} u\, w dx. \tag{3.1}$$

The condition (ii) from Theorem 3.1 and (3.1) justify the notation γu for $u \in L^2_{\mathrm{div}}(\Omega)$ as $u \cdot n$ and $\langle \gamma u, \phi\rangle$ for $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ as $\int_{\partial\Omega} u \cdot n\phi\, dS$. The details and proof of Theorem 3.1 are given in [19].

Basic results of functional analysis are used there and can be found in [20] and [21].

4. Energy and existence in each approximation step

Let us see that initial values of ε_p, y are explicit given by (2.1f), but initial values of u^{μ_c} and of A^{μ_c} seem to be unknown. However, let us introduce $t = 0$ to (2.1a), (2.1b), (2.1c) and to (2.1f):

$$\begin{aligned}
\operatorname{div}\sigma^{\mu_c}(0) &= -f(0),\\
\sigma^{\mu_c}(0) &= 2\mu(\varepsilon^{\mu_c}(0) - \varepsilon_p^0) + 2\mu_c(\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0))\\
&\qquad + \lambda\operatorname{tr}[\varepsilon^{\mu_c}(0)]\cdot\mathbb{I},\\
-l_c\Delta\operatorname{axl}(A^{\mu_c}(0)) &= \mu_c\ \operatorname{axl}(\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0)),\\
u(0) &= u_d(0), \qquad A(0) = A_d(0).
\end{aligned} \tag{4.1}$$

The equation above has unique solutions $u(0)$ and $A(0)$, which follow from the Lax–Milgram Theorem.

Next, we will present an existence and uniqueness result for the system (2.1). A proof of this result is quite similar to the one, which is presented in [2], where a result without isotropic hardening is obtained. The well-posedness of the model with hardening is commented in [8].

Theorem 4.1 (Existence and uniqueness result). *Let us assume that*

$$f \in W^{2,\infty}([0,T], L^2(\Omega,\mathbb{R}^3)), \qquad u_d \in W^{3,\infty}([0,T], H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)),$$
$$A_d \in W^{3,\infty}([0,T], H^{\frac{3}{2}}(\partial\Omega,\mathfrak{so}(3))), \qquad \varepsilon_p^0 \in L^2(\Omega,\mathrm{Sym}(3)), y^0 \in L^2(\Omega),$$

$F(2\mu(\varepsilon(u^{\mu_c}(0)) - \varepsilon_p^0), -\frac{\gamma}{\alpha}y^0) \in L^2(\Omega,\mathrm{Sym}(3))$ *and* $g(2\mu(\varepsilon(u^{\mu_c}(0)) - \varepsilon_p^0), -\frac{\gamma}{\alpha}y) \in L^2(\Omega)$, *where* $u^{\mu_c}(0)$ *and* $A^{\mu_c}(0)$ *are defined by the system* (4.1), *then there exists unique weak solution of* (2.1) *such that*

$$u^{\mu_c} \in W^{1,\infty}([0,T], H^1(\Omega,\mathbb{R}^3)), \qquad A^{\mu_c} \in W^{1,\infty}([0,T], H^2(\Omega,\mathfrak{so}(3)))$$
$$\varepsilon_p^{\mu_c} \in W^{1,\infty}([0,T], L^2(\Omega,\mathrm{Sym}(3))), \qquad y^{\mu_c} \in W^{1,\infty}([0,T], L^2(\Omega)).$$

We will use the physical structure of the problem (2.2) in this paper, since it turns out to be very useful in the proof of weak convergence. The energy of the system (2.1) is given by

$$\mathcal{E}^{\mu_c}(u,\varepsilon,\varepsilon_p,A,y)(t) = \int_\Omega \mu|\varepsilon-\varepsilon_p|^2 + \frac{1}{2}\lambda\,\mathrm{tr}[\varepsilon]^2 + \mu_c|\mathrm{skew}(\nabla u) - A|^2 + 2l_c|\nabla\,\mathrm{axl}(A)|^2 + \frac{1}{2}\frac{\gamma}{\alpha}y^2\,dx.$$

We need one more formula for the energy to complete the proof of our main theorem, which will be essential in a proof of strong convergence of stresses. The sum of energies of problems (2.2) and (2.3) without Cosserat effects is defined by

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)(t) = \int_\Omega \mu|\varepsilon-\varepsilon_p|^2 + \frac{1}{2}\lambda\,\mathrm{tr}[\varepsilon]^2 + 2l_c|\nabla\,\mathrm{axl}(A)|^2 + \frac{1}{2}\frac{\gamma}{\alpha}y^2\,dx. \tag{4.2}$$

Note that it does not depend on μ_c.

5. The proof of Theorem 2.1

We prove Theorem 2.1 in this section. The proof is based on three lemmas.

5.1. Sequence of initial values

We see that $u^{\mu_c}(0)$ and $A^{\mu_c}(0)$ are given by (4.1). We shall also notice that they possibly do depend on μ_c, so the first step in the analysis of our problem is to study how they depend on μ_c.

Lemma 5.1. *Let us assume that the assumptions of Theorem* 2.1 *are satisfied for data and for initial values, then*

$$u^{\mu_{c1}}(0) - u^{\mu_{c2}}(0) \to 0 \qquad in\ H^1(\Omega, \mathbb{R}^3),$$
$$A^{\mu_{c1}}(0) - A^{\mu_{c2}}(0) \to 0 \qquad in\ H^1(\Omega, \mathfrak{so}(3)),$$

when $\mu_{c1}, \mu_{c2} \to 0^+$.

P r o o f. First of all we shall prove some auxiliary inequality for $A^{\mu_c}(0)$ and $u^{\mu_c}(0)$.

Let us put $t = 0$ in (2.1a), (2.1b) and (2.1c). We can multiply equation (2.1a) by $u^{\mu_c}(0)$ and multiply (2.1c) by $\operatorname{axl}(A^{\mu_c}(0))$ and by number 4, and then add them each other and integrate:

$$\int_\Omega -\operatorname{div}\sigma^{\mu_c}(0)\cdot u^{\mu_c}(0) - 2l_c\Delta A^{\mu_c}(0)\cdot A^{\mu_c}(0)$$
$$- 2\mu_c(\operatorname{skew}(\nabla(u^{\mu_c}(0))) - A^{\mu_c}(0))\cdot A^{\mu_c}(0)\,dx = \int_\Omega f(0)\cdot u^{\mu_c}(0)\,dx.$$

When integrating by parts, we get

$$\begin{aligned}\int_\Omega f(0)\cdot u^{\mu_c}(0)\,dx &= \int_\Omega \sigma^{\mu_c}(0)\cdot\nabla u^{\mu_c}(0)dx - \int_{\partial\Omega}(\sigma^{\mu_c}(0)\cdot n)\cdot u_d(0)\,dS\\ &\quad + \int_\Omega 2l_c|\nabla A^{\mu_c}(0)|^2dx - 2l_c\int_{\partial\Omega}(\nabla A^{\mu_c}(0)\cdot n)\cdot A_d(0)\,dS\\ &\quad - \int_\Omega 2\mu_c(\operatorname{skew}(\nabla(u^{\mu_c}(0))) - A^{\mu_c}(0))\cdot A^{\mu_c}(0)\,dx\\ &= I_1 - I_2 + I_3 - 2l_cI_4 + I_5.\end{aligned}$$

We estimate

$$\begin{aligned}I_1 &= \int_\Omega 2\mu|\varepsilon^{\mu_c}(0)|^2 + \lambda(\operatorname{div}(u^{\mu_c}(0)))^2\,dx\\ &\quad + \int_\Omega 2\mu_c(\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0))\cdot\nabla u^{\mu_c}(0)\,dx - 2\mu\int_\Omega \varepsilon^{\mu_c}(0)\cdot\varepsilon_p^0\,dx\\ &\geq \int_\Omega 2\mu|\varepsilon^{\mu_c}(0)|^2 + \int_\Omega 2\mu_c(\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0))\cdot\nabla u^{\mu_c}(0)\,dx\\ &\quad - 2\mu\int_\Omega \varepsilon^{\mu_c}(0)\cdot\varepsilon_p^0\,dx.\end{aligned}$$

The first equality in the above inequality follows from Eq. (2.1b). Furthermore, we have

$$I_2 \leq C(\|\sigma^{\mu_c}(0)\|_{L^2(\Omega)} + \|\operatorname{div}\sigma^{\mu_c}(0)\|_{L^2(\Omega)})\|u_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega)} = C(\|\sigma^{\mu_c}(0)\|_{L^2(\Omega)} + \|f(0)\|_{L^2(\Omega)})\|u_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega}. \tag{5.1}$$

We can estimate $\|\sigma^{\mu_c}(0)\|_{L^2(\Omega)}$ as follows:

$$\|\sigma^{\mu_c}(0)\|_{L^2(\Omega)} \leq C(\|\varepsilon^{\mu_c}(0)\|_{L^2(\Omega)} + \|\varepsilon_p^0\|_{L^2(\Omega)} + \mu_c\|\nabla u^{\mu_c}(0)\|_{L^2(\Omega)} + \mu_c\|A^{\mu_c}(0)\|_{L^2(\Omega)}). \tag{5.2}$$

We insert (5.2) into (5.1) to finally get

$$I_2 \leq C(\|f(0)\|_{L^2(\Omega)} + \|\varepsilon^{\mu_c}(0)\|_{L^2(\Omega)} + \|\varepsilon_p^0\|_{L^2(\Omega)} + \mu_c\|\nabla u^{\mu_c}(0)\|_{L^2(\Omega)} + \mu_c\|A^{\mu_c}(0)\|_{L^2(\Omega)})\|u_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Next, we estimate the fourth integral

$$\begin{aligned} I_4 &\leq C(\|\nabla A^{\mu_c}(0)\|_{L^2(\Omega)} + \|\Delta A^{\mu_c}(0)\|_{L^2(\Omega)})\|A_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\overset{(2.1c)}{\leq} C(\|\nabla A^{\mu_c}(0)\|_{L^2(\Omega)} + \mu_c\|\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0)\|_{L^2(\Omega)})\|A_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega)}. \end{aligned}$$

Combining all these estimates, we obtain

$$\begin{aligned} &2\mu\|\varepsilon^{\mu_c}(0)\|^2_{L^2(\Omega)} + 2l_c\|\nabla A^{\mu_c}(0)\|^2_{L^2(\Omega)} + 2\mu_c\|\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0)\|^2_{L^2(\Omega)} \\ &\quad - 2\mu\int_\Omega \varepsilon^{\mu_c}(0)\cdot\varepsilon_p^0\,dx - C(\|f(0)\|_{L^2(\Omega)} + \|\varepsilon^{\mu_c}(0)\|_{L^2(\Omega)} \\ &\quad + \|\varepsilon_p^0\|_{L^2(\Omega)} + \mu_c\|\nabla u^{\mu_c}(0)\|_{L^2(\Omega)} + \mu_c\|A^{\mu_c}(0)\|_{L^2(\Omega)})\|u_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\quad - C(\|\nabla A^{\mu_c}(0)\|_{L^2(\Omega)} + \mu_c\|\operatorname{skew}(\nabla u^{\mu_c}(0)) - A^{\mu_c}(0)\|_{L^2(\Omega)})\|A_d(0)\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\qquad \leq \int_\Omega f(0)\cdot u^{\mu_c}(0)\,dx. \end{aligned}$$

Consequently,

$$\begin{aligned} 2\mu\|\varepsilon^{\mu_c}(0)\|^2_{L^2(\Omega)} &+ 2l_c\|\nabla A^{\mu_c}(0)\|^2_{L^2(\Omega)} \\ &\leq \int_\Omega f(0)\cdot u^{\mu_c}(0)dx + 2\mu\int_\Omega \varepsilon^{\mu_c}(0)\cdot\varepsilon_p^0\,dx \\ &\quad + C(\|\varepsilon^{\mu_c}(0)\|_{L^2(\Omega)} + \|\varepsilon_p^0\|_{L^2(\Omega)} + \mu_c\|\nabla u^{\mu_c}(0)\|_{L^2(\Omega)} \\ &\quad + \mu_c\|A^{\mu_c}(0)\|_{L^2(\Omega)} + \|\nabla A^{\mu_c}(0)\|_{L^2(\Omega)} + 1). \end{aligned}$$

By virtue of Young's inequality we have

$$\|u^{\mu_c}(0)\|^2_{H^1(\Omega)} + \|A^{\mu_c}(0)\|^2_{H^1(\Omega)} \le C(1 + (1+\mu_c)(\|u^{\mu_c}(0)\|_{H^1(\Omega)} + \|A^{\mu_c}(0)\|_{H^1(\Omega)})) \,,$$

and

$$\|u^{\mu_c}(0)\|^2_{H^1(\Omega)} + \|A^{\mu_c}(0)\|^2_{H^1(\Omega)} \le C(1+\mu_c)^2. \tag{5.3}$$

Now, we can prove our thesis. From (2.1), for $t = 0$, we obtain the following equation:

$$\begin{aligned}
&\int_\Omega 2\mu|\varepsilon^{\mu_{c_1}}(0) - \varepsilon^{\mu_{c_2}}(0)|^2 dx + \int_\Omega \lambda(\mathrm{div}(u^{\mu_{c_1}}(0) - u^{\mu_{c_2}}(0)))^2\, dx \\
&\quad + \int_\Omega 2(\mu_{c_1}(\mathrm{skew}(\nabla u^{\mu_{c_1}}(0)) - A^{\mu_{c_1}}(0)) - \mu_{c_2}(\,\mathrm{skew}(\nabla u^{\mu_{c_2}}(0)) - A^{\mu_{c_2}}(0))) \\
&\qquad\qquad \cdot (\nabla u^{\mu_{c_1}}(0) - \nabla u^{\mu_{c_2}}(0)) dx \\
&\quad + \int_\Omega 2l_c|\nabla A^{\mu_{c_1}}(0) - \nabla A^{\mu_{c_2}}(0)|^2\, dx \\
&\quad - \int_\Omega 2(\mu_{c_1}(\mathrm{skew}(\nabla u^{\mu_{c_1}}(0)) - A^{\mu_{c_1}}(0)) - \mu_{c_2}(\,\mathrm{skew}(\nabla u^{\mu_{c_2}}(0)) - A^{\mu_{c_2}}(0))) \\
&\qquad\qquad \cdot (A^{\mu_{c_1}}(0) - A^{\mu_{c_2}}(0))\, dx = 0.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\int_\Omega 2\mu|\varepsilon^{\mu_{c_1}}(0) - \varepsilon^{\mu_{c_2}}(0)|^2 dx + \int_\Omega \lambda(\mathrm{div}(u^{\mu_{c_1}}(0) - u^{\mu_{c_2}}(0)))^2\, dx \\
&\quad + \int_\Omega 2(\mu_{c_1}(\mathrm{skew}(\nabla u^{\mu_{c_1}}(0)) - A^{\mu_{c_1}}(0)) - \mu_{c_2}(\mathrm{skew}(\nabla u^{\mu_{c_2}}(0)) - A^{\mu_{c_2}}(0))) \\
&\qquad \cdot (\mathrm{skew}(\nabla u^{\mu_{c_1}}(0)) - \mathrm{skew}(\nabla u^{\mu_{c_2}}(0)) - (A^{\mu_{c_1}}(0) - A^{\mu_{c_2}}(0)))\, dx \\
&\qquad\qquad + \int_\Omega 2l_c|\nabla A^{\mu_{c_1}}(0) - \nabla A^{\mu_{c_2}}(0)|^2\, dx = 0.
\end{aligned}$$

Korn and Poincaré's inequality provides the bound

$$
\begin{aligned}
&\|u^{\mu_{c_1}}(0)-u^{\mu_{c_2}}(0)\|^2_{H^1(\Omega)}+\|A^{\mu_{c_1}}(0)-A^{\mu_{c_2}}\|^2_{H^1(\Omega)}\\
&\le -C\int_\Omega 2(\mu_{c_1}(\operatorname{skew}(\nabla u^{\mu_{c_1}}(0))-A^{\mu_{c_1}}(0)))-\mu_{c_2}(\operatorname{skew}(\nabla u^{\mu_{c_2}}(0))-A^{\mu_{c_2}}(0)))\\
&\qquad\qquad\cdot(\operatorname{skew}(\nabla u^{\mu_{c_1}}(0)-\nabla u^{\mu_{c_2}}(0))-(A^{\mu_{c_1}}(0)-A^{\mu_{c_2}}(0)))\,dx\\
&\le(\mu_{c_1}+\mu_{c_2})\int_\Omega(\operatorname{skew}(\nabla u^{\mu_{c_1}}(0))-A^{\mu_{c_1}}(0))\cdot(\operatorname{skew}(\nabla u^{\mu_{c_2}}(0))-A^{\mu_{c_2}}(0))dx.
\end{aligned}
$$

Applying (5.3) to the last inequality, we finish the proof of the lemma. ☐

5.2. Energy estimates

The next step in the proof of approximation is to get estimates for the time derivatives of the approximate sequence.

Lemma 5.2 (Energy estimate for time derivatives). *Let us suppose that the requirements of Theorem 2.1 are given, then there exists a constant C, such that the following inequality holds:*

$$\mathcal{E}^{\mu_c}(\dot{u^{\mu_c}},\dot{\varepsilon^{\mu_c}},\dot{\varepsilon_p^{\mu_c}},\dot{A^{\mu_c}},\dot{y^{\mu_c}})(t)\le C\qquad\textit{for all } 0\le t\le T$$

for all $\mu_c>0$.

P r o o f. For $h>0$ let us denote by $({u^{\mu_c}}_h(t),{\varepsilon^{\mu_c}}_h(t),{\varepsilon_p^{\mu_c}}_h(t),{A^{\mu_c}}_h(t),{y^{\mu_c}}_h(t))$ the shifted functions $(u^{\mu_c}(t{+}h),\varepsilon^{\mu_c}(t{+}h),\varepsilon_p^{\mu_c}(t{+}h),A^{\mu_c}(t{+}h))$ and calculate the energy evaluated on the differences $({u^{\mu_c}}_h-u^{\mu_c},{\varepsilon^{\mu_c}}_h-\varepsilon^{\mu_c},{\varepsilon_p^{\mu_c}}_h-\varepsilon_p^{\mu_c},{A^{\mu_c}}_h-A^{\mu_c})$. By calculating the time derivative of the energy, we get

$$
\begin{aligned}
(5.4)\qquad &\dot{\mathcal{E}}^{\mu_c}({u^{\mu_c}}_h-u^{\mu_c},{\varepsilon^{\mu_c}}_h-\varepsilon^{\mu_c},{\varepsilon_p^{\mu_c}}_h-\varepsilon_p^{\mu_c},{A^{\mu_c}}_h-A^{\mu_c},{y^{\mu_c}}_h-y^{\mu_c})(t)\\
&=\int_\Omega 2\mu({\varepsilon^{\mu_c}}_h-\varepsilon^{\mu_c}-({\varepsilon_p^{\mu_c}}_h-\varepsilon_p^{\mu_c}))\cdot(\dot{\varepsilon^{\mu_c}}_h-\dot{\varepsilon^{\mu_c}}-(\dot{\varepsilon_p^{\mu_c}}_h-\dot{\varepsilon_p^{\mu_c}}))\,dx\\
&\quad+\int_\Omega 2\mu_c(\operatorname{skew}(\nabla {u^{\mu_c}}_h-\nabla u^{\mu_c})-({A^{\mu_c}}_h-A^{\mu_c}))\\
&\quad\cdot(\operatorname{skew}(\nabla \dot{u^{\mu_c}}_h-\nabla\dot{u^{\mu_c}})-(\dot{A^{\mu_c}}_h-\dot{A^{\mu_c}}))\,dx\\
&\quad+\int_\Omega 4l_c\nabla\operatorname{axl}({A^{\mu_c}}_h-A^{\mu_c})\cdot\nabla\operatorname{axl}(\dot{A^{\mu_c}}_h-\dot{A^{\mu_c}})\,dx\\
&\quad+\int_\Omega\frac{\gamma}{\alpha}({y^{\mu_c}}_h-y^{\mu_c})\cdot(\dot{y^{\mu_c}}_h-\dot{y^{\mu_c}})+\lambda\operatorname{tr}[{\varepsilon^{\mu_c}}_h-\varepsilon^{\mu_c}]\operatorname{tr}[\dot{\varepsilon^{\mu_c}}_h-\dot{\varepsilon^{\mu_c}}]\,dx
\end{aligned}
$$

$$= \int_{\Omega} -(T^{\mu_c}_{Eh} - T^{\mu_c}_{E}) \cdot (\dot{\varepsilon_p^{\mu_c}}_h - \dot{\varepsilon_p^{\mu_c}}) + \frac{\gamma}{\alpha}(y^{\mu_c}{}_h - y^{\mu_c}) \cdot (\dot{y^{\mu_c}}_h - \dot{y^{\mu_c}})\, dx$$
$$+ \int_{\Omega} (\sigma^{\mu_c}_h - \sigma^{\mu_c}) \cdot (\nabla \dot{u^{\mu_c}}_h - \nabla \dot{u^{\mu_c}})\, dx$$
$$+ \int_{\Omega} 4 l_c \nabla \operatorname{axl}(A^{\mu_c}{}_h - A^{\mu_c}) \cdot \nabla \operatorname{axl}(\dot{A^{\mu_c}}_h - \dot{A^{\mu_c}})\, dx$$
$$- \int_{\Omega} 4\mu_c \operatorname{axl}(\operatorname{skew}(\nabla u^{\mu_c}{}_h - \nabla u^{\mu_c}) - (A^{\mu_c}{}_h - A^{\mu_c})) \cdot (\dot{A^{\mu_c}}_h - \dot{A^{\mu_c}})\, dx,$$

where $T^{\mu_c}_{Eh}(t) = T^{\mu_c}_E(t+h)$ and $\sigma^{\mu_c}_h(t) = \sigma^{\mu_c}(t+h)$. By the monotonicity of (F, g) the first term on the right-hand side of (5.4) is non-positive. Integrating the second and the fourth term by parts, and applying equations (2.1a) and (2.1c) we shall get

$$\dot{\mathcal{E}}^{\mu_c}(u^{\mu_c}{}_h - u^{\mu_c}, \varepsilon^{\mu_c}{}_h - \varepsilon^{\mu_c}, \varepsilon_p^{\mu_c}{}_h - \varepsilon_p^{\mu_c}, A^{\mu_c}{}_h - A^{\mu_c}, y^{\mu_c}{}_h - y^{\mu_c})(t)$$
$$\le \int_{\partial\Omega} (\dot{u}_{dh} - \dot{u}_d) \cdot (\sigma_h - \sigma) \cdot n\, dS + \int_{\Omega} (f_h - f) \cdot (\dot{u^{\mu_c}}_h - \dot{u^{\mu_c}})\, dx$$
$$+ \int_{\partial\Omega} 2 l_c (\dot{A}_{dh} - \dot{A}_d) \cdot (\nabla A^{\mu_c}{}_h - \nabla A^{\mu_c}) \cdot n\, dS,$$

where $f_h(t) = f(t+h)$, $u_{dh}(t) = u_d(t+h)$ and $A_{dh}(t) = A_d(t+h)$. Now, integrating the inequality above with respect to t we get:

$$\begin{aligned}
&\mathcal{E}^{\mu_c}(u^{\mu_c}{}_h - u^{\mu_c}, \varepsilon^{\mu_c}{}_h - \varepsilon^{\mu_c}, \varepsilon_p^{\mu_c}{}_h - \varepsilon_p^{\mu_c}, A^{\mu_c}{}_h - A^{\mu_c}, y^{\mu_c}{}_h - y^{\mu_c})(t) \\
&\quad \le \mathcal{E}^{\mu_c}(u^{\mu_c}{}_h - u^{\mu_c}, \varepsilon^{\mu_c}{}_h - \varepsilon^{\mu_c}, \varepsilon_p^{\mu_c}{}_h - \varepsilon_p^{\mu_c}, A^{\mu_c}{}_h - A^{\mu_c}, y^{\mu_c}{}_h - y^{\mu_c})(0) \\
&\quad + \int_0^t \int_{\partial\Omega} (\dot{u}_{dh} - \dot{u}_d) \cdot (\sigma_h - \sigma) \cdot n\, dS\, d\tau \\
&\quad + \int_0^t \int_{\Omega} (f_h - f) \cdot (\dot{u^{\mu_c}}_h - \dot{u^{\mu_c}})\, dx\, d\tau \\
&\quad + \int_0^t \int_{\partial\Omega} 2 l_c (\dot{A}_{dh} - \dot{A}_d) \cdot (\nabla A^{\mu_c}{}_h - \nabla A^{\mu_c}) \cdot n\, dS\, d\tau.
\end{aligned} \tag{5.5}$$

At this point, our plan is to shift, in the integral terms, the shift operator onto given data. We calculate this with details for the first integral only

$$\int_0^t \int_\Omega (f_h - f) \cdot (\dot{u^{\mu_c}}_h - \dot{u^{\mu_c}})\, dx\, d\tau \tag{5.6}$$
$$= \int_0^t \int_\Omega (f_h - f) \cdot \dot{u^{\mu_c}}_h\, dx\, d\tau - \int_0^t \int_\Omega (f_h - f) \cdot \dot{u^{\mu_c}}\, dx\, d\tau$$
$$= \int_h^{t+h} \int_\Omega (f - f_{-h}) \cdot \dot{u^{\mu_c}}\, dx\, d\tau - \int_0^t \int_\Omega (f_h - f) \cdot \dot{u^{\mu_c}}\, dx\, d\tau$$
$$= \int_t^{t+h} \int_\Omega (f_h - f) \cdot \dot{u^{\mu_c}}\, dx\, d\tau + \int_0^t \int_\Omega (2f - f_{-h} - f_h) \cdot \dot{u^{\mu_c}}\, dx\, d\tau$$
$$- \int_0^h \int_\Omega (f_h - f) \cdot \dot{u^{\mu_c}}\, dx\, d\tau.$$

In the same manner we transform the first and the third integral in (5.6). Next, we insert (5.6) and the results for the other terms into (5.5), divide by h^2 and pass to the limit $h \to 0^+$. Hence, we conclude with the following inequality:

$$\mathcal{E}^{\mu_c}(\dot{u^{\mu_c}}, \dot{\varepsilon^{\mu_c}}, \dot{\varepsilon_p^{\mu_c}}, \dot{A^{\mu_c}}, \dot{y^{\mu_c}})(t) \tag{5.7}$$
$$\leq \mathcal{E}^{\mu_c}(\dot{u^{\mu_c}}, \dot{\varepsilon^{\mu_c}}, \dot{\varepsilon_p^{\mu_c}}, \dot{A^{\mu_c}}, \dot{y^{\mu_c}})(0) + \int_\Omega \dot{f}(t) \cdot \dot{u^{\mu_c}}(t)\, dx - \int_\Omega \dot{f}(0) \cdot \dot{u^{\mu_c}}(0)\, dx$$
$$- \int_0^t \int_\Omega \ddot{f} \cdot \dot{u^{\mu_c}}\, dx\, d\tau + \int_{\partial\Omega} (\sigma^{\mu_c}(t) \cdot n) \cdot \partial_t^2 u_d(t)\, dS$$
$$- \int_{\partial\Omega} (\sigma^{\mu_c}(0) \cdot n) \cdot \partial_t^2 u_d(0) dS - \int_0^t \int_{\partial\Omega} (\sigma^{\mu_c} \cdot n) \cdot \partial_t^3 u_d\, dS$$
$$+ 4l_c \int_{\partial\Omega} (\nabla\, \mathrm{axl}(A^{\mu_c})(t) \cdot n) \cdot \partial_t^2 A_d(t)\, dS$$
$$- 4l_c \int_{\partial\Omega} (\nabla\, \mathrm{axl}(A^{\mu_c})(t) \cdot n) \cdot \partial_t^2 A_d(t)\, dS$$
$$- 4l_c \int_0^t \int_{\partial\Omega} (\nabla\, \mathrm{axl}(A^{\mu_c})(t) \cdot n) \cdot \partial_t^3 A_d\, dS\, d\tau.$$

To obtain the initial energy for time derivatives we observe that

$$\varepsilon_p^{\dot{\mu}_c}(0) = F\Big(T_E^{\mu_c}(0), -\frac{\gamma}{\alpha}y^0\Big)$$

and

$$y^{\dot{\mu}_c}(0) = g\Big(T_E^{\mu_c}(0), -\frac{\gamma}{\alpha}y^0\Big),$$

so by assumption they are bounded in $L^2(\Omega, \mathrm{Sym}(3))$ and $L^2(\Omega)$, respectively. Initial values $A^{\dot{\mu}_c}(0)$ and $u^{\dot{\mu}_c}(0)$ are solutions of the system

$$\begin{aligned}
\operatorname{div}\sigma^{\dot{\mu}_c} &= -\dot{f}(0),\\
\sigma^{\dot{\mu}_c} &= 2\mu\Big(\varepsilon(u^{\dot{\mu}_c}{}_0) - F^{\mu_c}\Big(2\mu(\varepsilon^{\mu_c}(0) - \varepsilon_p^0), -\frac{\gamma}{\alpha}y_0\Big)\Big)\\
&\quad + 2\mu_c(\mathrm{skew}(\nabla u^{\dot{\mu}_c}(0)) - A^{\dot{\mu}_c}(0)) + \lambda\,\mathrm{tr}[\varepsilon(u^{\dot{\mu}_c}(0))]\cdot\mathbb{I},\\
-l_c\Delta\,\mathrm{axl}(A^{\dot{\mu}_c}(0)) &= \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u^{\dot{\mu}_c}(0)) - A^{\dot{\mu}_c}(0)),\\
u^{\dot{\mu}_c}(0)|_{\partial\Omega} &= u_d(0), \qquad A^{\dot{\mu}_c}(0)|_{\partial\Omega} = A_d(0).
\end{aligned}$$

Consequently, the initial energy for the time derivatives is bounded.

Now, let $\tilde{u} \in H^1(\Omega)$ be such that $\tilde{u}|_{\partial\Omega} = \dot{u}_d$. By Korn's inequality we have

$$\begin{aligned}
\|u^{\dot{\mu}_c}\|_{L^2(\Omega)} &\le \|u^{\dot{\mu}_c} - \tilde{u}\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Omega)} \le C\|\varepsilon(u^{\dot{\mu}_c} - \tilde{u})\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Omega)}\\
&\le C\|\varepsilon^{\dot{\mu}_c}\|_{L^2(\Omega)} + C(t) \le C(\|\varepsilon_p^{\dot{\mu}_c}\|_{L^2(\Omega)} + \|\dot{T}_E^{\mu_c}\|_{L^2(\Omega)}) + C(t).
\end{aligned}$$

Since $\dot{T}_E^{\mu_c} = 2\mu(\varepsilon^{\dot{\mu}_c} - \varepsilon_p^{\dot{\mu}_c})$, we can see from (2.1d) that $|\varepsilon_p^{\dot{\mu}_c}| = \alpha|y^{\dot{\mu}_c}|$; thus,

$$\begin{aligned}
\|u^{\dot{\mu}_c}\|_{L^2(\Omega)} &\le C(\|\varepsilon_p^{\dot{\mu}_c}\|_{L^2(\Omega)} + \|\dot{T}_E^{\mu_c}\|_{L^2(\Omega)}) + C(t)\\
&= C(\alpha\|y^{\dot{\mu}_c}\|_{L^2(\Omega)} + \|\dot{T}_E^{\mu_c}\|_{L^2(\Omega)}) + C(t)\\
&\le C\mathcal{E}^{\mu_c\frac{1}{2}}(u^{\dot{\mu}_c}, \varepsilon^{\dot{\mu}_c}, \varepsilon_p^{\dot{\mu}_c}, A^{\dot{\mu}_c}, y^{\dot{\mu}_c}))(t) + C(t).
\end{aligned}$$

We can estimate the first term on the right side of (5.7),

$$\begin{aligned}
\int_\Omega \dot{f}(t)\cdot u^{\dot{\mu}_c}(t)\,dx &\le \|\dot{f}(t)\|_{L^2(\Omega)}\|u^{\dot{\mu}_c}(t)\|_{L^2(\Omega)}\\
&\overset{(5.7)}{\le} C\|\dot{f}(t)\|_{L^2(\Omega)}\mathcal{E}^{\mu_c\frac{1}{2}}(u^{\dot{\mu}_c}, \varepsilon^{\dot{\mu}_c}, \varepsilon_p^{\dot{\mu}_c}, A^{\dot{\mu}_c}, y^{\dot{\mu}_c})(t) + C(t),
\end{aligned}$$

The second and the third terms can be estimated similarly. We shall analyze the fourth integral term of (5.7)

$$\begin{aligned}
\int_{\partial\Omega} (\sigma^{\mu_c}(t)\cdot n)\cdot\partial_t^2 u_d(t)\,dS &\le C(\|\sigma^{\mu_c}\|_{L^2(\Omega)} + \|\operatorname{div}\sigma^{\mu_c}|_{L^2(\Omega)})\|\partial_t^2 u_d\|_{H^{\frac{1}{2}}(\partial\Omega)}\\
&\le C(\mathcal{E}^{\mu_c\frac{1}{2}}(u^{\mu_c}, \varepsilon^{\mu_c}, \varepsilon_p^{\mu_c}, A^{\mu_c}, y^{\mu_c}) + \|f\|_{L^2(\Omega)})\|\partial_t^2 u_d\|_{H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

The estimation of the seventh term of (5.7) yields

$$\begin{aligned}\int_{\partial\Omega} (\nabla \operatorname{axl}(A^{\mu_c})(t)\cdot n)\cdot \partial_t^2 A_d(t)\, dS \\ &\le C(\|\Delta A^{\mu_c}\|_{L^2(\Omega)} + \|\nabla A^{\mu_c}\|_{L^2(\Omega)})\|\partial_t^2 A_d\|_{H^{\frac12}(\partial\Omega)} \\ &\overset{(2.1c)}{\le} C\mathcal{E}^{\mu_c \frac12}(u^{\mu_c},\varepsilon^{\mu_c},\varepsilon_p^{\mu_c},A^{\mu_c},y^{\mu_c})(t)\|\partial_t^2 A_d\|_{H^{\frac12}(\partial\Omega)}.\end{aligned}$$

The other terms on the right-hand side of (5.7) can be calculated in the same fashion. Finally, we arrive at the following inequality:

$$\begin{aligned}\mathcal{E}^{\mu_c}(\dot{u^{\mu_c}},\dot{\varepsilon^{\mu_c}},\dot{\varepsilon_p^{\mu_c}},\dot{A^{\mu_c}},\dot{y^{\mu_c}})(t) &\le \mathcal{E}^{\mu_c}(\dot{u^{\mu_c}},\dot{\varepsilon^{\mu_c}},\dot{\varepsilon_p^{\mu_c}},\dot{A^{\mu_c}},\dot{y^{\mu_c}})(0) \\ &\quad + C(t)\mathcal{E}^{\mu_c \frac12}(u^{\mu_c},\varepsilon^{\mu_c},\varepsilon_p^{\mu_c},A^{\mu_c},y^{\mu_c})(t) \\ &\quad + \int_0^t D(t)\mathcal{E}^{\mu_c \frac12}(u^{\mu_c},\varepsilon^{\mu_c},\varepsilon_p^{\mu_c},A^{\mu_c},y^{\mu_c})(\tau)\, d\tau + E(t).\end{aligned}$$

Using Young's inequality we get

$$\mathcal{E}^{\mu_c}(\dot{u^{\mu_c}},\dot{\varepsilon^{\mu_c}},\dot{\varepsilon_p^{\mu_c}},\dot{A^{\mu_c}},\dot{y^{\mu_c}})(t) \le \int_0^t \mathcal{E}^{\mu_c}(u^{\mu_c},\varepsilon^{\mu_c},\varepsilon_p^{\mu_c},A^{\mu_c},y^{\mu_c})(\tau)\, d\tau + C(t).$$

Finally , Gronwall's lemma completes the proof. □

The energy estimate proved in the last theorem yields boundedness of $\dot{A^{\mu_c}}$ in the space $L^\infty((0,T),H^1(\Omega,\mathfrak{so}(3)))$ and of $\dot{y^{\mu_c}}$ in the space $L^\infty((0,T),L^2(\Omega))$. Moreover, using the fact that $|\dot{\varepsilon_p^{\mu_c}}| = \alpha|\dot{y^{\mu_c}}|$, we obtain that $\dot{\varepsilon_p^{\mu_c}}$ is bounded in the space $L^\infty((0,T),L^2(\Omega,\mathrm{Sym}(3))$. We can see that $\mathcal{E}^{\mu_c}$ controls $\dot{T}_E^{\mu_c}$, so consequently $\dot{u^{\mu_c}}$ is bounded in $L^\infty((0,T),\mathbb{R}^3)$. From the equalities

$$y^{\mu_c}(t) = \int_0^t \dot{y^{\mu_c}} d\tau + y^0, \qquad A^{\mu_c}(t) = \int_0^t \dot{A^{\mu_c}} d\tau + A^{\mu_c}(0),$$

$$u^{\mu_c}(t) = \int_0^t \dot{u^{\mu_c}} d\tau + u^{\mu_c}(0), \qquad \varepsilon_p^{\mu_c}(t) = \int_0^t \dot{\varepsilon_p^{\mu_c}} d\tau + \varepsilon_p^0,$$

we deduce the boundedness of $(u^{\mu_c}, A^{\mu_c}, {\varepsilon^{\mu_c}}_p, y^{\mu_c})$ in

$$\begin{aligned}W^{1,\infty}((0,T),H^1(\Omega,\mathbb{R}^3)) \times W^{1,\infty}((0,T),H^1(\Omega,\mathfrak{so}(3))) \\ \times W^{1,\infty}((0,T),L^2(\Omega,\mathrm{Sym}(3))) \times W^{1,\infty}((0,T),L^2(\Omega)).\end{aligned}$$

Hence, we have subsequences (still denoted by superscript μ_c) that

$$\begin{aligned}
u^{\mu_c} &\overset{*}{\rightharpoonup} u && \text{in } W^{1,\infty}((0,T),H^1(\Omega,\mathbb{R}^3)),\\
A^{\mu_c} &\overset{*}{\rightharpoonup} A && \text{in } W^{1,\infty}((0,T),H^1(\Omega,\mathfrak{so}(3))),\\
\varepsilon_p^{\mu_c} &\overset{*}{\rightharpoonup} \varepsilon_p && \text{in } W^{1,\infty}((0,T),L^2(\Omega,\mathrm{Sym}(3))),\\
y^{\mu_c} &\overset{*}{\rightharpoonup} y && \text{in } W^{1,\infty}((0,T),L^2(\Omega)).
\end{aligned}$$

Now, with standard procedure we see that the limit functions u, ε_p satisfy

$$\begin{aligned}
\operatorname{div}\sigma &= -f,\\
\sigma &= 2\mu(\varepsilon-\varepsilon_p)+\lambda\operatorname{tr}[\varepsilon]\cdot\mathbb{I},\\
u|_{\partial\Omega} &= u_d,
\end{aligned}$$

and A satisfies

$$\begin{aligned}
-l_c\Delta\operatorname{axl}(A) &= 0,\\
A|_{\partial\Omega} &= A_d.
\end{aligned}$$

Thus, to complete the proof of approximation we need to prove that the limit functions satisfy the evolution equation from (2.2). We would prove it, if we improved the weak convergence of the sequence $\{y^{\mu_c}\}$ and $\{T_E^{\mu_c}\}$.

LEMMA 5.3 (Strong convergence of stresses). *Under the assumptions of Theorem* 2.1, *we have*

$$\mathcal{E}(u^{\mu_{c1}}-u^{\mu_{c2}},\varepsilon^{\mu_{c1}}-\varepsilon^{\mu_{c2}},\varepsilon_p^{\mu_{c1}}-\varepsilon_p^{\mu_{c2}},A^{\mu_{c1}}-A^{\mu_{c2}},y^{\mu_{c1}}-y^{\mu_{c2}}(t)\to 0,$$

uniformly for $0\le t\le T$, *when* $\mu_{c_1},\mu_{c2}\to 0^+$.

P r o o f. Calculating the time derivative of the energy (4.2) evaluated on the differences of two solutions of (2.1) we obtain

$$\begin{aligned}
(5.8)\qquad &\dot{\mathcal{E}}(u^{\mu_{c1}}-u^{\mu_{c2}},\varepsilon^{\mu_{c1}}-\varepsilon^{\mu_{c2}},\varepsilon_p^{\mu_{c1}}-\varepsilon_p^{\mu_{c2}},A^{\mu_{c1}}-A^{\mu_{c2}},y^{\mu_{c1}}-y^{\mu_{c2}})(t)\\
&= \int_\Omega 2\mu(\varepsilon^{\mu_{c1}}-\varepsilon^{\mu_{c2}}-(\varepsilon_p^{\mu_{c1}}-\varepsilon_p^{\mu_{c2}}))\cdot(\dot\varepsilon^{\mu_{c1}}-\dot\varepsilon^{\mu_{c2}}-(\dot\varepsilon_p^{\mu_{c1}}-\dot\varepsilon_p^{\mu_{c2}}))\,dx\\
&\quad+\int_\Omega \lambda\operatorname{tr}[\varepsilon^{\mu_{c1}}-\varepsilon^{\mu_{c2}}]\operatorname{tr}[\dot\varepsilon^{\mu_{c1}}-\dot\varepsilon^{\mu_{c2}}]\,dx\\
&\quad+4l_c\int_\Omega(\nabla\operatorname{axl}(A^{\mu_{c1}})-\nabla\operatorname{axl}(A^{\mu_{c2}}))\cdot(\nabla\operatorname{axl}(\dot A^{\mu_{c1}})-\nabla\operatorname{axl}(\dot A^{\mu_{c2}}))\,dx\\
&\quad+\int_\Omega\frac{\gamma}{\alpha}(y^{\mu_{c1}}-y^{\mu_{c2}})\cdot(\dot y^{\mu_{c1}}-\dot y^{\mu_{c2}})\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega -(T_E^{\mu_{c_1}} - T^{\mu_{c_2}}) \cdot (\dot{\varepsilon}_p^{\mu_{c_1}} - \dot{\varepsilon}_p^{\mu_{c_2}}) + \frac{\gamma}{\alpha}(y^{\mu_{c_1}} - y^{\mu_{c_2}}) \cdot (\dot{y}^{\mu_{c_1}} - \dot{y}^{\mu_{c_2}})\, dx \\
&\quad + \int_\Omega (\sigma^{\mu_{c_1}} - \sigma^{\mu_{c_2}}) \cdot (\nabla \dot{u}^{\mu_{c_1}} - \nabla \dot{u}^{\mu_{c_2}})\, dx \\
&\quad - \int_\Omega 2(\mu_{c_1}(\operatorname{skew}(\nabla u^{\mu_{c_1}}) - A^{\mu_{c_1}}) - \mu_{c_2}(\operatorname{skew}(\nabla u^{\mu_{c_2}}) - A^{\mu_{c_2}})) \\
&\qquad\qquad \cdot \operatorname{skew}(\nabla \dot{u}^{\mu_{c_1}} - \nabla \dot{u}^{\mu_{c_2}})\, dx \\
&\quad + 4l_c \int_\Omega (\nabla \operatorname{axl}(A^{\mu_{c_1}}) - \nabla \operatorname{axl}(A^{\mu_{c_2}})) \cdot (\nabla \operatorname{axl}(A^{\mu_{c_1}}) - \nabla \operatorname{axl}(A^{\mu_{c_2}}))\, dx.
\end{aligned}
$$

The first integral on the right-hand side of (5.8) is non-positive. The second and the fourth term we integrate by parts. Since the boundary values for both solutions are the same, all boundary integrals are equal zero. Next, using equations (2.1a) and (2.1c), we conclude that

$$
\begin{aligned}
&\dot{\mathcal{E}}(u^{\mu_{c_1}} - u^{\mu_{c_2}}, \varepsilon^{\mu_{c_1}} - \varepsilon^{\mu_{c_2}}, \varepsilon_p^{\mu_{c_1}} - \varepsilon_p^{\mu_{c_2}}, A^{\mu_{c_1}} - A^{\mu_{c_2}}, y^{\mu_{c_1}} - y^{\mu_{c_2}})(t) \\
&\qquad \le \int_\Omega -2(\mu_{c_1}(\operatorname{skew}(\nabla u^{\mu_{c_1}}) - A^{\mu_{c_1}}) - \mu_{c_2}(\operatorname{skew}(\nabla u^{\mu_{c_2}}) - A^{\mu_{c_2}})) \\
&\qquad\qquad \cdot ((\operatorname{skew}(\nabla \dot{u}^{\mu_{c_1}}) - A^{\mu_{c_1}}) - (\operatorname{skew}(\nabla \dot{u}^{\mu_{c_2}}) - A^{\mu_{c_2}})\, dx.
\end{aligned}
$$

Now, by Theorem 5.2 we have

$$
\dot{\mathcal{E}}(u^{\mu_{c_1}} - u^{\mu_{c_2}}, \varepsilon^{\mu_{c_1}} - \varepsilon^{\mu_{c_2}}, \varepsilon_p^{\mu_{c_1}} - \varepsilon_p^{\mu_{c_2}}, A^{\mu_{c_1}} - A^{\mu_{c_2}}, y^{\mu_{c_1}} - y^{\mu_{c_2}})(t) \le C(\mu_{c_1} + \mu_{c_2}).
$$

Next, we integrate with respect to time and finally have

$$
\begin{aligned}
&\mathcal{E}(u^{\mu_{c_1}} - u^{\mu_{c_2}}, \varepsilon^{\mu_{c_1}} - \varepsilon^{\mu_{c_2}}, \varepsilon_p^{\mu_{c_1}} - \varepsilon_p^{\mu_{c_2}}, A^{\mu_{c_1}} - A^{\mu_{c_2}}, y^{\mu_{c_1}} - y^{\mu_{c_2}})(t) \\
&\le \mathcal{E}(u^{\mu_{c_1}} - u^{\mu_{c_2}}, \varepsilon^{\mu_{c_1}} - \varepsilon^{\mu_{c_2}}, \varepsilon_p^{\mu_{c_1}} - \varepsilon_p^{\mu_{c_2}}, A^{\mu_{c_1}} - A^{\mu_{c_2}}, y^{\mu_{c_1}} - y^{\mu_{c_2}})(0) + C(\mu_{c_1} + \mu_{c_2}).
\end{aligned}
$$

Using Theorem 5.1, we conclude that

$$
\mathcal{E}(u^{\mu_{c_1}} - u^{\mu_{c_2}}, \varepsilon^{\mu_{c_1}} - \varepsilon^{\mu_{c_2}}, \varepsilon_p^{\mu_{c_1}} - \varepsilon_p^{\mu_{c_2}}, A^{\mu_{c_1}} - A^{\mu_{c_2}}, y^{\mu_{c_1}} - y^{\mu_{c_2}})(0) \to 0,
$$

when $\mu_{c_1}, \mu_{c_2} \to 0^+$. It completes the proof. $\square$

So by Theorem 5.3 we have

$$
\begin{aligned}
y^{\mu_c} &\to y && \text{in } L^\infty((0,T), L^2(\Omega)), \\
T_E^{\mu_c} &\to T_E && \text{in } L^\infty((0,T), L^2(\Omega, \operatorname{Sym}(3))).
\end{aligned}
\tag{5.9}
$$

Moreover, by Theorem 5.2 we have (eventually going to a subsequence)

$$\dot{\varepsilon}_p^{\mu_c} = F^{\mu_c}\left(T_E^{\mu_c}, -\frac{\gamma}{\alpha}y^{\mu_c}\right) \overset{*}{\rightharpoonup} \dot{\varepsilon}_p \qquad \text{in } L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3))),$$

$$\dot{y^{\mu_c}} = g^{\mu_c}\left(T_E^{\mu_c}, -\frac{\gamma}{\alpha}y^{\mu_c}\right) \overset{*}{\rightharpoonup} \dot{y} \qquad \text{in } L^\infty((0,T), L^2(\Omega)).$$

Combining these results with the fact that the graph of a monotone field is weakly-strongly closed (see [22]) gives us that

$$\dot{\varepsilon}_p = F\left(T_E, -\frac{\gamma}{\alpha}y\right), \qquad \dot{y} = g\left(T_E, -\frac{\gamma}{\alpha}y\right).$$

Finally, we have proved Theorem 2.1.

Acknowledgment

The author would like to thank Przemysław Górka for reading a preliminary version of this article, for his fruitful comments and for correcting the English. He would also like to thank Sebastian Owczarek for his valuable suggestions and Paweł Wójcicki for correcting English. P. M. B. is also grateful to professor Krzysztof Chełmiński for correcting mistakes and for overall help. The author is also thankful to the Reviewers and professor Henryk Petryk for their precious comments.

References

1. E. Cosserat, F. Cosserat, *Théorie des corps déformables*, Librairie Scientifique A. Hermann et Fils, Paris, 1909.
2. P. Neff, K. Chełmiński, *Infinitesimal elastic-plastic Cosserat micropolar theory. Modeling and global existence in the rate-independent case*, Proc. Roy. Soc. Edinburgh A, **135**, 5, 1017–1039, 2005.
3. F.H. Norton, *The Creep of Steel at High Temperatures*, McGraw-Hill Company, London, 1929.
4. A. Bensoussan, J. Frehse, *Asymptotic behaviour of the time dependent Norton–Hoff law in plasticity theory and H^1-regularity*, Comment. Math. Univ. Carolin., **37**, 2, 285–304, 1996.
5. R. Temam, *A generalized Norton–Hoff model and the Prandtl–Reuss law of plasticity*, Arch. Rational Mech. Anal., **95**, 137–183, 1986.
6. K. Chełmiński, *Coercive limits for a subclass of monotone constitutive equations in the theory of inelastic material behaviour of metals*, Mat. Stos., **40**, 41–81, 1997.
7. D.H. Alber, *Materials with Memory: Initial-Boundary Value Problems for Constitutive Equations with Internal Variables*, Springer, Berlin, 1998.

8. P. Neff, K. Chełmiński, *A note on approximation of Prandtl-Reuss plasticity through Cosserat plasticity*, Quart. Appl. Math., **66**, 2, 351–357, 2008.

9. K. Chełmiński, P. Neff, H^1_{loc}*-stress and strain regularity in Cosserat plasticity*, ZAMM Z. Angew. Math. Mech., **89**, 4, 257–266, 2009.

10. P. Neff, K. Chełmiński, W. Müller, C. Wieners, *Numerical solution method for an infinitesimal elastic-plastic Cosserat model*, Math. Models Methods Appl. Sci., **17**, 1211–1239, 2007.

11. P. Neff, K. Chełmiński, *Well-posedness of dynamic Cosserat plasticity*, Appl. Math. Optimization, **56**, 19–35, 2007.

12. K. Chełmiński, P. Neff, S. Owczarek, *Poroplasticity with Cosserat effects*, ZAMM Z. Angew. Math. Mech., **92**, 6, 462–478, 2012.

13. J. Jeong, P. Neff, *Existence, uniqueness and stability in linear Cosserat elasticity for weakest curvature conditions*, Math. Mech. Solids, **15**, 1, 78–95, 2010.

14. J. Jeong, H. Ramézani, I. Münch, P. Neff, *A numerical study for linear isotropic Cosserat elasticity with conformally invariant curvature*, ZAMM Z. Angew. Math. Mech., **89**, 7, 552–569, 2009.

15. P. Neff, K.-I. Hong, J. Jeong, *The Reissner-Mindlin plate is the* Γ*-limit of Cosserat elasticity*, Math. Models Methods Appl. Sci., **20**, 9, 1553–1590, 2010.

16. P. Neff, J. Jeong, *A new paradigm: the linear isotropic Cosserat model with conformally invariant curvature energy*, ZAMM Z. Angew. Math. Mech., **89**, 2, 107–122, 2009.

17. P. Neff, J. Jeong, I. Münch, H. Ramézani, *Linear Cosserat elasticity, conformal curvature and bounded stiffness* [in:] G. A. Maugin, V. A. Metrikine [eds.], *Mechanics of Generalized Continua. One hundred years after the Cosserats*, Adv. Mech. Math., **21**, 55–63, 2010.

18. L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998.

19. R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.

20. P.D. Lax, *Functional Analysis*, Wiley-Interscience, New York, 2002.

21. W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, New York, 1973.

22. J.P. Aubin, A. Cellina, *Differential Inclusions. Set-valued Maps and Viability Theory*, Springer, Berlin, 1984.

Received May 17, 2013; revised version November 12, 2013.