

## Green's function for an anisotropic piezoelectric half-space bonded to a thin piezoelectric layer

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THE GREEN'S FUNCTION FOR AN ANISOTROPIC PIEZOELECTRIC HALF-SPACE bonded to a thin piezoelectric layer subject to a generalized line force and a generalized line dislocation is presented. The thickness of a thin layer is assumed to be small compared with a reference length. Thus, the existence of the layer is replaced by effective boundary conditions to avoid finding solutions in the layer. Combining with the Stroh formalism gives explicit solutions in a more compact form.

**Key words:** anisotropic piezoelectric half-space, thin piezoelectric layer, effective boundary conditions, Green's function, Stroh formalism.

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### 1. Introduction

PIEZOELECTRIC MATERIALS HAVE BEEN WIDELY USED in our daily life, and numerous applications rely on thorough analytical and numerical analyses. Green's functions play an important role in the solution of both analytical and numerical methods. The Green's function for a piezoelectric half-space bonded to a thin piezoelectric layer is an interesting and important problem in practice. NOWACKI *et al.* [1, 2] obtained exact solutions for a piezoelectric layer-substrate structure in a form of Fourier integrals with detailed discussions on the convergence.

Instead of using various plate and shell theories [3], if the layer is very thin compared with a reference length, it is desirable to replace its existence by effective boundary conditions to avoid finding the solution of the layer. This simplifies the analysis significantly. BOVIK [4] first replaced the existence of the layer by effective boundary conditions and considered the problem in case of an isotropic elastic layer. BATRA [5] considered a laminated plate with piezoelectric coatings. NIKLASSON *et al.* [6] studied the case when the layer is a monoclinic material with the symmetry plane parallel to the plane of the layer. ZHANG *et al.* [7] and BOSTRÖM and ZHANG [8] studied waves from a piezoelectric strip placed on an isotropic elastic half-space, with comparison between exact and effective

boundary condition solutions. BENVENISTE [9] and TING [10, 11] considered the case of a general anisotropic elastic material layer. JOHANSSON and NIKLASSON [12] considered a thin piezoelectric coating layer bonded to an elastic material. BENVENISTE [13] studied the case of a piezoelectric interface between two piezoelectric media.

By following the works of TING [10, 11], the Green's function for an anisotropic piezoelectric half-space bonded to a thin piezoelectric layer subject to a generalized line force and a generalized line dislocation is constructed. Effective boundary conditions and the Stroh formalism are adopted and solutions are explicitly given.

The basic governing equations for a piezoelectric layer are presented in Section 2. The effective boundary conditions are derived in Section 3. The Stroh formalism is briefly discussed in Section 4. Application of the effective boundary conditions in constructing the Green's function for a half-space bonded to a thin piezoelectric layer is given in Section 5. Mechanics of a thin piezoelectric layer are briefly discussed in Appendix for interested readers.

## 2. Basic equations

In a fixed rectangular coordinate system  $x_i$  ( $i = 1, 2, 3$ ), and in the absence of body forces and free charges, the equations of equilibrium for anisotropic piezoelectric materials are [14]

$$(2.1) \quad \sigma_{ij,j} = 0, \quad D_{i,i} = 0,$$

where  $\sigma_{ij}$  is the elastic stress,  $D_i$  is the electric displacement. A comma denotes differentiation with  $x_i$ . Repeated indices mean summation. The constitutive equations are given by

$$(2.2) \quad \begin{aligned} \sigma_{ij} &= C_{ijkl}u_{k,m} - e_{mij}E_m, \\ D_i &= e_{ikm}u_{k,m} + \omega_{im}E_m \quad (i, j, k, m = 1, 2, 3). \end{aligned}$$

Coefficients  $C_{ijkl}$ ,  $e_{mij}$  and  $\omega_{im}$  are, respectively, the elastic stiffness, piezoelectric stress constants and permittivity constants with the following symmetries:

$$(2.3) \quad C_{ijkl} = C_{jikm} = C_{kmi j}, \quad e_{mij} = e_{mji}, \quad \omega_{im} = \omega_{mi}.$$

$u_k$  is the elastic displacement and  $E_m$  is the electric field.  $C_{ijkl}$  and  $\omega_{im}$  are positive definite in the sense that

$$(2.4) \quad C_{ijkl}u_{i,j}u_{k,m} > 0, \quad \omega_{im}E_iE_m > 0,$$

for any real nonzero  $u_{i,j}$  and  $E_i$ . With

$$(2.5) \quad E_i = -\varphi_{,i},$$

where  $\varphi$  is the electrostatic potential, the constitutive equations (2.2) become

$$(2.6) \quad \begin{aligned} \sigma_{ij} &= C_{ijkl}u_{k,m} + e_{mij}\varphi_{,m}, \\ D_i &= e_{ikm}u_{k,m} - \omega_{im}\varphi_{,m} \quad (i, j, k, m = 1, 2, 3). \end{aligned}$$

By defining

$$(2.7) \quad \begin{aligned} \mathbf{t} &= \begin{bmatrix} \mathbf{t}^E \\ D_2 \end{bmatrix}, \quad \hat{\mathbf{t}} = \begin{bmatrix} \hat{\mathbf{K}}\hat{\mathbf{t}}^E \\ D_1 \end{bmatrix}, \quad \tilde{\mathbf{t}} = \begin{bmatrix} \tilde{\mathbf{K}}\tilde{\mathbf{t}}^E \\ D_3 \end{bmatrix}, \\ \mathbf{t}^E &= \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{bmatrix}, \quad \hat{\mathbf{t}}^E = \begin{bmatrix} \sigma_{11} \\ \sigma_{13} \\ \sigma_{33} \end{bmatrix}, \quad \hat{\mathbf{K}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

the equilibrium equations (2.1) and the constitutive equations (2.6) can be rewritten as

$$(2.8) \quad \mathbf{t}_{,2} + \mathbf{K}_1\mathbf{t}_{,1} + \mathbf{K}_3\mathbf{t}_{,3} + \hat{\mathbf{K}}_1\hat{\mathbf{t}}_{,1} + \hat{\mathbf{K}}_3\hat{\mathbf{t}}_{,3} + \tilde{\mathbf{K}}_3\tilde{\mathbf{t}}_{,3} = \mathbf{0},$$

and

$$(2.9) \quad \begin{aligned} \mathbf{t} &= \mathbf{C}_1\mathbf{u}_{,1} + \mathbf{C}_2\mathbf{u}_{,2} + \mathbf{C}_3\mathbf{u}_{,3}, \quad \hat{\mathbf{t}} = \hat{\mathbf{C}}_1\mathbf{u}_{,1} + \hat{\mathbf{C}}_2\mathbf{u}_{,2} + \hat{\mathbf{C}}_3\mathbf{u}_{,3}, \\ \tilde{\mathbf{t}} &= \tilde{\mathbf{C}}_1\mathbf{u}_{,1} + \tilde{\mathbf{C}}_2\mathbf{u}_{,2} + \tilde{\mathbf{C}}_3\mathbf{u}_{,3}, \end{aligned}$$

respectively. The generalized displacement vector  $\mathbf{u}$  is defined as

$$(2.10) \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}^E \\ \varphi \end{bmatrix}, \quad \mathbf{u}^E = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

and

$$(2.11) \quad \begin{aligned} \mathbf{K}_1 &= \begin{bmatrix} \mathbf{K}_1^E & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \hat{\mathbf{K}}_1 = \begin{bmatrix} \hat{\mathbf{K}}_1^E & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} \mathbf{K}_3^E & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \\ \hat{\mathbf{K}}_3 &= \begin{bmatrix} \hat{\mathbf{K}}\hat{\mathbf{K}}_3^E & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \tilde{\mathbf{K}}_3 = \begin{bmatrix} \tilde{\mathbf{K}}\tilde{\mathbf{K}}_3^E & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \\ \mathbf{K}_1^E &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{K}}_1^E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{K}_3^E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{K}}_3^E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

with  $\hat{\mathbf{K}}$  and  $\tilde{\mathbf{K}}$  defined in (2.7). Similarly,

$$\begin{aligned}
\mathbf{C}_1 &= \begin{bmatrix} \mathbf{C}_1^E & \mathbf{e}_{12} \\ \mathbf{e}_{21}^T & -\omega_{21} \end{bmatrix}, & \mathbf{C}_2 &= \begin{bmatrix} \mathbf{C}_2^E & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\omega_{22} \end{bmatrix}, & \mathbf{C}_3 &= \begin{bmatrix} \mathbf{C}_3^E & \mathbf{e}_{32} \\ \mathbf{e}_{23}^T & -\omega_{23} \end{bmatrix}, \\
\mathbf{C}_1^E &= \begin{bmatrix} C_{61} & C_{66} & C_{65} \\ C_{21} & C_{26} & C_{25} \\ C_{41} & C_{46} & C_{45} \end{bmatrix}, & \mathbf{C}_2^E &= \begin{bmatrix} C_{66} & C_{62} & C_{64} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{42} & C_{44} \end{bmatrix}, & \mathbf{C}_3^E &= \begin{bmatrix} C_{65} & C_{64} & C_{63} \\ C_{25} & C_{24} & C_{23} \\ C_{45} & C_{44} & C_{43} \end{bmatrix}, \\
(2.12) \quad \hat{\mathbf{C}}_1 &= \begin{bmatrix} \hat{\mathbf{K}}\hat{\mathbf{C}}_1^E & \hat{\mathbf{K}}\mathbf{e}_1 \\ \mathbf{e}_{11}^T & -\omega_{11} \end{bmatrix}, & \tilde{\mathbf{C}}_1 &= \begin{bmatrix} \tilde{\mathbf{K}}\tilde{\mathbf{C}}_1^E & \tilde{\mathbf{K}}\mathbf{e}_1 \\ \mathbf{e}_{31}^T & -\omega_{31} \end{bmatrix}, & \hat{\mathbf{C}}_1^E &= \begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{51} & C_{56} & C_{55} \\ C_{31} & C_{36} & C_{35} \end{bmatrix}, \\
\hat{\mathbf{C}}_2 &= \begin{bmatrix} \hat{\mathbf{K}}\hat{\mathbf{C}}_2^E & \hat{\mathbf{K}}\mathbf{e}_2 \\ \mathbf{e}_{12}^T & -\omega_{12} \end{bmatrix}, & \tilde{\mathbf{C}}_2 &= \begin{bmatrix} \tilde{\mathbf{K}}\tilde{\mathbf{C}}_2^E & \tilde{\mathbf{K}}\mathbf{e}_2 \\ \mathbf{e}_{32}^T & -\omega_{32} \end{bmatrix}, & \hat{\mathbf{C}}_2^E &= \begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{56} & C_{52} & C_{54} \\ C_{36} & C_{32} & C_{34} \end{bmatrix}, \\
\hat{\mathbf{C}}_3 &= \begin{bmatrix} \hat{\mathbf{K}}\hat{\mathbf{C}}_3^E & \hat{\mathbf{K}}\mathbf{e}_3 \\ \mathbf{e}_{13}^T & -\omega_{13} \end{bmatrix}, & \tilde{\mathbf{C}}_3 &= \begin{bmatrix} \tilde{\mathbf{K}}\tilde{\mathbf{C}}_3^E & \tilde{\mathbf{K}}\mathbf{e}_3 \\ \mathbf{e}_{33}^T & -\omega_{33} \end{bmatrix}, & \hat{\mathbf{C}}_3^E &= \begin{bmatrix} C_{15} & C_{14} & C_{13} \\ C_{55} & C_{54} & C_{53} \\ C_{35} & C_{34} & C_{33} \end{bmatrix},
\end{aligned}$$

$$(\mathbf{e}_{ij})_m = e_{ijm}, \quad \mathbf{e}_i^T = [e_{i11}, e_{i13}, e_{i33}],$$

by using the contracted notation [15] for  $C_{ijkm}$  and the superscript  $\mathbf{T}$  denotes the transpose. Note that matrices with superscript  $E$  in Eqs. (2.7), (2.11) and (2.12) all have their counterparts in [11] without the superscript. Equations (2.8) and (2.9) are the governing equations for  $\mathbf{u}$ ,  $\mathbf{t}$ ,  $\hat{\mathbf{t}}$ , and  $\tilde{\mathbf{t}}$ .

As the matrix  $\mathbf{C}_2$  is non-singular,  $\mathbf{u}_2$  can be eliminated in (2.9) and gives

$$(2.13) \quad \hat{\mathbf{t}} = \hat{\mathbf{E}}_2\mathbf{t} + \hat{\mathbf{E}}_1\mathbf{u}_{,1} + \hat{\mathbf{E}}_3\mathbf{u}_{,3}, \quad \tilde{\mathbf{t}} = \tilde{\mathbf{E}}_2\mathbf{t} + \tilde{\mathbf{E}}_1\mathbf{u}_{,1} + \tilde{\mathbf{E}}_3\mathbf{u}_{,3},$$

with

$$\begin{aligned}
(2.14) \quad \hat{\mathbf{E}}_2 &= \hat{\mathbf{C}}_2\mathbf{C}_2^{-1}, & \hat{\mathbf{E}}_1 &= \hat{\mathbf{C}}_1 - \hat{\mathbf{E}}_2\mathbf{C}_1, & \hat{\mathbf{E}}_3 &= \hat{\mathbf{C}}_3 - \hat{\mathbf{E}}_2\mathbf{C}_3, \\
\tilde{\mathbf{E}}_2 &= \tilde{\mathbf{C}}_2\mathbf{C}_2^{-1}, & \tilde{\mathbf{E}}_1 &= \tilde{\mathbf{C}}_1 - \tilde{\mathbf{E}}_2\mathbf{C}_1, & \tilde{\mathbf{E}}_3 &= \tilde{\mathbf{C}}_3 - \tilde{\mathbf{E}}_2\mathbf{C}_3.
\end{aligned}$$

Substitution of (2.13) into (2.8) gives

$$\begin{aligned}
(2.15) \quad \mathbf{t}_{,2} + (\mathbf{K}_1 + \hat{\mathbf{K}}_1\hat{\mathbf{E}}_2)\mathbf{t}_{,1} + (\mathbf{K}_3 + \hat{\mathbf{K}}_3\hat{\mathbf{E}}_2 + \tilde{\mathbf{K}}_3\tilde{\mathbf{E}}_2)\mathbf{t}_{,3} \\
+ \mathbf{G}_1\mathbf{u}_{,11} + \mathbf{G}_2\mathbf{u}_{,13} + \mathbf{G}_3\mathbf{u}_{,33} = \mathbf{0},
\end{aligned}$$

where

$$(2.16) \quad \mathbf{G}_1 = \hat{\mathbf{K}}_1\hat{\mathbf{E}}_1, \quad \mathbf{G}_2 = \hat{\mathbf{K}}_1\hat{\mathbf{E}}_3 + \hat{\mathbf{K}}_3\hat{\mathbf{E}}_1 + \tilde{\mathbf{K}}_3\tilde{\mathbf{E}}_1, \quad \mathbf{G}_3 = \hat{\mathbf{K}}_3\hat{\mathbf{E}}_3 + \tilde{\mathbf{K}}_3\tilde{\mathbf{E}}_3.$$

The above derivations did not consider the thickness of the layer so that the results are valid regardless of whether the layer is thin or not.

Consider a piezoelectric layer of thickness  $h$  that is parallel to the plane  $x_2 = 0$ . If the layer is bonded to a piezoelectric body, then  $\mathbf{u}$ ,  $\mathbf{t}$ ,  $\mathbf{u}_{,1}$ ,  $\mathbf{u}_{,3}$ ,  $\mathbf{t}_{,1}$ , and  $\mathbf{t}_{,3}$  are continuous across the interface but not  $\mathbf{u}_{,2}$ ,  $\mathbf{t}_{,2}$ ,  $\hat{\mathbf{t}}$ , and  $\tilde{\mathbf{t}}$ .

### 3. Effective boundary conditions

Consider a thin piezoelectric layer, occupying the region  $-h \leq x_2 \leq 0$ , bonded to a half-space  $x_2 \geq 0$  of different piezoelectric materials and being a traction-free and electrically open at the surface  $x_2 = -h$ . The vector  $\mathbf{t}$  at the interface  $x_2 = 0$  can be approximated by

$$(3.1) \quad \mathbf{t}|_{x_2=0} = h\mathbf{t}_{,2}|_{x_2=0}$$

when terms of order higher than  $h$  are ignored. Note that both  $\mathbf{t}$  and  $\mathbf{t}_{,2}$  in (3.1) are evaluated at  $x_2 = 0$  of the thin layer. Substituting (3.1) into (2.15), the equilibrium equations of the thin layer at  $x_2 = 0$  become

$$(3.2) \quad \frac{1}{h}\mathbf{t} + (\mathbf{K}_1 + \hat{\mathbf{K}}_1\hat{\mathbf{E}}_2)\mathbf{t}_{,1} + (\mathbf{K}_3 + \hat{\mathbf{K}}_3\hat{\mathbf{E}}_2 + \tilde{\mathbf{K}}_3\tilde{\mathbf{E}}_2)\mathbf{t}_{,3} \\ + \mathbf{G}_1\mathbf{u}_{,11} + \mathbf{G}_2\mathbf{u}_{,13} + \mathbf{G}_3\mathbf{u}_{,33} = \mathbf{0}.$$

Indeed, Eq. (3.2) is the effective boundary condition at  $x_2 = 0$  since all terms are continuous across the interface. For two-dimensional deformations, another form of Eq. (3.2) in term of a generalized stress function will be derived in Section 5.1.

In case of a rigid conductor, at  $x_2 = -h$ ,  $\mathbf{u}|_{x_2=-h} = \mathbf{0}$ ; hence,  $\mathbf{u}_{,2}$  of the thin layer at the interface  $x_2 = 0$  can be approximated by

$$(3.3) \quad \mathbf{u}_{,2}|_{x_2=0} = \frac{1}{h}\mathbf{u}|_{x_2=0}$$

when terms of order higher than  $h$  are ignored. Therefore, (2.9)<sub>1</sub> becomes

$$(3.4) \quad \mathbf{u} = h(\mathbf{C}_2^{-1}\mathbf{t} - \mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{u}_{,1} - \mathbf{C}_2^{-1}\mathbf{C}_3\mathbf{u}_{,3})$$

at  $x_2 = 0$ . Again, all terms in (3.4) are continuous across the interface.

### 4. The Stroh formalism

The Stroh formalism has been extensively studied [14, 16] for two-dimensional problems in which the generalized displacement vector  $\mathbf{u}$  depends on  $x_1$  and  $x_2$

only. By defining  $\Phi^T = [\phi_1, \phi_2, \phi_3, \phi_4]$  as the generalized stress function, the general solutions to (2.1) and (2.2) can be written as

$$(4.1) \quad \mathbf{u} = \text{Im}\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\}, \quad \Phi = \text{Im}\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}.$$

The stress and electrical displacement are given by

$$(4.2) \quad \sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \quad D_1 = -\phi_{4,2}, \quad D_2 = \phi_{4,1}.$$

As  $u_k$  and  $\varphi$  are independent of  $x_3$ , one gets

$$(4.3) \quad u_{3,3} = 0, \quad E_3 = -\varphi_{,3} = 0.$$

Note that  $\sigma_{33}$  and  $D_3$  can be determined by (2.6) after all other solutions are determined;  $\text{Im}$  stands for the imaginary part and  $\mathbf{q}$  is an arbitrary constant vector,  $\mathbf{A}$  and  $\mathbf{B}$  are  $4 \times 4$  matrices, and  $\langle f(z_*) \rangle$  is a diagonal matrix defined as

$$(4.4) \quad \mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4], \\ \langle f(z_*) \rangle = \text{diag} [f(z_1), f(z_2), f(z_3), f(z_4)].$$

The function  $f(z_\alpha)$  is an arbitrary function of

$$(4.5) \quad z_\alpha = x_1 + p_\alpha x_2 \quad (\alpha = 1, 2, 3, 4).$$

Let  $p_\alpha$  and  $\mathbf{a}_\alpha$  satisfy the eigenrelation

$$(4.6) \quad [\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}] \mathbf{a} = \mathbf{0},$$

with

$$(4.7) \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}^E & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\omega_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}^E & \mathbf{e}_{21} \\ \mathbf{e}_{12}^T & -\omega_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}^E & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\omega_{22} \end{bmatrix}, \\ (\mathbf{Q}^E)_{ik} = C_{i1k1}, \quad (\mathbf{R}^E)_{ik} = C_{i1k2}, \quad (\mathbf{T}^E)_{ik} = C_{i2k2},$$

and vector  $\mathbf{e}_{ij}$  defined in (2.12)<sub>16</sub>. It is known that  $p$  cannot be real and there are four pairs of complex conjugates for  $p$ .  $p_\alpha$  ( $\alpha = 1, 2, 3, 4$ ) are the  $p$  with positive imaginary part and the associated eigenvectors are  $\mathbf{a}_\alpha$  ( $\alpha = 1, 2, 3, 4$ ), respectively. The vectors  $\mathbf{b}_\alpha$  ( $\alpha = 1, 2, 3, 4$ ) are related to  $\mathbf{a}_\alpha$  by

$$(4.8) \quad \mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -(\mathbf{R} + p^{-1}\mathbf{Q})\mathbf{a}.$$

The second equality in (4.8) follows from (4.6). Rewriting (4.8) into matrix form gives

$$(4.9) \quad \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = p \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix},$$

where

$$(4.10) \quad \mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}.$$

### 5. A thin piezoelectric layer bonded to a piezoelectric half-space

Let a piezoelectric half-space  $x_2 \geq 0$  be subject to a generalized line force  $\hat{\mathbf{f}}$  and a generalized line dislocation  $\tilde{\mathbf{b}}$  applied at

$$(5.1) \quad x_1 = 0, \quad x_2 = d > 0.$$

Let  $\hat{\mathbf{f}}$  and  $\tilde{\mathbf{b}}$  be defined as

$$(5.2) \quad \hat{\mathbf{f}}^T = [\mathbf{f}^T, -\lambda], \quad \tilde{\mathbf{b}}^T = [\hat{\mathbf{b}}^T, \varphi],$$

respectively, with  $\mathbf{f}$  being a line force,  $\lambda$  being a line charge, and  $\hat{\mathbf{b}}$  representing a Burgers vector. A thin piezoelectric layer of thickness  $h$  is bonded to the half-space and occupies the region

$$(5.3) \quad -h \leq x_2 \leq 0.$$

The solutions for the half-space are given by

$$(5.4) \quad \mathbf{u} = \frac{1}{\pi} \text{Im}\{\mathbf{A}\langle \ln(z_* - p_*d) \rangle \mathbf{q}^\infty + \mathbf{A}\langle \ln(z_* - \bar{p}_\beta d) \rangle \mathbf{q}_\beta + h\mathbf{A}\mathbf{g}(z_*)\},$$

$$(5.5) \quad \Phi = \frac{1}{\pi} \text{Im}\{\mathbf{B}\langle \ln(z_* - p_*d) \rangle \mathbf{q}^\infty + \mathbf{B}\langle \ln(z_* - \bar{p}_\beta d) \rangle \mathbf{q}_\beta + h\mathbf{B}\mathbf{g}(z_*)\},$$

with

$$(5.6) \quad \mathbf{g}(z_*) = \begin{bmatrix} g_1(z_1) \\ g_2(z_2) \\ g_3(z_3) \\ g_4(z_4) \end{bmatrix}.$$

The last term of both (5.4) and (5.5) are the correction terms for the bonded thin layer. The overbar denotes the complex conjugate and the repeated indices on  $\beta$  imply summation with  $\beta = 1, 2, 3, 4$ .  $\mathbf{g}(z_*)$  are functions to be determined. When  $h = 0$ , Eqs. (5.4) and (5.5) reduce to the general solutions of a half-space without a layer [17], in which

$$(5.7) \quad \mathbf{q}^\infty = \mathbf{A}^T \hat{\mathbf{f}} + \mathbf{B}^T \tilde{\mathbf{b}}.$$

For a traction-free and electrically open surface at  $x_2 = 0$ ,

$$(5.8) \quad \mathbf{q}_\beta = \mathbf{B}^{-1} \bar{\mathbf{B}} \mathbf{I}_\beta \bar{\mathbf{q}}^\infty.$$

In case of a rigid conductor at  $x_2 = 0$ ,

$$(5.9) \quad \mathbf{q}_\beta = \mathbf{A}^{-1} \bar{\mathbf{A}} \mathbf{I}_\beta \bar{\mathbf{q}}^\infty.$$

Matrices  $\mathbf{I}_\beta$  are defined as

$$(5.10) \quad \begin{aligned} \mathbf{I}_1 &= \text{diag} [1, 0, 0, 0], & \mathbf{I}_2 &= \text{diag} [0, 1, 0, 0], \\ \mathbf{I}_3 &= \text{diag} [0, 0, 1, 0], & \mathbf{I}_4 &= \text{diag} [0, 0, 0, 1]. \end{aligned}$$

At the interface  $x_2 = 0$ , Eq. (5.4) becomes

$$(5.11) \quad \mathbf{u}|_{x_2=0} = \frac{1}{\pi} \text{Im}\{\mathbf{A}\langle \ln(x_1 - p_*d) \rangle \mathbf{q}^\infty + \mathbf{A}\mathbf{q}_\beta \ln(x_1 - \bar{p}_\beta d) + h\mathbf{A}\mathbf{g}(x_1)\},$$

and can be simplified to

$$(5.12) \quad \mathbf{u}|_{x_2=0} = \frac{1}{\pi} \text{Im}\{(\mathbf{A}\mathbf{q}_\beta - \bar{\mathbf{A}}\mathbf{I}_\beta \bar{\mathbf{q}}^\infty) \ln(x_1 - \bar{p}_\beta d) + h\mathbf{A}\mathbf{g}(x_1)\},$$

as

$$(5.13) \quad \begin{aligned} \text{Im}\{\mathbf{A}\langle \ln(x_1 - p_*d) \rangle \mathbf{q}^\infty\} &= \text{Im}\{\mathbf{A}\mathbf{I}_\beta \mathbf{q}^\infty \ln(x_1 - p_\beta d)\} \\ &= -\text{Im}\{\bar{\mathbf{A}}\mathbf{I}_\beta \bar{\mathbf{q}}^\infty \ln(x_1 - \bar{p}_\beta d)\}. \end{aligned}$$

Likewise, Eq. (5.5) can be reduced to

$$(5.14) \quad \Phi|_{x_2=0} = \frac{1}{\pi} \text{Im}\{(\mathbf{B}\mathbf{q}_\beta - \bar{\mathbf{B}}\mathbf{I}_\beta \bar{\mathbf{q}}^\infty) \ln(x_1 - \bar{p}_\beta d) + h\mathbf{B}\mathbf{g}(x_1)\}$$

at the interface  $x_2 = 0$ .

### 5.1. When the surface $x_2 = -h$ of the thin piezoelectric layer is traction free and electrically open

For two-dimensional deformations, (2.15) reduces to

$$(5.15) \quad \Phi_{,12} + \mathbf{D}_1 \Phi_{,11} + \mathbf{G}_1 \mathbf{u}_{,11} = \mathbf{0},$$

with

$$(5.16) \quad \mathbf{D}_1 = (\mathbf{K}_1 + \hat{\mathbf{K}}_1 \hat{\mathbf{E}}_2),$$

and in use of (2.7)<sub>1</sub> and (4.2)<sub>2,4</sub>. Without lost in generality, (5.15) can be rewritten as

$$(5.17) \quad \Phi_{,2} + \mathbf{D}_1 \Phi_{,1} + \mathbf{G}_1 \mathbf{u}_{,1} = \mathbf{0}.$$

When the surface  $x_2 = -h$  of the piezoelectric layer is traction free and electrically open,  $\Phi|_{x_2=-h} = \mathbf{0}$ . If terms of order higher than  $h$  are ignored,

$\Phi_{,2}$  of the thin layer at the interface  $x_2 = 0$  can be approximated by  $\Phi/h$ . Therefore, (5.17) becomes

$$(5.18) \quad \Phi + h(\mathbf{D}_1\Phi_{,1} + \mathbf{G}_1\mathbf{u}_{,1}) = \mathbf{0}$$

at  $x_2 = 0$ . Equation (5.18) is another form of Eq. (3.2) for two-dimensional deformations.

Substitution of Eqs. (5.12) and (5.14) into (5.18), and use of (5.8) leads to

$$(5.19) \quad \text{Im}\{h(\mathbf{D}_1\mathbf{B} + \mathbf{G}_1\mathbf{A})\mathbf{g}'(x_1) + \mathbf{B}\mathbf{g}(x_1) + \mathbf{G}_1(\mathbf{A}\mathbf{q}_\beta - \bar{\mathbf{A}}\mathbf{I}_\beta\bar{\mathbf{q}}^\infty)(x_1 - \bar{p}_\beta d)^{-1}\} = \mathbf{0},$$

or simply

$$(5.20) \quad h(\mathbf{D}_1\mathbf{B} + \mathbf{G}_1\mathbf{A})\mathbf{g}'(x_1) + \mathbf{B}\mathbf{g}(x_1) + \mathbf{G}_1(\mathbf{A}\mathbf{q}_\beta - \bar{\mathbf{A}}\mathbf{I}_\beta\bar{\mathbf{q}}^\infty)(x_1 - \bar{p}_\beta d)^{-1} = \mathbf{0}.$$

Rearranging of Eq. (5.20) gives

$$(5.21) \quad \mathbf{g}'(x_1) + \frac{\mathbf{F}^{-1}}{h}\mathbf{g}(x_1) = \frac{\mathbf{F}^{-1}}{h}\mathbf{m}_\beta(x_1 - \bar{p}_\beta d)^{-1},$$

with

$$(5.22) \quad \mathbf{F} = \mathbf{B}^{-1}(\mathbf{D}_1\mathbf{B} + \mathbf{G}_1\mathbf{A}), \quad \mathbf{m}_\beta = -\mathbf{B}^{-1}\mathbf{G}_1(\mathbf{A}\mathbf{q}_\beta - \bar{\mathbf{A}}\mathbf{I}_\beta\bar{\mathbf{q}}^\infty).$$

By assuming that  $\lambda_i$  and  $\mathbf{y}_i$  ( $i = 1, 2, 3, 4$ ) are eigenvalues and eigenvectors of  $\mathbf{F}$ , then

$$(5.23) \quad \mathbf{F} = \mathbf{Y}\langle\lambda_*\rangle\mathbf{Y}^{-1}, \quad \mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4],$$

and (5.22) can be rewritten as

$$(5.24) \quad \mathbf{Y}^{-1}\mathbf{g}'(x_1) + \left\langle\frac{\lambda_*^{-1}}{h}\right\rangle\mathbf{Y}^{-1}\mathbf{g}(x_1) = \left\langle\frac{\lambda_*^{-1}}{h}\right\rangle\mathbf{Y}^{-1}\mathbf{m}_\beta(x_1 - \bar{p}_\beta d)^{-1}.$$

Notice that Eq. (5.24) is a first-order differential equation for  $\mathbf{Y}^{-1}\mathbf{g}(x_1)$ . TING [11] derived a similar first-order differential equation for  $\mathbf{g}(x_1)$  in case of general anisotropic elasticity. Following [11], the solution for  $\mathbf{Y}^{-1}\mathbf{g}(x_1)$  is given by

$$(5.25) \quad \mathbf{Y}^{-1}\mathbf{g}(x_1) = e^{-\langle\frac{\lambda_*^{-1}}{h}\rangle x_1} \int e^{\langle\frac{\lambda_*^{-1}}{h}\rangle x_1} \left\langle\frac{\lambda_*^{-1}}{h}\right\rangle \mathbf{Y}^{-1}\mathbf{m}_\beta(x_1 - \bar{p}_\beta d)^{-1} dx_1.$$

By defining

$$(5.26) \quad \gamma = x_1 - \bar{p}_\beta d,$$

(5.25) reduces to

$$(5.27) \quad \mathbf{Y}^{-1}\mathbf{g}(x_1) = \int \left\langle e^{-\frac{\lambda_*^{-1}}{h}(x_1 - \bar{p}_\beta d - \gamma)} \frac{\lambda_*^{-1}}{h} \right\rangle \mathbf{Y}^{-1}\mathbf{m}_\beta \gamma^{-1} d\gamma,$$

and can be further simplified as

$$(5.28) \quad \mathbf{Y}^{-1}\mathbf{g}(x_1) = \langle \eta_*(x_1 - \bar{p}_\beta d) \rangle \mathbf{Y}^{-1}\mathbf{m}_\beta,$$

with

$$(5.29) \quad \eta_k(\xi) = \frac{\lambda_k^{-1}}{h} \int_0^\xi e^{-\frac{\lambda_k^{-1}}{h}(\xi - \gamma)} \gamma^{-1} d\gamma \quad (k \text{ is not summed}).$$

Therefore,  $\mathbf{g}(x_1)$  is obtained as

$$(5.30) \quad \mathbf{g}(x_1) = \mathbf{Y} \langle \eta_*(x_1 - \bar{p}_\beta d) \rangle \mathbf{Y}^{-1}\mathbf{m}_\beta,$$

which is exactly in the same form as (4.15) given in [11] except that dimensions are increased to include the piezoelectric tensors. An explicit solution for  $\mathbf{g}(z_*)$  is then given by

$$(5.31) \quad \mathbf{g}(z_*) = \mathbf{Z}(z_* - \bar{p}_\beta d) \mathbf{Y}^{-1}\mathbf{m}_\beta.$$

Accordingly,  $\mathbf{Y} \langle \eta_*(x_1 - \bar{p}_\beta d) \rangle$  is explicitly written as

$$(5.32) \quad \mathbf{Y} \langle \eta_*(x_1 - \bar{p}_\beta d) \rangle = \begin{bmatrix} Y_{11}\eta_1(x_1 - \bar{p}_\beta d) & Y_{12}\eta_2(x_1 - \bar{p}_\beta d) & Y_{13}\eta_3(x_1 - \bar{p}_\beta d) & Y_{14}\eta_4(x_1 - \bar{p}_\beta d) \\ Y_{21}\eta_1(x_1 - \bar{p}_\beta d) & Y_{22}\eta_2(x_1 - \bar{p}_\beta d) & Y_{23}\eta_3(x_1 - \bar{p}_\beta d) & Y_{24}\eta_4(x_1 - \bar{p}_\beta d) \\ Y_{31}\eta_1(x_1 - \bar{p}_\beta d) & Y_{32}\eta_2(x_1 - \bar{p}_\beta d) & Y_{33}\eta_3(x_1 - \bar{p}_\beta d) & Y_{34}\eta_4(x_1 - \bar{p}_\beta d) \\ Y_{41}\eta_1(x_1 - \bar{p}_\beta d) & Y_{42}\eta_2(x_1 - \bar{p}_\beta d) & Y_{43}\eta_3(x_1 - \bar{p}_\beta d) & Y_{44}\eta_4(x_1 - \bar{p}_\beta d) \end{bmatrix},$$

and  $\mathbf{Z}(z_* - \bar{p}_\beta d)$  is defined as

$$(5.33) \quad \mathbf{Z}(z_* - \bar{p}_\beta d) = \begin{bmatrix} Y_{11}\eta_1(z_1 - \bar{p}_\beta d) & Y_{12}\eta_2(z_1 - \bar{p}_\beta d) & Y_{13}\eta_3(z_1 - \bar{p}_\beta d) & Y_{14}\eta_4(z_1 - \bar{p}_\beta d) \\ Y_{21}\eta_1(z_2 - \bar{p}_\beta d) & Y_{22}\eta_2(z_2 - \bar{p}_\beta d) & Y_{23}\eta_3(z_2 - \bar{p}_\beta d) & Y_{24}\eta_4(z_2 - \bar{p}_\beta d) \\ Y_{31}\eta_1(z_3 - \bar{p}_\beta d) & Y_{32}\eta_2(z_3 - \bar{p}_\beta d) & Y_{33}\eta_3(z_3 - \bar{p}_\beta d) & Y_{34}\eta_4(z_3 - \bar{p}_\beta d) \\ Y_{41}\eta_1(z_4 - \bar{p}_\beta d) & Y_{42}\eta_2(z_4 - \bar{p}_\beta d) & Y_{43}\eta_3(z_4 - \bar{p}_\beta d) & Y_{44}\eta_4(z_4 - \bar{p}_\beta d) \end{bmatrix},$$

or simply

$$(5.34) \quad Z_{ij}(z_* - \bar{p}_\beta d) = Y_{ij}\eta_j(z_i - \bar{p}_\beta d) \quad (i, j \text{ are not summed}).$$

### 5.2. When the surface $x_2 = -h$ of the thin piezoelectric layer is a rigid conductor

In case of a rigid conductor at  $x_2 = -h$ ,  $\mathbf{u}|_{x_2=-h} = \mathbf{0}$  and hence  $\mathbf{u}_2$  of the thin layer at the interface  $x_2 = 0$  is approximated by (3.3). For two-dimensional deformations, the effective boundary condition (3.4) reduces to

$$(5.35) \quad \mathbf{u} = h(\mathbf{C}_2^{-1}\mathbf{t} - \mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{u}_1)$$

at  $x_2 = 0$ . In use of (2.7)<sub>1</sub> and (4.2)<sub>2,4</sub>, (5.35) becomes

$$(5.36) \quad \mathbf{u} = h(\mathbf{C}_2^{-1}\Phi_{,1} - \mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{u}_1).$$

Substitution of (5.12) and (5.14) into (5.36), and use of (5.9) leads to

$$(5.37) \quad \text{Im}\{\mathbf{A}\mathbf{g}(x_1) - \mathbf{C}_2^{-1}(\mathbf{B}\mathbf{q}_\beta - \bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}^\infty)(x_1 - \bar{p}_\beta d)^{-1} \\ - h\mathbf{C}_2^{-1}\mathbf{B}\mathbf{g}'(x_1) + h\mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{A}\mathbf{g}'(x_1)\} = \mathbf{0}.$$

Again, without loss in generality, (5.37) simply becomes

$$(5.38) \quad h(\mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{A} - \mathbf{C}_2^{-1}\mathbf{B})\mathbf{g}'(x_1) + \mathbf{A}\mathbf{g}(x_1) = \mathbf{C}_2^{-1}(\mathbf{B}\mathbf{q}_\beta - \bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}^\infty)(x_1 - \bar{p}_\beta d)^{-1}.$$

With

$$(5.39) \quad \hat{\mathbf{F}} = \mathbf{A}^{-1}(\mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{A} - \mathbf{C}_2^{-1}\mathbf{B}), \quad \hat{\mathbf{m}}_\beta = \mathbf{A}^{-1}\mathbf{C}_2^{-1}(\mathbf{B}\mathbf{q}_\beta - \bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}^\infty),$$

(5.38) is rewritten as

$$(5.40) \quad \mathbf{g}'(x_1) + \frac{\hat{\mathbf{F}}^{-1}}{h}\mathbf{g}(x_1) = \frac{\hat{\mathbf{F}}^{-1}}{h}\hat{\mathbf{m}}_\beta(x_1 - \bar{p}_\beta d)^{-1}.$$

Notice that (5.21) and (5.40) are in exactly the same form except  $\mathbf{F}$  and  $\mathbf{m}_\beta$  are replaced by  $\hat{\mathbf{F}}$  and  $\hat{\mathbf{m}}_\beta$ , respectively. Hence, an explicit solution for  $\mathbf{g}(z_*)$  is

$$(5.41) \quad \mathbf{g}(z_*) = \hat{\mathbf{Z}}(z_* - \bar{p}_\beta d)\hat{\mathbf{Y}}^{-1}\hat{\mathbf{m}}_\beta,$$

with

$$(5.42) \quad \hat{Z}_{ij}(z_* - \bar{p}_\beta d) = \hat{Y}_{ij}\hat{\eta}_j(z_i - \bar{p}_\beta d) \quad (i, j \text{ are not summed}),$$

$$(5.43) \quad \hat{\eta}_k(\xi) = \frac{\hat{\lambda}_k^{-1}}{h} \int_0^\xi e^{-\frac{\hat{\lambda}_k^{-1}}{h}(\xi-\gamma)}\gamma^{-1}d\gamma \quad (k \text{ is not summed}),$$

and  $\hat{\lambda}_i$  and  $\hat{\mathbf{y}}_i$  ( $i = 1, 2, 3, 4$ ) are eigenvalues and eigenvectors of  $\hat{\mathbf{F}}$  defined as

$$(5.44) \quad \hat{\mathbf{F}} = \hat{\mathbf{Y}}\langle\hat{\lambda}_*\rangle\hat{\mathbf{Y}}^{-1}, \quad \hat{\mathbf{Y}} = [\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \hat{\mathbf{y}}_3, \hat{\mathbf{y}}_4].$$

## 6. Conclusions

The effective boundary conditions for a thin piezoelectric layer occupying the region  $-h \leq x_2 \leq 0$  are derived and given in (3.2) (or (5.18)) and (3.4) (or (5.36)) when the layer is very thin compared with a reference length. In [6], the reference length was chosen as the wavelength of the elastic waves. Here, the reference length is  $d$  defined in (5.1)<sub>2</sub>. In case of  $h \sim d$ , solutions in the piezoelectric layer cannot be ignored and have to be constructed for accuracy. When  $h \gg d$ , a bi-material problem should be considered instead. For  $h \ll d$ , the problem simply reduces to a general half-space issue as given in [17]. The range of validity of the derived solutions depends on the actual material constants being considered. Numeral analysis might be employed and compared with the exact solutions [2].

By following the works of TING [10, 11] on general anisotropic elastic media, the Green's function for a piezoelectric half-space bonded to a thin piezoelectric layer is constructed. As the piezoelectric layer is replaced by a set of effective boundary conditions and the Stroh formalism is employed, the analysis is simplified and the results can easily be utilized in both analytical and numerical methods. Especially, a simpler FEM program can be written which does not need to handle the piezoelectric layer.

From (5.8) and (5.14), the generalized stress function at the interface  $x_2 = 0$  is simply

$$(6.1) \quad \Phi|_{x_2=0} = \frac{h}{\pi} \text{Im}\{\mathbf{B}\mathbf{g}(x_1)\},$$

when the surface  $x_2 = -h$  of the piezoelectric layer is traction free and electrically open. Similarly, when the surface  $x_2 = -h$  of the piezoelectric layer is a rigid conductor, from (5.9) and (5.12), the generalized displacement vector at  $x_2 = 0$  is

$$(6.2) \quad \mathbf{u}|_{x_2=0} = \frac{h}{\pi} \text{Im}\{\mathbf{A}\mathbf{g}(x_1)\}.$$

With the identities

$$(6.3) \quad \mathbf{C}_1^T = \mathbf{K}_1 \mathbf{C}_2 + \hat{\mathbf{K}}_1 \hat{\mathbf{C}}_2, \quad \mathbf{Q} = \hat{\mathbf{K}}_1 \hat{\mathbf{C}}_1 + \mathbf{K}_1 \mathbf{C}_1,$$

and the use of (2.14)<sub>1,2</sub>,  $\mathbf{D}_1$  given in (5.16) can be replaced by

$$(6.4) \quad \mathbf{D}_1 = \mathbf{C}_1^T \mathbf{C}_2^{-1},$$

and  $\mathbf{G}_1$  given in (2.16)<sub>1</sub> can be rewritten as

$$(6.5) \quad \mathbf{G}_1 = \mathbf{Q} - \mathbf{C}_1^T \mathbf{C}_2^{-1} \mathbf{C}_1.$$

With (4.10) and identities  $\mathbf{C}_1 = \mathbf{R}^T$ ,  $\mathbf{C}_2 = \mathbf{T}$ , (6.4) and (6.5) can be further simplified as

$$(6.6) \quad \mathbf{D}_1 = -\mathbf{N}_1^T, \quad \mathbf{G}_1 = -\mathbf{N}_3.$$

Similar equality expression given by (6.6)<sub>2</sub> had been derived by WANG and PAN [18].

Notice that most of the derivations given here have corresponding equations in [10, 11] when the piezoelectric tensors are set to zero. For example, the governing equations (2.8) and (2.9) can be reduced to (2.5) and (2.7) in [11] for general anisotropic elastic media, respectively, as follows.

By setting the piezoelectric tensors equal to zero, (2.8) becomes

$$(6.7) \quad \mathbf{t}_{,2}^E + \mathbf{K}_1^E \mathbf{t}_{,1}^E + \mathbf{K}_3^E \mathbf{t}_{,3}^E + \hat{\mathbf{K}}_1^E \hat{\mathbf{K}} \hat{\mathbf{t}}_{,1}^E + \hat{\mathbf{K}} \hat{\mathbf{K}}_3^E \hat{\mathbf{K}} \hat{\mathbf{t}}_{,3}^E + \tilde{\mathbf{K}} \tilde{\mathbf{K}}_3^E \tilde{\mathbf{K}} \tilde{\mathbf{t}}_{,3}^E = \mathbf{0}.$$

With the following identities,

$$(6.8) \quad \hat{\mathbf{K}}_1^E \hat{\mathbf{K}} = \hat{\mathbf{K}}_1^E, \quad \hat{\mathbf{K}} \hat{\mathbf{K}}_3^E \hat{\mathbf{K}} + \tilde{\mathbf{K}} \tilde{\mathbf{K}}_3^E \tilde{\mathbf{K}} = \hat{\mathbf{K}}_3^E,$$

(6.7) is reduced to (2.5) in [11]. Similarly, (2.9) becomes

$$(6.9) \quad \begin{aligned} \mathbf{t}^E &= \mathbf{C}_1^E \mathbf{u}_{,1}^E + \mathbf{C}_2^E \mathbf{u}_{,2}^E + \mathbf{C}_3^E \mathbf{u}_{,3}^E, \\ \hat{\mathbf{K}} \hat{\mathbf{t}}^E &= \hat{\mathbf{K}} \hat{\mathbf{C}}_1^E \mathbf{u}_{,1}^E + \hat{\mathbf{K}} \hat{\mathbf{C}}_2^E \mathbf{u}_{,2}^E + \hat{\mathbf{K}} \hat{\mathbf{C}}_3^E \mathbf{u}_{,3}^E, \\ \tilde{\mathbf{K}} \tilde{\mathbf{t}}^E &= \tilde{\mathbf{K}} \hat{\mathbf{C}}_1^E \mathbf{u}_{,1}^E + \tilde{\mathbf{K}} \hat{\mathbf{C}}_2^E \mathbf{u}_{,2}^E + \tilde{\mathbf{K}} \hat{\mathbf{C}}_3^E \mathbf{u}_{,3}^E, \end{aligned}$$

by ignoring the piezoelectric tensors. Again, with the identity,

$$(6.10) \quad \hat{\mathbf{K}} + \tilde{\mathbf{K}} = \mathbf{I},$$

(6.9) becomes (2.7) in [11].

Also, with (6.4) and (6.5), it is easy to see that the effective boundary conditions (5.18) and (5.36) can also be reduced to their general anisotropic elastic media counterparts (4.10) and (4.22) in [11], respectively.

## Appendix

By assuming that the thickness  $h$  of the layer is very thin, all quantities in terms of order higher than  $h$  can be ignored. Consider a thin piezoelectric layer located at  $-h/2 \leq x_2 \leq h/2$ . The surfaces at  $x_2 = \pm h/2$  are traction free and electrically open. This implies that  $\mathbf{t}|_{x_2=\pm h/2} = \mathbf{0}$ . Indeed,  $\mathbf{t} = \mathbf{0}$  for all  $x_2$  if terms of order higher than  $h$  are ignored. (2.9)<sub>1</sub>, (2.13), and (2.15) reduces to

$$(A1) \quad \mathbf{u}_{,2} = -\mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{u}_{,1} - \mathbf{C}_2^{-1}\mathbf{C}_3\mathbf{u}_{,3},$$

$$(A2) \quad \hat{\mathbf{t}} = \hat{\mathbf{E}}_1\mathbf{u}_{,1} + \hat{\mathbf{E}}_3\mathbf{u}_{,3}, \quad \tilde{\mathbf{t}} = \tilde{\mathbf{E}}_1\mathbf{u}_{,1} + \tilde{\mathbf{E}}_3\mathbf{u}_{,3},$$

$$(A3) \quad \mathbf{G}_1\mathbf{u}_{,11} + \mathbf{G}_2\mathbf{u}_{,13} + \mathbf{G}_3\mathbf{u}_{,33} = \mathbf{0},$$

respectively. After solving  $\mathbf{u}(x_1, x_3)$  from (A3),  $\hat{\mathbf{t}}(x_1, x_3)$ ,  $\tilde{\mathbf{t}}(x_1, x_3)$ , and  $\mathbf{u}_{,2}$  can be obtained from (A2)<sub>1</sub>, (A2)<sub>2</sub>, and (A1), respectively. Components  $u_{1,2}$  and  $u_{3,2}$  measure the shear strains and  $u_{2,2}$  gives the changes in thickness of the thin layer. The electric field normal to the thin layer is given by  $-\varphi_{,2}$ .

Detailed discussions in case of an anisotropic elastic thin layer had been given by TING [10]. More related materials for piezoelectricity can be found in [19].

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Received May 17, 2013; revised version October 28, 2013.

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