Arch. Mech., **66**, 4, pp. 257–267, Warszawa 2014

Application of fractional order theory of thermoelasticity to a 1D problem for a cylindrical cavity

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In this work, we apply the fractional order theory of thermoelasticity to a 1D problem of an infinitely long cylindrical cavity. Laplace transform techniques are used to solve the problem. Numerical results are computed and represented graphically for the temperature, displacement and stress distributions.

Key words: fractional calculus, cylindrical cavity, thermoelasticity.

1. Introduction

Biot [1] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. Lord and Shulman [2] introduced the theory of generalized thermoelasticity with one relaxation time by using the Maxwell–Cattaneo law of heat conduction instead of the conventional Fourier law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. Sherief and El-Maghraby solved some crack problems for this theory [3, 4]. Sherief and Hamza have obtained the solution of axisymmetric problems in spherical regions in [5] and in cylindrical regions in [6]. Sherief and Ezzat have obtained the solution in the form of series in [7]. Sherief *et al.*, in addition, extended this theory to deal with micropolar materials in [8]. This theory was extended to deal with viscoelastic effects in [9]. Recently, Sherief and Hussein developed the theory of generalized poro-thermoelasticity [10].

Fractional calculus has been used successfully to modify many existing models of physical processes [11]–[13]. One can state that the whole theory of fractional derivatives and integrals was established in the second half of the nineteenth century. Caputo and Mainardi [14, 15] and Caputo [16] obtained good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity. ADOLFSSON *et al.* [17] constructed a new fractional order model of viscoelasticity.

POVSTENKO [18] proposed a review of thermoelasticity that uses fractional heat conduction equation. He also proposed and investigated new models that use fractional derivative [19]–[22]. Recently, the fractional order theory of thermoelasticity was derived by SHERIEF *et al.* [23]. It is a generalization of both the coupled and the generalized theories of thermoelasticity. Other works in the subject are [24, 25].

The main reason behind the introduction of the fractional theory is that it predicts retarded response to physical stimuli, as is found in nature, as opposed to instantaneous response predicted by the generalized theory of thermoelasticity.

2. Formulation of the problem

In this work, we consider a 1D problem for an infinite medium with a cylindrical cavity of radius a, using the fractional theory of thermoelasticity. The surface of the cavity is taken to be traction-free and is subjected to a thermal shock that is a function of time.

The governing equations are given by [23]

$$(\lambda + 2\mu)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial e}{\partial r}\right) - \gamma\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \rho\frac{\partial^2 e}{\partial t^2}, \tag{2.1}$$

$$k\nabla^2 T = \frac{\partial}{\partial t}\left(1 + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho c_E T + \gamma T_0 e\right), \tag{2.2}$$

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e\delta_{ij} - \gamma\left(T - T_0\right)\delta_{ij}, \tag{2.3}$$

where T is the absolute temperature, ρ is the density, the constants λ and μ are Lamé's constants and $\gamma = \alpha_t(3\lambda + 2\mu)$where α_t is the coefficient of linear thermal expansion. T_0 is a reference temperature assumed to be such that $|(T - T_0)/T_0| \ll 1$ and α, τ_0 are constants such that $\tau_0 > 0$, $0 \le \alpha \le 1$, c_E is the specific heat per unit mass in the absence of deformation and k is the thermal conductivity, e is the cubical dilatation. σ_{ij} and $e_{ij}$are the components of the stress and strain tensors, respectively.

We shall use the following non-dimensional variables:

$$r' = c\eta r, \qquad u' = c\eta u, \qquad t' = c^2\eta t, \qquad \tau_0' = c^{2\alpha}\eta^\alpha\tau_0,$$
$$a' = c\eta a, \qquad \theta = \frac{\gamma(T - T_0)}{\lambda + 2\mu}, \qquad \sigma_{ij}' = \frac{\sigma_{ij}}{\mu},$$

where $c = \sqrt{\lambda + 2\mu/\rho}$, $\eta = \rho c_E/k$.

The governing equation, in the absence of body forces, in non-dimensional form is given by

$$\nabla^2 e - \nabla^2 \theta = \frac{\partial^2 e}{\partial t^2}, \tag{2.4}$$

$$\nabla^2 \theta = \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\frac{\partial \theta}{\partial t} + \varepsilon \frac{\partial e}{\partial t}\right), \tag{2.5}$$

$$\sigma_{rr} = \frac{\left(\beta^2 - 2\right)}{r} u + \beta^2 \frac{\partial u}{\partial r} - \beta^2 \theta, \tag{2.6}$$

where $\varepsilon = T_0 \gamma^2/(\lambda + 2\mu)k\eta$, $\beta^2 = (\lambda + 2\mu)/\mu$.

In the above equation, the time fractional derivative of order α used is taken to be in the sense of Caputo's fractional derivative.

We assume that the boundary conditions have the form

$$\theta = f(t) \qquad \text{at } r = a, \tag{2.7}$$

$$\sigma_{rr} = 0 \qquad \text{at } r = a. \tag{2.8}$$

We assume that the initial conditions are quiescent.

3. Solution in the Laplace transform domain

Applying the Laplace transform with parameter s defined by the relation

$$\overline{f}(x, s) = \int_0^\infty e^{-st} f(x, t) dt$$

to both sides of equations (2.4)–(2.6), we get the following equations:

$$(\nabla^2 - s^2)\overline{e} = \nabla^2 \overline{\theta}, \tag{3.1}$$

$$\left[\nabla^2 - s(1 + \tau_0 s^\alpha)\right] \overline{\theta} = (1 + \tau_0 s^\alpha) s \varepsilon \overline{e}, \tag{3.2}$$

$$\overline{\sigma}_{rr} = \frac{(\beta^2 - 2)}{r} \overline{u} + \beta^2 \frac{\partial \overline{u}}{\partial r} - \beta^2 \overline{\theta}. \tag{3.3}$$

Eliminating $\overline{\theta}$ between equations (3.1) and (3.2), we get

$$\left\{\nabla^4 - \nabla^2[s^2 + (1 + \varepsilon)s(1 + \tau_0 s^\alpha)] + s^3(1 + \tau_0 s^\alpha)\right\} \overline{e} = 0.$$

The above equation can be written as

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\overline{e} = 0, \tag{3.4}$$

where k_1^2 and k_2^2 are the roots with positive real parts of the characteristic equation

$$k^4 - k^2[s^2 + (1+\varepsilon)s(1+\tau_0 s^\alpha)] + s^3(1+\tau_0 s^\alpha) = 0.$$

The solution of equation (3.4) has the form

$$\overline{e} = \overline{e}_1 + \overline{e}_2,$$

where, $\overline{e}_i$ is the solution of the following equation:

$$(\nabla^2 - k_i^2)\overline{e}_i = 0.$$

The above equation can be written as

$$r^2 \frac{\partial^2 \overline{e}_i}{\partial r^2} + r \frac{\partial \overline{e}_i}{\partial r} - k_i^2 r^2 \overline{e}_i = 0. \tag{3.5}$$

The solution of equation (3.5) has the general form:

$$e_i = M_i I_0(k_i r) + A_i K_0(k_i r),$$

where A_i and M_i, $i = 1, 2$ are parameters to be determined from the boundary conditions and $I_0(z)$, $K_0(z)$ are the modified Bessel functions of the first and second kinds of order 0, respectively.

Similarly, we can show that

$$\theta_i = N_i I_0(k_i r) + \mathrm{B}_i K_0(k_i r).$$

Note that $I_0(z)$ is not bounded as $z \to \infty$, since the medium extends to infinity, we must set N_i and M_i equal to zero. We thus obtain

$$\overline{\theta} = \sum_{i=1}^{2} \mathrm{B}_i K_0(k_i r), \tag{3.6}$$

$$\overline{e} = \sum_{i=1}^{2} \mathrm{A}_i K_0(k_i r). \tag{3.7}$$

From equation (3.1), the solution can be written as

$$\overline{\theta} = \sum_{i=1}^{2} C_i (k_i^2 - s^2) K_0(k_i r), \tag{3.8}$$

$$\overline{e} = \sum_{i=1}^{2} \mathrm{C}_i k_i^2 K_0(k_i r). \tag{3.9}$$

The cubical dilatation e is given by

$$\overline{e} = \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u}) = \sum_{i=1}^{2} \mathrm{C}_i k_i^2 K_0(k_i r).$$

By integrating both sides of the above equation, we get

$$\overline{u} = -\sum_{i=1}^{2} \mathrm{C}_i k_i K_1(k_i r). \tag{3.10}$$

Substituting from equation (3.10) into equation (2.6), and using the following relation of the modified Bessel functions [26]:

$$\frac{dK_1(kr)}{dr} = -kK_0(kr) - \frac{K_1(kr)}{r}$$

we obtain

$$\overline{\sigma}_{rr} = \sum_{i=1}^{2} \mathrm{C}_i \left\{ \beta^2 s^2 K_0(k_i r) + \frac{2k_i}{r} K_1(k_i r) \right\}. \tag{3.11}$$

Applying the boundary conditions (2.7) and (2.8), we get the two equations

$$C_1(k_1^2 - s^2)K_0(k_1 r) + C_2(k_2^2 - s^2)K_0(k_2 r) = \overline{f}(s),$$

$$\mathrm{C}_1 \left[\beta^2 s^2 K_0(k_1 a) + \frac{2k_1}{a} K_1(k_1 a)\right] + \mathrm{C}_2 \left[\beta^2 s^2 K_0\,(k_2 a) + \frac{2k_2}{a} K_1(k_2 a)\right] = 0.$$

Solving the above equations, we obtain

$$\mathrm{C}_1 = \frac{\left[\beta^2 s^2 K_0(k_2 a) + \frac{2k_2}{a} K_1(k_2 a)\right]\overline{f}(s)}{\varGamma}, \tag{3.12}$$

$$\mathrm{C}_2 = \frac{-\left[\beta^2 s^2 K_0(k_1 a) + \frac{2k_1}{a} K_1(k_1 a)\right]\overline{f}(s)}{\varGamma}, \tag{3.13}$$

where

$$\begin{aligned}\varGamma = (k_1^2 - s^2)K_0(k_1 a)\left[\beta^2 s^2 K_0(k_2 a) + \frac{2k_2}{a} K_1(k_2 a)\right] \\ - (k_2^2 - s^2)K_0(k_2 a)\left[\beta^2 s^2 K_0(k_1 a) + \frac{2k_1}{a} K_1(k_1 a)\right].\end{aligned} \tag{3.14}$$

4. Inversion of the Laplace transform

We shall now outline the numerical inversion method used to find the solution in the physical domain. Let $\overline{f}(s)$ be the Laplace transform of a function $f(t)$. The inversion formula for Laplace transform can be written as [27]

$$f(t) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} e^{st} \overline{f}(s) ds,$$

where c is an arbitrary real number greater than all the real parts of the singularities of $\overline{f}(s)$. Taking $s = c + iy$, the above integral takes the form

$$f(t) = \frac{e^{ct}}{2\pi} \int\limits_{-\infty}^{\infty} e^{iyt} \overline{f}(c + iy) dy.$$

Expanding the function $h(t) = \exp(-ct) f(t)$ in a Fourier series in the interval $[0, 2T]$, we obtain the approximate formula [27]

$$f(t) = f_\infty(t) + E_D,$$

where

$$f_\infty(t) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k, \qquad 0 \le t \le 2T, \tag{4.1}$$

and

$$c_k = \frac{e^{ct}}{T} \mathrm{Re}[e^{ik\pi t/T} \overline{f}(c + ik\pi/T)]. \tag{4.2}$$

The discretization error E_D can be made arbitrarily small by choosing c large enough [27]. As the infinite series in (4.1) can only be summed up to a finite number N of terms, the approximate value of $f(t)$ becomes

$$f_N(t) = \frac{c_0}{2} + \sum_{k=1}^{N} c_k, \qquad 0 \le t \le 2T. \tag{4.3}$$

Using above formula to evaluate $f(t)$, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the "Korrecktur" method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and therefore to accelerate convergence.

The "Korrecktur"-method uses the following formula to evaluate the function $f(t)$:

$$f(t) = f_\infty(t) - e^{-2cT} f_\infty(2T + t) + E'_D,$$

where the discretization errors $|E'_D| \ll |E_D|$ [27]. Thus, the approximate value of $f(t)$ becomes

$$f_{NK}(t) = f_N(t) - e^{-2cT} f_{N'}(2T + t), \tag{4.4}$$

N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in (4.1). Let N be an odd natural number and let

$$s_m = \sum_{k=1}^{m} c_k,$$

be the sequence of partial sums of (4.1). We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m, \qquad m = 1, 2, 3, \ldots,$$

and

$$\varepsilon_{n+1,m} = \varepsilon_{n-1,m+1} + 1/(\varepsilon_{n,m+1} - \varepsilon_{n,m}), \qquad n, m = 1, 2, 3, \ldots.$$

It can be shown that [27] the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \ldots, \varepsilon_{N,1}$ converges to $f(t) + E_D - c_0/2$ faster than the sequence of partial sums s_m, $m = 1, 2, 3, \ldots$.

The actual procedure used to invert the Laplace transform consists of using equation (4.3) together with the ε-algorithm. The values of c and T are chosen according to the criteria outlined in [27].

5. Numerical results and discussion

The copper material was chosen for purposes of numerical evaluations. The constants of the problem are shown in Table 1.

Table 1.

$k = 386$ W/(m · K)	$\alpha_t = 1.78 \cdot 10^{-5}$ K^{-1}	$c_E = 381$ J/(kg · K)	$\eta = 8886.73$
$\mu = 3.86 \cdot 10^{10}$ kg/(m · s^2)	$\lambda = 7.76 \cdot 10^{10}$ kg/(m · s^2)	$\rho = 8954$ kg/m^3	$T_0 = 293$ K
$\varepsilon = 0.0168$	$\tau_0 = 0.025$ s		

The computations were carried out for a function $f(t)$ given by

$$f(t) = H(t),$$

which gives $\overline{f}(s) = 1/s$.

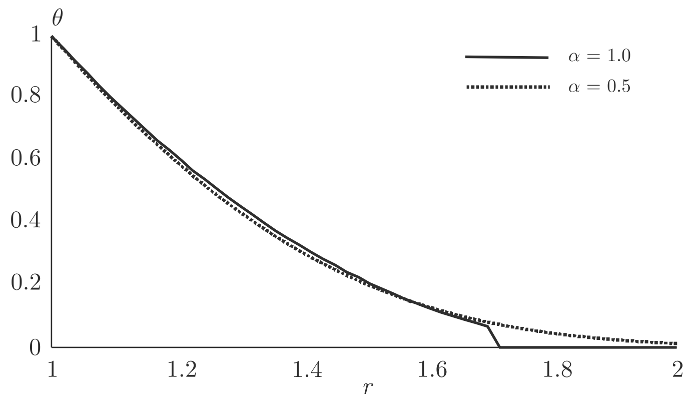

Fig. 1. Temperature distrubution for $t = 0.05$.

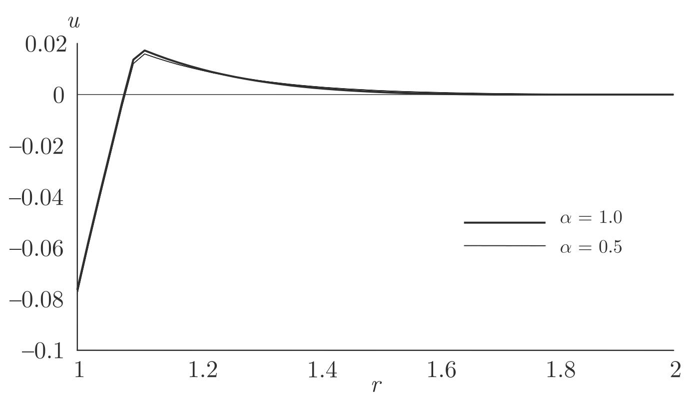

Fig. 2. Displacement distrubution for $t = 0.05$.

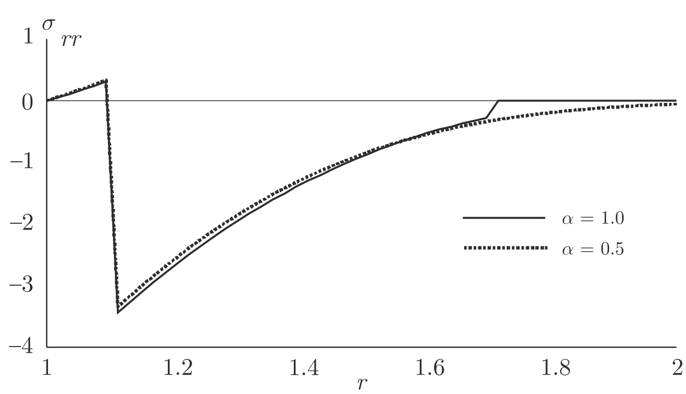

Fig. 3. Stress distrubution for $t = 0.05$.

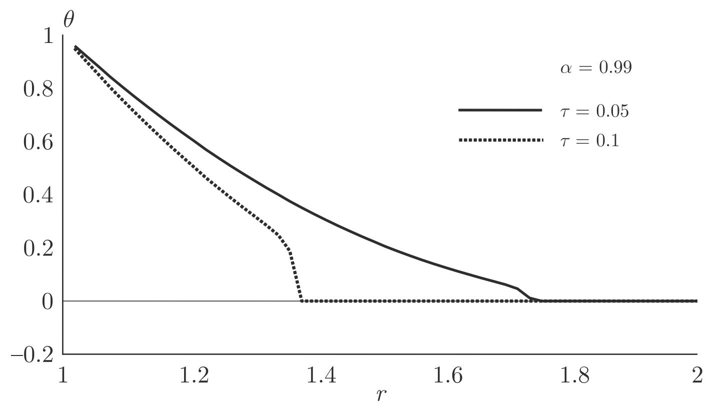

FIG. 4. Temperature distrubution for $\alpha = 0.99$.

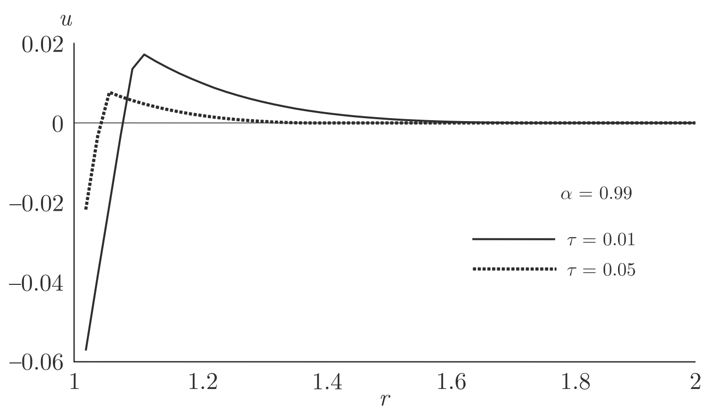

FIG. 5. Displacement distrubution for $\alpha = 0.99$.

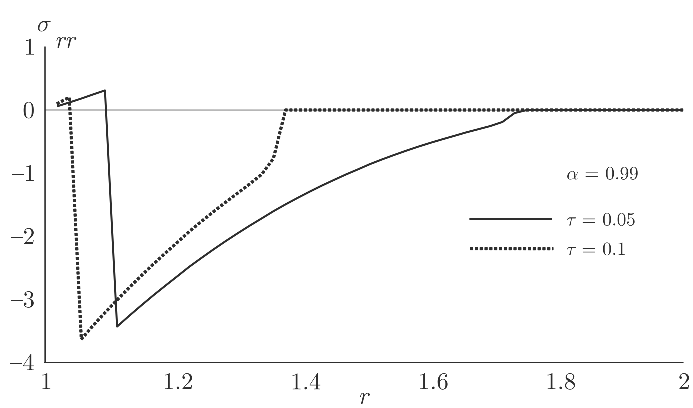

FIG. 6. Stress distrubution for $\alpha = 0.99$.

The computations were carried out for one value of time, namely $t = 0.05$ and two values of α, namely $\alpha = 0.5$ and $\alpha = 1$. The temperature displacement and stress distributions are obtained and plotted, as shown in Figs. 1, 2 and 3, respectively.

Next, the computations were carried out for one value of α, namely for $\alpha = 0.99$, and two values of time, namely $t = 0.05$ and $t = 0.1$. The temperature, displacement and stress distributions are obtained and plotted as shown in Figs. 4, 5 and 6, respectively..

For the pervious steps, the FORTRAN programming language was used on a personal computer. The accuracy maintained was five digits for the numerical program.

The computations show that:

For $\alpha = 0.5$ the solution behaves like the coupled theory of thermoelasticity where the velocity of the wave is infinite, but for $\alpha = 1$ the solution becomes that of the generalized theory of thermoelasticity.

For $\alpha = 0.99$ it is difficult to say whether the solution has a jump at the wave front or it is continuous with very fast changes. This aspect needs further investigation [20].

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Received December 10, 2013; revised version April 14, 2014.