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Brief Note

Cross-properties of the effective conductivity of the regular array of ideal conductors

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We present an accurate expression for the effective conductivity of a regular square-lattice arrangement of ideally conducting cylinders, valid for arbitrary concentrations. The formula smoothly interpolates between the two asymptotic expressions derived for low and high concentrations of the cylinders. Analogy with critical phenomena is suggested and taken to the extent of calculating the superconductivity critical exponent and the particle-phase threshold from the very long expansions in concentration. The obtained formula is valid for all concentrations including touching cylinders, hence it completely solves with high accuracy the problem of the effective conductivity for the square array.

Key words: inverse problem, R-linear problem, Laplace equation.

1. Introduction

We consider a two-dimensional composite corresponding to the regular square-lattice arrangement of ideally conducting unidirectional cylinders of radius r embedded into the matrix of a conducting material as shown in Fig. 1. The conductivity of the surrounding matrix is normalized to unity without loss of generality [23, 24].

The volume fraction (concentration) of cylinders is equal to $x = \pi r^2$ and corresponding periodicity cell is shown in Fig. 2. The effective conductivity does not depend on the linear sizes, hence it can be expressed through only one dimensionless geometrical parameter, the concentration x.

The effective conductivity along the unidirectional cylinders is infinite. The effective conductivity $\sigma(x)$ perpendicular to regular arrays of cylinders was dis-

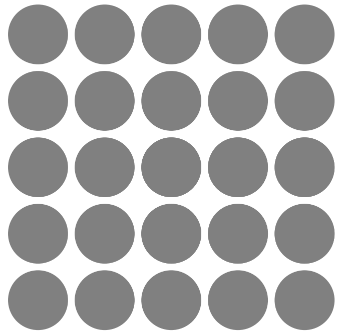

Fig. 1. Section perpendicular to the unidirectional cylinders.

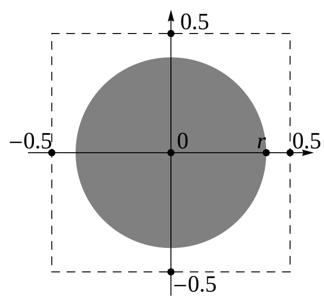

Fig. 2. Periodicity cell. Following [23, 24], we assume that it is the unit square and the radius r is given as a dimensionless value.

cussed by Maxwell [19] and Rayleigh [24] in the lowest orders in x. Their work was continued in [23], resulting in rather good numerical solutions.

Considerable efforts have been dedicated to the effective conductivity problem of regular composite when inclusions are ideal conductors and the concentration tends to the critical value $x_c = \pi/4$. Long series and rational approximations were proposed in [20–22, 25, 26] to estimate $\sigma(x)$ (see formula (2.2)). Two-point Padé approximants were applied in [1, 2, 29, 30] to derive upper and lower bounds on the effective conductivity. Asymptotically equivalent functions were constructed in [5] and applied to the percolating random media [3].

On the other hand, Keller [15] obtained in the regime of high concentrations, close to the particle-phase threshold x_c, the following expression for the effective conductivity:

$$\sigma(x) \simeq \frac{\pi^{3/2}}{2\sqrt{\pi/4 - x}}. \tag{1.1}$$

This formula was supplemented in [18] by a constant term

$$\sigma(x) \simeq \frac{\pi^{3/2}}{2\sqrt{\pi/4 - x}} - \pi + 1. \tag{1.2}$$

Formula (1.2) follows from equation (49) in [18] for a dipole coefficient when the conductivity of inclusions tends to infinity.

In all previous body of work such expressions for $\sigma(x)$ valid near x_c, were not matched with the series expansions at $x = 0$, though tight bounds were obtained in [2, 29, 30] for a sufficiently large x.

We suppose that previous failure to construct the global approximation, is related to the principal impossibility of uniform approximation by rational functions on the closed segment $[0, \frac{\pi}{4}]$ of the function $\sigma(x)$ having a fractional power singularity.

Our primary goal is to overcome such difficulty and derive an accurate, compact expression for the effective conductivity of such systems, valid for arbitrary concentrations. Moreover, we demonstrate that the sufficiently long series near $x = 0$ capture implicitly the critical value $x_c = \frac{\pi}{4}$ and even the type of singularity. This approach may be applied to the random materials as well [11].

The critical behavior of regular composites is practically never mentioned together with other critical phenomena [27]. It is remarkable that a relatively "different" Laplace equation for the potential, when complemented with a non-trivial boundary conditions in the regular domain of inclusions (see, e.g., [25]), behaves critically even without explicit non-linearity or randomness, typical to the phase transitions and percolation phenomena [27]. The series (2.2) and the critical behavior with the so-called superconductivity critical index $s = 1/2$ [31], and critical amplitude $A = \pi^{3/2}/2$, will be merged into a single, unified formula.

From the phase interchange theorem [16] it follows that in two dimensions, the superconductivity index is equal to the conductivity index [31]. We also demonstrate how the most typical characteristics of the critical phenomena, the threshold value and the superconductivity critical index, can be calculated directly from the series (2.2). To this end we primarily apply properly modified techniques of Padé approximants, widely used in the theory of critical phenomena [6, 7, 33].

The effective conductivity $\sigma(x)$ is an analytic function in x. In general case of a two-phase composite the so-called contrast parameter should be also included into consideration explicitly, see, e.g., [10].

We are interested here in the case of a high-contrast regular composites, when the conductivity of the inclusions is much larger than the conductivity of the host. That is, the highly conducting inclusions are replaced by the ideally conducting inclusions with infinite conductivity. In this case, the contrast parameter is equal to unity and remains implicit.

We also restrict our study to the two-dimensional case, which is still considerably interesting, both for practical [8] and physical reasons [23, 32]. In practice, the composite often consists of a uniform background-host reinforced by a large number (high concentration) of unidirectional rod- or fiber-like inclusions.

On the other hand, two-dimensional regular composites much closer resemble the two-dimensional random composites, than their respective 3D counterparts do [32]. The tendency to order in the two-dimensional random system of disks is a crucial feature in the theory of composites at high concentrations. The effective conductivity of random 2D high-contrast composites also exhibits critical behavior with critical index $s = 1.3$ [31].

One can even think about a dependence of the critical index on the degree of randomness, but the regular case should be studied first in the same framework as random. While classical renormalization group may not be useful in the random case [9], “the age-old method of series expansions" remains the only alternative. But it can be applied confidently only if the sufficiently long series in concentration were available.

The considered problem can be formulated as follows. Given the polynomial approximation of the function $\sigma(x)$, to estimate the convergence radius x_c of the Taylor series of $\sigma(x)$, and to determine parameters of the asymptotically equivalent approximation near $x = x_c$, for the sought quantities, we are going to obtain numerical sequences of approximations, and make conclusions based on convergence or semi-convergence of various sequences.

2. Critical point, square lattice

The normalized effective conductivity $\sigma(x)$ is exactly expressed by the complicated expansion [21, 22].

$$\sigma(x) = 1 + 2x + 2x^2 + \frac{2x^2}{\pi} \sum_{k=1}^{\infty} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \sigma_{m_1}^{(1)} \sigma_{m_2}^{(m_1)} \ldots \sigma_{m_k}^{(m_{k-1})} \sigma_1^{(m_k)} \left(\frac{x}{\pi}\right)^{2(m_1+m_2+\cdots+m_k)-k},$$

where

$$\sigma_k^{(n)} = \frac{(2n+2k-3)!}{(2n-1)!(2k-2)!} S_{2(n+k-1)}. \tag{2.1}$$

The lattice sums $S_{2(n+k-1)}$ can be computed by recurrence equations [21, 22]. Below, this expansion is presented in the truncated numerical form

$$\begin{aligned} \sigma(x) = {} & 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2.6116556664543236x^5 \\ & + 3.2233113329086476x^6 + 3.8349669993629707x^7 \\ & + 4.446622665817295x^8 + 5.272062706160579x^9 \\ & + 6.284564073656707x^{10} + 7.484126768305675x^{11} \end{aligned} \tag{2.2}$$

$$+\,8.870750790107483x^{12} + 11.376487693870311x^{13}$$
$$+\,14.200048348474557x^{14} + 17.39864131428396x^{15}$$
$$+\,21.029475151662286x^{16} + 26.277897666772887x^{17}$$
$$+\,32.66592466836545x^{18} + 40.31075089986001x^{19}$$
$$+\,49.34706707473475x^{20} + 62.10071128310876x^{21}$$
$$+\,77.45700559231325x^{22} + 95.88562104938396x^{23}$$
$$+\,117.89818526393154x^{24} + 149.04923618206843x^{25}$$
$$+\,186.9598285755466x^{26} + O(x^{27}).$$

Can we really extract a purely geometrical quantity, such as x_c, from the expansion (2.2) for the physical quantity $\sigma(x)$? The natural condition on x_c is to assume that it coincides with the point where $\sigma(x)$ diverges.

2.1. Padé approximants

Probably the simplest way to estimate the position of a critical point, is to apply the diagonal Padé approximants,

$$p_1^1(x) = \frac{A_1 x + 1}{B_1 x + 1}, \qquad p_2^2(x) = \frac{A_2 x^2 + A_1 x + 1}{B_2 x^2 + B_1 x + 1}, \ \ldots \tag{2.3}$$

Padé approximants locally are the best rational approximations of power series. Their poles determine singular points of the approximated functions [28]. Calculations with Padé approximants are straightforward and can be performed with $Mathematica^{®}$. They do not require any preliminary knowledge of the critical index and we have to find the position of a simple pole. There is a convergence within the approximations for the critical point generated by the sequence of Padé approximants, corresponding to their order increasing:

$$x_1 = 1, \quad x_2 = 1, \quad x_3 - n.a., \quad x_4 = 0.84447, \quad x_5 = 0.842471,$$
$$x_6 = 0.842781, \quad x_7 = 0.842471, \quad x_8 = 0.8446, \quad x_9 = 0.804535,$$
$$x_{10} = 0.804611, \quad x_{11} = 0.804535, \quad x_{12} = 0.80451, \quad x_{13} = 0.804536.$$

The percentage error given by the last approximant in the sequence equals to 2.43649%.

2.2. Quasi-rational Padé approximants

We assume that incorporating explicitly the known value of critical index, will lead to improved accuracy in the threshold estimates. Let us use asymptotically equivalent functions in the form of the quasi-rational (diagonal) Padé approximants [7],

$$P_1(x) = \sqrt{\frac{A_1x+1}{B_1x+1}}, \qquad P_2(x) = \sqrt{\frac{A_2x^2+A_1x+1}{B_2x^2+B_1x+1}}, \ldots \tag{2.4}$$

The corresponding sequence of approximate values for the critical point is given as follows:

$$x_1 = 1/2, \quad x_2 = 1, \quad x_3 = 0.728069, \quad x_4 = 0.75074, \quad x_5 - n.a., \quad x_6 - n.a.,$$
$$x_7 = 0.192259, \qquad x_8 = 0.779385, \qquad x_9 = 0.77997.$$

The percentage error given by the last approximant equals to 0.691135%. In the next order there are two solutions $x_{10}^{(1)} = 0.7781474$ and $x_{10}^{(2)} = 0.643305$, and the computations were stopped.

We suggest that further increase in accuracy is limited by "flatness" of the coefficients values in four starting orders of (2.2). We also consider another sequence of power-transformed Padé approximants, multiplied with the Clausius-Mossotti-type expression,

$$P_1^t(x) = \frac{(1-x)\sqrt{\frac{A_1x+1}{B_1x+1}}}{x+1}, \qquad P_2^t(x) = \frac{(1-x)\sqrt{\frac{A_2x^2+A_1x+1}{B_2x^2+B_1x+1}}}{x+1}, \ \ldots \tag{2.5}$$

The transformation which lifts the flatness, does improve convergence of the sequence of approximations for the threshold,

$$x_1 = 1/4, \quad x_2 - n.a, \quad x_3 = 0.568452, \quad x_4 - n.a,$$
$$x_5 = 0.826561, \quad x_6 = 0.803947, \quad x_7 = 0.827349,$$
$$x_8 = 0.750544, \quad x_9 = 0.762065, \quad x_{10} = 0.78493.$$

The percentage error achieved for the last point is equal 0.0596084%. The calculations were stopped here because of the second solution emergence at $x = 0.887047$. Although it does not interfere with the correct result x_{10}, such a branching gives a natural signal to stop.

3. Critical index s

The series from [22] approximated by (2.2) diverge as $x \to x_c$. Is it possible to evaluate the character of singularity as $x \to x_c$ from the series (2.2), assuming only that it is a power-law?

Let us apply the following transformation:

$$z = \frac{x}{x_c - x} \ \Leftrightarrow \ x = \frac{zx_c}{z+1} \tag{3.1}$$

to the original series (2.2). To such transformed series $M_1(z)$ let us apply the D-Log transformation (differentiate Log of $M_1(z)$) and call the transformed series $M(z)$. In terms of $M(z)$ one can readily obtain the sequence of approximations s_n for the critical index s,

$$s_n = \lim_{z\to\infty} (z\ PadeApproximant[M[z], n, n+1]). \tag{3.2}$$

Namely, $s_2 = 0.627161$, $s_4 = 0.633025$, $s_6 = 0.597334$, $s_8 = 0.550467$, $s_{10} = 0.551316$, $s_{12} = 0.535407$. One may expect that adding more terms to the expansion (2.2) will also improve an estimate for s.

In order to accelerate convergence of the method we suggest adapting the technique of corrected approximants [13]. Let us first estimate the critical index by simplest non-trivial factor approximant [14, 34],

$$f_3(x) = (B_1 x + 1)^{s_1} \left(1 - \frac{x}{x_c}\right)^{-s_0}, \tag{3.3}$$

where $s_0 = 0.59928, s_1 = 1.57496, B_1 = 0.785398$.

Let us divide original series (2.2) by $f_3(x)$, apply to the newly found series transformation (3.1), then apply D-Log transformation and call the transformed series $K(z)$. Finally, one can obtain the following sequence of corrected approximations for the critical index:

$$s_n = s_0 + \lim_{z\to\infty} (z\ PadeApproximant[K[z], n, n+1]). \tag{3.4}$$

The following corrected sequence of approximate values for the critical index can be calculated readily: $s_4 = 0.62284$, $s_5 = 0.135551$, $s_6 = 0.586051$, $s_7 = 0.559834$, $s_8 = 0.54658$, $s_9 = 0.553502$, $s_{10} = 0.549611$, $s_{11} = 0.555069$, $s_{12} = 0.503875$.

4. Crossover formula for all concentrations

Our suggestion for the conductivity formula valid for all concentrations is based on the following considerations. Let us first calculate the critical amplitude A. To this end let us again apply transformation (3.1) to the original series to obtain $M_1(z)$ as above. Then apply to $M_1(z)$ another transformation to get $T(z) = M_1(z)^{-1/s}$, in order to get rid of the square root behavior at infinity. In terms of $T(z)$ one can readily obtain the sequence of approximations A_n for the critical amplitude A,

$$A_n = x_c^s \lim_{z\to\infty} (z\ PadeApproximant[T[z], n, n+1])^{-s}; \tag{4.1}$$

$A_1 = 2.28682$, $A_2 = 2.24389$, $A_3 = 2.4418$, $A_4 = 2.57419$, $A_5 = 2.35515$, $A_6 = 2.34677$, $A_7 = 2.52728$, $A_8 = 2.63203$, $A_9 = 2.69504$, $A_{10} = 2.62364$, $A_{11} = 2.55292$, $A_{12} = 2.55224$.

The ninth member of the sequence gives the best result for amplitude. Corresponding approximant to $\sigma(x)$, satisfying 19 starting terms from (2.2), can be readily written as follows:

$$\sigma^* = \sqrt{\frac{zQ_3(z)}{Q_1(z)}}, \tag{4.2}$$

where

$$\begin{aligned} Q_1(z) &= z(z(z(z(zQ_2(z)+61.8908)+27.588)+7.03611)+0.773878),\\ Q_2(z) &= z(z(z((0.10813z+2.8373)z+16.885)+47.592)+80.664)+87.607,\\ Q_3(z) &= z(z(z(z(zQ_4(z)+325.32)+160.471)+51.0803)+9.46732),\\ Q_4(z) &= z(z(z(z(z+20.1693)+97.8524)+253.571)+411.488)+444.039. \end{aligned} \tag{4.3}$$

Substitution of (3.1) in (4.2) yields

$$\sigma^* = 2.09382\sqrt{\frac{xR_3(x)}{(\pi-4x)R_1(x)}}, \tag{4.4}$$

where

$$\begin{aligned} R_1(x) &= x(x(x(x(xR_2(x)-0.985604)+0.83488)-0.0555036)\\ &\qquad -0.473743),\\ R_2(x) &= x(x(x(x(1.39918\;-x)-0.632586)+0.648903)-0.882948)\\ &\qquad +1.45321,\\ R_3(x) &= x(x(x(x(xR_4(x)+0.53774)+0.19359)-0.49707)-0.96545)\\ &\qquad -0.33948,\\ R_4(x) &= x(x(x(x(x+1.44888)+0.504626)+0.0358506)+0.24819)\\ &\qquad +0.921202. \end{aligned} \tag{4.5}$$

For all practical purposes, this expression, which is a crossover between the low-concentration and high-concentration regimes, is indistinguishable from the modified Padé-based form,

$$\sigma^* = \sqrt{\frac{zS_3(z)}{S_1(z)}}, \tag{4.6}$$

where

$$
\begin{aligned}
S_1(z) &= z(z(z(z(zS_2(z) + 110.63) + 52.5492) + 14.5119) + 1.78014),\\
S_2(z) &= z(z(z((0.101321z + 3.76147)z + 24.5539) + 73.2536) + 130.155)\\
&\quad + 148.405,\\
S_3(z) &= z(z(z(z(zS_4(z) + 577.474) + 300.114) + 101.332) + 20.1044)\\
&\quad + 1.78014,\\
S_4(z) &= z(z(z(z(z + 27.4454) + 144.076) + 392.701) + 666.22)\\
&\quad + 751.704,
\end{aligned}
\tag{4.7}
$$

or

$$
\sigma^* = 1.91373\sqrt{\frac{xT_3(x)}{(\pi - 4x)T_1(x)}}, \tag{4.8}
$$

where

$$
\begin{aligned}
T_1(x) &= x(x(x(x(xT_2(x) + 0.477564) - 0.489113) + 1.0752) - 0.995924),\\
T_2(x) &= x(x(x(x(0.743958 - x) + 0.478232) - 0.398804) + 0.19758)\\
&\quad + 0.308858,\\
T_3(x) &= x(x(x(x(xT_4(x) + 1.21937) + 0.397015) - 0.388154)\\
&\quad - 1.40718) - 0.854307,\\
T_4(x) &= x(x(x(x(x + 2.24302) + 1.24708) + 0.216355) + 0.517846)\\
&\quad + 1.25935.
\end{aligned}
\tag{4.9}
$$

The approximant (4.8) matches starting 18 terms from (2.2) and asymptotic form (1.1) (18 + 1), not unlike [4]. There is a clear convergence to (18+1), in the sub-sequence of approximants (2+1), (6+1), (8+1), (16+1) , (18+1); while all other approximants are discontinuous.

From the crossover formula (4.8) one can readily obtain the higher-order coefficients [35], not employed in the final formula,

$$
\begin{aligned}
&a_{19} = 40.5543, \quad a_{20} = 49.768, \quad a_{21} = 62.9092, \quad a_{22} = 78.5999,\\
&a_{23} = 98.0859, \quad a_{24} = 121.136, \quad a_{25} = 152.608, \quad a_{26} = 190.893,
\end{aligned}
$$

in a fairly good agreement with the original series (2.2).

Analytical expression, deduced in [23], is the most appropriate for comparison with our suggestion,

$$
\sigma(x) \approx 1 + \frac{2x}{1 - x - 0.013362x^8 - \frac{0.305827x^4}{1-1.1403x^8}}. \tag{4.10}
$$

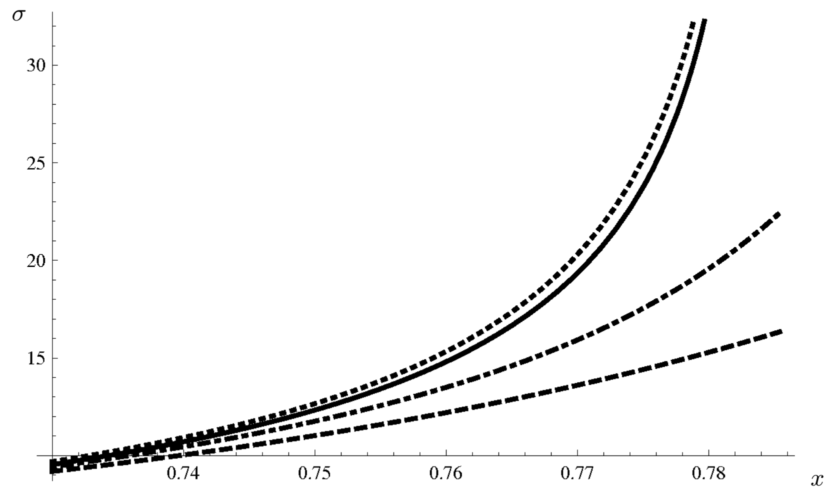

FIG. 3. Our formula (4.8) (solid line) is compared with (4.10) (dashed-dotted line), asymptotic expression (1.2) (dotted line) and expansion (2.2) (dashed line).

The bounds for $\sigma(x)$ from [2] give similar numerical results for not critically high concentrations. Formula (4.8) deviates from the expression (4.10) in the region of concentrations $(0.7, x_c)$. Both formulae are positioned significantly higher than expansion (2.2). Most importantly, formula (4.8) smoothly interpolates between the two asymptotic expressions through the whole crossover region of $(0.7, x_c)$, as shown in Fig. 3.

5. Expansion near the threshold

Throughout the paper we derived the expressions for the cross-properties extending the series from small x to large x on the basis of the polynomial formula (2.2). Alternatively, one can proceed to extend the series from the large x (close to x_c) to small x on the basis of (1.2) [12]. The simplest way to proceed is to look for the solution in the whole region $[0, x_c)$ in the form

$$\sigma = \alpha_1 (x_c - x)^{-1/2} + \alpha_2, \tag{5.1}$$

and obtain the unknowns from the two starting terms of (2.2), namely $\sigma \simeq 1+2x$. Then, $\alpha_1 = A$ (exact!), $\alpha_2 = (1-\pi)$, same form as obtained in [17]. See also [17] for a discussion of different approximate asymptotic expressions.

In order to improve (5.1), let us divide (series (2.2) by the approximant (5.1), apply to the new series transformation (3.1) and then use diagonal Padé (or two-point Padé) approximants. The following expression obtained from the simplest non-trivial two-point Padé approximant, satisfies five starting terms from (2.2) and Keller's asymptotic expression,

$$\sigma = \frac{2.05897(x(x(x+0.912641)+0.356584)+0.349627)(1.30004-\sqrt{0.785398-x})}{(x(x(x+0.847132)+0.342827)+0.336139)\sqrt{0.785398-x}}. \tag{5.2}$$

Another expression, obtained from the simplest non-trivial diagonal Padé approximant, satisfying the same five terms, but with slightly higher value of amplitude $A = 2.80931$,

$$\sigma = \frac{2.17708(x(x+0.414669)+0.18083)\left(1.30004-\sqrt{0.785398-x}\right)}{(x(x+0.42154)+0.183827)\sqrt{0.785398-x}}, \tag{5.3}$$

works even better than (5.2). Formula (5.3) is almost as good as (4.4).

5.1. Corrected critical index

Let us first estimate the critical index from the expression (5.1), not assuming that we know its value. To this end we have to employ one more coefficient from (2.2). Then,

$$\sigma = \frac{2.39749}{(0.785398-x)^{0.570796}} - 1.75194. \tag{5.4}$$

To correct the value of index we apply literally the procedure leading to the expression (3.4), starting with $s_0 = 0.570796$. The following sequence of approximate values for the critical index was found: $s_2 = 0.470413$, $s_3 = 0.523229$, $s_4 = 0.553386$, $s_5 = 0.52638$, $s_6 = 0.526343$, $s_7 = 0.513666$, $s_8 = 0.508803$, $s_9 = 0.514609$, $s_{10} = 0.511196$, $s_{11} = 0.51180$, $s_{12} = 0.458817$.

5.2. Corrected threshold

Let us, in addition, estimate the threshold from the expression (5.1), not assuming that we know its value in advance. To this end we again have to use the three starting terms from (2.2), $\sigma \simeq 1 + 2x + 2x^2$. All three unknowns,$x_c$, α_1, α_2 may be found explicitly, leading to the following approximant:

$$\sigma = \frac{3\sqrt{3}}{2\sqrt{\frac{3}{4}-x}} - 2, \tag{5.5}$$

with approximate threshold value of $x_0 = 3/4$. Let us look for the solution in the same form but with an exact, yet unknown threshold X_c,

$$\sigma' = \frac{3\sqrt{3}}{2\sqrt{X_c - x}} - 2. \tag{5.6}$$

From here one can express formally,

$$X_c = \frac{4x\sigma'(x)^2 + 16x\sigma'(x) + 16x + 27}{4(\sigma'(x)+2)^2}, \tag{5.7}$$

since $\sigma'(x)$ is also unknown. All we can do is to use for σ' the series (2.2), so that instead of a true threshold, we have an effective threshold,

$$\begin{aligned} X_c(x) = {} & 3/4 + x^3/9 + (2x^4)/27 - 0.268791x^5 + 0.0164609x^6 \\ & + 0.108801x^7 + 0.0933591x^8 - 0.0491905x^9 + 0.0306117x^{10} \\ & + 0.0816431x^{11} + \ldots, \end{aligned} \tag{5.8}$$

which should become a true threshold X_c as $x \to X_c$! Moreover, let us apply resummation procedure to the expansion (5.8) using factor approximants $F^*(x)$, and define the sought threshold X_c^* self - consistently,

$$X_c^* = \frac{3}{4} + \frac{x^3}{9} F^*(X_c^*), \tag{5.9}$$

as we approach the threshold the RHS should become the threshold. Since factor approximants are defined as F_k^* for arbitrary number of terms k, we will also have a sequence of $X_{c,k}^*$. For example,

$$\begin{aligned} F_2^* &= (7.92402x + 1)^{0.0841324}, \\ F_4^* &= (1 - 0.858374x)^{1.86127}(1.72735x + 1)^{1.31087}. \end{aligned} \tag{5.10}$$

Solving (5.9), we obtain $X_{c,2}^* = 0.823548$, $X_{c,4}^* = 0.770537$, $X_{c,6}^* = 0.778053$, There is no solution in the next even order.

One can also define a sequence of “odd" factor approximants [36], starting from $F_1 = (1 + (2x)/3)$ (where $2/3$ comes as the ratio of fourth- and third-order coefficients in the series (5.8)). The next-order odd approximant,

$$F_3^* = \frac{\left(\frac{2x}{3} + 1\right)^{31.0546}}{(0.425194x + 1)^{47.1229}}, \tag{5.11}$$

brings the most accurate value of the threshold, $X_{c,3}^* = 0.782355$, while in the higher orders, $X_{c,5}^* = 0.774622$, $X_{c,7}^* = 0.779692$. The percentage error achieved for the last point is equal -0.727%.

6. Concluding remarks

What is the significance of the long expansions in concentration for regular composites?

In this paper, we considered three different problems for the regular square lattice arrangements of ideally conducting cylinders and concluded that a series (2.2) is good enough to: calculate the position of a threshold for the effective conductivity, to calculate the value of a superconductivity critical index and to obtain an accurate crossover expression valid for arbitrary concentrations.

There are strong indications [10] that similar problems for random composites can be addressed along the same lines, although various techniques based on Padé approximants employed in the regular case are expected to fail. The obtained formula (4.2)–(4.3) is valid for all concentrations including touching cylinders, hence it completely solves the problem of the effective conductivity for the square array.

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