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An approximate analytic solution for isentropic flow by an inviscid gas model

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The aim of the present analysis is to apply the modified decomposition method (MDM) for the solution of isentropic flow of an inviscid gas model (IFIG). The modification form based on a new formula of Adomian's polynomials (APs). The new approach provides the solution in the form of a rapidly convergent series with easily computable components and not at grid points. The proof of convergence of MDM applied to such systems is introduced with a bound of the error. Using suitable initial values, the solution of the system has been calculated and represented graphically. An analytic continuous solution with high accuracy was obtained.

Key words: decomposition method, conservation laws, isentropic flow of inviscid gas, convergence, error analysis.

1. Introduction

The Adomian decomposition method (ADM) [1, 2] has been used to give analytic approximation for a large class of linear and nonlinear functional equations, including algebraic equations, differential equations, integral equations and integro-differential equations [3–6]. For nonlinear models, such as systems of conservation laws, the method has shown reliable results in supplying analytical approximation that converges rapidly [7–9]. Recently, a modification of the ADM [10] based on a new formula for APs was proposed. In this paper, we apply this modification to the model of IFIG.

The system of balance laws (conservation laws with source terms) given by

$$(1.1)\qquad \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} h_1(x,t) \\ h_2(x,t) \end{pmatrix}, \qquad (x,t)\in(\alpha,\beta)\times[0,T],$$

with initial condition

$$(1.2)\qquad \begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \qquad x\in(\alpha,\beta).$$

The functions $f(u,v)$ and $g(u,v)$ are the fluxes, which are assumed to be analytic functions, $h_1(x,t)$ and $h_2(x,t)$ are the source terms. The system (1.1) can be written in a more convenient and compact form by introducing the notations $U = (\,u\ v\,)^{\mathrm{T}}$, $F = (\,f\ g\,)^{\mathrm{T}}$ and $H = (\,h_1\ h_2\,)^{\mathrm{T}}$, the initial value problem (1.1)–(1.2) attains the form

$$U_t(x,t) + F(U(x,t))_x = H(x,t), \qquad U(x,0) = U_0(x). \tag{1.3}$$

The Jacobian matrix of $F(U)$ is

$$dF = \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}, \tag{1.4}$$

where f_u means $\partial f/\partial u$ and f_v means $\partial f/\partial v$ and so on. The system in (1.3) is said to be strictly hyperbolic in a domain $\Omega \subset \mathbb{R}^2$, if for every $(x,t) \in \Omega$ and for all possible solutions u and v of the system in Ω, the matrix (1.4) has two real distinct eigenvalues $\lambda_1(x,t,u,v)$, $\lambda_2(x,t,u,v)$. If there is a region $E \subset \mathbb{R}^2$ where the 2×2 matrix in (1.4) has no real eigenvalues the system is called elliptic in E. If neither Ω nor E is empty, then the system is of mixed type.

2. Procedure of the method

To illustrate the MDM method [10], we consider the following system in operator form:

$$L(U) + N(U) = H(x,t), \tag{2.1}$$

where $L = \partial/\partial t$ is a linear partial differential operator, N is a nonlinear analytic operator and $H(x,t)$ is bounded function on $(\alpha,\beta)\times[0,T]$. Operating the inverse operator $L^{-1} = \int_0^t (\cdot)d\tau$ on both sides of Eq. (2.1) yields

$$U(x,t) = U_0(x,t) + L^{-1}(H(x,t)) - L^{-1}(N(U)).$$

According to MDM, $U(x,t)$ is considered as the sum of series

$$U(x,t) = \sum_{k=0}^{\infty} U_k(x,t) = \left(\sum_{k=0}^{\infty} u_k(x,t),\ \ \sum_{k=0}^{\infty} v_k(x,t)\right)^{\mathrm{T}}, \tag{2.2}$$

and $N(U)$ as a sum of the series

$$\begin{aligned} N(U) &= (f(u,v), g(u,v))^{\mathrm{T}} \\ &= \sum_{k=0}^{\infty} P_k(x,t) = \left(\sum_{k=0}^{\infty} A_k(x,t), \sum_{k=0}^{\infty} B_k(x,t)\right)^{\mathrm{T}}, \end{aligned} \tag{2.3}$$

where A_k's and B_k are the APs. The APs are not unique and they can be generated from Taylor series expansion of nth-order partial differentiable function N about the first component U_0. To obtain this formula, using the partial sum

$$R_k = \Big(S_k, T_k\Big)^{\mathrm{T}} = N\Big(\sum_{i=0}^{k} U_i\Big) = \Big(f\Big(\sum_{i=0}^{k} u_i, \sum_{i=0}^{k} v_i\Big), g\Big(\sum_{k=0}^{n} u_k, \sum_{k=0}^{n} v_k\Big)\Big)^{\mathrm{T}}. \tag{2.4}$$

P_ks are defined as follows:

$$P_k = N(R_k) - \sum_{i=0}^{k-1} P_i = \Big(f(S_k, R_k) - \sum_{i=0}^{k-1} A_i, g(S_k, R_k) - \sum_{i=0}^{k-1} B_i\Big)^{\mathrm{T}}. \tag{2.5}$$

The method consists of the following scheme:

$$\begin{cases} U_0 = U(x,0) + L^{-1}H, \\ U_{k+1} = L^{-1}P_k, \quad k \geq 0. \end{cases} \tag{2.6}$$

We construct the solution $U(x,t)$ as

$$U(x,t) = (u(x,t), v(x,t))^{\mathrm{T}} = \lim_{k\to\infty} \Phi_k(x,t) = \lim_{k\to\infty} (\varphi_k(x,t), \phi_k(x,t))^{\mathrm{T}}, \tag{2.7}$$

where

$$\Phi_k = (\varphi_k, \phi_k)^{\mathrm{T}} = \Big(\sum_{i=0}^{k} u_k, \sum_{i=0}^{k} v_k\Big)^{\mathrm{T}}. \tag{2.8}$$

3. Convergence of solution

The convergence of the decomposition series has been investigated by several authors [11–14] analytically and numerically in [3–9, 15–17] and references therein. In this section, we study the convergence analysis presented in [10] applied to the general 2×2 nonlinear systems of partial differential equations (2.1). In [10], the authors have given new conditions for obtaining convergence of the decomposition series using fixed point theorem [18].

Let us consider the Hilbert space H, defined by $H = L^2((\alpha,\beta)\times[0,T])$ with the set of applications:

$$u : H \to \mathbb{R} \qquad \text{with} \int u^2 dx < \infty,$$

and the norm

$$\|u\|_H^2 = \int\limits_{(\alpha,\beta)\times[0,T]} u^2(\sigma,\tau)d\sigma d\tau.$$

Define $H^2 = H \oplus H$ by the direct sum of Hilbert spaces H and H, which is also Hilbert space with the set of applications

$$U = (u,v)^{\mathrm{T}} : ((\alpha,\beta)\times[0,T])^2 \to \mathbb{R}^2 \qquad \text{with} \int\limits_{(\alpha,\beta)\times[0,T]} (u^2+v^2)\, d\sigma d\tau < \infty,$$

and the norm

$$\begin{aligned} \|U\|^2 =: \|U\|_{H^2}^2 &= \int\limits_{(\alpha,\beta)\times[0,T]} (u^2(\sigma,\tau)+v^2(\sigma,\tau))d\sigma d\tau \\ &= \|u\|_H^2 + \|v\|_H^2. \end{aligned} \tag{3.1}$$

Suppose that N is Lipschitzian operator with Lipschitz constant C. The series solution (2.7) converges to the exact solution $U(x,t)$ if $\|U_1\|_{H^2} < \infty$, and if $\exists\delta \geq 0$, $\delta = CT$, such that $\delta < 1$. To verify the convergence proof: let $U \in H^2$, define the operator $E(U) : H^2 \to H^2$ by

$$E(U) = U_0(x,t) + L^{-1}H(x,t) - L^{-1}N(U). \tag{3.2}$$

Using arbitrary $U_1, U_2 \in H^2$, we have

$$\begin{aligned} \|E(U_1)-E(U_2)\| &= \|L^{-1}(N(U_1)-N(U_2))\| \leq L^{-1}\|(N(U_1)-N(U_2))\| \\ &\leq T\|(N(U_1)-N(U_2))\|. \end{aligned}$$

But, N is Lipchitzian with Lipchitz constant C, thus

$$\|E(U_1)-E(U_2)\| \leq CT\|U_1-U_2\| = \delta\|U_1-U_2\|$$

implies that E is a contraction operator. Hence, there is a unique solution of the problem (2.1). The sequence R_k in (2.4) is Cauchy in the Hilbert space H^2 since

$$\begin{aligned} \|R_k - R_m\| &= \left\| \sum_{i=m+1}^{k} U_i \right\| \leq L^{-1}\| \sum_{i=m+1}^{k} P_i \| \\ &= L^{-1}\|N(R_{k-1}) - N(R_{m-1})\| \\ &\leq TC\|R_{k-1} - R_{m-1}\| = \delta\|R_{k-1} - R_{m-1}\|, \end{aligned}$$

the triangle inequality was used, and since $\delta < 1$, we get

$$\|R_k - R_m\| \leq \frac{\delta^m}{1-\delta}\|R_1 - R_0\| = \frac{\delta^m}{1-\delta}\|U_1\|.$$

Now, to show that the convergent series solution (2.7) is itself the existing unique solution of (2.1), we have

$$\begin{aligned} L^{-1}N(U) &= L^{-1}\left(N\left(\sum_{i=0}^{\infty}U_i\right)\right) = L^{-1}\left(N(\lim_{k\to\infty}\sum_{i=0}^{k}U_i)\right) \\ &= L^{-1}(\lim_{k\to\infty}N(R_k)) = \lim_{k\to\infty}L^{-1}(N(R_k)). \end{aligned}$$

Moreover, if the truncated series (2.8) is used as an approximation to the exact solution $U(x,t)$, then the error bound can be found using

$$\left\| U(x,t) - \sum_{i=0}^{k} U_i(x,t) \right\|_H \leq \frac{\delta^k}{1-\delta}\|U_1(x,t)\|_H. \tag{3.3}$$

With the above notations, we have proved the following result:

THEOREM 1. *The series solution obtained by MDM applied to the nonlinear partial differential system*

$$L(U) + N(U) = H(x,t), \quad (x,t) \in (\alpha,\beta)\times[0,T],$$

converges to the exact solution $U(x,t) \in H^2$ if for Lipchitzian operator N, with Lipchitz constant C, there is $\delta = CT$ such that $\delta < 1$ and $\|U_1\| < \infty$.

In summary, by defining the convergence rate $\alpha_i = \|U_{i+1}\|_{H^2}/\|U_i\|_{H^2}$ if $\|U_i\|_{H^2} \neq 0$, and 0 if $\|U_i\|_{H^2} = 0$, Theorem 1 ensures the convergence of truncated series (2.8) to exact solution if $\alpha_i < 1$ for $i = 0, 1, \ldots, k$.

4. Solution of IFIG model

For purposes of illustration of the MDM for solving the IFIG equations, the computer application program "Mathematica" was used to execute the algorithms that were used with the numerical example.

EXAMPLE 1. The IFIG model [19] is an example of system of conservation laws of the form

$$\begin{aligned} u_t + uu_x + c^2v^{-1}v_x &= 0, \\ v_t + uv_x + vu_x &= 0, \end{aligned} \tag{4.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are the velocity and density of the gas respectively, and c is the local speed of sound. Given $c^2 = c^2(v) = a\gamma v^{\gamma-1}$ for a perfect gas where a is constant and the ratio of specific heats $\gamma = 1.4$ for air, we have

$$\frac{c^2}{v} = a\gamma v^{0.4}v^{-1} = 1.4av^{-0.6}. \tag{4.2}$$

This can be written as

$$u_t + \left(\frac{1}{2}u^2 + bv^{0.4}\right)_x = 0, \qquad v_t + (uv)_x = 0, \tag{4.3}$$

with $b = 3.5a$. The system is hyperbolic for $a > 0$ (equivalently for $b > 0$), and elliptic for $a < 0$, but never to be of mixed type.

Following the MDM analysis mentioned in Section 2, the system (4.3) in an operator form is

$$\begin{aligned} Lu + f(u, v)_x &= 0, \\ Lv + g(u, v)_x &= 0, \end{aligned} \tag{4.4}$$

where, $f(u, v) = 0.5u^2 + bv^{0.4}$ and $g(u, v) = uv$. Applying the inverse operator $L^{-1} = \int_0^t (\cdot)d\tau$ to the coupled equations (4.4) gives

$$\begin{aligned} u &= u(x, 0) - L^{-1}[f(u, v)_x], \\ v &= v(x, 0) - L^{-1}[g(u, v)_x]. \end{aligned} \tag{4.5}$$

Simply, take the equation in the compact form

$$U = U(x, 0) - L^{-1}[F(u, v)_x]. \tag{4.6}$$

Consider the system in (4.3) with $b = 1$ subject to the initial conditions

$$U(x, 0) = (x, 1)^{\mathrm{T}}, \qquad x \in [0, 3]. \tag{4.7}$$

We solve this model using our approach by finding the first 10 terms of the series (2.6). We give here the first few APs for $F(U)$ using formula (2.5) as

$$\begin{aligned} P_0 &= F(U_0) = (f(U_0), g(U_0))^{\mathrm{T}}, \\ P_1 &= F(R_1) - P_0, \\ P_2 &= F(R_2) - (P_0 + P_1), \\ P_3 &= F(R_3) - (P_0 + P_1 + P_2), \\ P_4 &= F(R_4) - (P_0 + P_1 + P_2 + P_3), \\ &\vdots \end{aligned}$$

and so on. For $t = 0$ the zeroth component U_0 is given by

$$U_0(x, t) = (x, 1)^{\mathrm{T}}.$$

We obtained in succession U_1, U_2, U_3, ..., U_{10} according to MDM scheme (2.6) to determine the individual terms of the series solution (2.8) as

$$U_1(x,t) = -\frac{1}{4}tU_0(x,t), \qquad U_2(x,t) = \frac{t^2}{8}\left(1-\frac{t}{6}\right)U_0(x,t), \quad \ldots.$$

Our approximate solution is given by

$$\Phi_{10} = \sum_{i=0}^{10} U_i(x,t). \tag{4.8}$$

Given $\|U_1\|_{H^2} = 0.5 < \infty$, and taking into account convergence rates α_ks, we have

$$\begin{aligned} \alpha_0 &= 0.288675 < 1, \\ \alpha_1 &= 0.333631 < 1, \\ \alpha_2 &= 0.211644 < 1, \\ \alpha_3 &= 0.179207 < 1, \\ \alpha_4 &= 0.144667 < 1, \\ &\vdots \end{aligned}$$

Choosing $\alpha = \max_i \alpha_i = \alpha_1$, Theorem 1 implies that the approximate solution (4.8) of the IFIG equations using MDM converges to unique solution. In general, decomposition series (2.8) converges rapidly in real physical problems. Figure 1 represents the approximate velocity and density of the gas for $(x,t) \in [0,3] \times [0,1]$.

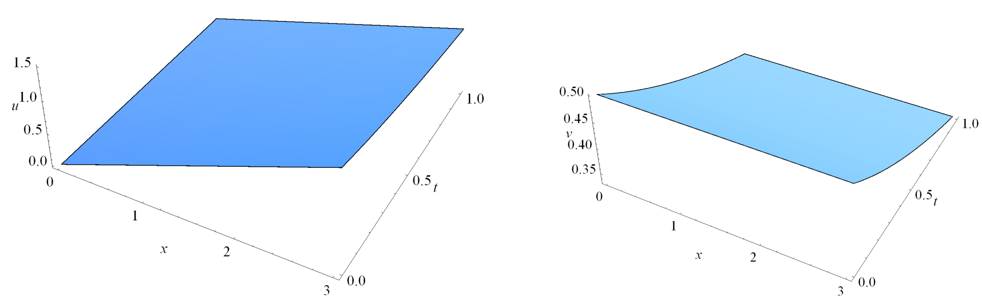

Fig. 1. The behavior of: a) the velocity of gas; b) the density of gas.

In this example, we cannot determine the error in comparison with exact solution since we do not know this solution, but an error bound can be determined using formula (3.3) to be

$$\left\| U(x,t) - \sum_{i=0}^{10} U_i(x,t) \right\|_H \leq 2.41145 \times 10^{-8}.$$

Moreover, substituting the obtained approximate solution (4.8) in the system (4.3) and comparing results to the zeros right-hand sides, Table 1 shows the absolute errors for $x \in \{1, 2, 3\}$ and $t = 0, 0.1, 0.2, \ldots, 1$ rounding to three significant digits. In addition, Figure 2 shows the errors in approximating the velocity and density of the gas for $(x,t) \in [0,3] \times [0,1]$.

Table 1. The absolute error for approximate values of the velocity and density.

t	$u(x,t)$			$v(x,t)$		
	x					
	1	2	3	1	2	3
0.1	2.78×10^{-17}	5.55×10^{-17}	0	5.55×10^{-17}	5.55×10^{-17}	5.55×10^{-17}
0.2	4.05×10^{-15}	8.10×10^{-15}	1.24×10^{-14}	4.14×10^{-15}	4.14×10^{-15}	4.14×10^{-15}
0.3	1.79×10^{-13}	3.57×10^{-13}	5.36×10^{-13}	1.79×10^{-13}	1.79×10^{-13}	1.79×10^{-13}
0.4	2.46×10^{-12}	4.92×10^{-12}	7.39×10^{-12}	2.46×10^{-12}	2.46×10^{-12}	2.46×10^{-12}
0.5	1.79×10^{-11}	3.59×10^{-11}	5.38×10^{-11}	1.79×10^{-11}	1.79×10^{-11}	1.79×10^{-11}
0.6	8.76×10^{-11}	1.75×10^{-10}	2.63×10^{-10}	8.76×10^{-11}	8.76×10^{-11}	8.76×10^{-11}
0.7	3.25×10^{-10}	6.51×10^{-10}	9.76×10^{-10}	3.25×10^{-10}	3.25×10^{-10}	3.25×10^{-10}
0.8	9.90×10^{-10}	1.98×10^{-9}	2.97×10^{-9}	9.90×10^{-10}	9.90×10^{-10}	9.90×10^{-10}
0.9	2.59×10^{-9}	5.18×10^{-9}	7.78×10^{-9}	2.59×10^{-9}	2.59×10^{-9}	2.59×10^{-9}
1.0	6.03×10^{-9}	1.21×10^{-8}	1.81×10^{-8}	6.03×10^{-9}	6.03×10^{-9}	6.03×10^{-9}

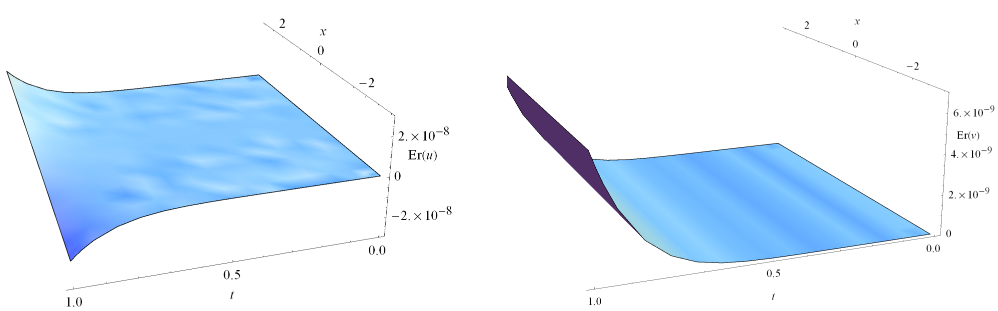

Fig. 2. The error of: a) velocity of gas; b) the density of gas.

5. Conclusions

In this paper, we consider non-linear coupled isentropic flow of inviscid gas equations (IFIG) for finding an analytic solution via modified decomposition method (MDM). In this example, a high accuracy analytic approximate solution was obtained. It may be concluded that the MDM methodology is very powerful and efficient technique in finding analytical solutions for wide classes of problems and can be easily extended to other non-linear evaluation equations, with the aid of *Mathematica.*

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